On symmetry properties of parabolic equations in bounded domains

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Abstract

We study symmetry properties of non-negative bounded solutions of fully nonlinear parabolic equations on bounded domains with Dirichlet boundary conditions. We propose sufficient conditions on the equation and domain, which guarantee asymptotic symmetry of solutions.

Keywords: Asymptotic symmetry, positive solutions, parabolic equations, moving hyperplanes

Contents

1 Introduction

In this paper we consider a fully nonlinear parabolic problem of the form

$$\left.\begin{aligned} \partial_t u = F(t,x,u,Du,D^2u), \qquad & (x,t) \in \Omega \times (0,\infty)\,, \\ u = 0, \qquad & (x,t) \in \partial\Omega \times (0,\infty)\,, \\ u \geq 0, \qquad & (x,t) \in \Omega \times (0,\infty)\,. \end{aligned}\right\} \tag{1.1}$$

Here, Dg and D^2g denote the gradient and Hess matrix of a function g. We assume that

(d1) $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain, convex in x_1, and symmetric with respect to the hyperplane

$$H_0 := \{x = (x_1, \cdots, x_N) : x_1 = 0\}\,.$$

The non-linearity F satisfies regularity, ellipticity and symmetry conditions (N1)–(N3) specified below. Our goal is to investigate symmetry and monotonicity properties of global solutions u as $t \to \infty$.

The first symmetry results for positive solutions of elliptic equations date back to the celebrated paper of Gidas, Ni, Nirenberg [13]. They showed that if u is a positive classical solution of the problem

$$\begin{aligned} \Delta u = f(u), \qquad & x \in \Omega\,, \\ u = 0, \qquad & x \in \partial\Omega\,, \end{aligned} \tag{1.2}$$

with a smooth domain Ω satisfying (d1) and a Lipschitz function f, then u is even in x_1 and $\partial_{x_1} u < 0$ in

$$\Omega_0 := \{x = (x_1, \cdots, x_N) \in \Omega : x_1 > 0\}\,.$$

The two main mathematical tools used in the proof were the maximum principle and the method of moving hyperplanes introduced by Alexandrov [1] and later developed by Serrin [24]. These results were further generalized to the fully nonlinear case by Li [19], to problems on non-smooth domains by Berestycki and Nirenberg [6] and Dancer [10]. Extensions in various directions including degenerate problems, problems on unbounded domains or cooperative systems of equations were done by many authors, see the surveys [5, 16, 21, 22].

The situation for parabolic problems is more complicated, since one cannot expect solutions to be symmetric, if the initial condition is not symmetric. However, it is possible that solutions 'symmetrize' as time approaches infinity, regardless of initial data. More precisely, we say that u is *asymptotically symmetric* if all functions in the ω-limit set of u:

$$\omega(u) := \{z : z = \lim_{k\to\infty} u(\cdot, t_k), \text{for some}\, t_k \to \infty\} \tag{1.3}$$

are even in x_1 and nonincreasing in Ω_0. The limit in (1.3) is in the supremum norm.

The first asymptotic symmetry results for parabolic problems appeared in [15], where Hess and Poláčik established asymptotic symmetry for classical, bounded, positive solutions of the problem

$$\begin{aligned} u_t - \Delta u = f(t,u), \qquad & (x,t) \in \Omega \times (0,\infty)\,, \\ u = 0, \qquad & (x,t) \in \partial\Omega \times (0,\infty)\,, \end{aligned} \tag{1.4}$$

where f is Hölder continuous in t and Lipschitz in u, and Ω is a smooth domain satisfying (d1). In addition, it was assumed that

$$\nu_1(x) > 0 \qquad (x = (x_1, x') \in \partial\Omega, x_1 > 0)\,, \tag{1.5}$$

where $(\nu_1(x), \nu'(x)) = \nu(x)$ is the exterior unit normal vector to $\partial\Omega$ at x. This geometric condition does not appear in the elliptic case but is essential in the parabolic one, as discussed below.

Independently to [15], Babin [2, 3] and later Babin and Sell [4] proved asymptotic symmetry of classical solutions of (1.1) where Ω satisfies (d1), F satisfies (N1)–(N3), and $F(t,x,0,0,0) \geq 0$ for all $(x,t) \in \Omega \times (0,\infty)$. In addition, it was assumed that the positive semiorbit $\phi^+(u) := \{u(\cdot,t) : t \in (0,\infty)\}$ is relatively compact in $C^{2,1}_{loc}(\Omega \times (0,\infty))$ and the solution is bounded away from 0 on compact subsets of Ω. The additional assumptions on F and u were removed in [23], where Poláčik showed that the classical bounded solution of (1.1), with Ω satisfying (d1) and F satisfying (N1)–(N3), is asymptotically symmetric if and only if either $\omega = \{0\}$ or there is $\phi \in \omega(u)$ with $\phi > 0$ in Ω. He also proposed two explicit sufficient conditions for asymptotic symmetry - Ω being a ball or

$$\liminf_{t\to\infty} F(x,t,0,0,0) \geq 0 \qquad (x \in \Omega)\,. \tag{1.6}$$

We remark that [23] contains an example (see [23, Example 2.3]) of a semilinear parabolic problem with a smooth nonlinearity and Ω being a rectangle, for which the asymptotic symmetry of solutions fails. Observe that the rectangle does not satisfy (1.5).

For further extensions including parabolic problems on unbounded domains, asymptotically symmetric equations, cooperative systems of equations, entire solutions, etc., we refer the reader to the survey [22].

In this paper we propose another explicit sufficient condition that guarantee asymptotic symmetry of solutions. To illustrate the results on a model problem, assume that Ω is a Lipschitz domain satisfying (d1) and:

(d2) For any $\delta^* > 0$ there is $\varepsilon > 0$ and a unit vector $v \in \mathbb{R}^N \setminus \{e_1\}$ such that

$$\mathrm{Cone}_{x,\varepsilon}(e_1, v) \subset \bar{\Omega} \qquad (x \in \partial\Omega, x_1 \geq \delta^*)\,.$$

Here, $\mathrm{Cone}_{x,\varepsilon}(r, s)$ be the part of the cone spanned by $-r, -s$ with the tip at x, which lies inside the ball of radius ε centered at x:

$$\mathrm{Cone}_{x,\varepsilon}(r, s) := \{y \in \mathbb{R}^N : x-y = \alpha r+\beta s, \alpha, \beta \geq 0, |x-y| \leq \varepsilon\}\,. \quad (1.7)$$

Let $f : (0, \infty) \times [0, \infty) \to \mathbb{R}$ be a continuous function such that

(f1) $f : (t, u) \mapsto f(t, u)$ is Lipschitz continuous in u uniformly with respect to t, meaning that there is $\beta_0 > 0$ such that

$$\sup_{t>0} |f(t, u) - f(t, \bar{u})| \leq \beta_0 |u - \bar{u}| \qquad (u, \bar{u} \in [0, \infty))\,.$$

(f2) $f(\cdot, 0)$ is a bounded function.

As a result we obtain.

Theorem 1.1. *If a Lipschitz domain Ω satisfies (d1), (d2), a function f satisfies (f1), (f2), and u is a global, nonnegative, bounded, classical solution of* (1.4)*, then u is asymptotically symmetric, that is, for each $z \in \omega(u)$*

$$z(x_1, x') = z(-x_1, x') \qquad ((x_1, x') \in \Omega),$$

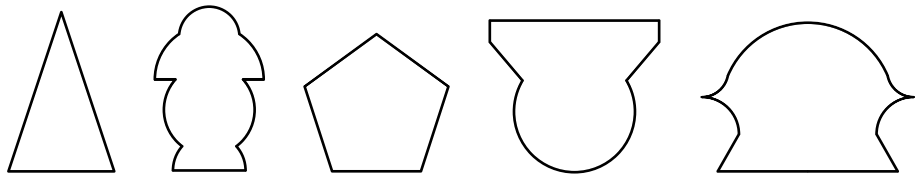
and either $z \equiv 0$ or z is strictly decreasing in Ω_0.

Examples of Lipschitz domains that satisfy (d1) and (d2), include (see the figures),

- symmetric domains, which are strictly convex in x_1, that is,

$$\alpha(x_1, x') + (1-\alpha)(x_1, y') \in \Omega \qquad (\alpha \in (0,1), (x_1, x'), (x_1, y') \in \partial\Omega)\,.$$

- some symmetric domains, which are not strictly convex in x_1 such as isosceles triangles, pentagons, pyramids, upper half balls, and so on.



Notice that a rectangle (the mentioned counterexample provided in [23]) does not satisfy (d2), but it is a 'borderline' case. Moreover, if Ω is a C^2 domain satisfying (d1), then (1.5) implies (d2). Hence, Theorem 1.1 is a generalization of results in [15].

The main contribution of our Theorem 1.1 and more general results in the next section, as compared to the results of [23], is that it gives a general, explicit, and easily verifiable condition, under which the asymptotic symmetry holds.

In the next section we extend Theorem 1.1 to fully nonlinear problems such as (1.1). That is, we formulate a sufficient condition for asymptotic symmetry only in terms of Ω and F. This condition covers a larger class of problems, compared to the explicit sufficient conditions from [23]. For example, if the F does not satisfy (1.6), then asymptotic symmetry of solutions of (1.1) was not discussed in [4], and [23] requires Ω to be a ball. For a general domain Ω, the asymptotic symmetry theorem of [23] applies only to solutions whose ω-limit set contains a positive function. We show that, if we in addition to (d1) and (N1)–(N3) assume (d2) and minor monotonicity assumptions on F, then the asymptotic symmetry holds.

As a by-product we obtain an improvement of the results in [8, 9, 11] on the question when a nonnegative, nontrivial solution of an elliptic problem is positive (cf. Corollary 2.4).

The method of the proof uses the framework introduced in [23], and we also use partial results of that paper. However, many arguments need refinements, extensions, or completely new approach. Some results or techniques might be of independent interest, for example the maximum principle on general, small, space-time domains.

The remainder of this paper is organized as follows. In Section 2 we formulate the assumptions and state the main results. Section 3 contains estimates for solutions of linear problems, geometric properties of non-smooth domains and we recall how the method of moving hyperplanes leads to linearization of nonlinear problems. In Sections 4 and 5 we give proofs of the symmetry results.

2 Main results

Let us introduce the following notation. Denote

$$H_\lambda := \{x = (x_1, x') \in \mathbb{R}^N : x_1 = \lambda\} \qquad (\lambda \in \mathbb{R})$$

and let $\mathcal{P}_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection in the hyperplane H_λ, that is, $\mathcal{P}_\lambda(x) = (2\lambda - x_1, x')$ for any $x = (x_1, x') \in \mathbb{R}^N$. Next, for any subset Ω of $\mathbb{R}^N$ define

$$\Omega_\lambda := \{x = (x_1, x') \in \Omega : x_1 > \lambda\}\,,$$

and

$$\ell := \sup\{x_1 : x \in \Omega\}\,.$$

Consider the problem (1.1) and assume the following hypotheses.

(D1) Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), such that $\Omega'_\lambda := \mathcal{P}_\lambda(\Omega_\lambda) \subset \Omega$ for all $\lambda \geq 0$.

(D2) For each $\lambda > 0$ the set Ω_λ has only finitely many connected components.

(D3) Ω is symmetric with respect to the hyperplane H_0.

Notice, that (D1) and (D3) are equivalent to (d1). We formulate them differently here to have a unified setting for directions other than e_1, where we cannot require Ω to be symmetric.

The hypothesis (D2) occurred already in [23] and it is still unknown if it is just technical or not. Based on the proofs in this paper it can be relaxed in several directions, although not completely removed. Observe that Lipschitz (and even Hölder) continuity of Ω implies (D2).

Let $\mathcal{T} \in \mathbb{R}^{N^2}$ be the matrix corresponding to $\mathcal{P}_0$, the reflection in the hyperplane H_0:

$$\mathcal{T}_{ij} := \delta_{ij} - 2\delta_{i1}\delta_{j1} \qquad (i, j = 1, \cdots, N)\,,$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

We identify the space of $N \times N$ matrices with $\mathbb{R}^{N^2}$. We then assume that the real valued function $F : (t, x, u, p, q) \mapsto \mathbb{R}$ is defined on $[0, \infty) \times \bar{\Omega} \times \mathcal{O}$, where $\mathcal{O}$ is an open convex subset of $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$ invariant under the transformation

$$Q : (u, p, q) \mapsto (u, p\mathcal{T}, \mathcal{T}q\mathcal{T}),$$

and it satisfies the following conditions.

(N1) *Regularity.* The function F is continuous, differentiable with respect to q and Lipschitz continuous in (u, p, q) uniformly with respect to $(x, t) \in \bar{\Omega} \times \mathbb{R}^+$. This means that there is $\beta_0 > 0$ such that

$$\sup_{x \in \Omega, t \geq 0} |F(t, x, u, p, q) - F(t, x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta_0 |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$
$$((x, t) \in \bar{\Omega} \times \mathbb{R}^+, \; (u, p, q), (\tilde{u}, \tilde{p}, \tilde{q}) \in \mathcal{O}).$$

(N2) *Ellipticity.* There is $\alpha_0 > 0$ such that for each $\xi \in \mathbb{R}^N$ and each $(t, x, u, p, q) \in [0, \infty) \times \bar{\Omega} \times \mathcal{O}$ one has

$$\frac{\partial F}{\partial q_{jk}}(t, x, u, p, q)\xi_j \xi_k \geq \alpha_0 |\xi|^2.$$

Here and also in the rest of this paper we use the summation convention, that is, when an index appears twice in a single term, then we are summing over all its possible values.

(N3) *Symmetry and monotonicity.* For each $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, and any $x = (x_1, x'), (\tilde{x}_1, x') \in \Omega$ with $\tilde{x}_1 > x_1 \geq 0$

$$F(t, \mathcal{T}x, Q(u, p, q)) = F(t, x, Q(u, p, q)) = F(t, x, u, p, q),$$
$$F(t, x_1, x', u, p, q) \geq F(t, \tilde{x}_1, x', u, p, q).$$

As an easy example or F that satisfies (N1) – (N3) is $F(t, x, u, p, q) = q_{ii} + f(t, u)$, where f satisfies (f1) and (f2).

As shown in [13], there are non-symmetric solutions of the problem (1.2), if f is merely Hölder continuous, thus we cannot relax (N1) in this direction. On the other hand, (N3) allows nontrivial generalizations. One can for example consider asymptotically symmetric or asymptotically monotone problems as in [12].

By a solution of (1.1) we mean a classical solution, that is, a function $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$, with $(u, Du, D^2u) \in \mathcal{O}$, which satisfies (1.1) everywhere. We only consider bounded global solutions, that is,

$$\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty\,, \tag{2.1}$$

such that the family of functions $\{u(\cdot,\cdot+s)\}_{s\geq 1}$, is equicontinuous on $\bar{\Omega}\times[0,1]$:

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},t,\bar{t}\in[0,1],\\ |t-\bar{t}|,|x-\bar{x}|<h\\ s\geq 1}} |u(x,t+s) - u(\bar{x},\bar{t}+s)| = 0\,. \tag{2.2}$$

Under these assumptions the positive semi-orbit $\{u(\cdot,t) : t \geq 0\}$ of u is relatively compact in the space $E := C(\bar{\Omega})$. Then its ω-limit set (for the definition see (1.3)) is nonempty and compact in E and

$$\lim_{t\to\infty} \mathrm{dist}_E(u(\cdot,t), \omega(u)) = 0\,, \tag{2.3}$$

where dist_E denotes the distance in E.

We remark, that by [23, Proposition 2.7], (2.2) follows from (2.1), if we in addition to (N1), (N2) assume Lipschitz continuity of Ω and boundedness of $F(t,x,0,0,0)$. Observe that, in this paper we do not discuss the existence of global solutions satisfying (2.1) and (2.2), but we rather investigate their properties once they exist.

We are ready to formulate our first main result.

Theorem 2.1. *Assume (D1) – (D3), (N1) – (N3) and let u be a nonnegative global solution of* (1.1) *satisfying* (2.1) *and* (2.2)*. Then there exists* $\lambda_0 \geq 0$ *such that for each* $z \in \omega(u)$ *the following is true:* z *is monotone nonincreasing in* x_1 *on* Ω_{λ_0} *and there is a connected component* U *of* Ω_{λ_0} *such that*

$$z(x_1,x') = z(2\lambda_0 - x_1, x') \qquad ((x_1,x') \in U)\,. \tag{2.4}$$

Moreover if $\lambda_0 > 0$ *there is* $\tilde{z} \in \omega(u)$ *and a connected component* $\tilde{U}$ *of* Ω_{λ_0} *such that*

$$\tilde{z}(x_1,x') = \tilde{z}(2\lambda_0 - x_1, x') \qquad ((x_1,x') \in \tilde{U})\,, \tag{2.5}$$

$$\tilde{z}(x) > 0 \qquad (x \in \tilde{U})\,. \tag{2.6}$$

If Ω_{λ_0} is connected then for each $z \in \omega(u)$ either $z \equiv 0$ in Ω_{λ_0} or z is strictly decreasing in x_1 in Ω_{λ_0}. The latter holds in the form $z_{x_1} < 0$ if $z_{x_1} \in C(\Omega_{\lambda_0})$ for some $z \in \omega(u)$.

This theorem is an improvement of [23, Theorem 2.4], as it gives more precise characterization (property (2.6)) of the function $\tilde{z}$, if $\lambda_0 > 0$. Property (2.6) is important in the proof of the next theorem.

Next we turn our attention to the question when λ_0 from the previous theorem is equal to 0, that is, when the solution is asymptotically symmetric. This is not always the case, even if u is strictly positive, as one can construct examples similar to [23, Example 2.2], for which $\lambda_0 = \frac{n-1}{n}\ell$ with $n \in \mathbb{N}$, $n \leq n_0$ and n_0 depends on $\alpha_0, \beta_0, N, \operatorname{diam}\Omega$.

However, we show that the solution is asymptotically symmetric if Ω and F satisfy analogous symmetry assumptions in a direction $\hat{v} \neq e_1$, as they satisfy in the direction e_1. Define

$$\lambda^*(v) = \inf\{\mu : \Omega_{\mu,v} \subset \Omega_{0,e_1} \text{ and } \Omega'_{\lambda,v} \subset \Omega, \text{ for each } \lambda > \mu\}\,. \tag{2.7}$$

Here

$$\Omega'_{\lambda,v} := \mathcal{P}_{\lambda,v}(\Omega_{\lambda,v})\,, \quad \text{where} \quad \Omega_{\lambda,v} := \{x \in \Omega : x \cdot v \geq \lambda|v|\}\,,$$

and $\mathcal{P}_{\lambda,v} : \mathbb{R}^N \to \mathbb{R}^N$ is the reflection in the hyperplane

$$H_{\lambda,v} := \{x \in \mathbb{R}^N : x \cdot v = \lambda\}\,.$$

We assume that $\lambda^*(v)$ is sufficiently small at least for one vector $v \neq e_1$. More precisely, we suppose that the following hypotheses hold for some $\delta^* > 0$. In our theorems we need δ^* less than or equal to a certain constant depending on N, α_0, β_0 and diam Ω (as specified in (4.12)).

(D4) There exists a unit vector $\hat{v} \in \mathbb{R}^N$ such that $0 < \hat{v} \cdot e_1 < 1$ and $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$.

(D5) $\Omega_{\lambda,v}$ has finitely many connected components for all vectors $v \in W := \{v \in \operatorname{Cone}_{0,1}(-e_1, -\hat{v}) : |v| = 1\}$ and all $\lambda \geq \lambda^*(v)$, where $\hat{v}$ is as in (D4) and $\operatorname{Cone}_{0,1}$ was defined in (1.7).

At the end of this section (in the proof of Theorem 1.1), we prove that Lipschitz continuity of Ω, (d1), and (d2) imply, that (D4) holds for each $\delta^* > 0$. Also, for Lipschitz domains $\Omega_{\lambda,v}$ has finitely many connected components for any v and λ, so that (D5) holds. However, even for Lipschitz domains satisfying (d1), the assumption (D4) is weaker than (d2) (consider for example n-gon with sufficiently large n).

In Lemma 3.10, we prove, that for any $v \in W$ sufficiently close to e_1, (D4) holds with $\hat{v}$ replaced by v. Hölder continuity of Ω provides a sufficient condition for (D5).

In addition to examples in the introduction, an example of a domain that satisfies (D1) – (D5) and bears all complications of a general domain is the union of finitely many overlapping balls or upper half balls centered at H_0. Generally, such domain is neither convex nor rotationally symmetric.

Let us turn our attention to the assumption on the nonlinearity F. For any unit vector $v \in \mathbb{R}^N$ denote $\mathcal{T}^v : \mathbb{R}^N \to \mathbb{R}^N$ the matrix that represents the reflection in the hyperplane $H_{0,v}$:

$$\mathcal{T}^v_{ij} = \delta_{ij} - 2v_i v_j, \qquad (i, j \in \{1, \cdots N\}),$$

and let Q_v be the transformation

$$Q_v : (u, p, q) \mapsto (u, p\mathcal{T}^v, \mathcal{T}^v q \mathcal{T}^v).$$

For $\hat{v}$, already fixed in (D1), suppose that the set $\mathcal{O}$ (defined in the paragraph before (N1)), is invariant under $Q_{\hat{v}}$. An easy argument shows that $\mathcal{O}$ is then invariant under Q_v for any $v \in W$ as well.

(N4) For each $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, and any $x, \tilde{x} \in \Omega_{\lambda^*(\hat{v}),\hat{v}}$ with $\tilde{x} = x + \xi\hat{v}$, $\xi \geq 0$,

$$F(t, \mathcal{T}^{\hat{v}} x, Q_{\hat{v}}(u, p, q)) = F(t, x, Q_{\hat{v}}(u, p, q)) = F(t, x, u, p, q),$$
$$F(t, x, u, p, q) \geq F(t, \tilde{x}, u, p, q).$$

Using (N3) and (N4), it is easy to prove that for any $v \in W$

$$F(t, \mathcal{T}^v x, Q_v(u, p, q)) = F(t, x, Q_v(u, p, q)) = F(t, x, u, p, q),$$
$$F(t, x, u, p, q) \geq F(t, \tilde{x}, u, p, q),$$

where $(t, u, p, q) \in [0, \infty) \times \mathcal{O}$, and $x, \tilde{x} \in \Omega_{\lambda^*(v),v}$ with $\tilde{x} = x + \xi v$, $\xi \geq 0$.

The function $F(t, x, u, p, q) = q_{ii} + f(u)$ satisfies (N4). For more a complex example, suppose without loss of generality (or use a rotation preserving e_1)

that $\hat{v}$ has the form $\hat{v} = \hat{\sigma}_1 e_1 + \hat{\sigma}_2 e_2$, where $\hat{\sigma}_1 \geq 0$ and $\tilde{\sigma}_2 > 0$. Then, F satisfies (N3) and (N4), if it depends only on

$$(t, |(x_1, x_2)|, x', u, p_1^2 + p_2^2, p_i, q_{11} + q_{22}, q_{ij}) \qquad (3 \leq i, j \leq n)\,.$$

We formulate the next main result.

Theorem 2.2. *There exists* $\delta^* = \delta^*(N, \alpha_0, \beta_0, diam\,\Omega) > 0$ *such that if (D1) – (D5), (N1) – (N4) hold, and* u *is a nonnegative global solution of* (1.1) *satisfying* (2.1), (2.2)*, then for each* $z \in \omega(u)$ *the function* z *is nonincreasing in* x_1 *in* Ω_0 *and*

$$z(x_1, x') = z(-x_1, x') \qquad ((x_1, x') \in \Omega_0)\,.$$

Moreover, $z \equiv 0$ *in* Ω *or* z *is strictly decreasing in* x_1 *in* Ω_0*. The latter holds in the form* $z_{x_1} < 0$ *if* $z_{x_1} \in C(\Omega_0)$*.*

Corollary 2.3. *Under the assumptions of the previous theorem either* $z \equiv 0$ *or* $z > 0$ *in* Ω*, for any* $z \in \omega(u)$*.*

When the problem (1.1) is time independent and u is an equilibrium, we obtain, from the corollary, an improvement of results in [8, 9, 11] to the nonsmooth domain with space dependent nonlinearity.

Corollary 2.4. *There exists* $\delta^* = \delta^*(N, \alpha_0, \beta_0, diam\,\Omega) > 0$ *such that, if* Ω *is a domain satisfying (D1) – (D5) and* $u : \Omega \to \mathbb{R}$ *is a classical nonnegative solution of*

$$\begin{aligned} F(x, u, Du, D^2u) = 0, &\qquad x \in \Omega\,, \\ u = 0, &\qquad x \in \partial\Omega\,, \end{aligned}$$

with F *satisfying (N1)–(N4), then either* $u \equiv 0$ *or* $u > 0$ *in* Ω*.*

Let us prove that Theorem 1.1 follows from the previous theorem.

Proof of Theorem 1.1. It is easy to check that (D1) and (D3) are equivalent to (d1). As mentioned above, Lipschitz continuity of Ω implies (D2) and (D5). Moreover, for any $\delta^* > 0$, (d2) implies the existence of v and $\varepsilon > 0$ such that $\mathrm{Cone}_{x,\varepsilon}(e_1, v) \subset \bar{\Omega}$ for any $x \in \partial\Omega$ with $x_1 \geq \delta^*$. Then a perturbation argument, (D1), and (D2) yield that (D4) holds for a unit vector $\hat{v} \in \mathrm{span}\{e_1, v\}$, which is sufficiently close to e_1.

In problem (1.4), one has $F(t,x,u,p,q) = q_{ii} + f(t,u)$, where the function f satisfies (f1) and (f2). In this case (N1) – (N4) hold. Finally, by [23, Proposition 2.7], Lipschitz continuity of Ω, (N1), (N2), boundedness of u and (f2) imply (2.1) and (2.2). Therefore the assumptions of Theorem 2.2 are satisfied and Theorem 1.1 follows. □

3 Linear equations

In this section we describe how one can derive linear equations from nonlinear problems using reflections in hyperplanes. We also introduce linear parabolic estimates as a preparation for the method of moving hyperplanes. In the last subsection we derive some properties of general symmetric domains, which are convex in two directions.

Recall the following standard notation. For an open set $Q \subset \mathbb{R}^{N+1}$ we denote by $\partial_P Q$ the parabolic boundary of Q (for the precise definition see [20]). Let

$$Q_M := \{(x,s) \in Q : s \in M\} \qquad (M \subset \mathbb{R}) \tag{3.1}$$

be a time cut of Q, and if $M = \{t\}$ we also write Q_t instead of $Q_{\{t\}}$.

For bounded sets U, U_1 in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\operatorname{diam} U$ stands for the diameter of U, and $|U|$ for its Lebesgue measure (if it is measurable). The open ball in $\mathbb{R}^N$ centered at x with radius r is denoted by $B(x,r)$. Symbols f^+ and f^- denote the positive and negative parts of a function f: $f^\pm := (|f| \pm f)/2 \geq 0$.

Denote $x^{\lambda,v} := \mathcal{P}_{\lambda,v} x$ and recall that we already defined

$$\begin{aligned}
H_{\lambda,v} &= \{x \in \mathbb{R}^N : (x,v) = \lambda|v|\} && (v \in \mathbb{R}^N, \lambda \in \mathbb{R}), \\
\Omega_{\lambda,v} &= \Omega \cap \{x \in \mathbb{R}^N : x \cdot v > \lambda|v|\} && (v \in \mathbb{R}^N, \lambda \in \mathbb{R}), \\
\Omega'_{\lambda,v} &= \mathcal{P}_{\lambda,v}(\Omega_{\lambda,v}) = \{x^{\lambda,v} : x \in \Omega_{\lambda,v}\} && (v \in \mathbb{R}^N, \lambda \in \mathbb{R}).
\end{aligned}$$

Since Ω is bounded, $\Omega \cap H_{\lambda,v} = \emptyset$ for any $v \in \mathbb{R}^N$ and sufficiently large λ. Equivalently, $\ell(v) < \infty$ for all $v \in \mathbb{R}^N$, where $\ell(v) := \sup\{\lambda : \Omega \cap H_{\lambda,v} \neq \emptyset\}$.

Convention 3.1. If $v = e_1$ we omit the argument v in $H_{\lambda,v}$, $\Omega_{\lambda,v}$, $x^{\lambda,v}$, $\Omega'_{\lambda,v}$, $\ell(v)$ and we simply write H_λ, Ω_λ, x^λ, Ω'_λ, ℓ instead.

We shall use the following definition.

Definition 3.2. Given an open set $Q \subset \mathbb{R}^{N+1}$, and positive numbers α_0, β_0, we say that an operator L of the form

$$L(x,t) = a_{km}(x,t)\frac{\partial^2}{\partial x_k \partial x_m} + b_k(x,t)\frac{\partial}{\partial x_k} + c(x,t) \tag{3.2}$$

belongs to $E(\alpha_0, \beta_0, Q)$ if its coefficients a_{km}, b_k and c are measurable functions defined on Q and they satisfy

$$\begin{aligned} &|a_{km}|, |b_k|, |c| \leq \beta_0 && (k,m = 1,\ldots,N)\,, \\ &a_{km}(x,t)\xi_k\xi_m \geq \alpha_0|\xi|^2 && ((x,t) \in Q,\ \xi \in \mathbb{R}^N)\,. \end{aligned}$$

3.1 Reflection in hyperplanes

Fix a unit vector $v \in W$, where W was defined in (D5).

If $v = e_1$, the results of this subsection were already published in [23, Section 3] with all necessary details and expressions using Hadamard's formulas. We only recall the most important steps for later references. Accordingly with Convention 3.1 we drop the index v since $v = e_1$.

Assume that Ω satisfies (D1), F satisfies (N1) – (N3) and u is a global solution of (1.1) satisfying (2.1) and (2.2). By (N3),

$$F(t, x^\lambda, Q(u,p,q)) \geq F(t,x,u,p,q) \quad ((t,x,u,p,q) \in [0,\infty) \times \Omega_\lambda \times \mathcal{O}, \lambda \geq 0)$$

and if we denote $u^\lambda(x,t) := u(x^\lambda, t)$, then

$$\partial_t u^\lambda \geq F(t,x,u^\lambda, Du^\lambda, D^2u^\lambda), \qquad (x,t) \in \Omega_\lambda \times (0,\infty)\,.$$

Hence, the function $w^\lambda : \bar{\Omega}_\lambda \times (0,\infty) \to \mathbb{R}$, $w^\lambda : (x,t) \mapsto u^\lambda(x,t) - u(x,t)$, $\lambda \in [0,\ell)$ satisfies

$$\begin{aligned} \partial_t w^\lambda(x,t) &\geq F(x,t,u^\lambda, Du^\lambda, D^2u^\lambda) - F(x,t,u,Du,D^2u) \\ &= L^\lambda(x,t)w^\lambda, \qquad (x,t) \in \Omega_\lambda \times (0,\infty)\,, \end{aligned} \tag{3.3}$$

where $L^\lambda \in E(\alpha_0, \beta_0, \Omega_\lambda \times (0,\infty))$. Moreover, w^λ satisfies the following boundary condition

$$w^\lambda(x,t) \geq 0, \qquad (x,t) \in \partial\Omega_\lambda \times (0,\infty)\,. \tag{3.4}$$

Next, consider $v \neq e_1$ and in addition to (D1) and (N1)–(N3) assume (D4) and (N4). By similar arguments as in the case $v = e_1$ we obtain that for any $\lambda \in (\lambda^*(v), \ell(v))$, the function $u^{\lambda,v}(x,t) := u(x^{\lambda,v}, t)$ satisfies

$$\partial_t u^{\lambda,v} \geq F(t, x, u^{\lambda,v}, Du^{\lambda,v}, D^2 u^{\lambda,v}), \qquad (x,t) \in \Omega_{\lambda,v} \times (0,\infty)\,,$$

and the function $w^{\lambda,v} : \bar{\Omega}_{\lambda,v} \times (0,\infty) \to \mathbb{R}$, $w^{\lambda,v} : (x,t) \mapsto u(x^{\lambda,v}, t) - u(x,t)$ satisfies

$$\partial_t w^{\lambda,v} \geq L^{\lambda,v}(x,t) w^{\lambda,v}, \qquad (x,t) \in \Omega_{\lambda,v} \times (0,\infty)\,, \tag{3.5}$$

$$w^{\lambda,v} \geq 0, \qquad (x,t) \in \partial\Omega_{\lambda,v} \times (0,\infty)\,, \tag{3.6}$$

where $L^{\lambda,v} \in E(\alpha_0, \beta_0, \Omega_{\lambda,v} \times (0,\infty))$.

3.2 Estimates of solutions

In this subsection, we derive several estimates for linear problems such as (3.3) or (3.5). Since the results might be of independent interest, we state them under more general assumptions than needed for symmetry theorems.

Let Q be an open subset (bounded or unbounded) of $\mathbb{R}^N \times (0,\infty)$ and let β_0, α_0 be positive constants. We consider a general linear parabolic inequality

$$v_t \geq L(x,t)v + f(x,t), \qquad (x,t) \in Q\,, \tag{3.7}$$

$$v \geq g(x,t), \qquad (x,t) \in \partial_P Q\,, \tag{3.8}$$

where $L \in E(\alpha_0, \beta_0, Q)$, $f \in L^{N+1}(Q)$ and $g \in C(\partial_P Q) \cap L^\infty(\partial_P Q)$. Denote by a_{ij}, b_i, c, $i, j \in \{1, 2, \cdots, n\}$, the coefficients of L

$$L(x,t) := a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t) \qquad ((x,t) \in Q)\,,$$

and let

$$M(x,t) := L(x,t) - c(x,t) \qquad ((x,t) \in Q)\,. \tag{3.9}$$

We say that v is a solution of (3.7) (or that it satisfies (3.7)) if it is an element of the space $W^{2,1}_{N+1,loc}(Q)$ and (3.7) is satisfied almost everywhere. If (3.7) is complemented by (3.8), we also require the solution to be continuous on $\bar{Q}$ and to satisfy the boundary inequalities everywhere.

One of the key tools in our paper is the maximum principle. If we mention the maximum or comparison principle, we refer to the following theorem with $f \equiv 0$. The proof of Theorem 3.3 can be found in [7] see also [17, 20, 25]. Recall that Q can be unbounded.

Theorem 3.3. *If $Q \subset \mathbb{R}^N \times [T_1, T_2]$ for some $T_1 < T_2$, $v \in C(\bar{Q})$ is a bounded solution of* (3.7) *with $L \in E(\alpha_0, \beta_0, Q)$, $c \leq 0$ and $f \in L^{N+1}(Q)$, then*

$$\sup_Q v^- \leq \sup_{\partial_P Q} v^- + C\|f^-\|_{L^{N+1}(Q)} ,$$

where C depends on $N, \alpha_0, \beta_0, T_2 - T_1$.

Corollary 3.4. *If $c \leq 0$ in the previous theorem is changed to $c \leq \beta$ for some $\beta > 0$, and all the other assumptions are retained, then*

$$\sup_Q v^- \leq e^{\beta(T_2 - T_1)} \left(\sup_{\partial_P Q} v^- + C\|f^-\|_{L^{N+1}(Q))} \right) .$$

where C depends on $N, \alpha_0, \beta_0, T_2 - T_1$.

Proof of Corollary 3.4. We see that the function $\tilde{v} := e^{-\beta t} v$ satisfies (3.7) with c replaced by $c - \beta$ and f changed to $e^{-\beta t} f$. Since $c - \beta \leq 0$ in Q, Theorem 3.3 yields

$$\sup_Q \tilde{v}^- \leq \sup_{\partial_P Q} \tilde{v}^- + C\|\tilde{f}^-\|_{L^{N+1}(Q)} ,$$

where $\tilde{f}(x,t) = e^{-kt} f(x,t)$ and C depends on $N, \alpha_0, \beta_0, T_2 - T_1$. Using the definition of $\tilde{v}$ we obtain the desired result. □

The following lemma states a version of the maximum principle on small domains. It was originally proved in [6] in elliptic setting with $f = g \equiv 0$. A generalization to parabolic problems on cylindrical domains was proved in [23] with $f = g \equiv 0$ and later in [12] for general f and g. Here, we present yet another extension to sets in space-time (not necessarily cylindrical). The proof is partly motivated by elliptic results in [7] and it is only based on Theorem 3.3. It does not rely on a construction of supersolutions as in [6, 23]. However, such construction is possible for general space-time sets by an application of parabolic Monge-Ampère equation, but we will not discuss this approach.

Lemma 3.5. *Given any $k > 0$ there exists $\delta = \delta(\alpha_0, \beta_0, N, k)$ such that $|Q_{[t,t+1]}| < \delta$ for any $t \in \mathbb{R}$ implies the following. If $v \in C(\bar{Q})$ is a solution of problem* (3.7), (3.8) *with $L \in E(\alpha_0, \beta_0, Q)$ then*

$$\begin{aligned} \|v^-\|_{L^\infty(Q_t)} \leq 2\max\{\|v^-\|_{L^\infty(Q_\tau)} e^{-k(t-\tau)}, \|g^-\|_{L^\infty(\partial_P Q_{[\tau,t]} \setminus Q_\tau)}\} \\ + C\|f^-\|_{L^{N+1}(Q_{[\tau,t]})} \qquad (\tau < t) , \end{aligned} \tag{3.10}$$

where C depends on N, β_0, α_0.

Proof. In the proof the constant C can vary from step to step, but it only depends on N, β_0, α_0. To simplify the notation let

$$\partial_S Q_{[a,b]} := \partial_P Q_{[a,b]} \setminus Q_a \qquad (a < b)\,.$$

Corollary 3.4 implies, that on a time interval of length at most one we have

$$\begin{aligned} \|v^-\|_{L^\infty(Q_{(t,t+s)})} &\le e^{\beta_0 s} \max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &\quad + Ce^{\beta_0 s}\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \qquad (t \in [\tau, T-1], s \in [0,1])\,. \end{aligned} \tag{3.11}$$

The function $w := e^{(k+\ln 2)t} v$ satisfies

$$\begin{aligned} w_t - (M(x,t) - c^-)w &\ge (c^+ + k + \ln 2)w + \tilde{f} \qquad &((x,t) \in Q)\,, \\ w(x,t) &\ge \tilde{g}(x,t) \qquad &((x,t) \in \partial_S Q)\,, \end{aligned}$$

where $\tilde{f}(x,t) := e^{(k+\ln 2)t} f(x,t)$, $\tilde{g}(x,t) := e^{(k+\ln 2)t} g(x,t)$ and M was defined in (3.9). Since $-c^- \le 0$, Theorem 3.3 yields

$$\begin{aligned} &\|v^-\|_{L^\infty(Q_{t+s})} = e^{-(k+\ln 2)(t+s)} \|w^-\|_{L^\infty(Q_{t+s})} \\ &\le e^{-(k+\ln 2)(t+s)} \max\{\|w^-\|_{L^\infty(Q_t)}, \|\tilde{g}^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ e^{-(k+\ln 2)(t+s)} C\|(c^+ + k + \ln 2)w^- + \tilde{f}^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ C\left[(\beta_0 + k + \ln 2)\delta^{\frac{1}{N+1}} \|v^-\|_{L^\infty(Q_{[t,t+s]})} + \|f^-\|_{L^{N+1}(Q_{[t,t+s]})}\right] \\ &\qquad\qquad\qquad\qquad (t \in [\tau, T-s], s \in [0,1])\,. \end{aligned} \tag{3.12}$$

If we choose δ such that

$$C(\beta_0 + k + \ln 2)\delta^{\frac{1}{N+1}} \le \frac{e^{-k}}{2} e^{-\beta_0}\,,$$

then by (3.12) and (3.11)

$$\begin{aligned} \|v^-\|_{L^\infty(Q_{t+s})} &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &\qquad + \frac{e^{-k}}{2} e^{-\beta_0} \|v^-\|_{L^\infty(Q_{[t,t+s]})} + C\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\le \max\{e^{-(k+\ln 2)s}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} \\ &+ \frac{e^{-k}}{2} \max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+s]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+s]})} \\ &\qquad\qquad\qquad\qquad (t \in [\tau, T-1])\,. \end{aligned} \tag{3.13}$$

In particular for $s = 1$

$$
\begin{aligned}
\|v^-\|_{L^\infty(Q_{t+1})} &\le \max\{\frac{e^{-k}}{2}\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} \\
&\quad + \frac{e^{-k}}{2}\max\{\|v^-\|_{L^\infty(Q_t)}, \|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \\
&\le \max\{e^{-k}\|v^-\|_{L^\infty(Q_t)}, 2\|g^-\|_{L^\infty(\partial_S Q_{[t,t+1]})}\} + C\|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t \in [\tau, T-1]) .
\end{aligned}
$$

Iterating the previous expression for any $j \in \mathbb{N}$ with $t + j \le T$ we obtain:

$$
\begin{aligned}
\|v^-\|_{L^\infty(Q_{t+j})} &\le \max\{e^{-kj}\|v^-\|_{L^\infty(Q_t)}, 2\|g^-\|_{L^\infty(\partial_S Q_{[t,t+j]})}\} \\
&\qquad + C\|f^-\|_{L^{N+1}(Q_{[t,t+j]})} \qquad (t \in [\tau, T-j]) .
\end{aligned}
\tag{3.14}
$$

Since any $t \in [\tau, T]$ can be expressed in the form $t = \tau + j + s$ where $j \in \mathbb{N} \cup \{0\}$ and $s \in [0, 1)$, (3.13) and (3.14) imply

$$
\begin{aligned}
\|v^-\|_{L^\infty(Q_t)} &\le \max\{e^{-kj}\|v^-\|_{L^\infty(Q_{\tau+s})}, 2\|g^-\|_{L^\infty(\partial_S Q_{[\tau+s,t]})}\} + C\|f^-\|_{L^{N+1}(Q_{[\tau+s,t]})} \\
&\le 2\max\{e^{-kt}\|v^-\|_{L^\infty(Q_\tau)}, \|g^-\|_{L^\infty(\partial_S Q_{[\tau,t]})}\} + C\|f^-\|_{L^{N+1}(Q_{[\tau,t]})} . \quad \square
\end{aligned}
$$

Remark 3.6. From the proof of Lemma 3.5 one can see that (3.10) can be changed to

$$
\begin{aligned}
\|v^-\|_{L^\infty(Q_t)} &\le 2\max\{\|v^-\|_{L^\infty(Q_\tau)} e^{-k(t-\tau)}, \|g^-\|_{L^\infty(\partial_P Q_{[\tau,t]} \setminus Q_\tau)}\} \\
&\qquad + C\frac{1}{1 + e^{-k}} \sup_{t \in [\tau, T-1]} \|f^-\|_{L^{N+1}(Q_{[t,t+1]})} \qquad (\tau < t) .
\end{aligned}
$$

For the reader's convenience and for an easier reference later on, we formulate the following lemma that was proved in [23].

Lemma 3.7. *For any $r > 0$ there exist a constant $\gamma = \gamma(r, N, \alpha_0, \beta_0) > 0$ and a smooth function h_r on $B(0, r)$ with*

$$
h_r(x) > 0 \quad (x \in B(0, r)), \quad h_r(x) = 0 \quad (x \in \partial B(0, r)),
$$

such that for any $x_0 \in \Omega$ with $B(x_0, r) \subset \Omega$ and any $L \in E(\alpha_0, \beta_0, B(x_0, r) \times (0, \infty))$, the function $\phi(x, t) = e^{-\gamma t} h_r(x - x_0)$ satisfies

$$
\begin{aligned}
\partial_t \phi - L(x,t)\phi < 0, &\qquad (x,t) \in B(x_0, r) \times (0, \infty) , \\
\phi = 0 \quad , &\qquad (x,t) \in \partial B(x_0, r) \times (0, \infty) .
\end{aligned}
\tag{3.15}
$$

As a consequence we have the following result.

Corollary 3.8. *Given $r > 0$, let $\gamma = \gamma(r, N, \alpha_0, \beta_0) > 0$ be as in Lemma 3.7. For fixed $x_0 \in \mathbb{R}^N$ and $\tau < T$ set $Q = B(x_0, r) \times (\tau, T)$, and assume that $v \in C(\bar{Q})$ satisfies (3.7), (3.8) with $g = f \equiv 0$, and $L \in E(\alpha_0, \beta_0, Q)$. If $v(\cdot, \tau) \geq q$ in $B(x_0, r)$ for some $q > 0$, then*

$$v(x,t) \geq \tilde{c}_r q e^{-\gamma(t-\tau)} \qquad \left((x,t) \in B\left(x_0, \frac{r}{2}\right) \times [\tau, T)\right),$$

where $0 < \tilde{c}_r \leq 1$ depends only on N, α_0, β_0 and r.

Proof. For the given r consider γ, h_r and ϕ as in Lemma 3.7. Then

$$\begin{aligned} v_t - L(x,t)v \geq 0 > \phi_t - L(x,t)\phi, \qquad & (x,t) \in B(x_0, r) \times (\tau, T), \\ v(x,t) \geq 0 = \phi(x,t), \qquad & (x,t) \in \partial B(x_0, r) \times [\tau, T], \end{aligned}$$

and

$$v(x,\tau) \geq q \frac{\phi(x,\tau)}{\|\phi(\cdot,\tau)\|_{L^\infty(B(x_0,r))}} \qquad (x \in B(x_0, r)).$$

An application of the comparison principle for v and ϕ gives

$$v(x,t) \geq q \frac{\phi(x,t)}{\|\phi(\cdot,\tau)\|_{L^\infty(B(x_0,r))}} \geq q e^{-\gamma(t-\tau)} \frac{h_r(x-x_0)}{\|h_r(\cdot - x_0)\|_{L^\infty(B(x_0,r))}} \qquad ((x,t) \in B(x_0, r) \times [\tau, T]). \tag{3.16}$$

Since $h > 0$ in $B(x_0, r)$, we obtain the desired result for

$$\tilde{c}_r = \frac{\inf_{x \in B(x_0, r/2)} h_r(x - x_0)}{\|h_r(\cdot - x_0)\|_{L^\infty(B(x_0,r))}}. \qquad \square$$

The next lemma, proved in [23, Lemma 3.4.], is a version of Krylov-Safonov Harnack inequality [17, 18] for sign changing supersolutions of parabolic problems (see also [14, 20]). Its formulation needs the following notation. For any open bounded subset S of $\mathbb{R}^{n+1}$ and any bounded, continuous function $f : S \to \mathbb{R}$ define

$$[f]_{p,S} := \left(\frac{1}{|S|} \int_S |f|^p dx\, dt\right)^{\frac{1}{p}} \qquad (p > 0).$$

Lemma 3.9. *Given $\Omega \subset \mathbb{R}^N$, $d > 0$, $\varepsilon > 0$, $\theta > 0$, there are positive constants κ, κ_1, p determined only by N, $diam\,\Omega, \alpha_0, \beta_0, d, \varepsilon$ and θ with the following properties. Let D and U be domains in Ω with $D \subset\subset U$, $dist(\bar{D}, \partial U) \geq d$, $|D| > \varepsilon$ and let $Q = U \times (\tau, \tau + 4\theta)$. Assume that $v \in C(\bar{Q})$ satisfies* (3.7) *with $L \in E(\alpha_0, \beta_0, Q)$ and $f \equiv 0$. Then*

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v(x,t) \geq \kappa[v^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \sup_{\partial_p(U\times(\tau,\tau+4\theta))} e^{4M\theta}v^-\,,$$

where $M = \sup_{U\times(\tau,\tau+4\theta)} c$.

3.3 Properties of $\Omega_{\lambda,v}$

In this purely geometrical subsection we assume that Ω is a bounded domain satisfying (D1), (D3) and (D4). Let us start with a lemma that extends property (D4) to all vectors in W, where W is as in (D5): $W = \mathrm{Cone}_{0,1}(-e_1, -\hat{v}) \cap \partial B(0,1)$.

Lemma 3.10. *If for some $\delta^* > 0$, Ω satisfies (D1), (D3) and (D4), then $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v}$ for any $v \in W$ sufficiently close to e_1.*

Proof. Fix $v \in W$ and let $\alpha, \beta \in [0,1]$ be such that $v = \alpha\hat{v} + \beta e_1$. If $\alpha = 0$ or $\beta = 0$ the statement follows directly from (D1) and (D4), thus we consider $\alpha, \beta > 0$. Since $v \neq \hat{v}$, $H_{\lambda^*(\hat{v}),\hat{v}} \cap H_{\lambda^*(v),v} \neq \emptyset$. Then we define Λ as

$$x \cdot e_1 = \frac{1}{\beta} x \cdot v - \frac{\alpha}{\beta} x \cdot \hat{v} = \frac{\lambda^*(v) - \alpha\lambda^*(\hat{v})}{\beta} =: \Lambda \qquad (x \in H_{\lambda^*(\hat{v}),\hat{v}} \cap H_{\lambda^*(v),v})\,. \quad (3.17)$$

First assume $\Lambda \leq \delta^*$. By (D4) we have $y \cdot e_1 > \delta^*$ and $y \cdot \hat{v} \geq \lambda^*(\hat{v})$ for any $y \in \Omega_{\delta^*,e_1}$. Then using $\Lambda \leq \delta^*$ we obtain

$$y \cdot v = \alpha(y \cdot \hat{v}) + \beta(y \cdot e_1) > \alpha\lambda^*(\hat{v}) + \beta\delta^* \geq \lambda^*(v) \qquad (y \in \Omega_{\delta^*,e_1})\,,$$

and therefore $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v}$, as desired.

We finish the proof, once we show that $\Lambda > \delta^*$ leads to a contradiction. Define

$$\varepsilon_0 := \beta\,(\Lambda - \delta^*) = \beta\left(\frac{\lambda^*(v) - \alpha\lambda^*(\hat{v})}{\beta} - \delta^*\right) > 0\,. \qquad (3.18)$$

By (3.18)

$$\alpha(x\cdot\hat v)+\beta(x\cdot e_1)=x\cdot v\geq\lambda^*(v)-\varepsilon_0\geq\alpha\lambda^*(\hat v)+\beta\delta^* \qquad (x\in\Omega_{\lambda^*(v)-\varepsilon_0,v}).$$

Thus either $x\cdot\hat v\geq\lambda^*(\hat v)$ or $x\cdot e_1\geq\delta^*$ and by (D4), any of these cases yields $x\in\Omega_{\lambda^*(\hat v),\hat v}$. Hence $\Omega_{\lambda^*(v)-\varepsilon_0,v}\subset\Omega_{\lambda^*(\hat v),\hat v}$ and in particular $\Omega_{\lambda^*(v)-\varepsilon,v}\subset\Omega_{\lambda^*(\hat v),\hat v}$ for any $\varepsilon\in(0,\varepsilon_0]$.

Next, the definition of $\lambda^*(v)$ implies the existence of ε with $0<\varepsilon\leq\varepsilon_0$, and a point $Q\in\Omega_{\lambda^*(v)-\varepsilon,v}$ such that $Q^{\lambda^*(v)-\varepsilon,v}\notin\Omega$. Let d_v and $d_{\hat v}$ be distances of Q to the hyperplanes $H_{\lambda^*(v)-\varepsilon,v}$ and $H_{\lambda^*(\hat v),\hat v}$ respectively and observe that $d_{\hat v}\geq d_v$. Then

$$Q-Q^{\lambda^*(v)-\varepsilon,v}=2d_v v\,,\quad Q-Q^{\lambda^*(\hat v),\hat v}=2d_{\hat v}\hat v$$

and consequently using $v=\alpha\hat v+\beta e_1$

$$Q-Q^{\lambda^*(v)-\varepsilon,v}=\alpha\frac{d_v}{d_{\hat v}}(Q-Q^{\lambda^*(\hat v),\hat v})+2\beta d_v e_1\,.$$

Since $\Omega_{\lambda^*(v)-\varepsilon,v}\subset\Omega_{\lambda^*(\hat v),\hat v}$, one has $\alpha\frac{d_v}{d_{\hat v}}\leq 1$ and the definition of $\lambda^*(\hat v)$ then implies $R:=Q-\alpha\frac{d_v}{d_{\hat v}}(Q-Q^{\lambda^*(\hat v),\hat v})\in\Omega$. We arrive to a contradiction by showing that $Q^{\lambda^*(v)-\varepsilon,v}=R-2\beta d_v e_1\in\Omega$ for v sufficiently close to e_1. Since $R\in\Omega$ and Ω is symmetric and convex in x_1, it is sufficient to prove that $(R+Q^{\lambda^*(v)-\varepsilon,v})\cdot e_1\geq 0$. Notice that $Q-\frac{d_{\hat v}}{e_1\cdot\hat v}e_1\in H_{\lambda^*(\hat v),\hat v}\cap\Omega\subset\bar\Omega_{0,e_1}$, and therefore

$$\left(Q-\frac{d_{\hat v}}{e_1\cdot\hat v}e_1\right)\cdot e_1\geq 0\,.$$

Then using the definitions of $R, Q^{\lambda^*(v)-\varepsilon,v}$, and $v=\alpha\hat v+\beta e_1$ we obtain

$$\begin{aligned}R+Q^{\lambda^*(v)-\varepsilon,v}&=Q-2\alpha d_v\hat v+Q^{\lambda^*(v)-\varepsilon,v}=2[Q-\alpha d_v\hat v-d_v v]\\&=2[(Q-\frac{d_{\hat v}}{e_1\cdot\hat v}e_1)+\frac{d_{\hat v}}{e_1\cdot\hat v}e_1-d_v(2\alpha\hat v+\beta e_1)]\,.\end{aligned}$$

Since $d_{\hat v}\geq d_v$ and $1>e_1\cdot\hat v>0$, one has $(R+Q^{\lambda^*(v)-\varepsilon,v})\cdot e_1\geq 0$ for any α sufficiently close to 0 (and β close to 1). □

The second lemma shows that the portion of $\partial\Omega$ that is close to H_{λ,e_1} is not symmetric with respect to this hyperplane.

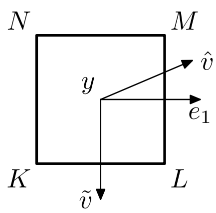
Lemma 3.11. *Given $\delta^* > 0$, consider a bounded domain Ω satisfying (D1), (D3) and (D4). Fix $\lambda \in (\delta^*, \ell(e_1))$ and let U be a connected component of Ω_{λ,e_1}. Then for any $\varepsilon > 0$ there is $z \in (\partial U)\setminus H_{\lambda,e_1}$ such that dist$(z, H_{\lambda,e_1}) < \varepsilon$ and $z^{\lambda,e_1} \in \Omega$.*

Proof. Recall, that in (D4) we fixed a unit vector $\hat{v} = (\hat{v}_1, \hat{v}')$ with $\hat{v}_1 \in (0,1)$ and $\Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$. Choose $x^* \in \bar{U}$ such that $x^* \cdot \hat{v} = \inf_{x\in\bar{U}} x \cdot \hat{v}$. Clearly $x^* \in \partial U \cap H_{\lambda,e_1}$. Since $\lambda > \delta^*$ and $\Omega_{\delta^*,e_1} \subset \bar{\Omega}_{\lambda^*(\hat{v}),\hat{v}}$, for any sufficiently small $\varepsilon > 0$ one has $B(x^*, \varepsilon) \cap \Omega \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$. We prove that for any such $\varepsilon > 0$ there is a point z in $B(x^*, \varepsilon)$ with the desired properties. Since $x^* \in \bar{U}$, for

$$\delta := \frac{\varepsilon}{2} \frac{\sqrt{1-\hat{v}_1^2}}{2\sqrt{1-\hat{v}_1^2}+1}\,, \tag{3.19}$$

there is $y \in U \cap B(x^*, \delta)$ and $0 < \rho < \delta$ with $B(y, \rho) \subset U \cap B(x^*, \delta)$. Consider the two-dimensional plane $\mathcal{S}$ passing through y spanned by vectors e_1 and $\hat{v}$.

Then the two-dimensional closed square (see figure) $\mathcal{C}(y, \rho) \subset \mathcal{S}$ centered at y with a side of length ρ perpendicular to e_1, is a subset of $B(y, \rho) \cap \mathcal{S} \subset U$. Let $\tilde{v} \in \mathcal{S}$ be the unit vector perpendicular to e_1 with $\tilde{v} \cdot \hat{v} < 0$. An elementary calculation shows $\tilde{v} \cdot \hat{v} = -\sqrt{1-\hat{v}_1^2}$. Translate $\mathcal{C}(y, \rho)$ along $\tilde{v}$ as far as it stays inside U, that is, define



$$\kappa_0 := \sup\{\mu : \mathcal{C}(y + \kappa\tilde{v}, \rho) \subset U \text{ for all } \kappa \in [0, \mu]\}.$$

Denote $\hat{y}$ and $\hat{K}, \hat{L}, \hat{M}, \hat{N}$ the center and the vertices of $\mathcal{C}(y + \kappa_0\tilde{v}, \rho)$ ($\hat{X}$ is the image of X in translation by vector $\kappa_0\tilde{v}$). We prove that $\hat{L}$ is a point with the desired properties.

First, since $\mathcal{C}(\hat{y}, \rho) \subset \bar{U}$, the definition of x^* yields $\hat{y} \cdot \hat{v} \geq x^* \cdot \hat{v}$ or equivalently $y \cdot \hat{v} + \kappa_0\tilde{v} \cdot \hat{v} \geq x^* \cdot \hat{v}$ and consequently

$$\kappa_0 \leq \frac{y \cdot \hat{v} - x^* \cdot \hat{v}}{-\tilde{v} \cdot \hat{v}} \leq \frac{|y - x^*||\hat{v}|}{-\tilde{v} \cdot \hat{v}} < \frac{\delta}{-\tilde{v} \cdot \hat{v}}\,.$$

Then (3.19) implies

$$\begin{aligned}|x - x^*| &\leq |x - \hat{y}| + |\hat{y} - y| + |y - x^*| \leq \rho + \kappa_0 + \delta \\ &\leq 2\delta + \frac{\delta}{-\tilde{v} \cdot \hat{v}} < \varepsilon \qquad (x \in \mathcal{C}(\hat{y}, \rho))\,.\end{aligned}$$

Thus $\mathcal{C}(\hat{y},\rho) \subset B(x^*,\varepsilon)$ and in particular

$$\operatorname{dist}(\hat{L}, H_{\lambda,e_1}) \leq \operatorname{dist}(\hat{L}, x^*) < \varepsilon\,.$$

Next, by the definition of κ_0, $\operatorname{Int}(\mathcal{C}(\hat{y},\rho)) \subset U$, where $\operatorname{Int}(\mathcal{C}(\hat{y},\rho))$ is the interior of $\mathcal{C}(\hat{y},\rho)$ in the topology of the plane $\mathcal{S}$. Moreover, there exists $\hat{z} \in \partial U$ that lies on the side connecting $\hat{K}$ and $\hat{L}$. Since $\mathcal{C}(\hat{y},\rho) \subset \Omega_{\lambda^*(\hat{v}),\hat{v}}$, there is $\hat{\lambda} > \lambda^*(\hat{v})$ such that $\hat{L} \in H_{\hat{\lambda},\hat{v}}$. Then $\mathcal{P}_{\hat{\lambda},\hat{v}}(\operatorname{Int}\mathcal{C}(\hat{y},\rho)) \subset \Omega$, and in particular $z \in \Omega$ for any $z \neq \hat{L}$ on the side connecting $\hat{K}$ and $\hat{L}$, sufficiently close to $\hat{L}$. Moreover, the convexity of Ω in e_1 implies $z \in \Omega$ for any $z \neq \hat{L}$ on the side connecting $\hat{K}$ and $\hat{L}$. Thus, $\hat{L} = \hat{z} \in \partial U$.

By the definition of ε, one has $\mathcal{P}_{\lambda,e_1}\hat{L} \in \bar{\Omega}_{0,e_1}$. If $\mathcal{P}_{\lambda,e_1}\hat{L} \in \partial\Omega$, then, by the convexity of Ω in e_1, the whole segment connecting $\hat{L}$ and $\mathcal{P}_{\lambda,e_1}\hat{L}$ is in $\partial\Omega$, a contradiction. Hence $\mathcal{P}_{\lambda,e_1}\hat{L} \in \Omega$.

Finally, since e_1 and $\tilde{v}$ are perpendicular,

$$\operatorname{dist}(\hat{L}, H_{\lambda,e_1}) = \operatorname{dist}(L, H_{\lambda,e_1}) > 0,$$

and therefore $\hat{L} \notin H_{\lambda,e_1}$. □

4 Proofs of the main results

In this section we assume that Ω satisfies (D1) and (D2) (not necessarily (D3)) and the nonlinearity F satisfies (N1) – (N3). At some places, where explicitly stated, we also assume (D3), (D4) or (N4). We remark that, even though (D2) is not needed in all results, we assume it throughout the section. Consider a classical solution u of (1.1) satisfying (2.1) and (2.2).

We use the notation introduced at the beginning of Section 2 and the following one. For any function $g : \Omega \to \mathbb{R}$, and any $\lambda \in [0,\ell)$ we set

$$V_\lambda g(x) := g(x^\lambda) - g(x), \qquad (x \in \Omega_\lambda)\,,$$

and for the solution u of (1.1) we define

$$w^\lambda(x,t) := u(x^\lambda,t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,\infty))\,.$$

As shown in Subsection 3.1, the function w^λ solves a linear problem (3.3), (3.4) with $L \in E(\alpha_0,\beta_0,\Omega_\lambda \times (0,\infty))$. Hence the results of Subsection 3.2 are applicable to w^λ. We use this observation below, often without notice.

We carry out the process of moving hyperplanes in the following way. Starting from $\lambda = \ell$ we move λ to the left as long as the following property is preserved

$$\lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0. \tag{4.1}$$

We show below that the process can get started and then we examine the limit of the process given by

$$\lambda_0 := \inf\{\mu > 0 : \lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0 \text{ for each } \lambda \in [\mu,\ell)\}. \tag{4.2}$$

Remark 4.1. Note, that by the relative compactness of $\{u(\cdot,t) : t \geq 0\}$ in $C(\bar{\Omega})$, (4.1) is equivalent to the following property:

$$V_\lambda z(x) \geq 0 \qquad (x \in \Omega_\lambda,\ z \in \omega(u),\ \lambda \in [\lambda_0,\ell))\,. \tag{4.3}$$

Further observe that each $z \in \omega(u)$ is nonincreasing in x_1 in Ω_{λ_0}. Indeed, if $(x_1,x'),(\tilde{x}_1,x') \in \Omega_{\lambda_0}$ and $x_1 > \tilde{x}_1$, then $V_\lambda z \geq 0$ with $\lambda = (x_1+\tilde{x}_1)/2 > \lambda_0$ gives $z(x_1,x') \geq z(\tilde{x}_1,x')$.

The following lemma shows that the process of moving hyperplanes can get started, that is, $\lambda_0 < \ell$. We do not include the proof here, since it follows from the proof of [23, Lemma 4.1].

Lemma 4.2. *For λ_0 defined in* (4.2) *we have $\lambda_0 < \ell$. Moreover, if $\delta = \delta(\alpha_0,\beta_0,N) > 0$ is such that Lemma 3.5 holds with $k = 1$, then $|\Omega_{\lambda_0}| \geq \delta$.*

Next, we investigate the properties of functions $V_\lambda z$ and z for $\lambda \in [\lambda_0,\ell)$, where $z \in \omega(u)$.

Lemma 4.3. *For any $\tilde{\lambda} \in [\lambda_0,\ell)$, $z \in \omega(u)$ and any connected component $U_{\tilde{\lambda}}$ of $\Omega_{\tilde{\lambda}}$ the following statements hold true:*

(i) either $V_{\tilde{\lambda}} z \equiv 0$ or $V_{\tilde{\lambda}} z > 0$ in $U_{\tilde{\lambda}}$,

(ii) either $z \equiv 0$ or $z > 0$ in $U_{\tilde{\lambda}}$,

(iii) $z \equiv 0$ in $U_{\tilde{\lambda}}$ implies $V_{\tilde{\lambda}} z \equiv 0$ in $U_{\tilde{\lambda}}$.

Proof. The statement (i) was already proved in [23, Lemma 4.2].

To prove (ii) it is sufficient to show that $z(x^*) = 0$ for some $x^* = (x_1^*,(x^*)') \in U_{\tilde{\lambda}}$ implies $z \equiv 0$ in $U_{\tilde{\lambda}}$.

Since $z(x^*) = 0$, the monotonicity of z (see Remark 4.1) yields

$$z(x_1, (x^*)') = 0 \qquad (x_1 \in [x_1^*, \Gamma_{x^*}]) ,$$

where

$$\Gamma_{x^*} := \sup\{x_1 : (x_1, (x^*)') \in \Omega\} > x_1^* .$$

Then for any $\lambda \in (x_1^*, \Gamma_{x^*})$, (i) with $\tilde{\lambda} = \lambda$ yields $V_\lambda z \equiv 0$ in $\Omega_\lambda \cap U_{\tilde{\lambda}}$, and therefore $z(x_1, (x^*)') = 0$ for all $x_1 \in [2x_1^* - \Gamma_{x^*}, \Gamma_{x^*}]$. Since $x_1^* > 2x_1^* - \Gamma_{x^*}$, we can iterate this argument with x^* replaced by $(2x_1^* - \Gamma_{x^*}, (x^*)')$ and obtain $z(x_1, (x^*)') = 0$ for each $x_1 \in [\tilde{\lambda}, \Gamma_{x^*}]$.

Consequently $V_\lambda z \equiv 0$ in $\Omega_\lambda \cap U_{\tilde{\lambda}}$ for all $\lambda \in [\tilde{\lambda}, \Gamma_{x^*}]$. To finish the proof of (ii), it is sufficient to show $\Lambda = \ell_{U_{\tilde{\lambda}}}$, where

$$\ell_{U_{\tilde{\lambda}}} := \sup\{x_1 : (x_1, x') \in U_{\tilde{\lambda}} \text{ for some } x' \in \mathbb{R}^{N-1}\} ,$$

and

$$\Lambda := \sup\{\mu \in (\tilde{\lambda}, \ell_{U_{\tilde{\lambda}}}) : V_\lambda z \equiv 0 \text{ in } U_{\tilde{\lambda}} \cap \Omega_\lambda \text{ for all } \lambda \in (\tilde{\lambda}, \mu)\} \geq \Gamma_{x^*} > \tilde{\lambda} .$$

Indeed, then, as in Remark 4.1, z is constant in x_1 in $U_{\tilde{\lambda}}$, and the boundary condition yields $z \equiv 0$ in $U_{\tilde{\lambda}}$ as desired.

For a contradiction assume that $\Lambda < \ell_{U_{\tilde{\lambda}}}$. Since $\tilde{\lambda} < (3\Lambda + \tilde{\lambda})/4 < \Lambda$, $V_{(3\Lambda+\tilde{\lambda})/4} z \equiv 0$ and consequently z is constant in x_1 for $x_1 \in (\tilde{\lambda}, (3\Lambda - \tilde{\lambda})/2)$. Thus by (i), $V_\lambda z \equiv 0$ for each $\lambda \in (\tilde{\lambda}, \min\{(3\Lambda - \tilde{\lambda})/2, \ell_{U_{\tilde{\lambda}}}\})$, a contradiction to the definition of Λ.

To prove (iii), observe that (using (i)) $V_\lambda z \equiv 0$ for each $\lambda \in (\tilde{\lambda}, \ell_{U_{\tilde{\lambda}}})$. Then the statement follows from the continuity. $\square$

The next proposition plays a central role in our arguments. The techniques are partly motivated by [23, Theorem 3.7], but the situation is more complicated here. Complications arise from the fact, that the solution u can be small on different connected components of Ω_{λ_0} at different times. A careful analysis of the interaction between different connected components of Ω_{λ_0} is required.

Proposition 4.4. *Assume $\lambda_0 > 0$. Then there is $z \in \omega(u)$ and a connected component U_{λ_0} of Ω_{λ_0} such that $V_{\lambda_0} z \equiv 0$ and $z > 0$ in U_{λ_0}.*

Proof. We proceed by a contradiction. That is (cf. Lemma 4.3), we assume:

For any $z \in \omega(u)$ and any connected component U_{λ_0} of Ω_{λ_0} either $V_{\lambda_0} z > 0$ or $z \equiv 0$ in U_{λ_0}. (4.4)

The definition of λ_0, yields the existence of an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to λ_0, $x_k \in \Omega_{\lambda_k}$ and $z_k \in \omega(u)$ such that $V_{\lambda_k} z_k(x_k) < 0$. Then, by (D2), we can fix a connected component U_{λ_0} of Ω_{λ_0} such that, for each $\lambda < \lambda_0$ there is $k \in \mathbb{N}$ with $x_k \in U_\lambda$, where

$$U_\mu \text{ is the connected component of } \Omega_\mu \text{ with } U_\mu \cap U_{\lambda_0} \neq \emptyset\,. \tag{4.5}$$

Next, define

$$U^*_{\lambda_0} := \bigcap_{\lambda<\lambda_0} U_\lambda\,,$$

and observe that $|U_\lambda \setminus U^*_{\lambda_0}|$ is arbitrary small if $\lambda > \lambda_0$ is sufficiently close to λ_0. (This property does not hold true, if we replace $U^*_{\lambda_0}$ by U_{λ_0}, consider for example Ω such that Ω_λ has two connected components for all $\lambda \geq \lambda_0$, but it is connected for $\lambda < \lambda_0$).

To continue we distinguish three cases

a) there is $z \in \omega(u)$ such that $V_{\lambda_0} z > 0$ on $U^*_{\lambda_0}$,

b) for all $z \in \omega(u)$, $z \equiv 0$ on $U^*_{\lambda_0}$,

c) for each $z \in \omega(u)$, $z \equiv 0$ on a connected component of $U^*_{\lambda_0}$, and for some $\tilde{z} \in \omega(u)$, $V_{\lambda_0}\tilde{z} > 0$ in a connected component $\tilde{U}_{\lambda_0}$ of $U^*_{\lambda_0}$

A contradiction with the definition of $\{x_k\}_{k\in\mathbb{N}}$ follows in a) from [23, Lemma 4.3], where we replace Ω_{λ_0} by $U^*_{\lambda_0}$ and Ω_λ by U_λ. In b) and c) it follows from the next two lemmas.

Lemma 4.5. *Assume that $\lambda_0 > 0$ and b) holds. Then $V_\lambda z \geq 0$ in U_λ for all $z \in \omega(u)$ and $\lambda < \lambda_0$ sufficiently close to λ_0.*

Lemma 4.6. *Assume that $\lambda_0 > 0$ and* (4.4) *holds. Then c) does not hold.*

The proofs of the lemmas are postponed till the next section. □

Now we address the question how big is the union of the connected components of Ω_{λ_0} on which the situation from Proposition 4.4 occurs.

Lemma 4.7. *Let $\delta = \delta(\alpha_0, \beta_0, N) > 0$ be such that Lemma 3.5 holds with $k = 1$. If $\Omega^*_{\lambda_0}$ denotes the union of connected components U of Ω_{λ_0}, for which there exists $z \in \omega(u)$ with $V_{\lambda_0} z \equiv 0$ and $z > 0$ in U, then*

$$\left|\Omega^*_{\lambda_0}\right| > \frac{\delta}{2}.$$

Proof. We proceed by contradiction, that is, we assume $\left|\Omega^*_{\lambda_0}\right| \le \frac{\delta}{2}$. Fix an open set $D \subset\subset \Omega_{\lambda_0} \setminus \Omega^*_{\lambda_0}$, convex in x_1, with $|\Omega_{\lambda_0} \setminus (\Omega^*_{\lambda_0} \cup D)| < \frac{\delta}{8}$, and fix $\varepsilon_0 > 0$ such that $|\Omega_{\lambda_0 - \varepsilon_0} \setminus \Omega_{\lambda_0}| < \frac{\delta}{8}$. Then $|\Omega_\lambda \setminus D| < \frac{3}{4}\delta$ for any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$.

By Proposition 4.4, the set $\Omega^*_{\lambda_0}$ is nonempty and we can choose its connected component $\tilde{U}$ and $\tilde{z} \in \omega(u)$ such that $V_{\lambda_0}\tilde{z} \equiv 0$ and $\tilde{z} > 0$ in $\tilde{U}$. Then for

$$C_\lambda := \frac{1}{6}\|(V_\lambda \tilde{z})^-\|_{L^\infty(\tilde{U})} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]) \tag{4.6}$$

we have $C_\lambda > 0$ for all $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0)$ and $C_\lambda \to 0$ as $\lambda \to \lambda_0$.
Claim. We can decrease $\varepsilon_0 > 0$ such that

$$|K_{z,\lambda}| < \frac{\delta}{4} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0), z \in \omega(u))\,,$$

where

$$K_{z,\lambda} := \{x \in D : V_\lambda z(x) < -C_\lambda\} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0), z \in \omega(u))\,.$$

We postpone the proof of this claim, and we finish the proof of the lemma first.

Fix any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0)$ and denote

$$Q := \{(x,t) \in \Omega_\lambda : w^\lambda(x,t) < -2C_\lambda\}\,.$$

Then (2.3) implies that for each sufficiently large t there is $z \in \omega(u)$ with $Q_t \cap D \subset K_{z,\lambda}$. Consequently for sufficiently large t

$$|Q_t| \le |\Omega_\lambda \setminus D| + |K_t| < \frac{3}{4}\delta + \frac{1}{4}\delta = \delta\,,$$

and therefore $|Q_{[t,t+1]}| < \delta$ (n+1-dimensional measure). Choose $(t_k)_{k\in\mathbb{N}}$, $t_k \to \infty$ such that $u(\cdot, t_k) \to \tilde{z}$ as $k \to \infty$. Then (4.6) gives $\|(w^\lambda(\cdot, t_k))^-\|_{L^\infty(\tilde{U})} \ge 5C^\lambda$ for any sufficiently large k.

Since, by the definition of Q and (3.4), one has $(w^\lambda)^- \le C_\lambda$ on $\partial_P Q$, Lemma 3.5 yields for any sufficiently large k

$$5C^\lambda \le \|(w^\lambda)^-\|_{L^\infty(Q_{t_k})} \le 2\max\{\|(w^\lambda)^-\|_{L^\infty(Q_{t_k-T})}e^{-T}, 2C_\lambda\},\ (0 < T < t_k)\,,$$

a contradiction for sufficiently large T.

Proof of the claim. For already fixed $\tilde{z}$ and $\tilde{U}$ denote

$$M := \sup_{\tilde{U}} \tilde{z} > 0\,. \tag{4.7}$$

Since $V_{\lambda_0} z \ge 0$ in Ω_{λ_0},

$$\begin{aligned} V_\lambda z(x_1, x') &= z(2\lambda - x_1, x') - z(x_1, x') \\ &= z(2\lambda - x_1, x') - z(2\lambda_0 - (2\lambda - x_1), x') + z(x_1 + 2(\lambda_0 - \lambda), x') - z(x_1, x') \\ &\ge z(x_1 + 2(\lambda_0 - \lambda), x') - z(x_1, x') \qquad ((x_1, x') \in \Omega(\lambda), z \in \omega(u))\,, \end{aligned} \tag{4.8}$$

where

$$\Omega(\lambda) := \{x = (x_1, x') \in \Omega_{\lambda_0} : (x_1 + 2(\lambda_0 - \lambda), x') \in \Omega_{\lambda_0}\} \qquad (\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0])\,.$$

Decrease $\varepsilon_0 > 0$ if necessary such that $D \subset \Omega(\lambda)$ for each $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$. We show that it is possible to decrease ε_0 such that $K_{z,\lambda} \ne \emptyset$ for some $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$, $z \in \omega(u)$, implies

$$\sup_D z < \frac{\delta M}{96[\mathrm{diam}(\Omega)]^N}\,. \tag{4.9}$$

Assume not, that is, assume that there is $(x_n)_{n\in\mathbb{N}} \subset D$, $(z_n)_{n\in\mathbb{N}} \subset \omega(u)$ and $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \nearrow \lambda_0$ as $n \to \infty$ such that $V_{\lambda_n} z_n(x_n) < -C_{\lambda_n}$ and (4.9) does not hold. After passing to a subsequence, we can assume $z_n \to z \in \omega(u)$ with convergence in $C(\bar{\Omega})$ and $x_n \to x_0 \in \bar{D}$, as $n \to \infty$. Then $V_{\lambda_0} z(x_0) \le 0$ and $\|z\|_{L^\infty(\bar{D})} > 0$. Consequently by Lemma 4.3 (i) and (ii) with $\tilde{\lambda} = \lambda_0$, $V_{\lambda_0} z \equiv 0$ and $z > 0$ in a connected component U of $\Omega_{\lambda_0} \setminus \Omega^*_{\lambda_0}$ for which $x_0 \in U$, a contradiction to the definition of $\Omega^*_{\lambda_0}$.

Next, by (4.8), monotonicity of z on Ω_{λ_0},convexity of D in x_1, and (4.9)

$$
\begin{aligned}
|K_{z,\lambda}| &= \int_D I_{\{x\in D: V_\lambda z(x) < -C_\lambda\}} dx \le \int_D I_{\{x\in D: z(x_1+2(\lambda_0-\lambda),x')-z(x_1,x')<-C_\lambda\}} dx \\
&= \iint_D I_{\{x\in D:[z(x_1,x')-z(x_1+2(\lambda_0-\lambda),x')]/C_\lambda>1\}} dx_1 dx' \\
&\le \frac{1}{C_\lambda} \iint_D z(x_1,x') - z(x_1+2(\lambda_0-\lambda),x') dx_1 dx' \\
&\le \frac{1}{C_\lambda}\int_D z\,dx - \frac{1}{C_\lambda}\int_{D_{2(\lambda_0-\lambda)}} z\,dx \le \frac{2(\lambda_0-\lambda)\sup_D z}{C_\lambda}[\operatorname{diam}(\Omega)]^{N-1} \\
&\le \frac{(\lambda_0-\lambda)\delta M}{48 C_\lambda \operatorname{diam}(\Omega)} \qquad (\lambda\in[\lambda_0-\varepsilon_0,\lambda_0], z\in\omega(u))\,,
\end{aligned}
\tag{4.10}
$$

where I_A is the indicator function of a set A and $D_\mu := \{x\in D : x-\mu e_1 \in D\}$. Finally, let us estimate C_λ. Decrease $\varepsilon_0 > 0$ one more time to obtain

$$
\sup_{\Omega(\lambda)\cap\tilde{U}} \tilde{z} \ge \frac{M}{2} \qquad (\lambda\in[\lambda_0-\varepsilon_0,\lambda_0])\,. \tag{4.11}
$$

Since $\tilde{z}^{\lambda_0}\equiv 0$ in $\tilde{U}$, the inequality (4.8) becomes an equality:

$$
\tilde{z}(x_1+2(\lambda_0-\lambda),x') - \tilde{z}(x_1,x') = V_\lambda \tilde{z}(x_1,x') \ge -6C_\lambda \qquad ((x_1,x')\in\Omega(\lambda)\cap\tilde{U})\,,
$$

where the last inequality follows from (4.6). Hence, the function $\tilde{z}$ cannot decrease in x_1 by more than $6C_\lambda$ on an interval of length $2(\lambda_0-\lambda)$. Moreover, $\tilde{z}(x_1+2(\lambda_0-\lambda),x') = 0$ for $(x_1,x')\in\partial\Omega(\lambda)\setminus H_{\lambda_0}$. Thus using (4.11) we obtain

$$
C_\lambda \ge \frac{(\lambda_0-\lambda)M}{12\operatorname{diam}(\Omega)} \qquad (\lambda\in[\lambda_0-\varepsilon_0,\lambda_0])\,.
$$

A substitution of this estimate into (4.10) yields the desired result. □

The next lemma deals with the strict monotonicity of functions in ω-limit set. The proof can be found in [23, Lemma 4.6].

Lemma 4.8. *Assume that Ω_{λ_0} is connected. Then for any $z\in\omega(u)$ we have either $z\equiv 0$ on Ω_{λ_0} or else $z>0$ and z is strictly decreasing in x_1 in Ω_{λ_0}. The latter holds in the form $z_{x_1}<0$ if $z_{x_1}\in C(\Omega_{\lambda_0})$.*

Once we proved all auxiliary results, it is rather standard to prove our first main theorem.

Proof of theorem 2.1. In addition to the assumptions of this section we also assume (D3). We show that λ_0 defined in (4.2) satisfies the assertions of the theorem. First, by Remark 4.1, each $z \in \omega(u)$ is nonincreasing in x_1 in Ω_{λ_0}.

Assume $\lambda_0 > 0$. By [23, Lemma 4.5], for each $z \in \omega(u)$ there is a connected component U of Ω_{λ_0} such that (2.4) holds true. Next, the existence of $\tilde{z} \in \omega(u)$ such that (2.5), (2.6) hold follows from Proposition 4.4 and the strict monotonicity follows from Lemma 4.8.

Assume $\lambda_0 = 0$. Since Ω_0 is connected, Lemma 4.3 (ii) and (iii) with $\tilde{\lambda} = \lambda_0 = 0$, imply that for each $z \in \omega(u)$ either $z \equiv 0$ and $V_{\lambda_0} z \equiv 0$ in Ω_0 or $z > 0$ and $V_{\lambda_0} z \geq 0$ in Ω_0. It means that either $z \equiv 0$ in Ω or $z > 0$ in Ω. If there is $z \in \omega(u)$ such that $z > 0$ in Ω, then the theorem follows from [23, Theorem 2.2].

If $\omega(u) = \{0\}$, the statement is trivial. □

Let us turn our attention to the second main result.

Proof of theorem 2.2. In the proof we do not apply Convention 3.1 and we indicate explicitly the dependence of all functions and sets on a vector $v \in W$. By Lemma 3.10 we can assume (if we change $\hat{v}$) that for each $v \in W$, (D4) holds with $\hat{v}$ replaced by v.

Let $\delta = \delta(\alpha_0, \beta_0, N) > 0$ be such that Lemma 3.5 holds with $k = 1$. We show that the theorem holds true for any $\delta^* > 0$ for which $|\Omega_{0,e_1} \setminus \Omega_{\delta^*,e_1}| < \frac{\delta}{2}$. (4.12)

In addition to the assumptions of this section we also assume (D3), (D5), (N4) and (D4) with already fixed δ^*. Analogously as before define

$$V_{\lambda,v}\zeta(x) := \zeta(x^{\lambda,v}) - \zeta(x) \qquad (x \in \Omega_{\lambda,v}, v \in W, \zeta \in \omega(u), \lambda > \lambda^*(v)).$$

Moreover for $w^{\lambda,v} = u(x^{\lambda,v}, t) - u(x,t)$ denote

$$\lambda_0(v) := \inf\{\mu > \lambda^*(v) : \lim_{t\to\infty} \|(w^{\lambda,v}(\cdot,t))^-\|_{L^\infty(\Omega_{\lambda,v})} = 0 \text{ for each } \lambda \in [\mu, \ell(v))\}.$$

We now use moving hyperplanes in a direction $v \in W$ in a similar way as we did in the direction e_1. The hyperplanes are now $H_{\lambda,v}$, for $\lambda \in (\lambda^*(v), \ell(v))$. Then one of the following statements is true:

(i) $\lambda_0(v) = \lambda^*(v)$,

(ii) $\lambda_0(v) \in (\lambda^*(v), \ell(v))$ and there exists a connected component $U(v)$ of $\Omega_{\lambda_0(v),v}$ and $\zeta \in \omega(u)$ such that $\zeta > 0$ and $V_{\lambda_0(v),v}\zeta \equiv 0$ in $U(v)$.

To prove this, we use arguments analogous of those used in Lemmas 4.2, 4.3, 4.7, and in Proposition 4.4, where we replace the direction e_1 with v and the assumption $\lambda_0(e_1) > 0$ with $\lambda_0(v) > \lambda^*(v)$. Also the assumption (D1) is replaced by Lemma 3.10, (D2) by (D5), and (N3) by (N4). The assumptions (N1), (N2) remain unchanged as they are independent of a direction, and (D3) was not supposed in these results.

To prove Theorem 2.2 we need to show that $\lambda_0(e_1) = 0$ (the rest of the statements follow from Theorem 2.1). We show that the assumption $\lambda_0(e_1) > 0$ leads to a contradiction.

First assume that $\lambda_0(e_1) > \delta^*$ and the condition (i) holds true for some $v \in W \setminus \{e_1\}$. Then, by Proposition 4.4, there is $\tilde{z} \in \omega(u)$ and a connected component $U(e_1)$ of $\Omega_{\lambda_0(e_1),e_1}$ such that $V_{\lambda_0(e_1),e_1}\tilde{z} \equiv 0$ and $\tilde{z} > 0$ in $U(e_1)$. Also, (i) and Lemma 3.10 imply

$$\Omega_{\lambda_0(e_1)-(\lambda_0(e_1)-\delta^*),e_1} = \Omega_{\delta^*,e_1} \subset \Omega_{\lambda^*(v),v} = \Omega_{\lambda_0(v),v}\,.$$

Consequently, Lemma 3.11 with $\varepsilon = (\lambda_0(e_1) - \delta^*)/2 > 0$ yields $\tilde{x} \in \partial\Omega \cap \partial U(e_1)$ such that $\tilde{x}^{\lambda_0(e_1),e_1} \in \Omega_{\lambda_0(e_1)-2\varepsilon,e_1} \subset \Omega_{\lambda_0(v),v}$.

Furthermore, $V_{\lambda_0(e_1),e_1}\tilde{z}(\tilde{x}) = 0$ and $\tilde{z}(\tilde{x}) = 0$, because $\tilde{x} \in \partial\Omega$. Therefore $\tilde{z}(\tilde{x}^{\lambda_0(e_1),e_1}) = 0$. Then, since $\tilde{x}^{\lambda_0(e_1),e_1} \in \Omega_{\lambda_0(v),v}$, Lemma 4.3 (ii) (in the direction v) with $\tilde{\lambda} = \lambda_0(v)$ yields $\tilde{z} \equiv 0$ in the connected component of $\Omega_{\lambda^*(v),v}$ that has a nonempty intersection with $U(e_1)$. Finally, Lemma 4.3 (ii) (in the direction e_1) with $\tilde{\lambda} = \lambda_0(e_1)$ implies $\tilde{z} \equiv 0$ in $U(e_1)$, a contradiction.

Next, we assume that $\lambda_0(e_1) > \delta^*$ and (ii) hold for all $v \in W$. Since W is uncountable and $|U(v)| > 0$ for each $v \in W$, there are $v, v' \in W$ such that $U(v) \cap U(v') \neq \emptyset$. Denote $\mathcal{A} := U(v) \cup \mathcal{P}_{\lambda_0(v),v} U(v)$, $\mathcal{B} := U(v') \cup \mathcal{P}_{\lambda_0(v'),v'} U(v')$ and without loss of generality assume $\mathcal{B} \not\subset \mathcal{A}$ (otherwise interchange v and v'). Then $\mathcal{B}\backslash\mathcal{A} \neq \emptyset$ and $\partial(\mathcal{B}\backslash\mathcal{A}) \subset \partial\mathcal{B} \cup \partial\mathcal{A}$. Next, we see that $\partial(\mathcal{B}\backslash\mathcal{A}) \not\subset \partial\mathcal{B}$, since otherwise we obtain $\mathcal{B} \setminus \mathcal{A} = \mathcal{B}$, a contradiction to $\emptyset \neq U(v) \cap U(v') \subset \mathcal{A} \cap \mathcal{B}$.

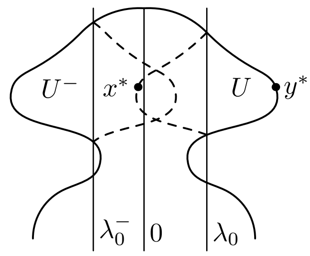
Thus, there is $\hat{x} \in \mathcal{B} \cap \partial\mathcal{A}$, or equivalently

$$\hat{x} \in \mathcal{B} \cap (\mathcal{P}_{\lambda_0(v),v} \partial U(v) \setminus \partial H_{\lambda_0(v),v})\,.$$

By ii) there is $\zeta \in \omega(u)$ such that $\zeta > 0$ in $U(v)$ and $\zeta^{\lambda_0(v),v} \equiv 0$ in $U(v)$ and in particular $\zeta(\hat{x}) = 0$. By the analogous arguments as in the previous case $\zeta \equiv 0$ in $\mathcal{B}$ and since $U(v) \cap U(v') \neq \emptyset$, also $\zeta \equiv 0$ in $U(v)$, a contradiction.

Finally, assume $0 < \lambda_0(e_1) \leq \delta^*$. If we start the process of the moving hyperplanes from the left (or we replace e_1 by $-e_1$), then using analogous arguments as above, we obtain $0 \ge \lambda_0^-(e_1) \ge -\delta^*$, where

$$\lambda_0^-(e_1) := \sup\{\mu < 0 : \lim_{t\to\infty} \|(w^\lambda(\cdot,t))^-\|_{L^\infty(\mathcal{P}_{0,e_1}(\Omega_{\lambda,e_1}))} = 0 \text{ for each } \lambda \in (-\ell,\mu]\}\,.$$



Without loss of generality assume $\lambda_0(e_1) \ge |\lambda_0^-(e_1)|$ (otherwise replace $u(x,t)$ by $u(x^{0,e_1},t)$). By Lemma 4.7, $|\Omega^*_{\lambda_0(e_1),e_1}| > \delta/2$ and since $|\Omega_{0,e_1} \setminus \Omega_{\lambda_0(e_1),e_1}| \le \delta/2$, one has $\mathcal{P}_{\lambda_0(e_1),e_1}(\Omega^*_{\lambda_0(e_1),e_1}) \not\subset \Omega_{0,e_1} \setminus \Omega_{\lambda_0(e_1),e_1}$. Thus, there exist a connected component U of $\Omega^*_{\lambda_0(e_1),e_1}$, and $y^* \in \partial U$ such that $x_1^* < 0$, where $x^* = (x_1^*, (x^*)') := \mathcal{P}_{\lambda_0(e_1),e_1} y^*$ (see figure). Moreover, since $U \subset \Omega^*_{\lambda_0(e_1),e_1}$, there is $z \in \omega(u)$ with $z^{\lambda_0(e_1),e_1} \equiv 0$ and $z > 0$ in U. In particular

$$z(x^*) = 0 \quad \text{and} \quad z(x_1,(x^*)') > 0 \qquad (x_1 \in (x_1^*, 0])\,. \tag{4.13}$$

Denote by U^- the connected component of $\Omega^-_{\lambda_0^-} := \{x \in \Omega : x_1 < \lambda_0^-\}$ that contains $\mathcal{P}_{0,e_1}(U)$. Since $\lambda_0(e_1) \ge |\lambda_0^-(e_1)|$, $x^* \in U^- \cup \mathcal{P}_{\lambda_0^-(e_1),e_1}(U^-)$ and consequently Lemma 4.3 (ii) and (iii) with $\tilde\lambda = -\lambda_0(e_1)$ (with x_1 changed to $-x_1$) yields $z \equiv 0$ in $U^- \cup \mathcal{P}_{\lambda_0^-(e_1),e_1}(U^-)$. In particular $z(x_1,(x^*)') = 0$ for all $x_1 \in (x_1^*, 0]$, a contradiction to (4.13).

Hence, in all cases we found a contradiction. Therefore $\lambda_0(e_1) = 0$ and we are done. □

5 Proofs of Lemma 4.5 and Lemma 4.6

The assumptions in this section are the same as in Section 4 and Proposition 4.4. In particular we assume that Ω satisfies (D1) and (D2) and the nonlinearity F satisfies (N1) – (N3). We consider a classical solution u of (1.1) satisfying (2.1) and (2.2). We also return to Convention 3.1, that is, we do not indicate the dependence of sets or functions on e_1. About λ_0 defined in (4.2) we assume $\lambda_0 > 0$. Recall the notation U_{λ_0}, $U^*_{\lambda_0}$ and U_λ from the proof of Proposition 4.4 (see the paragraph containing (4.5)).

Proof of Lemma 4.5. Let $\delta > 0$ be such that Lemma 3.5 holds with $k = 1$, and fix $\varepsilon_0 > 0$ such that $\lambda_0 > \varepsilon_0$ and $|\Omega_{\lambda_0-\varepsilon_0} \setminus \Omega_{\lambda_0}| < \delta$. We show that the conclusion of the lemma holds true for all $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$. For a contradiction assume that there is $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$, $C_\lambda > 0$ and a sequence $(x_j, t_j)_{j\in\mathbb{N}} \subset U_\lambda \times (0,\infty)$ with $t_j \to \infty$ as $j \to \infty$ and $w^\lambda(x_j, t_j) \le -C_\lambda$.

By b) and (2.3)

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(U^*_{\lambda_0})} = 0\,, \tag{5.1}$$

and in particular there is $T > 0$ such that $\|u(\cdot,t)\|_{L^\infty(U^*_{\lambda_0})} \le \frac{C_\lambda}{4}$ for any $t \ge T$. Consequently $w^\lambda(x,t) \ge -\frac{C_\lambda}{4}$ for all $(x,t) \in U^*_{\lambda_0} \times (T,\infty)$, and therefore $x_j \in U_\lambda \setminus U^*_{\lambda_0}$, $j \in \mathbb{N}$. Now, (5.1) and an application of Lemma 3.5 with $k = 1$ on the set $(U_\lambda \setminus U^*_{\lambda_0}) \times (T, t_j)$ yields for all sufficiently large $j \in \mathbb{N}$

$$\begin{aligned} C_\lambda \le (w^\lambda)^-(x_j,t_j) &\le \|(w^\lambda)^-(\cdot,t_j)\|_{L^\infty(U_\lambda\setminus U^*_{\lambda_0})} \\ &\le 2\max\left(\|(w^\lambda)^-(\cdot,T)\|_{L^\infty(U_\lambda\setminus U^*_{\lambda_0})} e^{T-t_j}, \frac{C_\lambda}{4}\right). \end{aligned}$$

Since w^λ is bounded and T is fixed, the right hand side is less than $\frac{C_\lambda}{2}$ for sufficiently large j, a contradiction. □

Proof of Lemma 4.6. We proceed by a contradiction, that is we assume $\lambda_0 > 0$, (4.4), and the condition c). For a domain $D \subset \Omega$, we define the inner radius of D to be

$$\operatorname{inrad}(D) := \{\rho > 0 : B(x_0,\rho) \subset D \text{ for some } x_0 \in D\}\,,$$

and if D is an open set, we let $\operatorname{inrad}(D)$ stand for the infimum of inner radii of all connected components of D.

Since $U^*_{\lambda_0}$ has finitely many connected components, $\operatorname{inrad}(U^*_{\lambda_0}) = 2r_0 = 2r_0(\lambda_0,\Omega) > 0$, and we can fix $\tilde z \in \omega(u)$ such that $V_{\lambda_0}\tilde z > 0$ holds on the largest number of connected components of $U^*_{\lambda_0}$.

Let $\gamma = \gamma(r_0,\alpha_0,\beta_0,N)$ and h_{r_0} be as in Lemma 3.7 corresponding to $r = r_0$ and choose $\delta = \delta(r_0,\alpha_0,\beta_0,N) > 0$ such that the conclusion of Lemma 3.5 holds true for $k = \gamma + 1$. Let $\varepsilon_0 > 0$ be such that $|U_{\lambda_0-\varepsilon_0} \setminus U^*_{\lambda_0}| < \delta/2$ and for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$, U_λ contains the same number of connected components of Ω_{λ_0} as $U^*_{\lambda_0}$.

Fix an open set $D \subset\subset U^*_{\lambda_0}$ satisfying $|U^*_{\lambda_0}\setminus D| < \delta/2$ such that $\operatorname{inrad}(D) \ge r_0$ and $D \cap V$ is a domain for any connected component V of $U^*_{\lambda_0}$. Then $|U_{\lambda_0-\varepsilon_0} \setminus D| < \delta$.

For already fixed $\tilde z$ denote by $\mathcal{U}^+$, $\mathcal{U}^0$ the set of connected components V of $U^*_{\lambda_0}$ such that $\tilde z^{\lambda_0} > 0$, $\tilde z \equiv 0$ in V, respectively. Then, by (4.4) and c), $\mathcal{U}^+$, $\mathcal{U}^0$ is a partition of the set of connected components of $U^*_{\lambda_0}$ and $\mathcal{U}^+$, $\mathcal{U}^0 \neq \emptyset$.

Next, fix an increasing sequence $(t_k)_{k\in\mathbb{N}}$ converging to ∞, with $u(\cdot,t_k) \to \tilde{z}$ in $C(\bar{\Omega})$ as $k \to \infty$. By the definition of $\mathcal{U}^+$, there is $q > 0$ such that for all sufficiently large n

$$w^{\lambda_0}(x,t_n) \geq 2q \qquad (x \in D \cap V, V \in \mathcal{U}^+)\,.$$

Then by the equicontinuity, with possibly decreased ε_0, there is $\vartheta > 0$ independent of n such that for all $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0]$

$$w^{\lambda}(x,t) \geq q \qquad ((x,t) \in (D \cap V) \times [t_n, t_n + 4\vartheta], V \in \mathcal{U}^+)\,. \tag{5.2}$$

Also, by the definition of $\mathcal{U}^0$

$$\lim_{n\to\infty} \|u(\cdot,t_n)\|_{L^\infty(V)} = 0 \qquad (V \in \mathcal{U}^0)\,. \tag{5.3}$$

Let κ_1, p be constants depending on $r_0, \vartheta, \alpha_0, \beta_0, N, \mathrm{diam}\Omega$ and dist $(D, \partial U^*_{\lambda_0})$ such that Lemma 3.9 holds true for $(D,U,\theta) = (D \cap V, V, \vartheta)$ where V is any connected component of $U^*_{\lambda_0}$. Notice that neither κ_1 nor p depend on ε_0 or n. Let $\tilde{c}_{r_0}$ be as in Corollary 3.8 and set

$$\nu := \frac{1}{4}\frac{\kappa_1^2 \tilde{c}_{r_0} \sigma_{r_0}^2}{\kappa_1 \tilde{c}_{r_0} \sigma_{r_0} + e^{4\beta_0\vartheta}} \quad \text{ where } \quad \sigma_{r_0} := \left(\frac{|B_{\frac{r_0}{2}}|}{|D|}\right)^{\frac{1}{p}} \leq 1\,. \tag{5.4}$$

A continuity argument, with possibly decreased ε_0 (see for example [23, Proof of Lemma 4.3]) implies for sufficiently large n

$$\|(w^\lambda)^-(\cdot,t_n)\|_{L^\infty(U_\lambda)} \leq \nu[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta,t_n+2\vartheta]} \qquad (V \in \mathcal{U}^+, \lambda \in (\lambda_0 - \varepsilon_0, \lambda_0))\,. \tag{5.5}$$

If necessary, decrease ε_0 again to obtain $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U^*_{\lambda_0}$ for each $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$. Now, fix any $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0)$.

Since U_λ is a domain, we can choose a domain $\tilde{D}$ with $\tilde{D} \subset\subset U_\lambda$ and $D \subset \tilde{D}$. Let $\tilde{\kappa}_1$, $\tilde{p}$ be constants depending on $r_0, \vartheta, \alpha_0, \beta_0, N, \mathrm{diam}\Omega$ and dist $(\tilde{D}, \partial U_\lambda)$ such that Lemma 3.9 holds true for $(D,U,\theta) = (\tilde{D}, U_\lambda, \vartheta)$. Finally, choose T such that

$$e^T \geq 2\frac{\kappa_1}{\tilde{\kappa}_1} \text{ and } T > 4\vartheta\,. \tag{5.6}$$

Define

$$T_n := \sup\{\tau : w^\lambda(x,t) > 0, (x,t) \in \bar{V} \times [t_n, t_n+\tau), V \in \mathcal{U}^+\} \qquad (n \in \mathbb{N})\,. \tag{5.7}$$

From (5.2), we have $T_n \geq 4\vartheta$.

Since inrad $(D) \geq r_0$, then for each $V \in \mathcal{U}^+$ there is $x_0^V \in D$ with $B_{r_0}^V := B(x_0^V, r_0) \subset D \cap V$. An application of Corollary 3.8 with $(r,v) = (r_0, w^{\lambda_0})$ and (5.2) imply

$$w^{\lambda_0}(x,t) \geq \tilde{c}_{r_0} q e^{-\gamma(t-t_n)} \qquad ((x,t) \in B^V_{\frac{r_0}{2}} \times [t_n, t_n+T_n], V \in \mathcal{U}^+)\,. \tag{5.8}$$

Next, we show that for $\tilde{T}_n := \min\{T_n, T\}$

$$\lim_{n\to\infty} \sup_{t \in [t_n, t_n+\tilde{T}_n]} \|w^\lambda(\cdot,t)\|_{L^\infty(V\cap D)} = 0 \qquad (V \in \mathcal{U}^0)\,. \tag{5.9}$$

Otherwise, by compactness, there exists $\hat{V} \in \mathcal{U}^0$, $d_0 > 0$, and a sequence $(y_m, s_m)_{m\in\mathbb{N}}$ with $(y_m, s_m) \in (D \cap \hat{V}) \times [t_{n_m}, t_{n_m} + \tilde{T}_{n_m}]$, $s_m \to \infty$ as $m \to \infty$ such that

$$|w^\lambda(y_0, s_m)| > d_0\,.$$

Passing to a subsequence we may assume $u(\cdot, s_m) \to \hat{z}$ in $C(\bar{\Omega})$ and $y_m \to y_0 \in \bar{D} \cap \hat{V}$ as $m \to \infty$ for some $\hat{z} \in \omega(u)$. Consequently

$$|V_\lambda \hat{z}(y_0)| \geq d_0\,. \tag{5.10}$$

Moreover, for each $V \in \mathcal{U}^+$, (5.8) yields $V_{\lambda_0}\hat{z} \geq \tilde{c}_{r_0} q e^{-\gamma T}$ on $B^V_{\frac{r_0}{2}}$ and therefore by Lemma 4.3 i), $V_{\lambda_0}\hat{z} > 0$ on V.

Hence $V_{\lambda_0}\hat{z} > 0$ on V for each $V \in \mathcal{U}^+$. But since $V_{\lambda_0}\tilde{z} > 0$ was true on the largest number of connected components of $U^*_{\lambda_0}$, $V_{\lambda_0}\hat{z} \not> 0$ in any $V \in \mathcal{U}^0$. Then (4.4) implies $\hat{z} \equiv 0$ for each $V \in \mathcal{U}^0$ and by Lemma 4.3 iii) also $V_{\lambda_0}\hat{z} \equiv 0$ on V for each $V \in \mathcal{U}^0$. Therefore in particular $\hat{z} \equiv 0$ on $\hat{V} \cup \mathcal{P}_{\lambda_0}\hat{V}$. Since $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U^*_{\lambda_0}$, one has $V_\lambda \hat{z} \equiv 0$ on $\bar{D} \cap \hat{V}$, a contradiction to (5.10).

Thus (5.9) holds, and in particular there exists n_0 such that

$$\|(w^\lambda)^-\|_{L^\infty((D\cap V)\times[t_n, t_n+\tilde{T}_n])} \leq \frac{\kappa_1 \tilde{c}_{r_0} \sigma_{r_0} q}{8 e^{4\beta_0\vartheta}} e^{-(\gamma+1)T} \qquad (V \in \mathcal{U}^0, n \geq n_0)\,. \tag{5.11}$$

Let us denote

$$\Gamma_0^n := \|(w^\lambda)^-(\cdot, t_n)\|_{L^\infty(U_\lambda)} \qquad (n \in \mathbb{N})\,.$$

An application of Lemma 3.5 on the set $(U_\lambda \setminus D) \times (t_n, t_n + \tilde{T}_n)$ and (5.11) yield

$$
\begin{aligned}
\|(w^\lambda)^-(\cdot,t)\|_{L^\infty(U_\lambda)} &\le 2\max\{e^{-(\gamma+1)(t-t_n)}\Gamma_0^n, e^{-(\gamma+1)T}\frac{\kappa_1 \tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\} \\
&\le 2e^{-(\gamma+1)(t-t_n)}\left(\Gamma_0^n + \frac{\kappa_1 \tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \qquad (t \in [t_n, t_n + \tilde{T}_n], n \ge n_0)\,.
\end{aligned} \tag{5.12}
$$

Next, since $\tilde{c}_{r_0} \le 1$, Lemma 3.9 with $(D, U, \theta) = (D \cap V, U_\lambda, \vartheta)$, $V \in \mathcal{U}^+$ and (5.2), (5.5), (5.12) imply

$$
\begin{aligned}
w^\lambda(x,t) &\ge \kappa_1 [w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta, t_n+2\vartheta]} - e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda \times (t_n, t_n+4\vartheta))} (w^\lambda)^- \\
&= \frac{\kappa_1}{2}[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta, t_n+2\vartheta]} + \frac{\kappa_1}{2}[w^\lambda]_{p,(D\cap V)\times[t_n+\vartheta, t_n+2\vartheta]} \\
&- e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda \times (t_n, t_n+4\vartheta))} (w^\lambda)^- \\
&\ge \kappa_1 \sigma_{r_0} \frac{1}{2}\left(\frac{\Gamma_0^n}{\nu} + q\right) - 2e^{4\beta_0\vartheta}\left(\Gamma_0^n + \frac{\kappa_1 \tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \\
&\ge \Gamma_0^n \left(\frac{\kappa_1 \sigma_{r_0}}{2\nu} - 2e^{4\beta_0\vartheta}\right) := \Gamma_1^n \\
&\qquad\qquad ((x,t) \in D \cap V \times (t_n + 3\vartheta, t_n + 4\vartheta), n \ge n_0)\,.
\end{aligned}
$$

This, (5.2) and Corollary 3.8 with $r = r_0$ and q replaced by $\frac{1}{2}(q + \Gamma_1^n)$ imply

$$
\begin{aligned}
w^\lambda(x,t) &\ge \frac{\tilde{c}_{r_0}}{2}(q + \Gamma_1^n)e^{-\gamma(t-t_n-4\vartheta)} \\
&\qquad ((x,t) \in B^V_{\frac{r_0}{2}} \times [t_n + 4\vartheta, t_n + \tilde{T}_n], V \in \mathcal{U}^+, n \ge n_0)\,.
\end{aligned} \tag{5.13}
$$

To obtain a contradiction, and finish the proof of the lemma, we show that neither $T_n \le T$ nor $T_n > T$ is possible for infinitely many n.

Case 1. There exist an infinite subset S of positive integers such that $\tilde{T}_n = T_n \le T$ for all $n \in S$. Then for any $V \in \mathcal{U}^+$, Lemma 3.9 with

$(D,U,\theta) = (D\cap V, U_\lambda, \vartheta)$, (5.13), (5.12) and (5.4) yield

$$\begin{aligned}
& w^\lambda(x,t_n+T_n) \\
& \quad \geq \kappa_1 [w^\lambda]_{p,(D\cap V)\times[t_n+T_n-3\vartheta,t_n+T_n-2\vartheta]} - e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda\times(t_n+T_n-4\vartheta,t_n+T_n))} (w^\lambda)^- \\
& \quad \geq \frac{\kappa_1 \tilde{c}_{r_0}\sigma_{r_0}}{2} e^{-\gamma T_n}(\Gamma_1^n + q) - 2e^{4\beta_0\vartheta} e^{-(\gamma+1)T_n} \left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \\
& \quad \geq e^{-\gamma T_n}\Gamma_0^n \left(\frac{\kappa_1^2\tilde{c}_{r_0}\sigma_{r_0}^2}{2\nu} - 2\kappa_1\tilde{c}_{r_0}\sigma_{r_0}e^{4\beta_0\vartheta} - 2e^{4\beta_0\vartheta}\right) + \frac{\kappa_1}{4} e^{-\gamma T_n}\tilde{q}\sigma_{r_0} \\
& \quad > 0 \qquad (x\in(\bar{D}\cap V), n\geq n_0)\,,
\end{aligned}$$

a contradiction to (5.7).

Case 2. For all sufficiently large n, $\tilde{T}_n = T < T_n$. By a similar argument as in Case 1, Lemma 3.9 with $(D,U,\theta) = (\tilde{D}, U_\lambda, \vartheta)$, (5.13), (5.12), (5.6) and (5.4) yield

$$\begin{aligned}
& w^\lambda(x,t_n+T) \\
& \quad \geq \tilde{\kappa}_1 [w^\lambda]_{p,(D\cap V)\times[t_n+T-3\vartheta,t_n+T-2\vartheta]} - e^{4\beta_0\vartheta} \sup_{\partial_P(U_\lambda\times(t_n+T-4\vartheta,t_n+T))} (w^\lambda)^- \\
& \quad \geq \frac{\tilde{\kappa}_1 \tilde{c}_{r_0}\sigma_{r_0}}{2} e^{-\gamma T}(\Gamma_1^n + q) - 2e^{4\beta_0\vartheta} e^{-(\gamma+1)T} \left(\Gamma_0^n + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{8e^{4\beta_0\vartheta}}\right) \\
& \quad \geq e^{-(\gamma+1)T}\Gamma_0^n \left(\frac{\kappa_1^2\tilde{c}_{r_0}\sigma_{r_0}^2}{2\nu} - 2\kappa_1\tilde{c}_{r_0}\sigma_{r_0}e^{4\beta_0\vartheta} - 2e^{4\beta_0\vartheta}\right) + \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{4} e^{-(\gamma+1)T} \\
& \quad \geq \frac{\kappa_1\tilde{c}_{r_0}\sigma_{r_0} q}{4} e^{-(\gamma+1)T} \qquad (x\in\tilde{D}, n\geq n_0)
\end{aligned} \tag{5.14}$$

Passing to a subsequence we may assume $u(\cdot,t_n+T)\to\hat{z}$ in $C(\bar{\Omega})$ as $n\to\infty$ for some $\hat{z}\in\omega(u)$. Then, by the same arguments as before ((5.8) and maximality property of $\tilde{z}$) we obtain $\hat{z}, \hat{z}^{\lambda_0} \equiv 0$ in V for all $V\in\mathcal{U}^0$. But, since $\mathcal{P}_\lambda D \subset \mathcal{P}_{\lambda_0} U_{\lambda_0}^*$, one has $\hat{z}^\lambda \equiv 0$ in $D\cap V\subset\tilde{D}$, $V\in\mathcal{U}^0$, a contradiction to (5.14).

This finishes the proof. □

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