EXISTENCE OF TRAVELING WAVES FOR A FOURTH ORDER SCHRÖDINGER EQUATION WITH MIXED DISPERSION IN THE HELMHOLTZ REGIME

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Abstract. In this paper, we study the existence of traveling waves for a fourth order Schrödinger equations with mixed dispersion, that is, solutions to

$$\Delta^2 u + \beta\Delta u + iV\nabla u + \alpha u = |u|^{p-2}u, \ in\ \mathbb{R}^N, \ N \geq 2.$$

We consider this equation in the Helmholtz regime, when the Fourier symbol P of our operator is strictly negative at some point. Under suitable assumptions, we prove the existence of solution using the dual method of Evequoz and Weth provided that $p \in (p_1, 2N/(N-4)_+)$. The real number p_1 depends on the number of principal curvature of M staying bounded away from 0, where M is the hypersurface defined by the roots of P. We also obtain estimates on the Green function of our operator and a $L^p - L^q$ resolvent estimate which can be of independent interest and can be extended to other operators.

1. Introduction

In this paper, we construct non-trivial complex valued solution to a fourth order nonlinear equation

$$\Delta^2 u + \beta\Delta u + iV\nabla u + \alpha u = |u|^{p-2}u, \qquad \text{in} \quad \mathbb{R}^N, \ N \geq 2, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, $V \in \mathbb{R}^N$, and $p > 2$. The equation (1.1) characterizes the profile of traveling wave solutions $\varphi(t,x) = e^{i\alpha t}u(x - vt)$ of the fourth order nonlinear Schrödinger equation with mixed dispersions

$$i\partial_t\varphi - \Delta^2\varphi - \beta\Delta\varphi + |\varphi|^{p-2}\varphi = 0, \ \varphi(0,x) = \varphi_0(x), \ (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.2}$$

The fourth order term in equation (1.2) has been introduced by Karpman and Shagalov [21] and it allows to regularize and stabilize solutions to the classical Schrödinger equation as observed through numerical simulations by Fibich, Ilan, and Papanicolaou [15]. Since the techniques for (1.2) are different compared the equation without Δ^2, let us only mention selected results for (1.2).

Well-posedness for (1.2) has been established by Pausader [30] (using the dispersive estimates of [3]) as well as some scattering results (we also refer to [26]). Recently, Boulenger and Lenzmann obtained (in)finite time blowing-up results in [9]. In the last few years, solitary waves solutions to (1.2), that is, solutions of the form $\varphi(t,x) = e^{i\alpha t}u(x)$, have been quite intensively studied. The profile of such waves satisfies (1.1) with $V \equiv 0$ and we refer to [4, 5, 6, 8, 14] for several results concerning the existence of ground states and normalized solutions as well as some of their qualitative properties. Notice that for the second order Schrödinger equation it is possible to remove the drift term $iV\nabla u$ by a suitable transformation. Such technique does not work for the fourth order equation, and therefore (1.1) with $V \neq 0$ should be studied separately.

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There are very few results for (1.1) in the literature. The first author in [11] showed the existence of normalized solutions, that is, solutions with fixed mass and the existence of solutions with fixed mass and momentum. If the mass (and momentum) is fixed, the parameter α (or the parameters α and V) appears as a Lagrange multiplier. Hence, compared to our setup the solution found in [11] solve (1.1) with α (and V) which is determined by mass (or momentum) constraint.

If the symbol of the operator on the left hand side to is positive, then solutions can be found by standard techniques as global minimizers of appropriate functionals. More precisely, in [20] (see also [10]) the authors investigated an analogue of (1.1) for more general operator of the form $P_V(D) = P(D) + iV\nabla$, where $P(D)$ is a self-adjoint, constant coefficient pseudo-differential operator defined by $(\widehat{P(D)u})(\xi) = p(\xi)\hat{u}(\xi)$, $\hat{u}$ being the Fourier transform u. The existence of ground states was obtained provided $2 < p < 2N/(N-2s)_+$ and $P_V(D)$ has a positive symbol (this is the case if for instance $\alpha > 0$ is large enough). Qualitative properties of these solutions were also investigated.

The main goal of the present manuscript is to obtain solutions to (1.1) when the symbol of the operator changes sign. In such case, the natural energy functional is not bounded from below, and therefore the grounds states do not exist. Also, it is not expected that the solutions belong to $L^2(\mathbb{R}^N)$, so they cannot be found as critical points of the energy. To solve this obstacle, we use the dual variational method due to Evequoz and Weth [13], who used it for the second order Helmholtz equation. Since for the second order equations one can remove the drift, the problem analyzed in [13] was radially symmetric, which greatly simplifies analysis compared to our situation. The differences are not only of technical nature, but we had to develop new approach to investigate mapping properties of the resolvent operator $\mathfrak{R}$, that is, the inverse of

$$L = \Delta^2 + \beta\Delta + iV\nabla + \alpha\,.$$

The main challenge stems from the fact that 0 is contained in the essential spectrum of L.

To be more precise, as in [13] we look for solution to

$$\mathfrak{R}(v) = |v|^{p'-2}v, \qquad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

where $v = |u|^{p-2}u$, $p' = p/(p-1)$ and $\mathfrak{R} = (\Delta^2 + \beta\Delta + iV\nabla + \alpha)^{-1}$ is a resolvent-type operator constructed by a limiting absorption principle. The advantage of (1.3) is that it has variational structure in $L^{p'}(\mathbb{R}^N)$ and one can use a mountain pass theorem to construct a non-trivial solution to (1.1). Our main result is the following.

Theorem 1.1. *Let P be the Fourier symbol of the differential operator in* (1.1) *without the drift term, that is, $P(x) = x^4 - \beta x^2 + \alpha$ and let*

$$M := \{\xi \in \mathbb{R}^N : P(|\xi|) - \xi V = 0\}.$$

Assume that :

- (A1) *there exists $\tilde{\xi} \in \mathbb{R}^N$ such that $P(|\tilde{\xi}|) - \tilde{\xi}V < 0$.*
- (A2) *$P'(|\xi|)\frac{\xi}{|\xi|} - V \neq 0$, for all $\xi \in M$.*
- (A3) *If $P'(|\xi|) = 0$, then $P(|\xi|) - |V||\xi| \neq 0$.*

Let $p_1 < p < 2N/(N-4)_+$, where $1/0 =: \infty$ and

$$p_1 = \begin{cases} \frac{2(N+1)}{N-1} & \text{if } 8\beta|V|^2 \leq (\beta^2 - 4\alpha)^2 \text{ and } \beta^2 - 4\alpha \neq 3|V|^{4/3}\,, \\ \frac{2N}{N-2} & \text{if } 8\beta|V|^2 > (\beta^2 - 4\alpha)^2 \text{ or } \beta^2 - 4\alpha = 3|V|^{4/3}\,. \end{cases}$$

Then, there exists a nontrivial solution $u \in W^{4,q}(\mathbb{R}^N) \cap C^{3,\alpha}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$*, for all* $q \in [p, \infty)$*,* $\alpha \in (0,1)$ *of*

$$\Delta^2 u + \beta \Delta u + iV\nabla u + \alpha u = |u|^{p-2}u, \qquad in \quad \mathbb{R}^N.$$

Let us make a few comments on this theorem. First note that our problem is rotationally symmetric with the axis of rotation parallel to V, and assumptions (A2) and (A3) are connected with the behavior on the axis of rotation. We refer to Remark 2.1 for necessary and sufficient conditions on α, β, and V that are equivalent to (A1), (A2), and (A3).

Assumption (A1) guarantees that we are in the Helmholtz case, that is, the symbol of the operator L changes sign, thus if removed, our main result is already covered in the literature. Assumption (A2) implies that M is a smooth manifold, and it is a mandatory condition for our method to work. Otherwise, $P(x)$ has a double root and then even the definition of the resolvent type operator would be problematic. We remark that if (A2) is not satisfied, then either M is a single point, or a union of two manifolds intersecting at the axis of rotation, see Figure 1 below. Note that these two manifolds might not be smooth at the intersection point, and therefore we cannot treat them separately with our techniques. On the other hand, (A2) is merely a point-wise condition and is satisfied generically.

The assumption (A3) allows us to control the number of non-vanishing curvatures of M located on the axis of rotation (see the case $h(t) = 0$ in the proof of Proposition 2.1 below). In particular, if (A3) is not satisfied, then all principal curvatures vanish at the axis of rotation, which leads to insufficient estimates for our purposes. Again, (A3) is a point-wise condition, and therefore it is satisfied generically.

The upper bound $\frac{2N}{(N-4)_+}$ on p is related to Sobolev embedding and is natural for our type problems. The lower bound p_1 is linked to the number k of principal curvatures of M staying bounded from 0, which in turn dictates the decay of the Green function. An interesting feature of our problem is that, depending on the coefficients of P, k is either $N-1$ (similar to the "classical" situation, where M is a $(N-1)$-sphere) or $N-2$. Let us remark that our method can be applied to more general, constant coefficient operators. More precisely, assume that the Fourier symbol $\tilde{P}$ of the operator $\tilde{L}$ is real and $\tilde{M} = \{x \in \mathbb{R}^N : \tilde{P}(x) = 0\}$ is a non-empty, regular, compact hypersurface with k principal curvature bounded away from 0. Then,

$$\tilde{L}(u) = |u|^{p-2}u \qquad \text{in} \quad \mathbb{R}^N, N \geq 2$$

admits non-trivial solution provided $2(k+2)/k < p < 2N/(N-s)_+$, where $s > 0$ is the order of the operator. We remark that we expect our solution to decay as $|x|^{-k/2}$ at infinity, in analogy with the situation for the nonlinear Helmholtz equation (see [24]).

Let us make some comments on the proof of Theorem 1.1. The main challenge is to construct the resolvent type operator $\mathfrak{R} = L^{-1}$ and to analyze its mapping properties. To proceed, we follow the approach of Gutierrez [17], which relies on two main ingredients. The first one is the decay of the Green function of the operator L, which is connected with k. This can be already seen from the classical Fourier restriction result of Littman [22] (see Theorem 2.1). To prove the decay, we follow the framework of Mandel [23], that was developed for a periodic, second order differential equation of Helmholtz type.

The second main ingredient is the Stein-Tomas Theorem (see Theorem 3.1 below proved in [2]), that enables us to prove $L^{2(k+2)/(k+4)}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ estimate on $\mathfrak{R}$ (see Theorem 3.2 below). In the literature, this bound is achieved with help of dyadic radial decomposition of the physical space which has radial Fourier image (analogous to Littlewood-Paley decomposition). However, our problem is not radially symmetric, and to have our results applicable to more general problems, we opted not to use rotational symmetry either.

Since in the non-symmetric case the technical difficulties arise either in physical or Fourier space, we again chose radial dyadic decomposition in the physical space. After applying the Stein-Tomas Theorem, we need an estimate on

$$\int_{-3c_1/4}^{3c_1/4} \max_{\xi \in M_\tau} (|\hat{G}_2^j(\xi)|^2) d\tau \leq C 2^j \,, \tag{1.4}$$

where G_2^j is roughly the Green function cut-off on to the annulus $B_{2^{j+1}} \backslash B_{2^{j-1}}$. This approach was employed in [17] and [7], in the radial case, where the dyadic decomposition has the same symmetry as $M = \mathbb{S}^{N-1}$. In the radial case, one obtains the left hand side of (1.4) without max, and then an application of Plancherel Theorem led to the desired result. However, in our non-radial case, the situation is more complicated and the Plancherel theorem, or Bernstein type inequality yields only insufficient bound by $C2^{Nj}$. To prove (1.4), we work directly in the Fourier space and we use a non-trivial cancellation properties that originate from the non-degeneracy assumption (A2). We stress that our final bound is the same as in the radial case, although we worked in more general setting.

To obtain the the bounds on $\mathfrak{R}$ as an operator between L^p spaces, we use duality, interpolation, and Riesz-Thorin theorem. Here, we observed that the radial case, or more generally the case $k = N - 1$, is very degenerate, and many calculations are simplified (see Theorem 3.3 below). Since for our main results we needed $L^p \to L^q$ mapping properties of $\mathfrak{R}$ for $p' = q$, we focus on such case and as a byproduct we obtained bounds on a bigger range. We also remark that the mapping properties of $\mathfrak{R}$ for $p' = q$ are independent of the dimension N, that is, they depend only on k. Soon after posting our manuscript online, the authors of [25], posted related Bochner-Riesz bounds for larger range of parameters p and q, which however coincides with ours in the case $p = q'$. Also, the assumption on the degeneracy of the manifold was not needed in [25], however it still does not seem to remove assumption (A2) below, since M would not be a manifold.

For the proof of non-trivial solutions of (1.1) we follow the framework from [13] and find mountain pass solutions using the dual variational formulation of (1.3). Some caution should be exercised, since we are working with complex valued functions, whereas [13] deals with real valued ones. With bounds on $\mathfrak{R}$ proved in the other sections, we can easily modify methods from the literature.

The plan of this paper is the following: in Section 2, we study the properties of M. In particular, we give conditions on the coefficients of our equation implying that the number of principal curvatures of M bounded away from 0 is either $N-1$ or $N-2$. Then, we use these properties to study the decay of the Green function of our operator. In Section 3, we study the mapping property of the resolvent type operator $\mathfrak{R} = (\Delta^2 + \beta\Delta + iv\nabla + \alpha)^{-1}$. Finally, we use them to construct non-trivial solutions to (1.1) using the dual variational method.

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2. Decay estimate of the Green function

The aim of this section is to analyse the decay of the Green function G of the operator $\Delta^2 + \beta\Delta + iV\nabla + \alpha$, that is,

$$\Delta^2 G + \beta\Delta G + iV\nabla G + \alpha G = \delta_0, \text{ in } \mathbb{R}^N \,,$$

where δ_0 is the Dirac measure located at the origin. Observe that formally, applying the Fourier transform to the previous equation, we obtain

$$G(x) = \int_{\mathbb{R}^N} \frac{1}{P(|\xi|) - \xi V} e^{ix\xi} d\xi,$$

where $P(x) = x^4 - \beta x^2 + \alpha$. This integral is not well defined for general P; however, due to our assumptions on $P(|\xi|) - \xi v$, we show non-trivial cancellation properties, which allow us to prove that G is well defined and decays at infinity. As usual for this kind of problem, we apply the "limiting absorption principle". Specifically, we define G as a limit of approximating Green functions G_ε, $\varepsilon \neq 0$ associated with the operator $\Delta^2 + \beta\Delta + iV\nabla + (\alpha - i\varepsilon)$, that is,

$$G_\varepsilon(x) = \int_{\mathbb{R}^N} \frac{1}{P(|\xi|) - \xi V - i\varepsilon} e^{ix\xi} d\xi.$$

Notice that the denominator in the integral does not vanish for any $\varepsilon \neq 0$. It is well-known that the decay of this integral depends only on the set of points where $P(|\xi|) - \xi V = 0$. More precisely, we prove that it depends on the number of non-zero principal curvatures M. Our first result gives optimal conditions on the coefficients of P so that the principal curvatures of M are strictly positive.

Proposition 2.1. *Let $P(x) = x^4 - \beta x^2 + \alpha$ and*

$$M := \{\xi \in \mathbb{R}^N : P(|\xi|) - \xi V = 0\}.$$

If (A1), (A2), and (A3) hold, then the following assertions are true.

- *If*

$$8\beta|V|^2 \leq (\beta^2 - 4\alpha)^2 \text{ and } \beta^2 - 4\alpha \neq 3|V|^{4/3}, \tag{2.1}$$

  then M is a regular, compact hypersurface with all principal curvatures bounded away from 0.
- *If*

$$8\beta|V|^2 > (\beta^2 - 4\alpha)^2 \text{ or } \beta^2 - 4\alpha = 3|V|^{4/3}, \tag{2.2}$$

  then exactly one of the principal curvature of M vanishes on a $N-2$-dimensional set.

Remark 2.1. *In this remark, we find necessary and sufficient conditions on the coefficients of our operator to satisfy assumptions (A1), (A2), and (A3). Let us begin with (A2). Assume by contradiction that there exists $\xi \in M$ such that $P'(|\xi|)\frac{\xi}{|\xi|} - V = 0$. Therefore ξ is a multiple of V so we set $\xi = \frac{V}{|V|}q$ and obtain that q satisfies $P'(q) - |V| = 0$, or equivalently*

$$4q^3 - 2\beta q - |V| = 0. \tag{2.3}$$

Since $\xi = \frac{V}{|V|}q \in M$, then $|\xi|^4 - \beta|\xi|^2 + \alpha - V\xi = 0$ and in addition to (2.3) we have

$$q^4 - \beta q^2 + \alpha - |V|q = 0\,.$$

Using the Euclidean algorithm for finding the greatest common divisor of these two polynomials, we obtain that they have a common root if and only if

$$256\alpha^3 - 128\alpha^2\beta^2 + 4\beta^3|V|^2 - 27|V|^4 + 16\alpha(\beta^4 - 9\beta|V|^2) = 0\,. \tag{2.4}$$

Although, technically complicated, it is easy to verify if a given set of parameters satisfies (2.4). Also, notice that (2.4) is a quadratic equation for $|V|^2$.

To satisfy assumption (A1), we need $P(|\xi|) - V\xi < 0$ for some ξ, and therefore we minimize the left hand side by choosing $\xi = \frac{V}{|V|}|\xi|$

$$|\xi|^4 - \beta|\xi|^2 + \alpha < |V||\xi|\,.$$

So our problem is equivalent to

$$|V| > \min_{x>0} \frac{x^4 - \beta x^2 + \alpha}{x} = \min_{x>0} g(x)\,.$$

If $\alpha < 0$, then $g(x) \to -\infty$ as $x \to 0$, and (A1) is trivially satisfied for any $|V|$. If $\alpha = 0$, then $g(0) = 0$, and (A1) is satisfied for any $V \neq 0$.

Finally, if $\alpha > 0$, then $g(x) \to \infty$ for $x \to \infty$ or $x \to 0$, then the global minimum is also a local minimum.

Evaluating $g'(x) = 0$, we find that the mininum is attained at x solving

$$\frac{3x^4 - \beta x^2 - \alpha}{x^2} = 0\,,$$

that is, at since $x \geq 0$

$$x_\pm = \sqrt{\frac{\beta \pm \sqrt{\beta^2 + 12\alpha}}{6}}\,.$$

Again, since $\alpha > 0$, then the only admissible root is x_+. Thus for $\alpha > 0$ after using $3x^4 - \beta x^2 - \alpha = 0$, we find (A1) holds if and only if

$$|V| > \frac{x_+^4 - \beta x_+^2 + \alpha}{x_+} = \frac{2}{3}\frac{-x_+^2 + 2\alpha}{x_+}.$$

Finally, (A3) holds if and only if

$$\alpha \neq 0 \qquad \textit{and} \qquad -\frac{\beta^2}{4} + \alpha \pm |V|\frac{\beta}{2} \neq 0.$$

Proof of Proposition 2.1. Since $P'(|\xi|)\frac{\xi}{|\xi|} - V \neq 0$, for all $\xi \in M$, and P has a superlinear growth, the implicit function theorem implies that $M = \{x : P(|x|) - Vx = 0\}$ is a smooth compact manifold. To simplify notation, we choose our coordinate axes such that $V = |V|e_1$ and parametrize M as a surface of revolution by $\xi = (\xi_1, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Since $(\xi_1, \tilde{\xi}) \in M$ satisfy $P((\xi_1^2 + |\tilde{\xi}|^2)^{\frac{1}{2}}) - |V|\xi_1 = 0$, by solving for $|\tilde{\xi}|$, we obtain $h_\pm(\xi_1) = |\tilde{\xi}|$ with $h_\pm(z) = \sqrt{(P_\pm^{-1}(|V|z))^2 - z^2}$ and

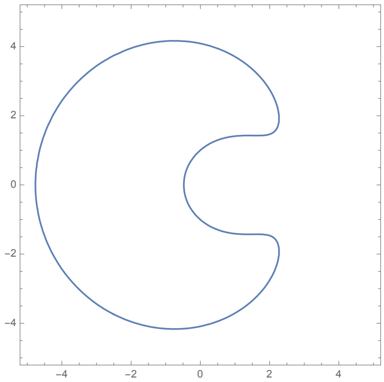
$$(P_\pm^{-1}(z))^2 = \frac{\beta \pm \sqrt{\beta^2 - 4(\alpha - z)}}{2}.$$

Before we proceed, let us clarify some details. Note that we choose h instead of $-h$, since $|\tilde{\xi}| \geq 0$. Also, in general $P^{-1}(z)$ has four solutions, but since P is even, they come in pairs differing by the sign. However, we only need to look $(P^{-1})^2$. Therefore it suffices to consider the two positive roots. Since we are looking at local properties (curvatures) we can treat two connected components (corresponding to h_+ and h_-) separately unless they touch. This happens only when a root of $P(|\xi|) - V\xi$ is degenerate, since otherwise M is a smooth manifold (see Figure 1).

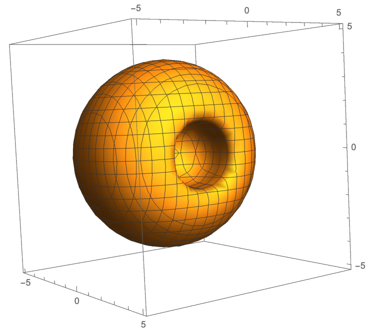
Since $P(x) \approx x^4$ for x large and P is bounded from below, we obtain that $P^{-1} \approx x^{1/4}$ for x large and P^{-1} is defined only for $x \geq Q$ for some $Q \in \mathbb{R}$. Since $P_\pm^{-1}(x) \geq |x|$ for any x in the domain of $h_\pm$, we get that the domain of $h_\pm$ is of the form $[A_\pm, B_\pm]$ for some $A_\pm, B_\pm \in \mathbb{R}$. Note that one or both branches might not exist in the situations where M is connected or empty.

Thus, we can parametrize M as

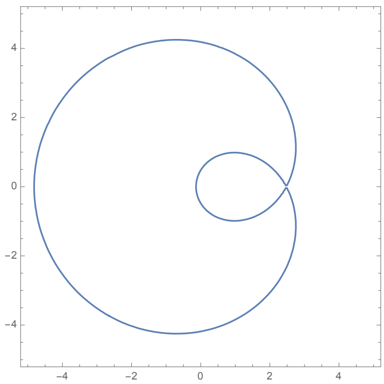
$$M_\pm = \{(t, h_\pm(t)\omega) : \omega \in \mathbb{S}^{N-2},\ A_\pm \leq t \leq B_\pm\}\,,$$



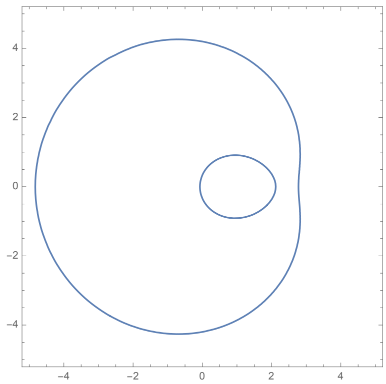
(A) Cross section of M with one component



(B) Visualization of M in 3D



(C) Degenerate case



(D) M with two components

FIGURE 1. Figures (A), (C), (D) are cross sections of M with different parameters. Figure (B) is surface of revolution corresponding to (A)

and then the principal curvatures κ_i, $i = 1, \dots, N-1$, of M are given (up to a sign) by

$$\kappa_i(t) = \begin{cases} \dfrac{1}{h(t)\sqrt{1+h'(t)^2}}, & i = 1, \dots, N-2, \\ \dfrac{-h''(t)}{(1+h'(t)^2)^{3/2}}, & i = N-1\,, \end{cases}$$

where h stands for $h_\pm$. Observe that κ_i, $i = 1, \dots, N-2$, are non-zero whenever $h(t) \neq 0$.

Before discussing κ_{N-1}, let us treat the case where t is such that $h(t) = 0$. By the rotational symmetry of M, we obtain that, at $(\xi_1, |\tilde{\xi}|) = (t, h(t)) = (t, 0)$, all principal curvatures are the same. Therefore it suffices to prove that one of them is non-zero. Choose a smooth curve $\gamma(\xi_2) = (\xi_1(\xi_2), \xi_2, 0) \in M$, $\xi_2 \in (-\varepsilon, \varepsilon)$ with $\xi_1'(0) = 0$. Such a curve exists by the rotational symmetry. If $\xi_1(0) = 0$, then $\gamma(0) \in M$ yields $P(0) - 0|V| = 0$ and since P is even $P'(0) = 0$, a contradiction to (A3). So, we assume $\xi_1(0) \neq 0$.

To prove that γ has nonzero curvature at 0, and therefore M has a nonzero curvature at $(\xi_1, 0)$, it suffices to prove that $\xi_1''(0) \neq 0$. However, since $\gamma \subset M$, we have

$$P(|(\xi_1(\xi_2),\xi_2,0)|) - |V|\xi_1(\xi_2) = 0\,.$$

Differentiating implicitly twice the previous equality and substituting $\xi_2 = 0$, $\xi_1'(0) = 0$, we obtain

$$-|V|\xi_1''(0) + P'(|\xi_1(0)|)\xi_1''(0)\frac{\xi_1(0)}{|\xi_1(0)|} + P'(|\xi_1(0)|)\frac{1}{|\xi_1(0)|} = 0\,.$$

Since $\xi_1(0) \neq 0$ and $V = |V|e_1$, we deduce from (A2) that

$$|P'(|\xi_1(0)|)\frac{\xi_1(0)}{|\xi_1(0)|} - |V|| = |\nabla(P(|\xi|) - V\xi)|\Big|_{\xi=(\xi_1(0),0)} \neq 0\,.$$

Finally, since by (A3), $P'(|\xi_1(0)|) = 0$ implies $P(|\xi_1(0)|) - |V|\xi_1(0) \neq 0$, that is, $(\xi_1(0),0) \notin M$, we obtain that $\xi_1''(0) \neq 0$. Thus M has $N-1$ non-vanishing curvatures at any point $(\xi_1,0) \in M$.

In the rest of the proof we assume $h(t) \neq 0$. To treat κ_{N-1}, recall that

$$h_\pm(z) = \sqrt{\frac{\beta \pm \sqrt{\beta^2 - 4(\alpha - |V|z)}}{2} - z^2}.$$

First, h_+ is a square root of a sum of two concave functions, at least one of them strictly concave, and therefore $h''(z) < 0$ whenever defined. In particular we obtain $(\kappa_+)_{N-1} \neq 0$.

To treat h_-, we set $z = |V|^{1/3}w$, $e = |V|^{-2/3}\beta$, and $k = (\beta^2 - 4\alpha)|V|^{-4/3}$ and (noticing that $h_- \geq 0$ when it is defined)

$$\begin{aligned}\sqrt{f(w)} &:= |V|^{-1/3}h_-(|V|^{1/3}w)\\ &= \sqrt{\frac{e - \sqrt{k+4w}}{2} - w^2}.\end{aligned}$$

Recall $h_-(z) > 0$ and notice that $h_-''(z) = 0$ if and only if

$$(f'(w))^2 = 2f''(w)f(w)\,,$$

or equivalently

$$\left(\frac{1}{\sqrt{k+4w}} + 2w\right)^2 = 2\left(\frac{2}{\sqrt{(k+4w)^3}} - 2\right)f(w)\,.$$

Observe that $k+4w = 0$ corresponds to the case $h(t) = 0$ treated separately above. Using the change of variables $y = (k+4w)^{1/2}$, the previous equality becomes

$$\left(\frac{1}{y} + \frac{y^2-k}{2}\right)^2 = 4(y^{-3}-1)f(w) = 4(y^{-3}-1)\left(\frac{e-y}{2} - \left(\frac{y^2-k}{4}\right)^2\right). \tag{2.5}$$

Since the left hand side of (2.5) is non-negative and $f(w) > 0$, there are no solutions if $y > 1$. Next, for $g = \frac{e}{2} - \frac{k^2}{16}$, the factorization of (2.5) gives

$$\frac{3}{y^2} + \frac{k^2}{4} - \frac{3}{4}y - \frac{3}{2}\frac{k}{y} = 4g\left(\frac{1}{y^3} - 1\right)\,.$$

A multiplication by $4y^2$ yields

$$12 + k^2y^2 - 3y^3 - 6ky = \frac{16}{y}g\left(1 - y^3\right)\,,$$

and therefore

$$(ky-3)^2 + 3(1-y^3) = \frac{16}{y}g(1-y^3). \tag{2.6}$$

Hence, if $g \leq 0$ or equivalently when $8\beta V^2 \leq (\beta^2 - 4\alpha)^2$, then (2.6) has no solution $y < 0$ since the left hand side is positive and the right hand side is non-positive. Finally when $y = 1$, (2.6) has a solution only if $k = 3$, that is, $\beta^2 - 4\alpha = 3V^{4/3}$. Overall, we have proved that $h'' \neq 0$ provided that

$$(2.7) \qquad 8\beta V^2 \leq (\beta^2 - 4\alpha)^2 \qquad \text{and} \quad \beta^2 - 4\alpha \neq 3V^{4/3}.$$

Let us remark that if (2.7) does not hold, then (2.5) has a solution. Indeed, if $\beta^2 - 4\alpha = 3V^{4/3}$, then $k = 3$ and $y = 1$ is a solution of (2.5). A direct substitution yields that $f > 0$ in such point if and only if $e > \frac{3}{2}$. Also, if $g > 0$, then the right hand side of (2.5) is of order g/y^3 and the left hand side is of order $1/y^2$ around zero. Let y_0 be the smallest positive root of the right hand side of (2.5) and notice that $y_0 \leq 1$. Since the left hand side is always non-negative, by the intermediate value theorem, there is a solution y^* of (2.5). In addition, since $y_0 \leq 1$ was the first zero, we have that $f(w^*) \geq 0$ with w^* corresponding to y^*, and consequently there is a point on M with vanishing curvature.

Finally notice that (2.6) is equivalent to a polynomial of at most fourth degree, and therefore for any set of parameters, the set of points on M with one principal curvature vanishing, consists of a union of at most four $N-2$ dimensional surfaces. □

Proposition (2.1) allows us to use the following Fourier restriction result proved in [22] and [19]. We refer to [19] if $k = N-1$ and [22] if $f \equiv 1$.

Theorem 2.1. *Let $M \subset \mathbb{R}^N$ be a smooth compact and closed hypersurface with k principal curvatures bounded away from 0. Then, for any smooth f in a neighbourhood of M, one has*

$$\left| \int_M e^{ix\xi} f(\xi) d\mathcal{H}^{N-1}(\xi) \right| \leq C(1 + |x|)^{-\frac{k}{2}}.$$

Instead of considering the integral over $\mathbb{R}^N$, we restrict the integration domain to $M_\tau := \{\xi \in \mathbb{R}^N : F(\xi) = P(|\xi|) - \xi V = \tau\}$, for $|\tau| \leq \rho$ sufficiently small. Specifically, we define a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\operatorname{supp}(\chi) \subset [-\rho, \rho]$ and $\chi \equiv 1$ on $(-\rho/2, \rho/2)$. Define $\tilde{\chi}(\xi) = \chi(F(\xi))$ and consider separately

$$G_\varepsilon^1(x) = \int_{\mathbb{R}^N} \frac{1 - \tilde{\chi}(\xi)}{P(|\xi|) - \xi V - i\varepsilon} e^{ix\xi} d\xi$$
$$G_\varepsilon^2(x) = \int_{\mathbb{R}^N} \frac{\tilde{\chi}(\xi)}{P(|\xi|) - \xi V - i\varepsilon} e^{ix\xi} d\xi.$$

First, we observe that G_ε^1 is the Fourier transform of a function Q, with the property that $D^\gamma Q \in L^1(\mathbb{R}^N)$ for any sufficiently large $|\gamma|$, where γ is a multi-index. Thus, $|G_\varepsilon^1(x)| \leq C_\gamma |x|^{-|\gamma|}$ for large $|\gamma|$, and therefore G_ε^1 has arbitrarily (polynomial) fast decay at infinity. To estimate G_ε^2 we use the coarea formula

$$G_\varepsilon^2(\xi) = \int_{\mathbb{R}} \frac{\chi(\tau)}{\tau - i\varepsilon} \left(\int_{M_\tau} \frac{e^{ix\xi}}{|\nabla F(\xi)|} d\mathcal{H}^{N-1}(\xi) \right) d\tau = \int_{-\rho}^{\rho} \frac{\chi(\tau)}{\tau - i\varepsilon} a_x(\tau) d\tau,$$

where

$$a_x(\tau) = \int_{M_\tau} \frac{e^{ix\xi}}{|\nabla F(\xi)|} d\mathcal{H}^{N-1}(\xi).$$

Then, (A2) yields $\nabla F(\xi) \neq 0$ for $\xi \in M_0$. Therefore by continuity, decreasing $\rho > 0$ if necessary, we can assume that $\nabla F(\xi) \neq 0$ for any $\xi \in M_0$ and $|\tau| < \rho$. Since F is smooth, we have in addition $\left\| \dfrac{1}{\nabla F(\xi)} \right\|_{C^{\gamma, \tilde{\alpha}}(U)} \leq C$, for all $\gamma > 0$ and $\tilde{\alpha} \in (0, 1)$.

Hence, from Proposition 2.1 and Theorem 2.1, it follows that

$$|a_x(0)| \leq C(1+|x|)^{-\frac{k}{2}}, \tag{2.8}$$

with

$$k = \begin{cases} N-1, & \text{if } (2.1) \text{ holds,} \\ N-2, & \text{if } (2.2) \text{ holds.} \end{cases} \tag{2.9}$$

Proceeding for instance as in [23] (see [23, (43)]), one can also show that if $t > 0$ is small enough, then for some $b > 0$,

$$|a_x(t) - a_x(0)| \leq Ct^b(1+|x|)^{-k/2}. \tag{2.10}$$

Observe that we already obtained in (2.8) the decay of a_x as a function of x. The Hölder continuity in time t follows by trivial modifications from [23].

Next, we to use the following proposition

Proposition 2.2. [23, Proposition 7] *Fix $\lambda \in \mathbb{R}$ and $\rho \in (0,\infty]$ and assume that $a : [\lambda-\rho, \lambda+\rho] \to \mathbb{R}$ is measurable such that $|a(\lambda+t) - a(\lambda)| \leq \omega(|t|)$ where $t \to \omega(t)/t$ is integrable over $(0,\rho)$. Then, for any $\varepsilon > 0$,*

$$\begin{aligned} &\left| \int_{\lambda-\rho}^{\lambda+\rho} \frac{a(\tau)}{\tau - \lambda \mp i\varepsilon} d\tau - p.v. \int_{\lambda-\rho}^{\lambda+\rho} \frac{a(\tau)}{\tau-\lambda} d\tau \mp i\pi a(\lambda) \right| \\ &\leq \int_0^\rho \frac{2\varepsilon}{\sqrt{t^2+\varepsilon^2}} \frac{\omega(t)}{t} dt + \left(\pi - 2\arctan\left(\frac{\rho}{\varepsilon}\right)\right) |a(\lambda)|, \end{aligned}$$

and

$$\left| \int_{\lambda-\rho}^{\lambda+\rho} \frac{a(\tau)}{\tau-\lambda \mp i\varepsilon} d\tau \right| \leq 2\pi \left(\int_0^\rho \frac{\omega(t)}{t} dt + |a(\lambda)| \right).$$

We have all ingredients to estimate the decay of G.

Proposition 2.3. *Assume that (A1), (A2) and (A3) hold. Let k be as in (2.9), that is, k is the number of principal curvatures of M staying bounded away from 0. Then, for $|x| \geq 1$,*

$$|G(x)| \leq C(1+|x|)^{-k/2}, \tag{2.11}$$

and, for $|x| \leq 1$,

$$|G(x)| \leq C \begin{cases} |x|^{-(N-4)}, & \text{if } N > 4, \\ |\log|x||, & \text{if } N = 4, \\ 1, & \text{if } N < 4. \end{cases} \tag{2.12}$$

Proof. The estimate (2.12) is classical (see for example [31, Theorem 5.7]) so we focus on (2.11), where we follow the framework from [23, Proof of Proposition 3].

By (2.10), we obtain that χa_x satisfies $|(\chi a_x)(t) - (\chi a_x)(0)| \leq \omega_x(t) = Ct^b(1+|x|)^{-k/2}$ with integrable $t \to \omega_x(t)/t$ on $(0,\rho)$. If we define

$$G^2_\pm(x) = p.v. \int_{-\rho}^{\rho} \frac{\chi(\tau)}{\tau} a_x(\tau) d\tau \pm i\pi a_x(0),$$

then Proposition 2.2 with $a = \chi a_x$, $\varepsilon = 1$, $\lambda = 0$ and (2.8), yield

$$\begin{aligned}|G^2_\pm(x)| &\le \left|G^2_\pm(x) - \int_{-\rho}^{\rho} \frac{a(\tau)}{\tau \mp i} d\tau\right| + \left|\int_{-\rho}^{\rho} \frac{a(\tau)}{\tau \mp i} d\tau\right| \\ &\le C\left(\int_0^\rho \frac{\omega_x(t)}{t} dt + |a_x(0|\right) \\ &\le C(1+|x|)^{-k/2}\int_0^\rho t^{b-1}dt + C(1+|x|)^{-k/2} \\ &\le C(1+|x|)^{-k/2}.\end{aligned}$$

Also, assuming that $\varepsilon > 0$, the first estimate of Proposition 2.2, and (2.8) imply

$$\begin{aligned}|G^2_\varepsilon(x) - G^2_+(x)| &\le \int_0^\rho \frac{2\varepsilon}{\sqrt{\varepsilon^2+t^2}} \frac{\omega_x(t)}{t} dt + \left(\pi - 2\arctan\left(\frac{\rho}{\varepsilon}\right)\right)|a_x(0)| \\ &\le C(1+|x|)^{-k/2}\int_0^\rho \frac{\varepsilon}{\sqrt{\varepsilon^2+t^2}} t^{b-1}dt + C(1+|x|)^{-k/2}\frac{2\varepsilon}{\rho} \\ &\le C\varepsilon^b(1+|x|)^{-k/2}\end{aligned}$$

and analogous inequality for $\varepsilon < 0$ after replacing G^2_+ by G^2_-. Therefore, since $b > 0$, we deduce that G^2_ε converges pointwise to $G^2_\pm$ when $\varepsilon \to 0^\pm$.

Overall, we get

$$\begin{aligned}|G(x)| = \limsup_{\epsilon\to 0}|G_\epsilon(x)| &\le \limsup_{\epsilon\to 0}|G^1_\epsilon(x)| + |G^2_\epsilon(x)| \\ &\le \limsup_{\epsilon\to 0^\pm}|G^1_\epsilon(x)| + |G^2_\epsilon(x) - G^2_\pm| + |G^2_\pm| \le C(1+|x|)^{-k/2},\end{aligned}$$

as desired. □

3. Resolvent estimate and application to the nonlinear Helmholtz equation

In this section we investigate the boundedness of the resolvent $\mathfrak{R}$" = "$(\Delta^2+\beta\Delta+iV\nabla+\alpha)^{-1}$ as an operator from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$. The obtained estimates are crucially used in the construction of solution to the nonlinear Helmholtz equation. As in previous sections, we define $\mathfrak{R}$, by the limit absorption principle, that is,

$$\mathfrak{R}f := \lim_{\varepsilon\to 0^+}\mathfrak{R}_{i\varepsilon}f,$$

where

$$(\mathfrak{R}_{i\varepsilon}f)(x) := \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{ix\xi}\frac{\hat{f}(\xi)}{|\xi|^4 - \beta|\xi|^2 + V\xi + \alpha - i\varepsilon}.$$

To study the mapping properties of $\mathfrak{R}$, we use ideas from [17, Proof of Theorem 6] (see also [12, Theorem 2.1] and [7, Theorem 3.3]). Our proof relies on the decay properties of the Green function established in the previous section and on the Stein-Tomas Theorem, which was proved in [2] and improves [29, 27]. We also refer to [18] for an example proving that the used results are sharp in some sense.

Theorem 3.1. [2] *Let $1 \le p \le \frac{2(k+2)}{k+4}$ and M be a smooth, closed and compact manifold with k non-null principal curvatures. Then, there is a $C > 0$ such that for all $g \in \mathcal{S}(\mathbb{R}^N)$ the following inequality holds:*

$$\Big(\int_M |\hat{g}(\omega)|^2\, d\sigma(\omega)\Big)^{1/2} \le C\|g\|_{L^p(\mathbb{R}^N)}.$$

Before we proceed, let us introduce some notation. Let $\psi \in \mathcal{S}(\mathbb{R}^N)$ be a function such that $\hat{\psi} \in C_c^\infty(\mathbb{R}^N)$ satisfies $0 \le \hat{\psi} \le 1$ and

$$\hat{\psi}(\xi) = \begin{cases} 1, & \text{if } \operatorname{dist}(\xi, M) \le c_1, \\ 0, & \text{if } \operatorname{dist}(\xi, M) \ge 2c_1, \end{cases}$$

for some c_1 small enough detailed below. As above, define $G_1 := (1 - \psi) * G$ and $G_2 := \psi * G$ and $\mathfrak{R}_i f = G_i * f$ for $i = 1, 2$. First, we obtain estimates for $\mathfrak{R}_1$.

Proposition 3.1. *For every $c_1 > 0$, any $p, q \in (1, \infty)$ with $q > p$ and*

$$\frac{1}{p} - \frac{1}{q} \begin{cases} \le 1 & \text{if } N < 4, \\ < 1 & \text{if } N = 4, \\ \le \frac{4}{N} & \text{if } N \ge 4, \end{cases} \tag{3.1}$$

the operator $\mathfrak{R}_1$ extends to a bounded linear operator $\mathfrak{R} : L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$, that is, there exists $C = C_{p,q}$ such that

$$\|\mathfrak{R} f\|_{L^q(\mathbb{R}^N)} \le C \|f\|_{L^p(\mathbb{R}^N)} .$$

Proof. By Proposition 2.3, we have

$$|G_1(x)| \le \begin{cases} C \min\{|x|^{4-N}, |x|^{-N}\} & \text{if } N > 4, \\ C \min\{1 + |\log |x||, |x|^{-N}\} & \text{if } N = 4, \\ C \min\{1, |x|^{-N}\} & \text{if } N < 4, \end{cases} \qquad \text{for all } x \in \mathbb{R}^N \setminus \{0\} . \tag{3.2}$$

Fix $p, q \in (1, \infty)$ as in (3.1) and assume $\frac{1}{p} - \frac{1}{q} < \frac{4}{N}$ if $N \ge 4$. Set $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$. Since $q > p$, then $r \in \left(1, \frac{N}{(N-4)_+}\right)$, where we set $a/0 = \infty$. Since G_1 belongs to the weak Lebesgue space $L^{r,w}(\mathbb{R}^N)$, the Young's convolution inequality for weak Lebesgue spaces, (see [16, Theorem 1.4.24]) gives us

$$\|G_1 * f\|_{L^q(\mathbb{R}^N)} \le \|G_1\|_{L^{r,w}(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N)} \le C \|f\|_{L^p(\mathbb{R}^N)} ,$$

as desired. If $N > 4$, and $\frac{1}{p} - \frac{1}{q} = \frac{4}{N}$, then $r < \infty$ and we proceed as above. □

In the next result we establish the crucial bound on $\mathfrak{R}_2$. Since ψ in the definition of G_2 is a Schwartz function such that $\hat{\psi} \in C_c^\infty(\mathbb{R}^N)$, then its convolution yields a bounded function, and by Proposition 2.3,

$$|G_2(x)| \le C(1 + |x|)^{-\frac{k}{2}} \quad \text{for all } x \in \mathbb{R}^N . \tag{3.3}$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ if $|x| \ge 2$. For $j \in \mathbb{N}$ we define $\eta_j(x) := \eta(x/2^j) - \eta(x/2^{j-1})$ and $\eta_0 := \eta$. Therefore, by (3.3),

$$G_2 = \sum_{j=0}^{\infty} G_2^j \quad \text{with } G_2^j := G_2 \eta_j \text{ so that } |G_2^j(x)| \le C 2^{-jk/2} 1_{[2^{j-1}, 2^{j+1}]}(|x|), \tag{3.4}$$

and

$$G_2 * f = \sum_{j=0}^{\infty} G_2^j * f .$$

Theorem 3.2. *Assume that (A1), (A2), and (A3) hold and k is as in Theorem 3.1. Then, for any sufficiently small $c_1 > 0$ there is $C > 0$ such that*

$$\|G_2^j * f\|_{L^2(\mathbb{R}^N)} \le C 2^{j/2} \|f\|_{L^{2(k+2)/(k+4)}(\mathbb{R}^N)}, \tag{3.5}$$

for any $j \ge 1$ and $f \in L^{2(k+2)/(k+4)}(\mathbb{R}^N)$.

Proof. Using Plancherel's Theorem and the coarea formula, we obtain, for $f \in \mathcal{S}(\mathbb{R}^N)$,

$$\|G_2^j * f\|^2_{L^2(\mathbb{R}^N)} \leq C \int_{-2c_1}^{2c_1} \max_{\xi \in M_\tau} (|\hat{G}_2^j(\xi)|^2) \int_{M_\tau} \frac{|\hat{f}(\xi)|^2}{|\nabla F(\xi)|} d\sigma(\xi)\, d\tau.$$

Since F is a smooth function with $\nabla F \neq 0$ on M_0, then by making c_1 smaller if necessary, we may assume $\nabla F \neq 0$ on M_τ for $|\tau| < 3c_1$. Then, $\xi \mapsto |\nabla F(\xi)|^{-1}$ is also smooth on M_τ for $|\tau| < 3c_1$ and since M_τ, for $|\tau|$ sufficiently small, is smooth closed and compact manifold with k principal curvatures bounded away from 0, the Stein-Tomas Theorem (see Theorem 3.1) implies

$$\|G_2^j * f\|^2_{L^2(\mathbb{R}^N)} \leq C \|f\|^2_{L^{2(k+2)/(k+4)}(\mathbb{R}^N)} \int_{-2c_1}^{2c_1} \max_{\xi \in M_\tau} (|\hat{G}_2^j(\xi)|^2) d\tau.$$

The coarea formula, yields

$$\hat{G}_2^j(\xi) = 2^{(j-1)N} \int_{-2c_1}^{2c_1} \frac{1}{\tau} \int_{M_\tau} \frac{\hat{\psi}(s)\hat{\eta}_1(2^{j-1}(\xi - s))}{|\nabla F(s)|} d\sigma(s) d\tau\,.$$

Let us define the map $\phi_\tau : M_\tau \to M_0$ such that $\phi_\tau(w) = w + \tau m(\tau)\nu(w)$, where $\nu(w)$ is the unit exterior normal vector to M_0 at w and m is the smallest number such that $\phi_\tau(w) \in M_\tau$. Since M_τ is a level surface of F, then $\nabla F(w) = \nu(w)$, and by Taylor theorem, for any sufficiently small c_1, there is a constant c_2 such that $c_2^{-1} \leq m(\tau) \leq c_2$ and $|m'(\tau)| \leq c_2$ for any $|\tau| < 3c_1$. Thus, ϕ_τ is a smooth bijection, and therefore we can introduce the change of variables $s = \phi_\tau(w)$ and obtain

$$\hat{G}_2^j(\xi) = 2^{(j-1)N} \int_{-2c_1}^{2c_1} \frac{1}{\tau} \int_{M_0} \frac{\hat{\psi}(\phi_\tau(w))\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau(w)))}{|\nabla F(\phi_\tau(w))|} |J_\tau(w)| d\sigma(w) d\tau\,,$$

where $|J_\tau(w)|$ is the Jacobian of the transformation ϕ_τ. Using the oddness of the function $\tau \to 1/\tau$, we have

$$\begin{aligned} \hat{G}_2^j(\xi) = 2^{(j-1)N} \int_0^{2c_1} \frac{1}{\tau} \int_{M_0} \Big(& \frac{\hat{\psi}(\phi_\tau(w))\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau(w)))}{|\nabla F(\phi_\tau(w))|} |J_\tau(w)| \\ & - \frac{\hat{\psi}(\phi_{-\tau}(w))\hat{\eta}_1(2^{j-1}(\xi - \phi_{-\tau}(w)))}{|\nabla F(\phi_{-\tau}(w))|} |J_{-\tau}(w)| d\sigma(w) d\tau. \end{aligned}$$

Next, write $\hat{G}_2^j$ as

$$\hat{G}_2^j(\xi) = 2^{(j-1)N} \int_0^{2c_1} \frac{1}{\tau} \int_{M_0} (I + II + III + IV) d\sigma(w) d\tau,$$

where, after omitting the argument w,

$$I = \frac{(\hat{\psi}(\phi_\tau) - \hat{\psi}(\phi_{-\tau}))\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau))}{|\nabla F(\phi_\tau)|} |J_\tau|,$$

$$II = \frac{\hat{\psi}(\phi_{-\tau})(\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau)) - \hat{\eta}_1(2^{j-1}(\xi - \phi_{-\tau})))}{|\nabla F(\phi_\tau)|} |J_\tau|,$$

$$III = (\hat{\psi}(\phi_{-\tau})\hat{\eta}_1(2^{j-1}(\xi - \phi_{-\tau}))) \left(\frac{1}{|\nabla F(\phi_\tau)|} - \frac{1}{|\nabla F(\phi_{-\tau})|} \right) |J_\tau|,$$

and

$$IV = \frac{\hat{\psi}(\phi_{-\tau})\hat{\eta}_1(2^{j-1}(\xi - \phi_{-\tau}))}{|\nabla F(\phi_{-\tau})|} (|J_\tau| - |J_{-\tau}|).$$

First we estimate I. Since $\hat{\psi}$ is smooth and m is bounded, by the mean value theorem

$$|\hat{\psi}(\phi_\tau(w)) - \hat{\psi}(\phi_{-\tau}(w))| \leq C|\phi_\tau(w) - \phi_{-\tau}(w)| = C|\tau m(\tau) + \tau m(-\tau)| \leq C\tau\,,$$

and consequently the boundedness of $|\nabla F|^{-1}$ and a return to the original variables imply

$$\begin{aligned}
2^{(j-1)N} & \int_0^{2c_1} \frac{1}{\tau} \int_{M_0} I d\sigma(w) d\tau \\
& \leq C 2^{jN} \int_0^{2c_1} \int_{M_0} |\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau(w)))| \frac{|J_\tau(w)|}{|\nabla F(\phi_\tau(w))|} d\sigma(w) d\tau \\
& = C 2^{jN} \int_0^{2c_1} \int_{M_\tau} \frac{|\hat{\eta}_1(2^{j-1}(\xi - s))|}{|\nabla F(s)|} d\sigma(s) d\tau \\
& \leq C 2^{jN} \int_{\mathbb{R}^N} |\hat{\eta}_1(2^{j-1}(\xi - z))| dz = C 2^{jN} \int_{\mathbb{R}^N} |\hat{\eta}_1(2^{j-1} z)| dz \\
& \leq C \,,
\end{aligned}$$

where in the last step we used that $\hat{\eta}_1$ is a Schwartz function and in particular integrable.

To estimate III we use that $|\nabla F|^{-1}$ is smooth, and therefore we get

$$\left| \frac{1}{|\nabla F(\phi_\tau(w))|} - \frac{1}{|\nabla F(\phi_{-\tau}(w))|} \right| \leq C\tau,$$

and the estimate for III follows as above.

Also, since ϕ is smooth and non-degenerate, then J_τ (proportional to $\nabla\phi$) is smooth and $|J_\tau(w)| \geq c > 0$ for any $|\tau| \leq 3c_1$. Therefore,

$$||J_\tau(w)| - |J_{-\tau}(w)|| \leq C\tau |J_\tau(w)|$$

and the estimate for IV follows as above.

Finally, for II, by the mean value theorem, we have

$$\begin{aligned}
\hat{\eta}_1(2^{j-1}(\xi - \phi_\tau(w))) - \hat{\eta}_1(2^{j-1}(\xi - \phi_{-\tau}(w))) &= \int_{-1}^1 \frac{d}{dr} \hat{\eta}_1(2^{j-1}(\xi - \phi_{r\tau}(w))) dr \\
&= \int_{-1}^1 \nabla\hat{\eta}_1(2^{j-1}(\xi - \phi_{r\tau}(w))) \nu(w)(\tau m(r\tau) + r\tau^2 m'(r\tau)) dr \,.
\end{aligned}$$

Then, since $|m|, |m'| \leq C$ and $|J_{\tau r}| \geq c > 0$ for $|\tau| \leq 2c_1$ and any $|r| \leq 1$, by following steps above we obtain

$$\begin{aligned}
2^{(j-1)N} & \int_0^{2c_1} \frac{1}{\tau} \int_{M_0} IV d\sigma(w) d\tau \\
& \leq C 2^{jN} \int_{-1}^1 \int_0^{2c_1} \int_{M_0} |\nabla\hat{\eta}_1(2^{j-1}(\xi - \phi_{r\tau}(w)))| d\sigma(w) d\tau dr \\
& \leq C 2^{jN} \int_{-1}^1 \int_0^{2c_1} \int_{M_0} |\nabla\hat{\eta}_1(2^{j-1}(\xi - \phi_{r\tau}(w)))||J_{\tau r}(w)| d\sigma(w) d\tau dr \\
& = C 2^{jN} \int_{-1}^1 \int_0^{2c_1} \int_{M_{\tau r}} |\nabla\hat{\eta}_1(2^{j-1}(\xi - s))| d\sigma(s) d\tau dr \\
& \leq C 2^{jN} \int_{-1}^1 \int_{\mathbb{R}^N} |\nabla\hat{\eta}_1(2^{j-1}(\xi - z))| dz dr \\
& \leq C 2^{jN} \int_{\mathbb{R}^N} |\nabla\hat{\eta}_1(2^{j-1}(\xi - z))| dz
\end{aligned}$$

and the rest of the proof follows analogously as above with η replaced by $\nabla\eta$.

Overall, we showed

$$\|G_2^j * f\|^2_{L^2(\mathbb{R}^N)} \leq C 2^j \|f\|^2_{L^{2(k+2)/(k+4)}(\mathbb{R}^N)}.$$

and the proof is finished. □

Theorem 3.3. *Assume that (A1), (A2), and (A3) hold and k is as in Theorem 3.1. Then, for any sufficiently small $c_1 > 0$ there is $C = C(p,q) > 0$ such that*

$$\|G_2 * f\|_{L^q(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}$$

provided p and q satisfy

$$\begin{aligned} &\frac{1}{p} - \frac{1}{q} > \frac{2}{2+k}\,, \\ &\frac{1}{q} + \frac{(k+2)(N-k-1)}{kN}\frac{1}{p} > \frac{4N + 2kN - 4 - 6k - k^2}{2kN}\,, \\ &\frac{(k+2)(N-1-k)}{kN}\frac{1}{q} + \frac{1}{p} > 1 - \frac{k}{2N}\,. \end{aligned} \tag{3.6}$$

If $p = q'$, then (3.6) reduces to $q > \frac{2(k+2)}{k}$.

Proof. Using (3.4), for any $\tilde p, \tilde q, r \in [1,\infty]$ such that $1 + \frac{1}{\tilde q} = \frac{1}{r} + \frac{1}{\tilde p}$, Young inequality gives

$$\begin{aligned} \|G_2^j * f\|_{L^{\tilde q}(\mathbb{R}^N)} &\le \|G_2^j\|_{L^r(\mathbb{R}^N)}\|f\|_{L^{\tilde p}(\mathbb{R}^N)} \\ &\le C2^{j\left(-\frac{k}{2}+\frac{N}{r}\right)}\|f\|_{L^{\tilde p}(\mathbb{R}^N)} \\ &= C2^{j\left(-\frac{k}{2}+N+\frac{N}{\tilde q}-\frac{N}{\tilde p}\right)}\|f\|_{L^{\tilde p}(\mathbb{R}^N)}\,. \end{aligned} \tag{3.7}$$

Fix $q \ge 2$, $1 \le p \le \frac{2(k+2)}{k+4}$, and $\tilde p \le p$, $\tilde q \ge q$, and $\theta \in [0,1]$ such that

$$\frac{1}{p} = \frac{\theta}{\tilde p} + \frac{(1-\theta)(k+4)}{2(k+2)}\,, \qquad \frac{1}{q} = \frac{\theta}{\tilde q} + \frac{1-\theta}{2}\,.$$

By substituting $\tilde p = 1$ and $\tilde q = \infty$, we observe that such choice is possible if and only if $1 \ge \theta \ge \max\{1 - \frac{2}{q}, \frac{2(k+2)-p(k+4)}{kp}\}$. Then, from the Riesz-Thorin theorem, (3.5) and (3.7), it follows that

$$\|G_2^j * f\|_{L^q(\mathbb{R}^N)} \le C2^{j\left(\frac{1}{2}+\theta N\left(1-\frac{k+1}{2N}+\frac{1}{\tilde q}-\frac{1}{\tilde p}\right)\right)}\|f\|_{L^p(\mathbb{R}^N)} \qquad (j \in \mathbb{Z}),$$

which after substitution for $\tilde p$ and $\tilde q$ becomes

$$\|G_2^j * f\|_{L^q(\mathbb{R}^N)} \le C2^{j\left(\frac{2N+2+k}{2(k+2)}+\frac{N}{q}-\frac{N}{p}+\theta N\left(\frac{3}{2}-\frac{k+1}{2N}-\frac{k+4}{2(k+2)}\right)\right)}\|f\|_{L^p(\mathbb{R}^N)} \qquad (j \in \mathbb{Z})\,.$$

Denote

$$A = \frac{2N+2+k}{2N(k+2)}\,, \qquad B = \left(\frac{3}{2} - \frac{k+1}{2N} - \frac{k+4}{2(k+2)}\right)\,.$$

Since the factor of θ is positive, to make the exponent negative, we need to choose θ as small as possible and such θ exists if and only if

$$A + \frac{1}{q} - \frac{1}{p} + \left(1 - \frac{2}{q}\right)B < 0, \text{ and } A + \frac{1}{q} - \frac{1}{p} + \left(\frac{2(k+2)}{kp} - \frac{k+4}{k}\right)B < 0\,. \tag{3.8}$$

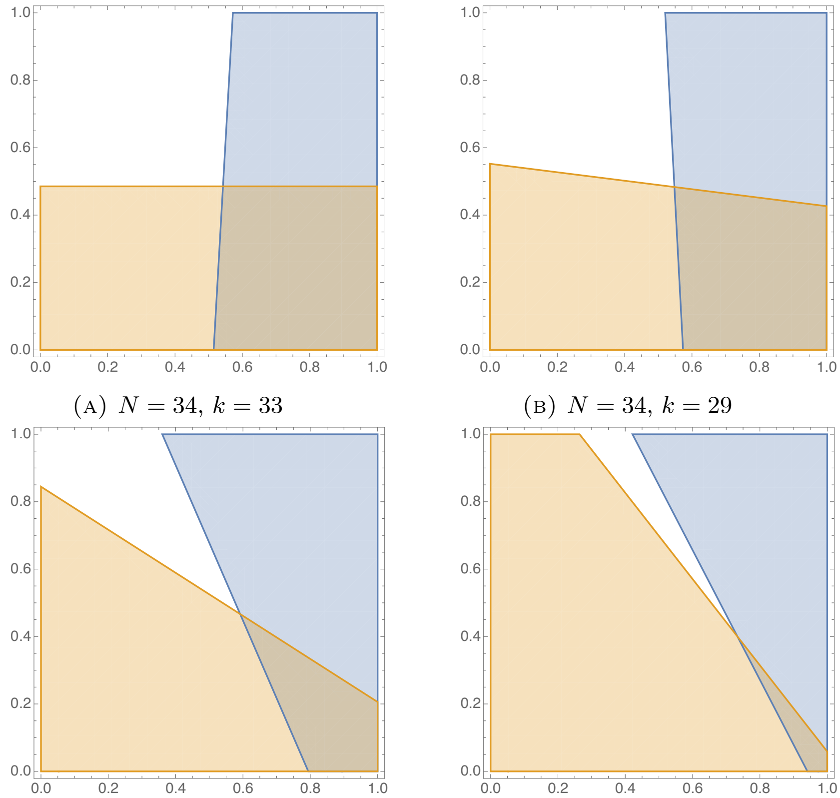
Then, (3.8) guarantee that

$$\|G_2 * f\|_{L^q(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}\,.$$

To visualize (3.8) we substitute $x = 1/p$ and $y = 1/q$ and get

$$A + y - x + (1-2y)\,B < 0, \text{ and } A + y - x + \left(\frac{2(k+2)}{k}x - \frac{k+4}{k}\right)B < 0.$$

Therefore the admissible region Γ in xy-plane lies between two lines and inside the square $[0,1]\times[0,1]$, see Figure 2 for sample regions.



(A) $N = 34$, $k = 33$ (B) $N = 34$, $k = 29$

(C) $N = 34$, $k = 14$ (D) $N = 34$, $k = 4$

FIGURE 2. The region Γ in dimension $N = 34$ with selected values of k

Since $\hat{G}_2$ is real, the convolution with G_2 is self-adjoint operator, and we obtain an estimate for the adjoint

$$\|G_2 * f\|_{L^{p'}(\mathbb{R}^N)} \leq C\|f\|_{L^{q'}(\mathbb{R}^N)}$$

whenever $p \leq \frac{2(k+2)}{k+4}$ and $q \geq 2$ satisfy (3.8). Noticing that if (p, q) is admissible, then (q', p') is admissible as well. Equivalently, if (x, y) is admissible, then $(1-y, 1-x)$ is admissible as well. Thus, since Γ is an admissible region, then the reflection of Γ, denoted Γ', with respect to the line $x + y = 1$ that lies inside the unit square is also an admissible region. By the Riesz-Thorin theorem, we can interpolate, which in the xy-plane means that the convex hull of $\Gamma \cup \Gamma'$ is also an admissible region.

Let us provide quantitative calculations. Using that $A + B = 1 - \frac{k}{2N}$ and calculating the intersections of lines, we obtain that he region Γ is the quadrilateral (see Figure 3) with the vertices

$$P = \left(1 - \frac{k}{2N}, 0\right), \ Q = (1, 0), \ R = \left(1, \frac{k}{2N}\right), \ S = \left(\frac{4 + 6k + k^2}{4 + 6k + 2k^2}, \frac{k}{2(k+1)}\right).$$

Perhaps surprisingly S is independent of N. Since the P and R are symmetric with respect to the line $x + y = 1$, then Γ' is a quadrilateral bounded by the vertices P, Q, R and

$$S' = \left(\frac{k+2}{2(k+1)}, \frac{k^2}{4 + 6k + 2k^2}\right).$$

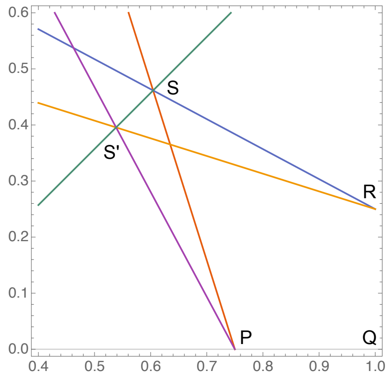


FIGURE 3. Admissible region for $N = 24$ and $k = 12$

Finally, the convex hull of $\Gamma \cup \Gamma'$ is a pentagon with the vertices P, Q, R, S, S'. The sides of this pentagon lie on the lines

$$y = 0, \qquad x = 1, \qquad A + y - x + \left(\frac{2(k+2)}{k} x - \frac{k+4}{k} \right) B = 0$$

$$A + y - x + \left(1 - \frac{2(k+2)}{k} y \right) B = 0, \qquad x - y = \frac{2}{2+k} .$$

Transforming back to p and q and omitting trivial conditions $p \geq 1$ and $q \leq \infty$, we obtain that

$$\|G_2 * f\|_{L^q(\mathbb{R}^N)} \leq C \|f\|_{L^p(\mathbb{R}^N)}$$

if (3.6) is valid.

Finally, $p = q'$, is equivalent to $x = 1 - y$, which is the axis of symmetry of $\Gamma \cup \Gamma'$. Thus, any point on axis of symmetry that lies inside the square $[0, 1] \times [0, 1]$ satisfying $x - y > \frac{2}{2+k}$ belongs to the convex hull of $\Gamma \cup \Gamma'$, and the last assertion follows.

□

Finally, we use Theorem 3.2 to prove the existence of solution to the nonlinear Helmholtz equation using the dual variational method of Evequoz and Weth [13]. The main idea of the method is to rewrite (1.1) as $u = \mathfrak{R}(|u|^{p-2}u)$ and after the substitution $v = |u|^{p-2}u$, we are looking for a function $v \in L^{p'}(\mathbb{R}^N)$ satisfying

$$\mathfrak{R}(v) = |v|^{p'-2}v, \qquad \text{in } \mathbb{R}^N, \tag{3.9}$$

for $\frac{2(k+2)}{k} < p < \dfrac{2N}{(N-4)_+}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Equation (3.9) admits a variational structure and solutions can be found as critical points of the functional $J \in C^1(L^{p'}(\mathbb{R}^N), \mathbb{R})$ defined by

$$J(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} dx - \frac{1}{2} \int_{\mathbb{R}^N} \bar{v} \mathfrak{R} v dx ,$$

where $\bar{v}$ is the complex conjugate of v. Note that J is real valued since $\hat{G}$ is real, and therefore by the Plancherel theorem

$$\int_{\mathbb{R}^N} \bar{v} \mathfrak{R} v dx = \int_{\mathbb{R}^N} \bar{\hat{v}} \hat{G} \hat{v} d\xi = \int_{\mathbb{R}^N} \hat{G} |\hat{v}|^2 d\xi .$$

Proof of Theorem 1.1. Since $p > 2$, then $p' < 2$ and by a standard scaling argument (see e.g. [13, Lemma 4.2]), one can show that J has the mountain pass geometry, and therefore

$$c = \inf_{\gamma \in P} \max_{t \in [0,1]} J(\gamma(t)) > 0,$$

where $P = \{\gamma \in C([0,1], L^{p'}(\mathbb{R}^N)) : \ \gamma(0) = 0 \ \text{ and } J(\gamma(1)) < 0\}$ Moreover, there exists a Palais-Smale sequence $(v_n) \subset L^{p'}(\mathbb{R}^N)$, that is, (v_n) satisfies $\sup_n |J(v_n)| < \infty$ and $J'(v_n) \to 0$ in $L^{p'}(\mathbb{R}^N)$. Since $p' < 2$ as in [13, Lemma 4.2] we have that (v_n) is bounded in $L^{p'}(\mathbb{R}^N)$. In addition, since $\hat{G}$ is real, the convolution is a self-adjoint operator, and therefore

$$\left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \bar{v}_n (G * v_n) dx = J(v_n) - \frac{1}{p'} J'(v_n)[v_n] \to c \qquad \text{as} \quad n \to \infty \,. \tag{3.10}$$

Next, we show a non-vanishing property. More precisely, we prove that there exist $R > 0$, $\zeta > 0$, and a sequence $(x_n)_n \subset \mathbb{R}^N$ such that, up to a subsequence

$$\int_{B_R(x_n)} |v_n|^{p'} dx \geq \zeta \quad \text{for all} \quad n. \tag{3.11}$$

First, notice that it is sufficient to prove (3.11) for sequence v_n belonging to $\mathcal{S}(\mathbb{R}^N)$ the class of Schwartz function. Otherwise, we replace v_n by $\tilde{v}_n \in \mathcal{S}(\mathbb{R}^N)$ with $\|v_n - \tilde{v}_n\|_{L^{p'}} \leq \frac{1}{n}$. Arguing as in [13, proof of Theorem 3.1], we obtain that (3.10) holds true with v_n and c replaced respectively by $\tilde{v}_n$ and $c/2$, and (3.11) holds for v_n if we prove it for $\tilde{v}_n$. We proceed by contradiction and assume that

$$\lim_{n \to \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B_\rho(y)} |v_n|^{p'} dx \right) = 0 \text{ for all} \quad \rho > 0. \tag{3.12}$$

The same decomposition as in the proof of Theorem 3.2 yields

$$\int_{\mathbb{R}^N} \bar{v}_n \mathfrak{R} v_n = \int_{\mathbb{R}^N} \bar{v}_n G_1 * v_n dx + \int_{\mathbb{R}^N} \bar{v}_n G_2 * v_n dx. \tag{3.13}$$

Using the estimate (3.2) for G_1, we proceed exactly as in [13, Lemma 3.2], just replacing $N-2$ by $N-4$ to show that

$$\int_{\mathbb{R}^N} \bar{v}_n [G_1 * v_n] dx \to 0 \ \textit{as} \ n \to \infty. \tag{3.14}$$

For fixed $R = 2^M > 0$ specified below, denote $M_R = \mathbb{R}^N \backslash B_R$ and decompose

$$\int_{\mathbb{R}^N} \bar{v}_n [G_2 * v_n] dx = \int_{\mathbb{R}^N} \bar{v}_n [(1_{B_R} G_2) * v_n] dx + \int_{\mathbb{R}^N} \bar{v}_n [(1_{M_R} G_2) * v_n] dx.$$

By the second half of [13, Proof of Lemma 3.4] which only uses the boundedness of G_2, we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \bar{v}_n [(1_{B_R} G_2) * v_n] dx = 0 \qquad \text{for any} \quad R > 0. \tag{3.15}$$

To estimate $\int_{\mathbb{R}^N} \bar{v}_n [(1_{M_R} G_2) * v_n] dx$, denote $P_R = 1_{M_R} G_2$, $R \geq 4$ and for η_j as in the proof of Theorem 3.2, define

$$P_j(x) = P_R(x) \eta_j(x), \ j \in \mathbb{N}, \qquad \text{and therefore } P_R = \sum_{j=[\log_2 R]}^{\infty} P^j.$$

Notice that $P^j = 0$ for $j \leq M-1$ and $P^j = G_2^j$ for $j \geq M+1$, where $G_2^j = G_2 \eta_j$ was defined in the proof of Theorem 3.2. Since P^M can be treated as $1_{M_R} G_2$ above, we only need to estimate $P^j = G_2^j$ for $j \geq M+1$. By (3.3),

$$\|P^j\|_{L^\infty(\mathbb{R}^N)} \leq C 2^{-jk/2}, \qquad \text{for any} \quad j \geq M+1. \tag{3.16}$$

Fix any $j > M$ and denote $d = \frac{2(k+2)}{k+4}$. Then, from (3.5) follows

$$\|P^j * v_n\|_{L^2(\mathbb{R}^N)} \leq 2^{j/2}\|v_n\|_{L^d(\mathbb{R}^N)},$$

and by duality (convolution with kernel that has real fourier transform is self-adjoint),

$$\|P^j * v_n\|_{L^{d'}(\mathbb{R}^N)} \leq 2^{j/2}\|v_n\|_{L^2(\mathbb{R}^N)}.$$

Since

$$\frac{1}{s} = \frac{1-\theta}{b} + \frac{\theta}{c} \qquad \text{implies } \frac{1}{s'} = \frac{1-\theta}{c'} + \frac{\theta}{b'}$$

if $\theta = \frac{1}{2}$, we obtain that for $(b, c) = (d, 2)$ that $s = \frac{2(k+2)}{k+1}$. Consequently, the Riesz-Thorin theorem yields

$$\|P^j * v_n\|_{L^s(\mathbb{R}^N)} \leq C2^{j/2}\|v_n\|_{L^{s'}(\mathbb{R}^N)}.$$

On the other hand, from Young's inequality and (3.16) follows

$$\|P^j * v_n\|_{L^\infty(\mathbb{R}^N)} \leq C2^{-jk/2}\|v_n\|_{L^1(\mathbb{R}^N)}.$$

Since

$$\frac{1}{p} = \frac{1-\theta}{s} + \frac{\theta}{\infty} \qquad \text{implies } \frac{1}{p'} = \frac{1-\theta}{s'} + \frac{\theta}{1}$$

we obtain from the Riesz-Thorin theorem that, for any $p \geq s$ and $j > M$,

$$\|P^j * v_n\|_{L^p(\mathbb{R}^N)} \leq C2^{j(\frac{k+2}{p} - \frac{k}{2})}\|v_n\|_{L^{p'}(\mathbb{R}^N)}.$$

Notice that, by assumption, $\frac{k+2}{p} - \frac{k}{2} < 0$, so a summation with respect to $j > M$ implies

$$\begin{aligned}\|(1_{M_R}G_2) * v_n\|_{L^p(\mathbb{R}^N)} &\leq C\|v_n\|_{L^{p'}(\mathbb{R}^N)} \sum_{j=M+1}^{\infty} 2^{j(\frac{k+2}{p} - \frac{k}{2})} \\ &\leq C2^{M(\frac{k+2}{p} - \frac{k}{2})}\|v_n\|_{L^{p'}(\mathbb{R}^N)},\end{aligned}$$

and consequently

$$\sup_{n\in\mathbb{N}} \left| \int_{\mathbb{R}^N} \bar{v}_n[(1_{M_R}G_2) * v_n]dx \right| \leq C2^{M(\frac{(k+2)}{p} - \frac{k}{2})} \sup_{n\in\mathbb{N}} \|v_n\|^2_{L^{p'}(\mathbb{R}^N)} \to 0 \tag{3.17}$$

as $R \to \infty$. Thus, we have proved that if (3.12) holds then, using (3.13), (3.14), (3.15), and (3.17),

$$\int_{\mathbb{R}^N} v_n \mathfrak{R} v_n dx \to 0, \ \textit{as } n \to 0.$$

This contradicts (3.10), and therefore (3.11) holds. Thus, denoting $u_n(x) = v_n(x - x_n)$, (u_n) is a bounded Palais-Smale sequence of J which weakly converges to some $u \in L^{p'}(\mathbb{R}^N)$. By proceeding as for instance in [7, Theorem 4.1], for any $R > 0$ any any smooth φ compactly supported in B_R one has

$$\begin{aligned}&\left| \int_{\mathbb{R}^N} (|u_n|^{p'-2}u_n - |u_m|^{p'-2}u_m)\varphi dx \right| \\ &\quad = \left| J'(u_n)[\varphi] - J'(m_n)[\varphi] + \int_{B_R} \mathfrak{R}(u_n - u_m)\varphi \right| \\ &\quad \leq (\|J'(u_n)\| + \|J'(u_m)\|)\|\varphi\|_{L^{p'}(\mathbb{R}^N)} + \|1_{B_R}\mathfrak{R}(u_n - u_m)\|_{L^p(\mathbb{R}^N)}\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.\end{aligned}$$

Since $p < \frac{2N}{(N-4)_+}$, $W^{4,p'}$ is compactly embedded in L^p, and by local regularity results (see [1, Theorem 14.1′]) one has

$$\|1_{B_R}\mathfrak{R}w\|_{L^p(\mathbb{R}^N)} \leq C\|1_{B_R}\mathfrak{R}w\|_{W^{4,p'}(\mathbb{R}^N)} \leq C_R\|w\|_{L^{p'}(B_R)}.$$

Therefore, by compactness $(\Re(u_n-u_m))_{m,n}$ converges strongly to zero as $m,n\to\infty$ in L^p. In addition, both $\|J'(u_n)\|$ and $\|J'(u_m)\|$ converge to zero as $n,m\to 0$. Thus $|u_n|^{p'-2}u_n$ strongly converges to $|u|^{p'-2}u$ locally in L^p. By (3.11), we have

$$\zeta \leq \int_{B_R(x_n)} |v_n|^{p'}dx = \int_{B_R} |u_n|^{p'}dx = \int_{B_R} |u_n|^{p'-2}u_n\overline{u}_n dx \to \int_{B_R} |u|^{p'}dx$$

as $n\to\infty$, were we used that $|u_n|^{p'-2}u_n \to |u|^{p'-2}u$ strongly in $L^p(B_R)$ and $u_n \rightharpoonup u$ weakly in $L^{p'}(\mathbb{R}^N)$. By standard arguments (see for example [7, Theorem 4.1]), we obtain that $u\in L^{p'}(\mathbb{R}^N)$ is a non-trivial critical point of J. Also, by [1, Theorem 14.1′], see for example [7, Proposition 5.1]), we obtain that $u\in W^{4,p}_{\mathrm{loc}}(\mathbb{R}^N)$, and therefore by (3.11), u is a nontrivial (strong) solution of (1.1).

To obtain the global estimates, we proceed as in [7, Proposition 5.2 and Theorem 5.1]. By a bootstrap argument and using again that $p<\frac{2N}{(N-4)_+}$, we obtain that u is bounded (by a function of $\|u\|_{L^{p'}(\mathbb{R}^N)}<\infty$, which is bounded), for details see [7, proof of Theorem 5.1], and consequently by interpolation, $u\in L^q(\mathbb{R}^N)$ for any $q\in[p,\infty]$. Finally, by using [28, Corollary on page 559], we obtain that $u\in W^{4,q}(\mathbb{R}^N)$ for any $q\in[p,\infty)$ and the Hölder regularity follows from embeddings of Sobolev into Hölder spaces.

This concludes the proof. □

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