Hydrodynamic stability in the presence of a stochastic forcing: a case study in convection

Juraj Földes[1], Nathan Glatt-Holtz[2], Geordie Richards[3], and Jared P. Whitehead[2]†

[1]Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

[2]Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

[3]Mechanical and Aerospace Engineering Department, Utah State University, Logan, UT 84322, USA

[4]Mathematics Department, Brigham Young University, Provo, UT 84601, USA

We investigate the stability of a statistically stationary conductive state for Rayleigh-Bénard convection between stress-free plates that arises due to a bulk stochastic internal heating. This setup may be seen as a generalization to a stochastic setting of the seminal 1916 study of Lord Rayleigh. Our results indicate that stochastic forcing at small magnitude has a stabilizing effect, while strong stochastic forcing has a destabilizing effect. The methodology put forth in this article, which combines rigorous analysis with careful computation, also provides an approach to hydrodynamic stability for a variety of systems subject to a large scale stochastic forcing.

Key words: Stochastic Convection, Hydrodynamic Stability.

1. Introduction

Rayleigh Bénard convection, the buoyancy driven motion of a fluid under the influence of a gravitational field, is ubiquitous in nature. It is one of the driving forces in a variety of situations ranging from boiling a pot of water, to geophysical processes, to pattern formation in stellar dynamics. Yet, despite remarkable advances in mathematical, computational, and experimental analysis, fundamental aspects of Rayleigh-Bénard convection remain poorly understood (Ahlers *et al.* 2009; Chillà & Schumacher 2012).

The seminal work of Lord Rayleigh (1916), inspired by the experiments of Bénard (1900), quantified the onset of convection in terms of the instability of purely conductive solutions of the Boussinesq equations. This work established that, when the Rayleigh number Ra (a dimensionless parameter proportional to the boundary heating) is less than a critical value Ra_c, then the purely conductive state is globally attractive.

Various modifications to Rayleigh's original model, including alternative boundary conditions and different sources of heat (see Goluskin (2015*a*) for example) have been considered in the past century. In particular it is natural to extend Rayleigh's stability analysis to convective flows driven by stochastic forcing. Indeed, many sources of heat in physical models are inherently noisy. For example, in the earth's mantle, radioactive decay is an important source of heating, and in large stars thermonuclear reactions destabilize

† Email address for correspondence: whitehead@mathematics.byu.edu

the density gradient of convective cells; both of these processes are inherently stochastic Schubert *et al.* (2001); Kippenhahn & Weigert (1994).

This work develops a methodology to investigate hydrodynamic stability for stochastically driven models of Rayleigh Bénard convection in the spirit of Rayleigh's original analysis. Our approach relies on a combination of energy stability methods, ergodic theory, and numerical computation. We focus here on a Boussinesq system with horizontally stratified stochastic heating. Nevertheless our approach applies to a larger class of randomly forced convective models with various physical interpretations which we will address in future studies. The work presented here is also closely related to a line of research by the authors on the ergodic theory and dynamical properties of stochastically driven models for Rayleigh-Bénard convection (Földes *et al.* (2015); Földes *et al.* (2015, 2016)).

While our framework is new to the best of our knowledge, significant previous efforts have been made to incorporate random perturbations into models of convection. In an effort to understand the influence of thermal fluctuations, Swift & Hohenberg (1977); Ahlers *et al.* (1981) considered the Boussinesq system modulated by a singular (i.e. active at spatial frequencies) small noise in the bulk, and derived a reduced model for describing flow statistics. This model leads to accurate predictions of the rate of heat transfer near the onset of convection (Hohenberg & Swift (1992)), but requires stochastic forcing stronger than the predicted thermal fluctuations.

One significant difficulty in the approach initiated in Swift & Hohenberg (1977) is that the presence of a generic stochastic source eliminates the existence of a traditionally defined conductive state for which the velocity field is zero. See Swift & Hohenberg (1977); Ahlers *et al.* (1981); Meyer *et al.* (1991); Hohenberg & Swift (1992) and containing references. For example, Venturi *et al.* (2012) considered the 2-dimensional Boussinesq equations with stochastic horizontal boundary conditions, and identified a substitute "quasi-conductive regime", for which the velocity of solutions is non-zero but small. Through numerical analysis, they identified that these quasi-conductive states maintain stability for Ra $\sim 0 - 4000$ thus reaching beyond the classical bifurcation point of Ra$_c$ = 2585.02 for the deterministic Boussinesq system with this geometry.†

It is interesting to compare these numerical stability results with the theoretical development of rigorous ergodic theorems for stochastically forced Navier-Stokes equations, and related systems. For example, Hairer & Mattingly (2006, 2008) have established that for periodic 2-dimensional Navier-Stokes equations with bulk stochastic forcing, the system possesses a unique ergodic invariant measure provided the stochastically forced modes satisfy a modest geometric constraint. Moreover, these results have been extended to stochastic Boussinesq equations by the authors with various boundary conditions and parameter constraints (Földes *et al.* (2015); Földes *et al.* (2015, 2016)). In these contexts, the unique invariant measure is almost surely globally attractive in a statistical sense, but when stochastic forcing is more degenerate, e.g. restricted to a single spatial direction (horizontally stratified), and when the Rayleigh number Ra is large, the longtime statistics are much less clear.

In contrast to the approaches described above, we observe that when the stochastic forcing is horizontally stratified there is still a well-defined statistically stationary conductive state. The existence of this state provides the starting point of our work, and our primary goal is to investigate the onset of convection as a bifurcation from this conductive profile. As in Rayleigh (1916), we consider only stress-free velocity boundary conditions and fixed temperatures at the top and bottom plates. In this article we will focus only

† Note however that this is not the same framework that was considered in Rayleigh (1916).

on a bulk stochastic heat source, but the methodology is applicable to other situations, particularly when the stochastic perturbation appears in the boundary conditions as we will illustrate in future work.

Our approach to this problem may be summarized as follows. We begin by observing that the conductive state $\tau(t, z)$ (a random process dependent on the vertical spatial variable) satisfies a linear stochastic partial differential equation for which the unique stationary distribution can be computed explicitly. From an evolution equation for the fluctuations about $\tau(t, z)$ we derive a constrained optimization problem which provides a sufficient condition for decay of the fluctuations at an explicit random rate, denoted by $\lambda(\tau(t))$, which depends on the conductive state. In this way we adapted energy method from hydrodynamic stability (Drazin & Reid 2004) to our setting, and analyze the stability of the conductive state by solving a stochastic eigenvalue problem.

A crucial simplification both analytically an computationally follows from the observation that the system is stable about the conductive state provided that $\liminf_{t \to \infty} \frac{1}{t} \int_0^t \lambda(\tau(s)) ds > 0$, almost surely. However since τ is an ergodic process this expression is equivalent to integration of λ against the stationary law of τ. Fixing the non-dimensional stochastic heating strength H, through the use of the Dedalus computational package (Burns *et al.* 2017), we identify a critical Rayleigh number Ra_c such that the above holds for any $\mathrm{Ra} \leqslant \mathrm{Ra}_c$.

Our work demonstrates that for small H the critical Rayleigh number is comparable in value to the number obtained in Rayleigh (1916). However, we identify a rapid transition when the non-dimensional strength of the stochastic heating H is $O(1)$, where the critical Rayleigh number Ra_c quickly decays to zero, and hence the stability of the conductive state is no longer guaranteed for any value of Ra when H is sufficiently large.

The results are presented as follows: section 2 introduces the equations of motion, their non-dimensionalization and while section 3 sketches the derivation of the nonlineary stability. Section 4 discusses the numerical and algorithmic implementation of this calculation including convergence checks and criteria. Section 5 contains the results including sample distributions of the critical horizontal wave number. Finally in section 6 we draw some broad conclusions and discuss the potential extension of our method to further problems where stochasticity is present in a hydrodynamic setting.

2. Equations of motion

We are interested in the three-dimensional Boussinesq equations driven by a bulk stochastic forcing in the temperature equation:

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} + \frac{1}{\rho} \nabla \tilde{p} = g\alpha \boldsymbol{k} \tilde{T} + \nu \Delta \tilde{\boldsymbol{u}}, \qquad \nabla \cdot \tilde{\boldsymbol{u}} = 0 \tag{2.1}$$

$$d\tilde{T} + \left(\tilde{\boldsymbol{u}} \cdot \nabla \tilde{T} - \kappa \Delta \tilde{T} \right) d\tilde{t} = \gamma \sum_{k=1}^{M} \sigma_k dW^k, \tag{2.2}$$

where $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v}, \tilde{w})^T$ is the three-dimensional velocity vector field, $\tilde{p}$ is the pressure, and $\tilde{T}$ is the temperature field. In this model the parameters are ρ a reference density, g the gravitational constant, α the thermal expansion coefficient, κ the thermal diffusivity, and ν the kinematic viscosity. We are interested in a horizontally periodic box $\mathcal{D}$ of height h, complemented with stress-free conditions for $\tilde{\boldsymbol{u}}$, that is $\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0$ and $\frac{\partial \tilde{\boldsymbol{u}}_H}{\partial \boldsymbol{n}} = 0$ on the top and bottom plates where $\tilde{\boldsymbol{u}}_H$ is the horizontal components of the velocity parallel to the top and bottom plates, and a prescribed temperature difference: $\tilde{T}(z = 0) = T_1 > 0$ and $\tilde{T}(z = h) = 0$.

The parameter γ is the strength of a mean zero stochastic term that consists of M independent Brownian motions W^k acting on M spatially orthogonal directions (in the L^2 norm) given by $\{\sigma_k\}$. Details on the mathematical setting of (2.1)-(2.2) can be found in Földes *et al.* (2016) (see Da Prato & Zabczyk (1992); Kuksin & Shirikyan (2012) as well). The limit $M \to \infty$ represents noise at all the spatial scales of the system which is very roughly speaking the setting considered in Swift & Hohenberg (1977); Ahlers *et al.* (1981); Meyer *et al.* (1991); Hohenberg & Swift (1992). Generically, we are interested in stochastic forcing on physically relevant spatial scales, i.e. we will not consider forcing at scales below a given cutoff length scale. As discussed in the introduction, we restrict the σ_k to depend on the vertical coordinate z only.

2.1. *Non-dimensionalization*

As usual it is useful to consider (2.1)-(2.2) in dimensionless units. We non-dimensionalize by h spatially, ν/h^2 temporally, and T_1 for the temperature. This gives the following equivalent system (we use the same labels for the non-dimensional system, modulo the "tilde"):

$$\frac{1}{\mathrm{Pr}} \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) + \nabla p = \mathrm{Ra} \boldsymbol{k} T + \Delta \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{2.3}$$

$$dT + (\boldsymbol{u} \cdot \nabla T - \Delta T) \, dt = H \sum_{k=1}^{M} \sigma_k dW^k, \tag{2.4}$$

where the non-dimensional parameters are the Prandtl number $\mathrm{Pr} = \frac{\nu}{\kappa}$ a kinematic property of the fluid, the Rayleigh number $\mathrm{Ra} = \frac{\alpha g T_1 h^3}{\nu \kappa}$ and the heating parameter $H = \frac{\gamma}{T_1 \sqrt{\kappa h}}$. In these units the temperature on the bottom and the top of the box are $T(z=0) = 1$ and $T(z=1) = 0$. The stress-free boundaries on the fluid velocity are as described above namely $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ on the top and bottom boundaries.†

The Prandtl number varies greatly depending on the material properties of the fluid. For instance in air $\mathrm{Pr} \approx 0.7$, for water $\mathrm{Pr} \approx 7$, and analysis of the earth's mantle indicates that $\mathrm{Pr} \approx 10^{24}$ which is well approximated as infinity (Wang 2004, 2005, 2007, 2008; Földes *et al.* 2015). The Rayleigh number, representing the strength of the boundary driven forcing, has a wide range in applications. In particular the Rayleigh numbers of key geophysical and astrophysical interest range from 10^6 to 10^{20}.

The heating parameter H is the relative impact of the stochastic internal heating to the boundary driven heating, weighted appropriately by the cell height and thermal diffusivity. The parameter H also has a significant range of physically relevant values, although not as much as the other two parameters of interest. Clearly $H \sim 0$ is realized when the boundary forcing dominates the internal stochastic heating (or $\gamma = 0$), regardless of the cell height and thermal diffusivity of the fluid. Thus we are interested in determining which positive, large values of H are physically viable. It is difficult to physically compare γ relative to T_1, but we do expect that the stochastic effects will be less significant (Hohenberg & Swift (1992)), i.e. we assume that the most influential

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† In our recent related works we used a different non-dimensionalization that emphasized the role of the bulk stochastic heating over the deterministic boundary forcing. The current non-dimensionalization is more consistent with traditional studies of Rayleigh-Bénard convection in that $H = 0$ exactly recovers the original deterministic system proposed by Rayleigh (1916). In Földes *et al.* (2016) we consider two 'Rayleigh parameters' Ra and $\tilde{\mathrm{Ra}}$ whose product yield the Rayleigh number in this manuscript. The reciprocal of $\tilde{\mathrm{Ra}}$ from Földes *et al.* (2016) recovers the stochastic heating number H considered here.

Physical Setting	h (m)	κ (m^2/s)	maximal H
Earth's mantle	10^6	10^{-7}	3
Earth's oceans	10^3	10^{-7}	10
entire troposphere (Earth's atmosphere)	10^4	10^{-5}	3
convective updraft (Earth's atmosphere)	10^2	10^{-5}	3
convective zone in the sun	10^8	10^{-3}	0.003

TABLE 1. The relevant parameters used to determine the maximal value of H, the stochastic internal heating parameter, assuming that $\frac{\gamma}{T_1} \sim 0.1$ at maximum. The physically motivated situations here are by no means exhaustive, but they do indicate a relative maximal value of H that we may motivate physically.

noise can be is on the order of $\frac{\gamma}{T_1} \sim 0.1$, however this assumption is not required of our mathematical analysis. The other two quantities κ and h are properties of the system. Table 1 displays these values for several different physically relevant situations wherein Rayleigh-Bénard convection is used as a first order model. The given value of H is computed assuming that $\gamma/T_1 = 0.1$, and thus may be adjusted, dependent on the relative strengths of the stochastic forcing to the boundary forcing. From this table we can see that H is justifiably in the range from 0 to $O(10)$.

2.2. *The conductive state in the presence of a stochastic heat source*

The conductive state for (2.3)-(2.4) occurs when $\boldsymbol{u} = 0$, but we must retain time dependence of the temperature profile in order to modulate the forcing term. Moreover, to maintain $\boldsymbol{u} = 0$ the temperature field cannot be a function of the horizontal variables, since the buoyancy term in (2.3) needs to be absorbed into the pressure gradient. Hence, we seek a temperature field $\tau(z,t)$ that is a solution of $d\tau - \dfrac{\partial^2 \tau}{\partial z^2} dt = H \sum_{k=1}^{M} \sigma_k dW^k$, and satisfies the non-homogeneous boundary condition $\tau(z = 0) = 1$ and $\tau(z = 1) = 0$. To completely determine the solution to the above equation, we first need to specify σ_k. For the current investigation, we select σ_k to be the vertically dependent eigenfunctions of the Laplace operator on the domain $\mathcal{D}$, i.e. $\sigma_k(z) = \sqrt{2}\sin(\pi k z)$. This specification is convenient as it provides an ideal method to identify length scales in the forcing itself, as given by the vertical wave-number k.

With this choice of σ_k, the conductive state is easily found by breaking the stochastic partial differential equation above into its component spatial frequencies. The solution is the sum of Orhnstein-Uhlenbeck processes:

$$\tau(z,t) = 1 - z + \sum_{k=1}^{\infty} e^{-k^2\pi^2 t}\tau_k(0)\sigma_k(z) + H \sum_{k=1}^{M} \left[\int_0^t e^{-k^2\pi^2(t-s)} dW^k(s) \right] \sigma_k(z), \quad (2.5)$$

where $\tau_k(0)$ is the kth coefficient of the sine series of the initial condition $\tau(z,0)$. The stationary distribution (see Da Prato & Zabczyk (1996)) for this conductive state is $\widetilde{\tau}(z) = 1 - z + \sum_{k=1}^{M} \gamma_k \sigma_k(z)$, where the γ_k are normally distributed with mean 0 and variance $\frac{H^2}{2k^2\pi^2}$, i.e. $\gamma_k \sim \mathcal{N}\left(0, \frac{H^2}{2k^2\pi^2}\right)$. We first note that this simple solution is ergodic, and in fact the stationary distribution of the conductive profile is simply the deterministic conductive state plus mean zero stochastic fluctuations.

3. Nonlinear stability of the conductive state

As the conductive state in this context is time dependent, we will not consider linear stability but will focus entirely on nonlinear stability via the energy method (see Drazin & Reid (2004) and Goluskin (2015a,b) for example). To this end we decompose the temperature field as $T(x, y, z, t) = \tau(z, t) + \theta(x, y, z, t)$ so that (2.3)–(2.4) becomes

$$\frac{1}{\text{Pr}} \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) + \nabla \tilde{p} = \text{Ra} \boldsymbol{k} \theta + \Delta \boldsymbol{u},$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta + w \frac{\partial \tau}{\partial z} = \Delta \theta,$$

where the pressure term $\tilde{p}$ has been modified to absorb the buoyancy term from the conductive profile τ. Note that this system is stochastic only through the presence of τ. We compute the evolution of the energy (L^2 norm of θ and $\boldsymbol{u}$) as

$$\frac{1}{2} \frac{d}{dt} \left(\|\theta\|^2 + \frac{1}{\text{Pr} \, \text{Ra}} \|\boldsymbol{u}\|^2 \right) = -\mathcal{Q}(\boldsymbol{u}, \theta, \tau), \tag{3.1}$$

$$\text{where} \quad \mathcal{Q}(\boldsymbol{u}, \theta, \tau) = \|\nabla \theta\|^2 + \frac{1}{\text{Ra}} \|\nabla \boldsymbol{u}\|^2 + \int_{\mathcal{D}} w\theta \left(\frac{\partial \tau}{\partial z} - 1 \right) d\boldsymbol{x}, \tag{3.2}$$

and we define the L^2 norm as $\|f\|^2 = \int_{\mathcal{D}} |f|^2 d\boldsymbol{x}$.

For fixed τ, $\mathcal{Q}$ is a quadratic form in $\boldsymbol{u}$ and θ, and following the energy stability method (Drazin & Reid 2004) we consider $\lambda(\tau) = \min_{\boldsymbol{u}, \theta} \dfrac{\mathcal{Q}(\boldsymbol{u}, \theta, \tau)}{\|\theta\|^2 + (\text{PrRa})^{-1}\|\boldsymbol{u}\|^2}$, which is a random quantity depending on the parameters Pr, Ra, and H. The evolution of the energy (3.1) then shows that $\tilde{\tau}$ from the stationary distribution is the unique statistically steady state of (2.3)-(2.4) provided $\liminf_{t \to \infty} \frac{1}{t} \int_0^t \lambda(\tau(s)) ds > 0$ almost surely, independent of the initial condition. Invoking geometric ergodicity (Da Prato & Zabczyk 1996) we have that $\lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda(\tau(s)) ds = \mathbb{E}\lambda(\tilde{\tau})$, where $\tilde{\tau}$ is the stationary distribution of the conductive state and $\mathbb{E}$ denotes the statistical mean.

We will consider a range of physically plausible values of H and for each value, determine a 'critical' Ra so that $\mathbb{E}\lambda(\tilde{\tau}) = 0$. Any Ra below this critical value will guarantee stability of the system as we will demonstrate rigorously in future work.

3.1. Comparison to a deterministic system

As a rough comparison of the effect of stochastic and deterministic forcing, we will contrast the calculated critical Rayleigh number Ra_c with one that arises from inserting a deterministic forcing of the same magnitude. That is, we replace (2.4) with $\dfrac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T - \Delta T = \tilde{H} \sum_{k=1}^{M} \sigma_k$, where $\tilde{H} = \frac{\gamma \sqrt{h}}{T_1 \kappa}$ is the proper non-dimensional measure of a deterministic bulk heating. In what follows, we use only these non-dimensional versions of the deterministic and stochastic problems, and make a comparison as if $H = \tilde{H}$.

4. Algorithmic description

The computation of the marginally stable parameters is performed in the following fashion. This is a root-finding problem where the Prandtl Pr and stochastic heating H numbers are fixed parameters, and the Rayleigh number Ra is the independent variable. We approximate $\mathbb{E}\lambda(\tilde{\tau})$ from a sample mean, and seek Ra_c so that $\widehat{\lambda}(\text{Ra}_c) = 0$,

where $\widehat{\lambda}(\mathrm{Ra}) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \lambda(\widetilde{\tau}_k)$, and each $\widetilde{\tau}_k$ is drawn from the independent, identically distributed distribution defined by the stationary distribution of the conductive state. The number of samples $\mathcal{N}$ is a parameter at the discretion of the user. Due to numerical considerations, we seek Ra_c such that $|\widehat{\lambda}(\mathrm{Ra}_c)| \leqslant \epsilon$, where ϵ is an additional user prescribed parameter.

For each realization $\widetilde{\tau}_k$ of the stationary distribution, we consider the Euler-Lagrange equations for the minimization principle of $\lambda(\tau)$ and identify $\lambda(\widetilde{\tau}_k)$ as the solution of a one-dimensional eigenvalue problem which is solved numerically via the Dedalus software package (Burns *et al.* 2017). Starting with the Euler-Lagrange equations for $\lambda(\tau)$, we take the curl of the corresponding momentum equation twice, and apply the horizontal Fourier transform to arrive at the system of equations

$$-\frac{\lambda}{\mathrm{Pr}\,\mathrm{Ra}} \left(\partial_{zz}^2 - m^2 \right) \hat{w}_m = \frac{1}{\mathrm{Ra}} \left(\partial_{zz}^2 - m^2 \right)^2 \hat{w}_m + \frac{1}{2} \left(\partial_z \tau - 1 \right) m^2 \hat{\theta}_m, \qquad (4.1)$$

$$\lambda \hat{\theta}_m = - \left(\partial_{zz}^2 - m^2 \right) \hat{\theta}_m + \frac{1}{2} \left(\partial_z \tau - 1 \right) \hat{w}_m, \qquad (4.2)$$

where the $\hat{\cdot}_m$ denotes the Fourier terms corresponding to the horizontal wavenumber with magnitude m. The system also satisfies the boundary conditions:

$$\hat{w}_m(0) = \hat{w}_m(1) = \partial_{zz}^2 \hat{w}_m(0) = \partial_{zz}^2 \hat{w}_m(1) = 0, \qquad \hat{\theta}_m(0) = \hat{\theta}_m(1) = 0, \qquad (4.3)$$

for stress-free boundaries.

The numerical implementation of this calculation uses the bisection root finding method on $\widehat{\lambda}$ to locate Ra_c for each specified value of Pr and H, taking previously computed values as an initial guess. This leads to the following parameters in the algorithm that must be selected.

 (i) *Forced modes.* We chose $M = 8$ for the reported results meaning that the first 8 vertical modes are forced stochastically. Although not shown here, these results qualitatively compare with cases where the forcing is selected as a subset of the first 8 modes.

 (ii) *Sample size.* We choose $\mathcal{N} = 192$ Monte Carlo generated samples to compute $\widehat{\lambda}$. As a control, similar calculations were carried out for $\mathcal{N} = 384$, but the change in Ra_c was less than 5% in all cases.

 (iii) *Level of discretization.* We chose to discretize the eigenvalue problem with $N_z = 64$ vertical Chebyshev modes. Results held true up to 6 significant digits for $N_z = 128$ as long as the highest forced mode was less than 8, i.e. $M \leqslant 8$.

 (iv) *Absolute error.* As the algorithm is searching for the root of a highly nonlinear function, we must specify a tolerance for finding that root. Using $\epsilon = 10^{-3}$ allows for a determination of Ra_c at $H = 0$ that is accurate up to 8 significant digits. We select $\epsilon = 10^{-3}$ for the results reported here. Using $\epsilon = 10^{-2}$ resulted in differences of Ra_c of less than 10%.

5. Results

The value of the critical Rayleigh number for the stochastic and deterministic systems, normalized by the critical Rayleigh number Ra_c^* for (2.4) with $H = 0$, are plotted in Figure 1. Note that for H small, Ra_c is larger than the critical Rayleigh number in the deterministic case, indicating that the stochastic forcing has a stabilizing effect at small values of H. In contrast, once H exceeds $O(1)$, Ra_c quickly drops to zero, and the stochastic term is destabilizing. For $H > O(10)$ the conductive state is completely destabilized. Qualitatively the same picture emerges when a different number of modes are forced or the Prandtl number is varied. More details will appear in a future study.

To fully investigate the transition from the conductive to the convective regime, we consider the distribution of the growth factors λ from the 192 samples for each value of H at the transitional value of $\mathrm{Ra} = \mathrm{Ra}_c$. A few representative histograms demonstrating this information are presented in Figure 2. We have chosen these particular values of H as they represent different qualitative regions in Figure 1. We prove in a follow up study that the maximal positive value of λ is bounded above uniformly in H, and that the variance of λ is increasing with H for fixed Ra. For $H \lesssim 2.5$, the variance is small and the distribution of λ is approximately Gaussian. For $H \in (2.5, 6)$ with Ra fixed, as the variance of λ increases, there is a noticeable transition where a significant bulk of the distribution nears the upper bound, which

FIGURE 1. The normalized (by the critical value when $H = 0$) critical Rayleigh number for a prescribed relative strength of the internal heating H for both a purely deterministic (supposing that $\tilde{H} = H$), and a stochastic setting. The Prandtl number is 1 here, and the first 8 vertical modes are forced in both the deterministic and stochastic problems. In addition, $\mathcal{N} = 192$ samples are used for the stochastic problem, $N_z = 64$ Chebyshev modes are used in the eigenvalue solver, and the tolerance is set to $\epsilon = 10^{-3}$.

is independent of H. This causes the mean of λ to decrease, and results in a negatively skewed distribution. This skew is clearly visible in the lower plots of Figure 2. In order to maintain the mean zero stability condition, Ra_c must be lowered, becoming negligibly small for large values of H. This transition is more pronounced in the stochastic setting than in the deterministic case due to this particular distribution of the growth factor λ. In the deterministic case, the solution to the eigenvalue problem reveals that λ decreases steadily with H for fixed Ra, and the corresponding decrease in Ra to maintain $\lambda = 0$ is similarly steady.

A linear stability analysis of the deterministic system shows that there is a gap between the onset of instability and guaranteed nonlinear stability, indicating the potential for subcritical convection (see Busse (2014) for example) to occur in this parameter regime. Although linear stability is not available for (2.4), preliminary direct numerical simulations indicate that a similar gap appears between the guaranteed stability region, and the onset of convection in the stochastic system. These simulations indicate that the linear instability occurs for $\mathrm{Ra} > \mathrm{Ra}_c$, but we have not yet verified whether subcriticality arises in this setting.

6. Conclusions and stability of generic stochastically driven hydrodynamic systems

We have investigated the nonlinear stability of a convective system with additive stochastic white noise on the first 8 vertical modes of the system. When the stochastic heating is weak, it has a relatively stabilizing effect, then transitions to strongly destabilizing as the strength of the noise increases. This is an effect of the distribution of the growth factor λ and its dependence on the strength of the stochastic heating.

The results discussed above have demonstrated the need to better quantify the role that stochasticity plays in physically relevant fluid systems. If the internal heat source were modeled as a deterministic bulk forcing, then as described above, the stability of the resultant conductive state would be very different, particularly for the physically relevant
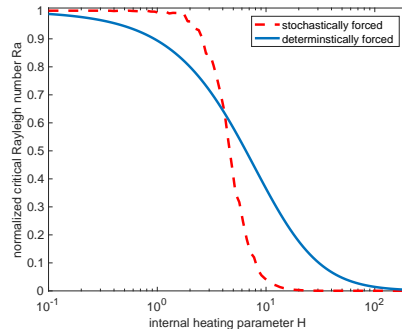
setting of $H = o(1)$. This implies that at least in this idealized convective setting, if there is an inherently noisy source, we will miss some of the fundamental physics by modeling the system in a purely deterministic fashion. We are not claiming here that the precise nature of the noise we have chosen is the 'best' way to model noisy convection, but we do insist that accounting for noise in such physical systems is necessary to achieve physically realistic results.

Further considerations of the onset of convection in such stochastic settings are natural extensions of the current work. For instance, is there an analogue of the finite amplitude equations in this context, and if so, is their derivation and application the same or similar? Do coherent structures such as the roll-states present in low Rayleigh number deterministic convection exist in the stochastic setting, and if so are they defined only in a mean sense? If such structures exist, can similar statements be made regarding their stability, or does the stochastic nature of the problem preclude the utility of such investigations? Further analysis and computation is required to answer these questions, and ascertain the influence that noise can play in fully developed turbulent convection.

The methodology developed in this article applies not only to Rayleigh-Bénard convection under these constraints (stress-free boundaries etc.) considered here, but is applicable to any hydrodynamic system driven by a stochastic forcing where a basic (time dependent) state still exists. In particular, we can extend this approach to convection in the presence of a stochastically forced boundary condition on the temperature and to Couette shear flow, both of which will appear in a forthcoming article.

FIGURE 2. Histograms of the growth factor λ from which the critical Rayleigh number is calculated at $H = 1.761$, $\mathrm{Ra}_c = 650.08$ (top left), $H = 2.6017$, $\mathrm{Ra}_c = 591.9$ (top middle), $H = 3.555$, $\mathrm{Ra}_c = 497.27$ (top right), $H = 4.493$, $\mathrm{Ra}_c = 353.06$ (bottom left), $H = 6.1395$, $\mathrm{Ra}_c = 162.787$ (bottom middle), and $H = 9.0704$, $\mathrm{Ra}_c = 36.664$ (bottom right). Note that the differences in scale for each of these histograms. This data is collected for $\mathrm{Pr} = 1$ and $M = 8$.

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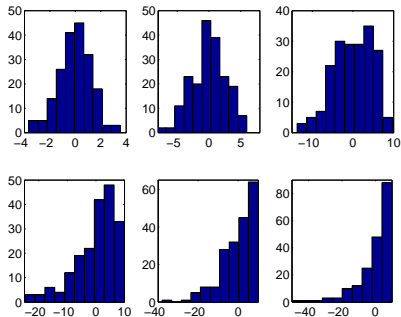
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