On Serrin's symmetry result in nonsmooth domains and its applications

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Abstract

The goal of this paper is to show that Serrin's result on overdetermined problems holds true for symmetric non-smooth domains. Specifically, we show that if a non-smooth domain D satisfies appropriate symmetry and convexity assumptions, and there exists a positive solution to a general overdetermined problem on D, then D must be a ball. As an application, we improve results on symmetry of non-negative solutions of Dirichlet problems.

1 Introduction

In this paper we consider the overdetermined fully nonlinear radially symmetric problem

$$F(\Delta u, u_{x_i} u_{x_i x_j} u_{x_j}, |\nabla u|^2, u) = 0\,, \qquad x \in D\,, \tag{1.1}$$
$$u = 0\,, \quad |\nabla u| = c\,, \qquad x \in \partial D\,, \tag{1.2}$$
$$u > 0\,, \qquad x \in D\,. \tag{1.3}$$

Here, and also in the rest of the paper we use the summation convention, that is, when an index appears twice in a single term, then we are summing over all its possible values.

In a celebrated paper [27], Serrin showed that if the problem is quasilinear. Specifically, if u satisfies conditions (1.2), (1.3), and the equation

$$a(u, |\nabla u|^2)\Delta u + h(u, |\nabla u|^2) u_{x_i} u_{x_i x_j} u_{x_j} + f(u, |\nabla u|^2) = 0, \tag{1.4}$$

for some $a, h, f \in C^1_{loc}(\mathbb{R}^2)$, D satisfies the interior sphere condition, and $u \in C^2(\bar{D})$, then D is a ball. The proof uses the strong maximum principle, the corner point lemma, and the method of moving hyperplanes introduced by Alexandrov [1] for problems in geometry. Some of these techniques rely on sufficient regularity of D and on the smoothness of u in the closure of D. We also remark that quasilinear structure was essential for the proof.

There is a vast literature that uses Serrin's framework and generalizes the symmetry results in various directions, such as systems of quasilinear equations, exterior domains, ring shaped domains, other boundary conditions etc. Most of these extensions (the others are discussed below) require quasilinear structure, smoothness of ∂D, and $u \in C^2(\bar{D})$, or an additional regularity on F, usually analyticity. For references, we direct the interested reader to the survey [26].

Another fruitful direction for proving the symmetry of D for overdetermined problems was given in the paper of Weinberger [30]. His method, based on integral estimates, does not require either regularity of D or $u \in C^2(\bar{D})$; however, it was applied only to equation

$$\Delta u = -1 \,.$$

To generalize Weinberger's result one can replace the Laplacian operator by a possibly degenerate elliptic operator, generalize the domain, or boundary condition, but the constant right hand side seems to be essential for the method. We again refer the reader to the survey [26].

There are other approaches showing that D is a ball if u solves certain overdetermined problem. These, as the Serrin's and Weinberger's method, usually depend either on the regularity of D and smoothness of u on $\bar{D}$, or they require a special structure of the equation.

Quasilinear overdetermined problem (1.4), (1.2), (1.3) on nonsmooth domains was already studied by Vogel [29] in the case $c > 0$. As a result he showed that D is a ball, provided $F \in C^1_{loc}(\mathbb{R})$ and $u \in C^2(\Omega)$ satisfies (1.1), where the boundary conditions (1.2) are satisfied in the following sense:

$$u \to 0 \,, \quad \text{and} \quad |\nabla u| \to c \,, \quad \text{uniformly as} \quad x \to \partial D \,. \tag{1.5}$$

However, the very important case $c = 0$, for non-smooth domains, was not discussed in [29]. First, note that if $c = 0$ and $u \in C^1(D) \cap C(\bar{D})$, then (1.5) and the mean value theorem imply $u \in C^1(\bar{D})$. To illustrate the relevance of the case $c = 0$, assume that D is a planar domain and there exist a point $A \in \partial D$ and two smooth curves $\gamma_i : [0,1] \to \partial D$, $(i = 1, 2)$ such that $\gamma_i(0) = A$, and tangent vectors $\tau_i := \gamma_i'(0)$ at A are linearly independent. Since $u \in C^1(\bar{D})$ and $u = 0$ on ∂D it is easy to show that $\nabla u(A) \cdot \tau_i = 0$, and consequently the linear independence of τ_1 and τ_2 yields $|\nabla u(A)| = c = 0$. This example can be easily generalized to other domains with non-smooth boundaries and domains in higher dimensions.

We remark that Sirakov [28, Remark 1] claimed the Serrin's result for $u \in C^1(\bar{\Omega})$ and smooth Ω under implicit geometrical conditions on Ω; however, the proof or precise assumptions were not provided.

Our first goal is to show that if $c = 0$, Serrin's result holds for the overdetermined problem (1.1)–(1.3) on symmetric nonsmooth domains. This will generalize Serrin's result to fully nonlinear problems on non-smooth domains.

We remark that one can assume that u is not C^1 on $\bar{D}$, or that the Neumann condition in (1.2) is not satisfied everywhere. Such quasilinear problem was studied by Prajapat [25], when D is not smooth at a single point and u does not satisfy a Neumann condition at that point. Other results when u satisfies semilinear problem and either Dirichlet or Neumann boundary conditions are assumed only on a part of the boundary were obtained by Fragalà et al. [10, 11].

Recently Poláčik [24] showed the importance of the case $c = 0$ for the proof of symmetry of solutions satisfying Dirichlet problems. Before we describe Poláčik's idea let us mention older results on the reflectional symmetry.

Using the method of moving hyperplanes, Gidas et. al. [12] proved the symmetry of positive solutions on symmetric domains. Specifically, if $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, convex in x_1, and symmetric with respect to the hyperplane

$$H_0 := \{x \in \mathbb{R}^N : x_1 = 0\} ,$$

and f is a Lipschitz function, then a positive classical solution u of

$$\Delta u + f(u) = 0, \qquad x \in \Omega , \tag{1.6}$$

$$u = 0, \qquad x \in \partial\Omega , \tag{1.7}$$

is even in x_1 and nonincreasing in the set

$$\Omega_0 := \{x \in \Omega : x_1 > 0\} .$$

Later, the results of Gidas et al. were generalized by Li [19] to fully nonlinear problems. Berestycki and Nirenberg [4], and Dancer [8] extended the symmetry results to nonsmooth domains. We refer the reader to the surveys [3, 20, 23] for more results, references, and generalizations.

The situation is more interesting if we merely assume that u is non-negative. If $\Omega \in \mathbb{R}^N$ is convex in x_1 and symmetric with respect to H_0, then the symmetry and monotonicity result of [12] is equivalent to the statement:

$$\text{If } u \geq 0 \text{ in } \Omega, \text{ then either } u \equiv 0 \text{ or } u > 0 \text{ in } \Omega. \tag{1.8}$$

If $N = 1$, then (1.8) does not hold true as shown in [12]. Indeed, the function $u = 1 - \cos x$ satisfies $u'' + u - 1 = 0$ on $\Omega = (-2\pi, 2\pi)$ with the Dirichlet boundary condition. Although u is symmetric, it is not decreasing for $x > 0$.

If $N \geq 2$, we were not able to locate in the literature any example that would contradict (1.8) for solutions of (1.6), (1.7). In this paper we show that if such an example exists for symmetric Ω, then Ω must have very special shape. We remark that Poláčik [21, 22] constructed several counterexamples to (1.8) on various domains if f depends on u and also on $x' := (x_2, x_2, \cdots, x_N)$. For such general problem one might investigate symmetry of non-monotonic solutions as in [21].

If u satisfies (1.6), (1.7) and $f(0) \geq 0$, then the maximum principle implies (1.8). For general f, (1.8) was proved by Castro and Shivaji [6] when Ω is a ball, and more general conditions were given by Hess and Poláčik [14], Damascelli et al. [7], and the present author [9].

Recently Poláčik [24] showed (1.8), provided Ω is a C^2 domain which is convex in x_1 and symmetric about H_0, and $u \in C^2(\bar{\Omega})$ satisfies (1.4) and Dirichlet boundary conditions. The proof in [24] is of our interest since it connects symmetry and monotonicity properties of solutions to certain overdetermined problems. Let us briefly describe the idea of the proof, more details are mentioned in the proofs below. If a non-negative u is not symmetric or monotonic, then by a use of the method of moving hyperplanes one obtains an overdetermined problem (1.1) with $c = 0$ on a subdomain $D \subset \Omega$ which is convex in x_1 and symmetric with respect to a hyperplane $H_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$ for some $\lambda > 0$. Using that Ω is of class C^2 and $u \in C^2(\bar{\Omega})$, one proves that D is a smooth domain, and consequently by Serrin's result D, must be a ball. Finally, the unique continuation argument yields that $u \equiv 0$.

Motivated by [24] we first focus on the problem (1.1) with $c = 0$ and D having an additional reflectional symmetry and convexity. Since Ω is not smooth, it is not natural to assume $u \in C^2(\bar{D})$; however, to satisfy the boundary conditions in (1.1) we assume (1.5) with $c = 0$, which yields $u \in C^1(\bar{D})$. Note that this assumption is satisfied for irregular domains as well (see Remark 1.11 below).

Let us formulate assumptions on the nonlinearity $F : (d, q, p, u) \in \mathbb{R}^4 \to \mathbb{R}$.

(N1) (*Regularity.*) Assume F is locally Lipschitz continuous in (p, u) and C^1 in (d, q).

(N2) (*Ellipticity.*) There is $\alpha_0 > 0$ such that

$$F_d(d, q, |p|^2, u) + F_q(d, q, |p|^2, u)(p_i \xi_i)^2 \geq \alpha_0 > 0 \qquad ((d, q, u) \in \mathbb{R}^3, p, \xi \in \mathbb{R}^N, |\xi| = 1)\,. \quad (1.9)$$

Before we state our first main result, denote $S := \{1, 2, \cdots, N\}$ and let $(e_i)_{i \in S}$ be the standard basis of $\mathbb{R}^N$.

Theorem 1.1. *Let $D \subset \mathbb{R}^N$ be a bounded domain, symmetric with respect to the hyperplane $H_0^1 := \{x \in \mathbb{R}^N : x_1 = 0\}$ and convex in x_1 and F satisfy (N1) and (N2). If $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ satisfies* (1.1)*,* (1.2)*,* (1.3) *with $c = 0$, and one of the assumptions*

(a) $u \in C^2(\bar{D})$,

(b) $|\nabla u|^2$ is Lipschitz in $\bar{\Omega}$ and $\nabla F(\cdot, q, p, u)$ is globally bounded,

(c) F is independent of q and $\nabla F(\cdot, q, p, u)$ is globally bounded,

then D is a ball, and u is radially symmetric and radially decreasing.

Since the equation (1.1) is radially symmetric and translational invariant, we can replace the assumption on the convexity and symmetry in x_1 by any direction. As we are only interested in the behavior of F on the range of u (and its derivatives), it suffices to assume local Lipschitz continuity in p and u, and if (a) is assumed, then we only require F to be locally Lipschitz. Similarly one can weaken ellipticity condition (1.9) and boundedness assumption on ∇F in (b) and (c).

Since $|\nabla u| = 0$ on $\partial\Omega$, the assumption on the Lipschitz continuity of $|\nabla u|^2$ is weaker than the boundedness of D^2u. As mentioned above, the assumption $u \in C^1(\bar{\Omega})$ can be replaced by $u \in C(\bar{\Omega})$ and (1.5). If u satisfies quasilinear problem (1.4), by combining Theorem 1.1 with the Vogel's result one can remove the assumption $c = 0$.

The case (a) is a direct improvement of Serrin's result (if $c = 0$) to the fully nonlinear problems on non-smooth, symmetric domains. Cases (b) and (c) show that the $C^2(\bar{\Omega})$ regularity of u can be weakened as well. If the equation (1.1) is semilinear, that is, $F(d, q, |p|^2), u) = d + f(|p|^2, u)$, then (c) is automatically satisfied.

Remark 1.2. From the proof, immediately follows that if F also depends on $\tilde{x} := (x_3, x_4, \cdots, x_N)$, and the remaining assumptions are considered to be uniform in $\tilde{x}$, then u and Ω are rotationally symmetric with respect to rotations rotating e_1, e_2 only.

Remark 1.3. If F_d and F_q are Lipschitz in $\mathbb{R}^4$ and $u_{x_i x_j}$ is Lipschitz on Ω, then instead of (1.3) one can only assume that u is non-negative, nontrivial function ($u \not\equiv 0$); if the problem is quasilinear (see (1.4)) then we do not have to assume Lipschitz continuity of $u_{x_i x_j}$. Indeed, by [21, Theorem 2.2], there are finitely many disjoint sets $G_i \subset \Omega$ such that $\bar{\Omega} = \cup \bar{G}_i$, each G_i is convex in x_1 and symmetric with respect to a hyperplane with the normal vector e_1. Moreover, for each i, $u > 0$ in G_i and $u = 0$ on ∂G_i.

Since $u \geq 0$ in Ω, any $x \in \partial G_i \cap \Omega$ is a local minimum of u, and therefore $\nabla u(x) = 0$. Also by (1.2), $\nabla u(x) = 0$ for each $x \in \partial G_i \setminus \Omega \subset \partial\Omega$. Clearly $u \in C^1(\bar{G}_i)$ and if one of (a)–(c) in Theorem 1.1 is true then it also holds true for u restricted to G_i. Therefore we can use Theorem 1.1 with Ω replaced by G_i and conclude that G_i is a ball. However, since Ω convex in x_1, it cannot be a union of finitely many balls unless Ω is a ball.

Remark 1.4. Notice that the overdetermined problem (1.1), (1.2), (1.3) has a solution if D is a ball and $c = 0$. For example if J_0 is the Bessel function of the first kind, then $\Delta J_0(|x|) = \mu J_0(|x|)$ for appropriate μ. Let ξ be the first positive zero of J_0'. Then $u(x) = J_0(|x|) - J_0(\xi)$ satisfies $\Delta u = \mu(u + J_0(\xi))$ and $u(x) = \nabla u(x) = 0$ for $|x| = \xi$. Also, since J_0 decreases on $(0, \xi)$, one has $u > 0$.

If the domain is not symmetric and convex in x_1 (or in any other direction), then we show Serrin's result only for quasilinear problem and Ω which is not smooth on a lower dimensional set. Analogously as above we say that (1.4) is elliptic if

$$a(u, |p|^2) + h(u, |p|^2)(p_i\xi_i)^2 \geq \alpha_0 > 0, \qquad (u \in \mathbb{R}, p, \xi \in \mathbb{R}^N, |\xi| = 1)\,. \quad (1.10)$$

Proposition 1.5. *Let $D \subset \mathbb{R}^N$ be a bounded domain such that $\partial D \setminus X$ is of class C^2 for $X \subset H_0^1$. If there exist locally Lipschitz functions a, h satisfying* (1.10)*, and $u \in C^1(\bar{D}) \cap C^2(\bar{D} \setminus X)$ satisfying* (1.4), (1.2), (1.3)*, then D is a ball.*

As above we can replace the assumption $X \subset H_0^1$ by $X \subset H$, where H is any hyperplane in $\mathbb{R}^N$. Also ellipticity condition (1.10) can be considered only on the range of u and ∇u.

As in [24] we apply Theorem 1.1, to the study of symmetry of nonnegative solutions on reflectional symmetric domains and we improve the main result of [24] to a wide range of non-smooth domains and fully nonlinear equations. From a private communication the author learned that P. Poláčik proved more general independent results for two dimensional, nonsmooth domains.

First we state the result for fully nonlinear equations on smooth domains.

Corollary 1.6. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded C^2 which is convex in x_1 and symmetric with respect to H_0. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be a C^1 function satisfying (1.9). If $u \in C^3(\bar{\Omega})$ is a nonnegative solution of*

$$\begin{aligned} F(\Delta u, u_{x_i} u_{x_i x_j} u_{x_j}, |u|^2, u) &= 0, \qquad x \in \Omega\,, \\ u &= 0, \qquad x \in \partial\Omega\,, \end{aligned}$$

then either $u \equiv 0$, or $u > 0$.

The proof of Corollary 1.6 is an straightforward modification of the proof [24, Theorem 1.1], where we use Theorem 1.1 instead of Serrin's result [27]. The C^3 regularity of u is merely assumed because of the unique continuation method that requires Lipschitz continuity of leading coefficients (cp. [16]).

The assumption $u \in C^3(\bar{\Omega})$ (and even $u \in C^3(\bar{\Omega})$) is usually too restrictive for problems on non-smooth domains. Under natural assumption $u \in C(\bar{\Omega})$, we will state our results for semilinear problems only. Interested reader can formulate analogous results for quasilinear or fully nonlinear problems under stronger regularity assumptions on u. Denote $\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$ and let $\mathcal{P}_\lambda(x_1, x') := (2\lambda - x_1, x')$ be the reflection with respect to the hyperplane H_λ, where $x' = (x_2, \cdots, x_N)$.

Definition 1.7. *We say that a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies condition (A) if the following holds. If there is $A \in \partial\Omega$ and $\varepsilon > 0$ such that $A - te_1 \in \partial\Omega$ for any $t \in [0, \varepsilon]$, then either*

(a) $\partial\Omega$ is of class $C^{2,\beta}$ at A for some $\beta > 0$, or

(b) for any connected component D of Ω_{A_1} with $A \in \bar{D}$, there exists $X \in \partial D$ with $X_1 > A_1$ such that $\mathcal{P}_{A_1} X \in \Omega$ and Ω is not $C^{2,\alpha}$ at X for some $0 \le \alpha < 1$.

The definition asserts that if $\partial\Omega$ contains a segment in direction e_1, then either $\partial\Omega$ is smooth along the segment, or $\partial\Omega$ has a point of nonsmoothness to the right of the segment. Of course Ω trivially satisfies (A) if $\partial\Omega$ does not contain any such segment.

Definition 1.8. *Assume D is a connected component of Ω_λ $(\lambda > 0)$ and $M := \partial D \cap \{x : x_1 > \lambda\}$ is of class C^2. We say that u satisfies condition (B) if $\nabla u = 0$ on M implies $u \in C^1(\bar{D})$.*

We will discuss sufficient condition (B) in Remark 1.11. We are ready to state the next main result.

Theorem 1.9. *Let $f \in C^{1,\alpha}_{loc}(\mathbb{R})$ for some $\alpha \in (0,1)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain, which is convex in x_1, symmetric with respect to $H_0 := \{x : x_1 = 0\}$, and satisfies the condition (A). Assume that a non-negative function $u \in C(\bar{\Omega})$ satisfies* (1.6)*,* (1.7)*, and the assumption (B). Then u is even in x_1, and either $u \equiv 0$, or $u > 0$ and $u_{x_1}(x) < 0$ for $x \in \Omega$ with $x_1 > 0$.*

Remark 1.10. We mainly assume $f \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, instead of $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^N)$ because of Schauder's boundary estimates on smooth domains.

Without additional assumptions on u or Ω, we can generalize Theorem 1.9 to quasi-linear equations of the form

$$\Delta u + F(u, |\nabla u|^2) = 0\,, \tag{1.11}$$

assuming $F \in C^{1,\alpha}$. The proof is analogous, but more technical.

It is not known whether the assumption (A) is technical or not. We remark that if the problem is semilinear, then one can replace $C^{2,\alpha}$ in Definition 1.7 (b) by $C^{3,\alpha}$, as mentioned in the proof.

Remark 1.11. Standard interior Schauder estimates imply $u \in C^1(D)$ and since $M \in C^{1,\alpha}$ $(\alpha < 1)$, [17, 18] yield that $u \in C^1(D \cup M)$. Hence assumption (B) really requires u to be C^1 on $\partial D \cap \partial\Omega \cap H_\lambda$.

If D does not have the relevant part of the boundary of class C^2, then (B) is trivially satisfied. In particular (B) is satisfied if $\partial\Omega$ is not of class C^2 (in fact $C^{2,\alpha}$ for some $\alpha < 1$, as seen from the proof) at y for all $y = (y_1, y') \in \partial\Omega$ with $y_1 = \ell := \sup\{x_1 : x \in \Omega\}$. In this case (A) is trivially satisfied as well. Moreover, it suffices to assume that such y exists with $\ell - y_1 < \delta$ for sufficiently small δ (see Lemma 2.4).

If $\partial\Omega$ is of class $C^{1,\alpha}$, then again [17, 18] imply $u \in C^1(\bar{\Omega})$ and in particular $u \in C^1(\bar{D})$.

If $\partial\Omega$ is not of class $C^{1,\alpha}$ we have the following result.

Corollary 1.12. *If Ω is piecewise $C^{2,\alpha}$ domain, then Theorem 1.9 holds true without assumption (B).*

Since the investigation of the regularity of solutions on nonsmooth domains is not primary goal of this paper, we leave formulation of other sufficient conditions that guarantee (B) to an interested reader.

2 Proofs of main results

In this section we use the notation and definitions from the introduction and the following one from [13].

Definition 2.1. *Given $k \in \{0, 1, \cdots\}$, $0 \leq \alpha \leq 1$, we say that a bounded domain Ω in $\mathbb{R}^N$ is $C^{k,\alpha}$ at $x_0 \in \partial\Omega$, if there exist a ball B centered at x_0 and a one-to-one mapping Ψ of B onto $D \subset \mathbb{R}^N$ such that*

$$(i)\Psi(B \cap \Omega) \subset \mathbb{R}^N_+; \quad (ii)\Psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^N_+; \quad (iii)\Psi \in C^{k,\alpha}(B), \Psi^{-1} \in C^{k,\alpha}(D).$$

Here, $C^{k,\alpha}$ denotes the standard space of Hölder continuous functions (see [13]). We say that a bounded domain Ω in $\mathbb{R}^N$ is of class $C^{k,\alpha}$, if it is of class $C^{k,\alpha}$ at any $x_0 \in \partial\Omega$.

Remark 2.2. Note that if Ω is of class $C^{k,\alpha}$ at $x_0 \in \partial\Omega$, then it is of class $C^{k,\alpha}$ at any $x \in \partial\Omega \cap B$, where B is as in Definition 2.1.

2.1 Proofs for overdetermined problems

Proof of Theorem 1.1. Since D is symmetric with respect to H^1_0 and convex in x_1, the symmetry result of [4] implies that u is even in x_1 and $u_{x_1} < 0$ on $D^+ := \{x \in D : x_1 > 0\}$. Also, the symmetry implies $u_{x_1} = 0$ on $H^1_0 \cap D$.

If we differentiate (1.1) with respect to x_1, and we use the Neumann boundary conditions from (1.2), then $v := u_{x_1}$ satisfies

$$\begin{aligned} a_{ij}(x)v_{x_ix_j} + b_i(x)v_{x_i} + c(x)v = \lambda v\,, \qquad & x \in D\,, \\ v = 0\,, \qquad & x \in \partial D\,, \end{aligned} \tag{2.1}$$

where $\lambda = 0$,

$$\begin{aligned} a_{ij} &:= F_d[u]\delta_{ij} + F_q[u]u_{x_i}u_{x_j} \\ b_i(x) &:= F_q[u](|\nabla u|^2)_{x_i} + F_p[u]u_{x_i}, \\ c(x) &:= F_u[u], \\ F_h[u] &:= F_h(\Delta u, u_{x_i}u_{x_ix_j}u_{x_j}, |u|^2, u), \qquad h = d, q, p, \text{or } u \end{aligned}$$

As usual $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Since F is C^1 in d and q, a_{ij} are continuous functions. Moreover, (1.9) guarantees ellipticity of (2.1).

If (a) holds, that is, if $u \in C^2(\bar{\Omega})$, then the argument of F is bounded, and therefore local Lipschitz continuity implies the boundedness of a_{ij}, b_i, and c.

If (b) holds true, then $u_{x_i}u_{x_ix_j}u_{x_j} = \frac{1}{2}\nabla u \cdot \nabla(|\nabla u|^2)$, and therefore the arguments of F in variables q, p, and u are bounded. Now, boundedness of ∇F in d and local Lipschitz continuity yield the boundedness of the coefficients a_{ij}, b_i, and c.

If (c) holds true, then the boundedness of a_{ij}, b_i, and c is immediate.

Also, $v \in C^1(\Omega) \cap C(\bar{\Omega})$, and therefore $v \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$. By standard interior regularity theory $v \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ for each $p > 1$.

Since $u_{x_1} = 0$ on $H^1_0 \cap D$, $\lambda = 0$ is the first eigenvalue of (2.1), with D replaced by D^+, corresponding to the principal (negative) eigenfunction u_{x_1}. Fix arbitrary $j \in \{2, \cdots, N\}$, denote $\partial_\phi := x_j\partial_{x_1} - x_1\partial_{x_j}$ and $w := \partial_\phi u$. Then

$$\begin{aligned} 2\nabla u \nabla(\partial_\phi u) - \partial_\phi(|\nabla u|^2) &= 2u_{x_k}(\delta_{jk}u_{x_1} - \delta_{1k}u_{x_j}) = 0 \\ \Delta(\partial_\phi u) - \partial_\phi(\Delta u) &= 2(\delta_{kj}\partial_{x_1x_k}u - \delta_{k1}\partial_{x_jx_k}u) = 0\,, \end{aligned}$$

and

$$\begin{aligned}&w_{x_k}u_{x_kx_l}u_{x_l}+u_{x_k}u_{x_kx_l}w_{x_l}+u_{x_k}w_{x_kx_l}u_{x_l}-\partial_\phi[u_{x_k}u_{x_kx_l}u_{x_l}]\\&=[\delta_{jk}u_{x_1}-\delta_{1k}u_{x_j}]u_{x_kx_l}u_{x_l}+[\delta_{jl}u_{x_1}-\delta_{1l}u_{x_j}]u_{x_k}u_{x_kx_l}\\&+[(\delta_{jk}u_{x_1}-\delta_{1k}u_{x_j})_{x_l}+(\delta_{jl}u_{x_1}-\delta_{1l}u_{x_j})_{x_k}]u_{x_k}u_{x_l}\\&=0\,.\end{aligned}$$

Hence, after differentiating (1.1) with respect to ∂_ϕ we obtain that $w(x) := x_j u_{x_1}(x) - x_1 u_{x_j}(x)$ also satisfies (2.1) with $\lambda = 0$. Since $u_{x_1} = 0$ on $H_0^1 \cap D$ one has $w(x) = 0$ for each $x \in H_0^1 \cap D$, and therefore $w(x) = 0$ for each $x \in \partial D_k^+$.

Thus, by [5, Corollary 2.2], w is a constant multiple of a principal eigenfunction, that is, $w = c_j u_{x_1}$ for some $c_j \in \mathbb{R}$. Equivalently $(x_j - c_j)u_{x_1}(x) - x_1 u_{x_j}(x) = 0$.

Solving this first order partial differential equation by the method of characteristics, one obtains that u, instead of x_1, x_j, depends only on $x_1^2 + (x_j - c_j)^2$. After translation by c_j in x_j (which does not change the form of the problem (1.1)) we obtain that u is rotationally symmetric with respect to rotations that rotate the plane spanned by e_1 and e_j only.

Using the rotation of the e_1, e_N plane, we obtain

$$u(x) = u(\sqrt{x_1^2 + x_N^2}, x_2, \cdots, x_{N-1}, 0)\,.$$

Next, using the rotation of the e_1, e_{N-1} plane (which does not change x_N coordinate), we have $u(x_1, \cdots, x_N) = u(\sqrt{x_1^2 + x_N^2 + x_{N-1}^2}, x_2, \cdots, x_{N-2}, 0, 0)$. Proceeding by induction we conclude $u(x) = u(\sqrt{x_1^2 + \cdots + x_N^2}, 0, \cdots, 0)$, and therefore u is radially symmetric.

Positivity of u in D and Dirichlet boundary conditions imply that D is radially symmetric as well. Monotonicity of u in r follow from [4]. □

Proof of Proposition 1.5. Let us assume that D is not a ball.

To prove the statement we use the method of moving hyperplanes in the same way as in [27] (notice that [27] also discusses quasilinear equations (1.4)).

The hyperplanes are $H_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$ and we start with $\lambda \gg 1$. Denote $D_\lambda := \{x \in D : x_1 > \lambda\}$ and let D'_λ be the reflection of D_λ with respect to H_λ.

We decrease λ until either $\lambda = 0$, or $\lambda > 0$ and one of the following events occurs:

(i) H_λ is orthogonal to ∂D at some point (since ∂D is smooth at any $x \in \partial D$ with $x_1 > 0$, orthogonality is defined by $\nu_1(x) = 0$, where $\nu(x)$ is a unit normal vector to ∂D at x),

(ii) D'_λ is internally tangent to the boundary of D at some point $P \notin H_\lambda$, and $P \notin X$,

(iii) D'_λ is internally tangent to the boundary of D at $P \in X$.

Assume $\lambda > 0$. Since $\partial D \setminus X$ is smooth, $D_\lambda \neq \emptyset$. Now, (i) and (ii), analogously as in [27] (relevant parts of ∂D are smooth and u is C^2 there), yield that D is symmetric with respect to H_λ. Since $\lambda > 0$ this means that D is smooth everywhere. In particular D satisfies an interior sphere condition at any point of X, and by [27, Theorem 1], D is a ball.

If (iii) occurs, then D satisfies an interior sphere condition at P, namely the interior sphere touching $\partial D'_\lambda$ at X. Now, Hopf's boundary lemma implies, in the same way as in [27], that D is symmetric with respect to H_λ, and we again conclude that D is a ball.

Assume $\lambda = 0$. Then $D'_\lambda \subset \bar{D}$ for each $\lambda \geq 0$, and therefore D_0 is convex in x_1 and $D'_0 \subset \bar{D}$. By analogous arguments, if the method of moving hyperplanes starts from $\lambda' \ll -1$ (from left), then we obtain that $D_0^- := \{x \in D : x_1 < 0\}$ is convex in x_1 and $\mathcal{P}_0 D_0^- \subset \bar{D}$. Hence, $\mathcal{P}_0 D_0^- \subset \bar{D}_0$ and $\mathcal{P}_0 D_0 \subset \bar{D}_0^-$, and therefore D is symmetric with respect to H_0.

Finally, Theorem 1.1 implies that D is a ball. □

2.2 Proofs for Dirichlet problem

Proof of Theorem 1.9. If $u \equiv 0$, the statement of Theorem 1.9 is trivial. Hence, for the rest of the proof assume $u \not\equiv 0$. Since u is bounded in $\bar{\Omega}$, f' is bounded by a constant β_0 on the range of u. We split the proof of the theorem into several lemmas.

Lemma 2.3. *The statement of Theorem 1.9 holds true if we in addition assume* $f(0) \geq 0$.

Proof. By a standard application of the maximum principle, we obtain $u \equiv 0$ or $u > 0$ in Ω. If $u \equiv 0$, Theorem 1.9 is trivial, otherwise it follows from [4]. □

Define

$$
\begin{aligned}
\ell &:= \sup\{x_1 : x = (x_1, x') \in \Omega\}, \\
H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\} && (\lambda \in \mathbb{R}), \\
x^\lambda &:= (2\lambda - x_1, x') && (x = (x_1, x') \in \mathbb{R}^N, \lambda \in \mathbb{R}), \\
\Omega_\lambda &:= \{x \in \Omega : x_1 > \lambda\} && (\lambda \in [0, \ell)), \\
w^\lambda(x) &:= u(x^\lambda) - u(x) && (x \in \Omega_\lambda, \lambda \in [0, \ell)).
\end{aligned}
$$

Moreover, let $\mathcal{P}_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection about H_λ, that is, $\mathcal{P}_\lambda(x) := x^\lambda$. Since Ω is symmetric and convex in x_1, w^λ is well defined on Ω_λ for $\lambda \in [0, \ell)$. By Hadamard's formula and non-negativity of u in Ω, w^λ satisfies

$$
\Delta w^\lambda = c^\lambda(x) w^\lambda, \qquad x \in \Omega_\lambda \tag{2.2}
$$

and

$$
w^\lambda \geq 0, \qquad x \in \partial\Omega_\lambda, \tag{2.3}
$$

where

$$c^\lambda(x) := \int_0^1 f'(su(x^\lambda) + (1-s)u(x))\, ds$$

is a measurable function, bounded by β_0.

Lemma 2.4. *There exists $\delta = \delta(\beta_0, \Omega) > 0$ such that $w^\lambda(x) \geq 0$ for each $x \in \Omega_\lambda$ and $\lambda \in (\ell - \delta, \ell)$.*

Proof. The lemma immediately follows from the maximum principle on small domains (see [4]). ☐

Define

$$\lambda_\infty := \inf\{\mu > 0 : w^\lambda(x) \geq 0 \text{ for each } x \in \Omega_\lambda, \lambda \in [\mu, \ell)\}\,. \tag{2.4}$$

Lemma 2.5. *If $\lambda_\infty > 0$, then there exists a connected component D of Ω_{λ_∞} such that $w^{\lambda_\infty} \equiv 0$ in D and $w^\lambda > 0$ in Ω_λ for each $\lambda \in (\lambda_\infty, \ell)$. Moreover, $u > 0$ in Ω_{λ_∞}.*

Proof. This is a well known result (see [4, 9, 24]). ☐

Lemma 2.6. *Assume $\lambda_\infty > 0$ and let D be as in Lemma 2.5. Then there is $y^* \in \partial D \cap \partial\Omega$ such that $\partial\Omega$ is not $C^{2,\alpha}$ at y^* for any $\alpha \in (0,1)$.*

Proof. For a contradiction assume $\lambda_\infty > 0$ and for each $y \in \partial\Omega \cap \partial D$ there is $\alpha \in (0,1)$ such that $\partial\Omega$ is $C^{2,\alpha}$ at y. By Remark 2.2, being $C^{2,\alpha}$ at a point is an open property, and since $\partial\Omega \cap \partial D$ is compact, there exists $\alpha > 0$ such that $\partial\Omega$ is $C^{2,\alpha}$ at any point $y \in \partial\Omega$ with $\mathrm{dist}(y, \partial D) < \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Now, by standard Schauder estimates, $u \in C^2(\bar{D})$ and we obtain a contradiction in a similar way as in [24, proof of Theorem 1.1], since the relevant part of $\partial\Omega$ is smooth. ☐

Lemma 2.7. *Assume $\lambda_\infty > 0$ and let D be as in Lemma 2.5. Then for any $y^* \in \partial D \setminus H_{\lambda_\infty}$ with $y_1^* > \lambda_\infty$ there is $\beta > 0$ such that $\partial\Omega$ is $C^{2,\beta}$ at y^*.*

Proof. For any fixed $y^* \in \partial D \setminus H_{\lambda_\infty}$, the convexity of Ω in x_1 yields $(y^*)^{\lambda_\infty} \in \bar{\Omega}$.

First assume $z := (y^*)^{\lambda_\infty} \in \Omega$, and denote $\Gamma := \mathcal{P}_{\lambda_\infty}(\partial D \setminus H_{\lambda_\infty}) \cap \Omega$ a set containing z. Then, $w^{\lambda_\infty} \equiv 0$ in D yield $u(x) = 0$ for any $x \in \Gamma$. Since $u \geq 0$ in Ω, any $x \in \Gamma$ is a local minimum of u, and therefore $\nabla u(x) = 0$ for each $x \in \Gamma$.

If $u_{x_i x_i}(z) = 0$ for each $i \in \{1, \cdots, N\}$, then $0 = \Delta u(z) = f(u(z)) = f(0)$ and Lemma 2.3 implies $\lambda_\infty = 0$, a contradiction. Otherwise, $u_{x_i x_i}(z) \neq 0$ for some $i \in S$. Fix such i and denote $v := u_{x_i}$. Then $v(z) = 0$, $\nabla v(z) \neq 0$, and v solves

$$\Delta v = V(x) v, \qquad x \in \Omega\,,$$

where $V(x) := f'(u(x))$ belongs to $C^\alpha(\Omega)$. Now, by [15, Theorem 2], the nodal set $\mathcal{N} := \{x \in \Omega : v(x) = 0\}$ of v is a $C^{3,\alpha}$ hypersurface in a neighborhood U of

z. We remark that if u is a solution of the quasi-linear problem (1.11), then v satisfies

$$\Delta v = b(x_1, x_2) \cdot \nabla v + V(x_1, x_2)v, \qquad (x_1, x_2) \in \Omega\,, \tag{2.5}$$

with $b \in C^\alpha(\Omega, \mathbb{R}^N)$, $V \in C^\alpha(\Omega)$, and we need to replace [15, Theorem 2] by Schauder's estimates which imply $\Gamma \in C^{2,\alpha}$. Higher regularity of Γ for (2.5) was announced in [15], but we were not able to locate it in the literature.

Let $\Gamma^* := \Gamma \cap V^*$ and $\mathcal{N}^* := \mathcal{N} \cap V^*$ where $V^* \subset \Omega \setminus \bar{\Omega}_{\lambda_\infty}$ is sufficiently small neighborhood of z such that $V^* \setminus \mathcal{N}^*$ has two connected components. This choice is possible since $\mathcal{N}$ is smooth. We finish the proof by showing that $\Gamma^* = \mathcal{N}^*$. Then Γ^* is of class $C^{3,\alpha}$ and by the definition of Γ^*, $\partial\Omega$ is of class $C^{3,\alpha}$ in a neighborhood of y^*.

Observe, that we already showed $\Gamma^* \subset \mathcal{N}^*$. Assume $\mathcal{N}^* \not\subset \Gamma^*$. Since $\Gamma^* \subsetneq \mathcal{N}^*$, one has that $V^* \setminus \Gamma^* \subset \Omega$ is connected, and therefore $\mathcal{P}_{\lambda_\infty}(V^* \setminus \Gamma^*)$ is connected as well. Since $\mathcal{P}_{\lambda_\infty}(V^* \setminus \Gamma^*) \subset \mathbb{R}^N \setminus \partial\Omega$, one has $\mathcal{P}_{\lambda_\infty}(V^* \setminus \Gamma^*) \subset \Omega_{\lambda_\infty}$. Using $w^{\lambda_\infty} \equiv 0$, we obtain $u(x) = 0$ for each $x \in \mathcal{P}_{\lambda_\infty}(\mathcal{N}^* \setminus \Gamma^*) \subset \Omega_{\lambda_\infty}$, a contradiction to Lemma 2.5.

Next, assume $z = (y^*)^{\lambda_\infty} \in \partial\Omega$, then $\lambda_\infty > 0$ and the convexity of Ω in x_1 yield that either $z_1 \geq 0$ and then the segment connecting y^* and z lies in $\partial\Omega$, or $z_1 < 0$ and then the segment connecting y^* and $\mathcal{P}_0\mathcal{P}_{\lambda_\infty}(y^*)$ lies in $\partial\Omega$. In either case, Ω satisfies condition (A) with $y^* = A$.

If (a) in Definition 1.7 holds true, then $\partial\Omega$ is of class $C^{2,\beta}$ at y^*. If (b) holds true, we obtain a contradiction with the first part of the proof, when y^* is replaced by $X \in \partial D$, defined in Definition 1.7 (b) (observe that $\mathcal{P}_{\lambda_\infty} X \in \Omega$). □

Lemma 2.8. $\lambda_\infty = 0$.

Proof. For a contradiction assume $\lambda_\infty > 0$. Let D be as in Lemma 2.5. Denote $D^* := \mathrm{Int}(\bar{D} \cup \mathcal{P}_{\lambda_\infty}(\bar{D}))$ and $\partial^0 D := \partial D \setminus H_{\lambda_\infty}$.

First, D^* is by definition convex in x_1 and symmetric with respect to H_{λ_∞}. Next, $u \in C(\bar{\Omega})$ solves (1.6) in D^* and, by the definition of D, $u = 0$ on ∂D^*. Theorem 1.1 implies that D^* is a ball, if we show that $u \in C^1(\bar{D}^*)$ and $\nabla u = 0$ on ∂D^*.

By Lemma 2.7 and Schauder estimates we obtain $u \in C^2(D^* \cup \partial^0 D)$, and $\partial^0 D^* \in C^2$. Then, by [24, Claim 3.6 (i)], one has $\nu_1(x) > 0$ for each $x \in \partial^0 D$. Note that the proof of [24, Claim 3.6 (i)] requires $u \in C^2(D^* \cup \partial^0 D)$ and $\partial^0 D \in C^2$ only.

Since $\nu_1(x) > 0$ for each $x \in \partial D^*$ with $x_1 > \lambda_\infty > 0$, the convexity of Ω in x_1 gives $\partial D^* \cap \{x : x_1 < \lambda_\infty\} \subset \Omega$. Then similarly as in the proof of Lemma 2.7, one has $u \in C^1(\bar{D}^* \setminus H_{\lambda_\infty})$ and $\nabla u = 0$ on $\partial \bar{D}^* \setminus H_{\lambda_\infty}$.

Consequently, by Definition 1.8 (property (B)), one has $u \in C^1(\bar{D}^*)$.

Finally, Theorem 1.1 yields that D^* is a ball and u is radially symmetric on D^*. Using the unique continuation method as in [24, Proof of Theorem 1.1], we obtain $u \equiv 0$, a contradiction. □

Now, we finish the proof of Theorem 1.9. From Lemma 2.8 and the definition of λ_∞ follows that u is nonincreasing in Ω_0 and $w^0 \geq 0$. In particular, $u \equiv 0$ or

$u > 0$ in Ω. In the first case Theorem 1.9 is trivial, in the second one it follows from [4, Theorem 1.3]. □

Proof of Corollary 1.12. By Lemma 2.8, $\lambda := \lambda_\infty > 0$ only if $u \in C^2(D^*) \cap C(\bar{D}^*)$ satisfies the problem

$$\begin{aligned} \Delta u + F(|\nabla u|^2, u) = 0\,, &\qquad x \in D^*\,, \\ u = 0\,, &\qquad x \in \partial D^*\,, \\ u > 0\,, &\qquad x \in D^*\,, \end{aligned}$$

where D^* is a bounded, symmetric domain, convex in x_1, and $\partial D^* \setminus H_\lambda \in C^2$. To obtain a contradiction (to the assumption $\lambda > 0$), as in Lemma 2.8, it suffices to show $u \in C^1(\bar{D}^*)$.

Fix $x \in \partial D^* \cap H_\lambda$ such that ∂D^* is not $C^{1,\alpha}$ at x (otherwise trivially u is C^1 at x and $\nabla u(x) = 0$). Since $\partial\Omega$ is piecewise $C^{2,\alpha}$, the limit

$$\lim_{\substack{y \to x \\ y_1 > x_1}} \nu(y) =: \mu \in \mathbb{R}^N$$

exists, where $\nu(y)$ is the outer unit normal to $\partial\Omega$ at y. Moreover, $\mu_1 \geq 0$ by the convexity of Ω in x_1. Assume $\mu_1 = 0$. Since $\partial\Omega$ is piecewise C^2, then as in [24, Claim 3.7] one obtains that ∂D is of class C^2 at x, a contradiction.

Otherwise $\mu_1 > 0$. Then we straighten $\partial\Omega$ by a smooth map such that in a neighborhood of x, $\partial\Omega$ transforms to part of two hyperplanes with angle less than π (cp. [2, Proof of Theorem 1]). Then [2, Theorem 2] applies and we conclude that u is C^1 at x as desired.

Therefore $u \in C^1(\bar{D}^*)$ and Corollary 1.12 follows from Theorem 1.9. □

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