LIOUVILLE THEOREMS, A PRIORI ESTIMATES, AND BLOW-UP RATES FOR SOLUTIONS OF INDEFINITE SUPERLINEAR PARABOLIC PROBLEMS

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Abstract. In this paper, we establish new nonlinear Liouville theorems for parabolic problems on half spaces. Based on the Liouville theorems, we derive estimates for the blow-up of positive solutions of indefinite parabolic problems and we investigate the complete blow-up of these solutions. We also discuss the a priori estimates for indefinite elliptic problems.

Keywords: A priori estimates, Liouville theorems, blow-up rates, positive solutions, indefinite parabolic problems

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1. Introduction

In this paper we consider the problem

$$\begin{aligned} u_t &= \Delta u + a(x)|u|^{p-1}u, && (x,t) \in \Omega \times (0,T)\,, \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T)\,, \end{aligned} \tag{1.1}$$

which, if needed, is completed with an initial condition

$$u(\cdot,0) = u_0(\cdot) \in L^\infty(\Omega)\,. \tag{1.2}$$

We assume that Ω is a smooth domain in $\mathbb{R}^N$ and $p > 1$. Furthermore, we suppose that $a : \bar\Omega \to \mathbb{R}$ belongs to $C^2(\bar\Omega)$ and

$$\text{if} \quad \lim_{k\to\infty} a(x_k) = 0, \quad \text{then} \quad \limsup_{k\to\infty} |\nabla a(x_k)| > 0\,. \tag{1.3}$$

Here, $C^k(D)$ denotes the space of k-times differentiable, bounded functions on $D \subset \mathbb{R}^N$, with bounded, continuous derivatives up to k^{th} order.

If Ω is bounded and if we denote

$$\Gamma := \{x \in \bar\Omega : a(x) = 0\}, \tag{1.4}$$

$$\Omega^+ := \{x \in \Omega : a(x) > 0\}, \tag{1.5}$$

$$\Omega^- := \{x \in \Omega : a(x) < 0\}\,, \tag{1.6}$$

then (1.3) is equivalent to

$$|\nabla a(x)| \neq 0 \qquad (x \in \Gamma)\,, \tag{1.7}$$

that is, a has nondegenerate zeros in $\bar{\Omega}$. Since u_0 and a are bounded, standard results [21] yield the unique, strong solution of the problem (1.1), (1.2), with the maximal existence time $T_{\max} \in (0, \infty]$. Moreover, by regularity results, if $T_{\max} < \infty$, then $\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to T_{\max}$. We do not indicate the dependence of $T_{\max}$ on u_0 if no confusion seems possible. Here and in the rest of the paper we assume $T \in (0, T_{\max}]$.

As a main result of this paper, we derive an upper bound for the blow-up rate of nonnegative solutions of (1.1). The blow-up rates and related a priori estimates were studied under various assumptions on a, Ω and u in [1, 10, 11, 17, 13, 14, 15, 22, 26, 27, 28, 36, 34, 35], see also references therein. We just briefly describe the results directly connected to our results. First, Friedman and McLeod [11] studied blowing up solutions ($T_{\max} < \infty$) of the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u, && (x,t) \in \Omega \times (0,T)\,, \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T)\,, \end{aligned} \tag{1.8}$$

with $T = T_{\max}$, and the initial condition (1.2). They proved

$$|u(x,t)| \leq C(1 + (T_{\max} - t)^{-\frac{1}{p-1}}) \qquad (x \in \Omega)\,, \tag{1.9}$$

where Ω is a bounded convex domain, $p > 1$, and u is positive, increasing (in time) solution of (1.8). These results were generalized by Giga and Kohn [13] and later by Giga et al. [14, 15]. With the help of localized energy estimates and iterative arguments, they proved that (1.9) holds true if Ω is a bounded convex domain or $\Omega = \mathbb{R}^N$, u is, not necessarily positive, solution of (1.8), (1.2), and $1 < p < p_S$, where

$$p_S = p_S(N) := \begin{cases} \infty & N \leq 2\,, \\ \frac{N+2}{N-2} & N \geq 3\,. \end{cases}$$

In [9] Fila and Souplet employed scaling and Fujita type results to remove the assumption on convexity of Ω and they established (1.9) for all positive solutions of (1.8), (1.2), and $1 < p \leq 1 + \frac{2}{N+1}$.

Finally, Poláčik et al. [26] investigated positive solutions of (1.8) with sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ and $1 < p < p_B$, where

$$p_B = p_B(N) := \begin{cases} \infty & N \leq 1\,, \\ \frac{N(N+2)}{(N-1)^2} & N \geq 2\,. \end{cases} \tag{1.10}$$

Using scaling, doubling lemma and Liouville theorems they obtained

$$u(x,t) \leq C(1 + t^{-\frac{1}{(p-1)}} + (T-t)^{-\frac{1}{(p-1)}}) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.11}$$

where C is a universal constant depending only on p, N and Ω. We remark that the estimates for the initial blow-up rate were previously established by Bidaut-Véron [5] (see also [3]) for $1 < p < p_B$ and $\Omega = \mathbb{R}^N$. Some estimates on the initial blow-up rates for bounded Ω were proved by Quittner et al. [29].

The first a priori estimates for positive solutions of (1.1), (1.2) with sign-changing a were derived in the form (see [27] and references therein)

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(\|u_0\|_{L^\infty(\Omega)}, \delta, N, p, \Omega, a) \qquad (t \in [0, T_{\max} - \delta], \delta > 0, T_{\max} < \infty)\,. \tag{1.12}$$

Later, Xing [36] obtained an upper estimate for the blow-up rate, of positive solutions of (1.1), (1.2)

$$u(x,t) \le C(1 + (T_{\max} - t)^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0, T_{\max}), T_{\max} < \infty)$$

when Ω is bounded, $1 < p < p_B$ and $\Gamma \subset \Omega$, that is, when a does not vanish on $\partial\Omega$. Here C depends on $\|u_0\|_{L^\infty(\Omega)}$, N, p, Ω, a.

The next theorem refines the results in [36] in various directions. It includes unbounded domains and it allows for a very general behavior of a on $\partial\Omega$. In addition it also yields an estimate for the initial blow-up rate. Denote $\nu_\Omega(x)$ the unit outward normal vector to $\partial\Omega$ at x.

Theorem 1.1. *Let Ω be a uniformly regular domain of class C^2 in $\mathbb{R}^N$ (cf. [2]) and let $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies (1.3) and*

$$\left| \frac{\nabla a(x_0)}{|\nabla a(x_0)|} - \nu_\Omega(x_0) \right| \ge \tilde{c} > 0 \qquad (x_0 \in \Gamma \cap \partial\Omega)\,. \tag{1.13}$$

Then every nonnegative solution u of (1.1) satisfies

$$u(x,t) \le C(1 + t^{-\frac{3}{2(p-1)}} + (T-t)^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.14}$$

where C depends on N, p, Ω and a.

Remark 1.2. (a) The nonlinearity $|u|^{p-1}u$ in (1.1) can be replaced by $f(u)$ with

$$\lim_{v \to \infty} \frac{f(v)}{v^p} = \ell > 0\,.$$

Then (1.14) holds with C depending on N, f, Ω and a. Also, we can add lower order terms to the right hand side, that is, we can add a function $g : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{u \to \infty} \sup_{(x,t) \in \Omega \times (0,T)} \frac{g(x,t,u)}{u^p} = 0\,.$$

Then (1.14) holds with C depending on N, p, Ω, a and g.
(b) For the blowing-up solutions ($T_{\max} < \infty$) of (1.8) one has (cf. [28, Proposition 23.1]) $\sup_{x \in \mathbb{R}^N} u(x,t) \ge C(T_{\max} - t)^{-\frac{1}{p-1}}$. This shows the optimality of the final blow up estimate in (1.11) for $a \equiv 1$. However, it is not known whether or not the weaker estimate (1.14) is optimal for sign changing a. Below, we show that under additional assumptions the stronger estimate (1.11) holds true even if a changes sign.
(c) If a also depends on t and $p > \frac{N+2}{N}$, the initial blow-up estimate in (1.14) does not hold even if $0 \le a \le 1$ (see e.g. [32, 33]). If Ω is bounded, then (1.13) is equivalent to $\frac{\nabla a(x_0)}{|\nabla a(x_0)|} \neq \nu_\Omega(x_0)$ for any $x_0 \in \Gamma \cap \partial\Omega$. It is not known if this assumption is technical or not.
(d) Universal estimates of the form (1.11) or (1.14) are not true for $p \ge p_S$, $N \ge 3$,

$\Omega = \mathbb{R}^N$, due to the existence of arbitrarily large stationary radial solutions of (1.1). We require $p < p_B < p_S$ mainly because the Liouville theorem for the problem

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,, \tag{1.15}$$

with $p_B \le p < p_S$ is not known. If one proves such a Liouville theorem for some $p \in [p_B, p_S)$, then the conclusion of Theorem 1.1 would hold for the same p as well.
(e) If we restrict ourselves to the class of radial solutions (of course now Ω and a are radially symmetric), then similarly as in [26], one can prove Theorem 1.1 for each $1 < p < p_S$. This is possible, since the Liouville theorem is known for nonnegative radial solution of (1.15) for any $1 < p < p_S$ (see [24]).
(f) If a nonnegative solution u of (1.1) is global ($T_{\max} = \infty$), then after letting $T \to \infty$ in (1.14), we obtain

$$u(x,t) \le C(1 + t^{-\frac{3}{2(p-1)}}) \qquad ((x,t) \in \Omega \times (0,\infty))\,. \tag{1.16}$$

In particular u is bounded on $\Omega \times (1,\infty)$. For previous results, see [5, 26].

Remark 1.3. Observe that (1.14) is equivalent to

$$M(x,t) \le C(1 + d^{-1}(t)) \qquad ((x,t) \in \Omega \times (0,T))\,, \tag{1.17}$$

where

$$M := u^{\frac{(p-1)}{3}} \quad \text{and} \quad d(t) := \min\{t, T-t\}^{\frac{1}{2}}\,.$$

Also, for each $x \in \Omega$, one has $d(t) = d_P[(x,t),\Theta]$, where $\Theta := \Omega \times \{0,T\}$ and d_P denotes the parabolic distance:

$$d_P[(x,t),(y,s)] = |x-y| + |t-s|^{\frac{1}{2}} \qquad ((x,t),(y,s) \in \Omega \times (0,T))\,. \tag{1.18}$$

In this notation we obtain yet another form of (1.14)

$$u(x,t) \le C(1 + d_P^{-3/(p-1)}[(x,t),\Theta]) \qquad ((x,t) \in \Omega \times (0,T))\,.$$

If u is a stationary solution of (1.1), that is, if u solves

$$\begin{aligned} 0 &= \Delta u + a(x)|u|^{p-1}u, && x \in \Omega\,, \\ u &= 0, && x \in \partial\Omega\,, \end{aligned} \tag{1.19}$$

we obtain the following corollary.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a uniformly regular domain of class C^2 (cf.* [2]*),* $1 < p < p_S$*, and* $a \in C^2(\bar{\Omega})$ *that satisfies* (1.3) *and* (1.13)*. If u is a nonnegative solution of* (1.19)*, then $u \le C(p,N,\Omega,a)$.*

This corollary extends the results of Du and Li [7] (see also references therein), as it allows a to vanish on $\partial\Omega$. If $1 < p < p_B(N)$, then since $T_{\max} = \infty$, Corollary 1.4 follows from (1.16). If we merely assume $1 < p < p_S(N)$, then one has to reprove Theorem 1.1 for solutions of (1.19). The only difference is the application of elliptic Liouville theorems [12], instead of parabolic ones, whenever $p < p_B$ is required.

The next propositions shows that final blow-up rates in Theorem 1.1 (and main results in [36]) can be improved if $a > 0$ and Ω is a convex bounded set. Notice that a is allowed to vanish on $\partial\Omega$. In this case, the universal bounds (1.12) were already obtained in [27].

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth, convex set and let $1 < p < p_B$. Assume $a \in C^2(\bar{\Omega})$ satisfies* (1.7) *and $a(x) > 0$ for $x \in \Omega$. Then a nonnegative solution u of* (1.1), (1.2) *satisfies*

$$u(x,t) \le C(1 + (T-t)^{-\frac{1}{p-1}}) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.20}$$

where C depends on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.

If a is changes sign in Ω, we formulate a sufficient conditions for (1.20) only in the one-dimensional case. However, one can generalize the following propositions to higher dimensional case if Ω is convex and certain monotonicity of a and u_0 near $\partial\Omega$ is assumed.

Proposition 1.6. *Let $N = 1$ and $\Omega = (0,1)$. Suppose that $a \in C^2([0,1])$ and has exactly one nondegenerate zero $\mu \in [0,1]$, that is, $a(\mu) = 0$ and $a'(\mu) \neq 0$. If*

$$\operatorname{sign}[a(x)](u_0(2\mu - x) - u_0(x)) \le 0 \qquad (x \in (\max\{0, 2\mu - 1\}, \mu)),$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

Proposition 1.7. *Let $N = 1$ and $\Omega = (0,1)$. Suppose that $a \in C^2([0,1])$ and has exactly two nondegenerate zero $\mu_1 < \mu_2$ in $[0,1]$, that is, $a(\mu_i) = 0$ and $a'(\mu_i) \neq 0$ for $i = 1,2$. If $\max\{\mu_1, 1-\mu_2\} < \mu_2 - \mu_1$ and*

$$\begin{aligned} a(x) < 0, \quad u_0(2\mu_1 - x) &\ge u_0(x) \qquad (x \in (0,\mu_1)), \\ u_0(2\mu_2 - x) &\ge u_0(x) \qquad (x \in (\mu_2, 1)), \end{aligned}$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

One can also employ Liouville theorems and universal estimates in the investigation of the complete blow-up and the continuity of blow-up time. Let us recall these notions and explain the results.

Let u be a nonnegative solution of (1.1), (1.2) with $T_{\max} < \infty$. Let u_k, $(k \in \mathbb{N})$ be the solution of the approximation problem

$$\begin{aligned} (u_k)_t - \Delta u_k &= f_k(x, u_k), \qquad && (x,t) \in \Omega \times (0,\infty), \\ u_k &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \\ u_k(x,0) &= u_0(x) \ge 0, && x \in \Omega, \end{aligned}$$

where

$$f_k(x,v) := \begin{cases} a(x)\min\{v^p, k\} & \text{if } a(x) \ge 0, v \in \mathbb{R}, \\ a(x)v^p & \text{if } a(x) < 0, v \in \mathbb{R}. \end{cases}$$

Since f_k is bounded from above, nonnegative solution u_k exists globally (for all positive times). Since $f_k \le f_{k+1}$, the maximum principle implies $u_{k+1}(x,t) \ge u_k(x,t)$ for any $(x,t) \in \Omega \times (0,\infty)$. Thus

$$\bar{u}(x,t) := \lim_{k\to\infty} u_k(x,t) \in [0,\infty] \qquad ((x,t) \in \Omega \times [0,\infty))$$

is well defined. Moreover, $\bar{u}(x,t) = u(x,t)$ for any $(x,t) \in \bar{\Omega} \times [0, T_{\max})$. We say that u *blows-up completely* in $D \subset \Omega$ at T, if $\bar{u}(x,t) = \infty$ for any $x \in D$ and $t > T$.

Theorem 1.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies (1.7) and (1.13). If $T_{\max} < \infty$ for a nonnegative solution u of (1.1), (1.2), then u blows-up completely in Ω^+ at $T_{\max}$. In addition, the function*

$$T : \{u_0 \in L^\infty(\Omega) : u_0 \geq 0\} \to (0, \infty], \qquad T : u_0 \mapsto T_{\max}(u_0)$$

is continuous.

If $a \equiv 1$, Baras and Cohen [4] proved complete blow-up of nonnegative solutions of (1.8), (1.2) in Ω at $T_{\max} < \infty$ for each $1 < p < p_S$ (see also [28]). However, for $p > p_S$, $N \leq 10$, and Ω being a ball, there exist radial solutions of (1.8) that do not blow-up completely in Ω at $T_{\max}$. For further discussion see [28] and references therein.

If a changes sign, then one cannot expect the complete blow-up in the whole Ω, since $\bar{u}$ stays bounded in Ω^- for any $t > 0$ (see [20]). Quittner and Simondon [27] proved the complete blow-up of u in Ω^+ at $T_{\max} < \infty$ for $1 < p \leq 1 + 3/(N+1)$ and $\Gamma \subset \Omega$. Later Poláčik and Quittner [23] replaced the former assumption by $1 < p < p_B$ and proved Theorem 1.8 under an additional assumption $\Gamma \subset \Omega$.

The rest of the paper is organized as follows. In Section 2 we state and prove parabolic Liouville theorems. In Section 3 we formulate doubling lemma and we prove our main results.

2. Liouville theorems

Since some results in this section can be of independent interest, we formulate them in more general setting, than required for the proofs of the main results. Let us define

$$\mathbb{R}^N_\lambda := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > \lambda\} \qquad (\lambda \in \mathbb{R}), \tag{2.1}$$

$$H_\lambda := \partial\mathbb{R}^N_\lambda = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = \lambda\} \qquad (\lambda \in \mathbb{R}). \tag{2.2}$$

The following two lemmas were proved in [36] for increasing function f. Here, we propose a simpler proofs that remove this unnecessary assumption. The elliptic counterparts can be found in [8, 30, 31], see also references therein.

Lemma 2.1. *Let f be a continuous function with $f(v) > 0$ for any $v > 0$. If $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded solution of*

$$u_t - \Delta u = -f(u), \qquad (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

then $u \equiv 0$.

Proof. We proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geq C^* := \frac{1}{2} \sup_{(x,t) \in \mathbb{R}^N \times \mathbb{R}} u(x, t) > 0.$$

For each $\varepsilon > 0$ denote

$$v_\varepsilon(x, t) := u(x, t) - \varepsilon|x - x^*|^2 - \varepsilon(\sqrt{(t - t^*)^2 + 1} - 1) \qquad ((x, t) \in \mathbb{R}^N \times \mathbb{R}).$$

Since $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N \times \mathbb{R}$ with

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} v_\varepsilon(x,t)\,.$$

Then for each $\varepsilon > 0$

$$2C^* \ge u(x_\varepsilon, t_\varepsilon) \ge v_\varepsilon(x_\varepsilon, t_\varepsilon) \ge v_\varepsilon(x^*, t^*) = u(x^*, t^*) \ge C^* > 0\,,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0\,, \qquad \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \le 0\,.$$

Consequently,

$$\begin{aligned}
0 &\le (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon\frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\
&= -f(u(x_\varepsilon, t_\varepsilon)) - \varepsilon\frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\
&\le -\inf_{2C^*\ge v\ge C^*} f(v) + \varepsilon + 2\varepsilon N \qquad (\varepsilon > 0)\,.
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. ☐

Lemma 2.2. *Suppose $f \in C^1$ satisfies $f(0) = 0$ and $f(v) > 0$ for any $v > 0$. Let h be a continuous function with $h(x_1) < 0$ for each $x_1 > 0$, and $\limsup_{x_1\to\infty} h(x_1) < 0$. If u is a nonnegative bounded solution of the problem*

$$\begin{aligned}
u_t - \Delta u &= h(x_1)f(u), && (x,t) \in \mathbb{R}^N_0 \times \mathbb{R}\,,\\
u &= 0, && (x,t) \in H_0 \times \mathbb{R}\,,
\end{aligned}$$

then $u \equiv 0$.

Proof. The proof is similar to the proof of Lemma 2.1. We again proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N_0 \times \mathbb{R}$ such that

$$u(x^*, t^*) \ge C^* := \frac{1}{2}\sup_{(x,t)\in\mathbb{R}^N_0\times\mathbb{R}} u(x,t) > 0\,.$$

It is easy to see that there exists a function $\phi \in C^2(\mathbb{R}^N \times \mathbb{R})$ with

$$\begin{gathered}
\phi(x,t) \ge 0, \quad |\nabla\phi(x,t)| \le 1, \quad |\phi_t - \Delta\phi| \le 1 \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R})\,,\\
\phi(0,0) = 0, \quad \phi(x,t) \to \infty \quad \text{if } |x| \to \infty \quad \text{or } t \to \pm\infty\,.
\end{gathered}$$

For each $\varepsilon \in (0,1)$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon\phi(x - x^*, t - t^*) \qquad ((x,t) \in \mathbb{R}^N_0 \times \mathbb{R})\,.$$

Since u is bounded, $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$. Moreover, $v_\varepsilon(x,t) \le 0 < v_\varepsilon(x^*, t^*)$ for any $(x,t) \in H_0 \times \mathbb{R}$, and therefore there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N_0 \times \mathbb{R}$ such that

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}^N_0\times\mathbb{R}} v_\varepsilon(x,t)\,.$$

Consequently,

$$2C^* \geq u(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x_\varepsilon, t_\varepsilon) \geq v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geq C^* > 0\,,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad (\Delta v_\varepsilon)(x_\varepsilon, t_\varepsilon) \leq 0\,.$$

Observe that u satisfies

$$u_t = \Delta u + h(x_1)\frac{f(u)}{u}u = \Delta u + c(x,t)u\,.$$

Since $f \in C^1$, $f(0) = 0$, and u is bounded, c is a bounded function in $\{(x,t) \in \mathbb{R}^N_0 \times \mathbb{R} : x_1 < 2\}$. Hence, standard parabolic regularity (see for example [19, Theorem 1.15]) implies

$$|\nabla u(x,t)| \leq C \qquad ((x,t) \in \bar{\mathbb{R}}^N_0 \times \mathbb{R}, x_1 < 1)\,,$$

and consequently

$$|\nabla v_\varepsilon(x,t)| \leq C + 1 \qquad ((x,t) \in \bar{\mathbb{R}}^N_0 \times \mathbb{R}, x_1 < 1)\,,$$

where C is independent of $\varepsilon \in (0,1)$. Furthermore, $v_\varepsilon(x_\varepsilon, t_\varepsilon) \geq C^* > 0$ and $v_\varepsilon(x,t) \leq 0$ for all $(x,t) \in H_0 \times \mathbb{R}$ yield $\mathrm{dist}(x_\varepsilon, H_0) = (x_\varepsilon)_1 \geq c_0$, where c_0 is a constant independent of ε. Finally,

$$\begin{aligned} 0 &\leq (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\ &= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon[\phi_t(x_\varepsilon, t_\varepsilon) - \Delta\phi(x_\varepsilon, t_\varepsilon)] \\ &\leq h((x_\varepsilon)_1) f(u(x_\varepsilon, t_\varepsilon)) + \varepsilon \\ &\leq \sup_{y \geq c_0} h(y) \inf_{2C^* \geq v \geq C^*} f(v) + \varepsilon\,. \end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Next, consider the problem

$$\begin{aligned} u_t - \Delta u &= h(x \cdot v) f(u), && (x,t) \in \Omega \times \mathbb{R}\,, \\ u &= 0, && (x,t) \in \partial\Omega \times \mathbb{R}\,, \end{aligned} \tag{2.3}$$

where

(v1) $v = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$ is a unit vector with $v_1 > 0$ and $v_i = 0$ for $i \geq 3$.

About Ω, we assume that

(d1) Ω is a subset of $\mathbb{R}^N$, convex and unbounded in x_1, that is, $x + \xi e_1 \in \Omega$ for any $x \in \Omega$ and $\xi > 0$.

(d2) there is a constant d^* such that $x_2 v_2 \leq d^*$ for any $x = (x_1, x_2, \cdots, x_N) \in \Omega$.

Next, the function $h : \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis.

(h1) h is continuous, nondecreasing, and strictly increasing on $(0, \infty)$.

(h2) $h(0) = 0$ and $\lim_{y\to\infty} h(y) = \infty$.

About f we assume

(f1) $f \in C^1([0,\infty))$, with $f(0) = f'(0) = 0$, and $f(v) > 0$, $f'(v) \geq 0$ for each $v > 0$.

The following theorem is a generalization of elliptic [7] and parabolic [23] results proved for $v = e_1$ and $\Omega = \mathbb{R}^N$. The general framework of the proof is similar to one used in [7, 23].

Theorem 2.3. *If (v1), (d1), (d2), (h1), (h2), and (f1) hold true, then the only nonnegative, bounded solution u of* (2.3) *is* $u \equiv 0$.

As a corollary we obtain Liouville theorem for indefinite problems on half spaces.

Corollary 2.4. *Given unit vectors $b, v \in \mathbb{R}^N$ and a constant c^*, let $\Omega := \{x \in \mathbb{R}^N : x \cdot b > c^*\}$. Consider functions h and f that satisfy (h1), (h2) and (f1) respectively. Let u be a nonnegative, bounded solution of* (2.3). *If $v \neq -b$, then $u \equiv 0$.*

Remark 2.5. The statement of Corollary 2.4 still holds true if $v = -b$, $c^* \geq 0$ and h in addition to (h1), (h2) satisfies $h(y) < 0$ for $y < 0$. This follows after suitable rotation and translation, from Lemma 2.2. However, if $v = -b$ and $c^* < 0$, then there are nontrivial, nonnegative solutions of (2.3). This result will be published elsewhere.

Proof of Corollary 2.4. We rotate the coordinates such that $b = e_2$, $v_1 \geq 0$, and $v_i = 0$ for $i \geq 3$. Then $\Omega = \{x \in \mathbb{R}^N : x_2 > c^*\}$ and (d1) holds true. Notice that (2.3), (h1), (h2), and (f1) are invariant under rotations.

If $v_1 > 0$ and $v_2 \leq 0$, then (v1) and (d2) are satisfied with $d^* = c^* v_2$, and the corollary follows from Theorem 2.3.

If $v_2 > 0$, consider another rotation that maps v to e_1 and fixes the space spanned by $\{e_3, \cdots, e_N\}$. Then (v1) and (d2) are clearly satisfied with $d^* = 0$. Also, Ω is transformed to $\Omega' := \{x \in \mathbb{R}^N : x \cdot b' > c^*\}$, where $b' = (v_2, v_1, 0, \cdots, 0)$. In particular $b'_1 > 0$ and (d1) holds. Now, the corollary follows from Theorem 2.3.

If $v_1 = 0$ and $v_2 \leq 0$, then $v = -e_2 = -b$, a contradiction to our assumptions. □

Before we proceed, define $Lu := u_t - \Delta u$ and $M := \sup_\Omega u$. Furthermore, given $\lambda \in \mathbb{R}$ set

$$
\begin{aligned}
&\Sigma_\lambda := \{x \in \Omega : x_1 < \lambda\}, \\
&x^\lambda := (2\lambda - x_1, x_2, \cdots, x_N) \qquad (x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N), \\
&w_\lambda(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \bar{\Sigma}_\lambda \times \mathbb{R}), \\
&\lambda(t) := \sup\{\mu : w_\lambda(x,t) \geq 0 \text{ for all } x \in \Sigma_\lambda \text{ and } \lambda < \mu\}, \\
&\lambda^* := \inf\{\lambda(t) : t \in \mathbb{R}\}\,.
\end{aligned} \tag{2.4}
$$

Observe that (d1) implies $x^\lambda \in \Omega$ for any $x \in \bar{\Sigma}_\lambda$, and therefore w_λ is well defined. Moreover, since u is nonnegative in Ω and vanishes on $\partial\Omega$,

$$w_\lambda(x,t) = u(x^\lambda, t) - u(x,t) = u(x^\lambda, t) \geq 0 \qquad ((x,t) \in (\partial\Omega \cap \bar{\Sigma}_\lambda) \times \mathbb{R})\,.$$

Clearly $w_\lambda(x,t) = 0$ if $(x,t) \in (\Omega \cap \partial\Sigma_\lambda) \times \mathbb{R}$, and therefore

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times \mathbb{R})\,. \tag{2.5}$$

We divide the proof of Theorem 2.3 into several lemmas, in which we implicitly suppose the assumptions of the theorem.

First, notice that $v_1 > 0$ implies

$$x^\lambda \cdot v - x \cdot v = 2(\lambda - x_1)v_1 \geq 0 \qquad (x \in \Sigma_\lambda)\,, \tag{2.6}$$

and consequently by (h1)

$$h(x \cdot v) \leq h(x^\lambda \cdot v) \qquad (x \in \Sigma_\lambda)\,. \tag{2.7}$$

Lemma 2.6. *If there are $\lambda \in \mathbb{R}$, $\tilde{x} \in \Sigma_\lambda$ and $\tilde{t} \in \mathbb{R}$ with $h(\tilde{x} \cdot v) \leq 0$ and $w_\lambda(\tilde{x}, \tilde{t}) \leq 0$, then $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$. Moreover, if $\tilde{x}_1 \leq \frac{-d^*}{v_1}$, then $w_\lambda(\tilde{x}, \tilde{t}) \leq 0$ implies $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$.*

Proof. The positivity and monotonicity of f, and (2.7) yield

$$\begin{aligned} Lw_\lambda(\tilde{x}, \tilde{t}) &= h(\tilde{x}^\lambda \cdot v) f(u(\tilde{x}^\lambda, \tilde{t})) - h(\tilde{x} \cdot v) f(u(\tilde{x}, \tilde{t})) \\ &\geq h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda, \tilde{t})) - f(u(\tilde{x}, \tilde{t}))] \geq 0\,, \end{aligned}$$

and the first statement follows. Next, assume $\tilde{x}_1 \leq -\frac{d^*}{v_1}$. Then $v_1 > 0$ and (d2) imply

$$\tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leq \tilde{x}_1 v_1 + d^* \leq 0\,,$$

and by (h1) and (h2) one has $h(\tilde{x} \cdot v) \leq 0$. Now, the second statement follows from the first one. □

Lemma 2.7. $\lambda(t) \geq \frac{-d^*}{v_1}$ *for all $t \in \mathbb{R}$.*

Proof. We proceed by a contradiction, that is, we assume the existence of $\lambda < \frac{-d^*}{v_1}$ and $(\tilde{x}, \tilde{t}) \in \Sigma_\lambda \times \mathbb{R}$ with $w_\lambda(\tilde{x}, \tilde{t}) < 0$. Then, $Lw_\lambda(\tilde{x}, \tilde{t}) \geq 0$ by the second statement of Lemma 2.6. One can easily verify that for any sufficiently smooth function $g : (-\infty, \lambda] \to (0, \infty)$

$$g(x_1) L\bar{w}_\lambda(x, t) = Lw_\lambda(x, t) + 2(\partial_{x_1} \bar{w}_\lambda(x, t)) g'(x_1) + \bar{w}_\lambda(x, t) g''(x_1) \qquad ((x, t) \in \Sigma_\lambda \times (0, \infty))\,, \tag{2.8}$$

where $\bar{w}_\lambda(x, t) := w_\lambda(x, t)/g(x_1)$. If we set

$$g(y) := \ln(\lambda + 1 - y) + 1 \qquad (y \in (-\infty, \lambda])\,,$$

then $g > 0$ and for already fixed $\tilde{x}$ and $\tilde{t}$ we have

$$L\bar{w}_\lambda(\tilde{x}, \tilde{t}) \geq 2(\partial_{x_1} \bar{w}_\lambda(\tilde{x}, \tilde{t})) \frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} + \bar{w}_\lambda(\tilde{x}, \tilde{t}) \frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}\,. \tag{2.9}$$

Consider the solution of the problem

$$\begin{aligned} z_t - z_{yy} &= F(y, z, z_y), & (y, t) &\in \mathbb{R} \times (0, \infty)\,, \\ z(y, 0) &= -M, & y &\in \mathbb{R}\,, \end{aligned} \tag{2.10}$$

where

$$F(y, z, z_y) = \begin{cases} 2z_y g'/g & y < \lambda - 1, \\ 2z_y g'/g - az & y \in [\lambda - 1, \lambda], \\ 0 & y > \lambda, \end{cases}$$

and $a := -g''(\lambda-1)/g(\lambda-1) > 0$. Then, the maximum principle implies $z(y,t) < 0$ for all $(y,t) \in \mathbb{R} \times (0,\infty)$, and since $F(y,-M,0) \geq 0$, z is increasing in t. Also, for any $T \geq 0$ the function $Z : (x,t) \mapsto z(x_1, t+T)$ satisfies

$$L[Z] \leq 2\frac{g'(x_1)}{g(x_1)}\partial_{x_1} Z + \frac{g''(x_1)}{g(x_1)} Z \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty), x_1 < \lambda)\,.$$

Then, the maximum principle on the set where $\bar{w}_\lambda \leq 0$ yields $\bar{w}_\lambda(\tilde{x},\tilde{t}) \geq Z(\tilde{x},\tilde{t}) = z(\tilde{x}_1, \tilde{t}+T)$ for any $T > 0$.

Since z is increasing in t, $\tilde{z}(y) := \lim_{t\to\infty} z(y,t)$ is well defined for each $y \in \mathbb{R}$ and

$$-\tilde{z}_{yy} = F(y, \tilde{z}, \tilde{z}_y), \qquad y \in \mathbb{R}\,.$$

An analysis of this problem (for details see [23, Claim 2]) implies $\tilde{z} \equiv 0$. Thus, $\bar{w}_\lambda(\tilde{x},\tilde{t}) \geq z(\tilde{x}_1, \tilde{t}+T) \to 0$ as $T \to \infty$, a contradiction. □

Lemma 2.8. *The mapping $t \mapsto \lambda(t)$ is nondecreasing. If $\lambda(t_1) = \infty$, this means that $\lambda(t_2) = \infty$ for all $t_1 \leq t_2$.*

Proof. Fix $t_0 \in \mathbb{R}$ and $\lambda < \lambda(t_0)$. Then

$$w_\lambda(x,t_0) \geq 0 \qquad (x \in \Sigma_\lambda)\,,$$

and by (2.5)

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times [t_0,\infty))\,.$$

Next, (2.7) and the mean value theorem imply

$$\begin{aligned} Lw_\lambda(x,t) &= h(x^\lambda \cdot v) f(u(x^\lambda,t)) - h(x\cdot v) f(u(x,t)) \\ &\geq h(x\cdot v)[f(u(x^\lambda,t)) - f(u(x,t))] \\ &= h(x\cdot v) f'(\theta(x,t)) w_\lambda(x,t), \qquad (x,t) \in \Sigma_\lambda \times [t_0,\infty)\,, \end{aligned}$$

where $\theta(x,t)$ is a number between $u(x,t)$, and $u(x^\lambda,t)$. In particular $\theta : (x,t) \mapsto [0,\infty)$ is a bounded function. Since by (d2)

$$x\cdot v = x_1 v_1 + x_2 v_2 \leq x_1 v_1 + d^* \leq \lambda + d^* \qquad (x \in \Sigma_\lambda),$$

one has $h(x\cdot v) \leq h(\lambda + d^*)$ for each $x \in \Sigma_\lambda$. Now, the maximum principle, with the coefficient $c(x,t) := h(x\cdot v) f'(\theta(x,t))$ being possibly unbounded from below (see [6, 18]), gives $w_\lambda(x,t) \geq 0$ for all $(x,t) \in \Sigma_\lambda \times [t_0,\infty)$. Since $\lambda < \lambda(t_0)$ was arbitrary, $\lambda(t) \geq \lambda(t_0)$ for each $t \geq t_0$. □

Lemma 2.9. *$\lambda^* = \infty$, or equivalently u is nondecreasing in x_1.*

Proof. We proceed by a contradiction, that is, we suppose $\lambda^* < \infty$. Lemma 2.7 guarantees $\lambda^* \geq \frac{-d^*}{v_1}$. By the definition of λ^* and Lemma 2.8, there exit $\lambda_k \searrow \lambda^*$ and $t_k \searrow -\infty$ with

$$\inf_{x\in\Sigma_{\lambda_k}} w_{\lambda_k}(x,t_k) < 0\,.$$

Since u is bounded, there is $M > 0$ with $u \leq M$. Consequently, by (f1), there exists C_f such that $f' \leq C_f$ on $[0,M]$. Set $b_2 := h(\lambda^* v_1 + d^* + 1)C_f > 0$ and choose $1 > \delta > 0$ with

$$2\delta^{-2} \geq 3^3(2b_2+1)\,. \qquad (2.11)$$

Since $f'(0) = 0$, we can fix $\eta > 0$ with

$$f'(z) \le \frac{\delta}{h(\lambda^* + d^* + 1)(\lambda^* + 1 + \frac{d^*}{v_1})^3} \qquad (z \in [0, \eta]). \tag{2.12}$$

Let ε with $0 < \varepsilon < \delta$ be sufficiently small (as specified below), and fix k such that $\lambda_k < \lambda^* + \varepsilon$. To simplify the notation set $\lambda := \lambda_k$ and denote

$$g(y) := 2 - \frac{\delta}{\delta + \lambda - y} \qquad (y \in (-\infty, \lambda]),$$
$$\bar{w}_\lambda(x,t) := \frac{w_\lambda(x,t)}{g(x_1)} \qquad ((x,t) \in \Sigma_\lambda \times \mathbb{R})\,.$$

Observe that $g''(y) \le 0$ and $g(y) > 0$ for any $y \le \lambda$. For already fixed λ, define

$$S := \{(x,t) \in \Sigma_\lambda \times \mathbb{R} : w_\lambda(x,t) \le 0\}\,.$$

Case 1. If $(\tilde{x}, \tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x}, \tilde{t}) \ge 0$, then (2.8) and the concavity of g yield

$$L\bar{w}_\lambda(\tilde{x}, \tilde{t}) \ge 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x}, \tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}\,.$$

Case 2. If $(\tilde{x}, \tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x}, \tilde{t}) < 0$, then Lemma 2.6 yields $h(\tilde{x} \cdot v) > 0$. Consequently (h1) and (d2) yield

$$0 \le \tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \le \tilde{x}_1 v_1 + d^* \le \lambda^* + d^* + 1\,. \tag{2.13}$$

Also, Lemma 2.6 implies $\tilde{x}_1 > \frac{-d^*}{v_1}$, and therefore

$$\tilde{x}^\lambda \cdot v = (2\lambda - \tilde{x}_1)v_1 + \tilde{x}_2 v_2 \le 2\lambda v_1 + 2d^* \le 2\lambda^* + 2d^* + 1\,. \tag{2.14}$$

Now, (2.7) implies $h(\tilde{x}^\lambda \cdot v) \ge h(\tilde{x} \cdot v) > 0$ and (h1), (2.13), (2.14) yield

$$h(-1) \le h(x \cdot v) \le h(2(\lambda^* + d^*) + 2) \qquad ((x,t) \in \mathbb{R}^{N+1}, d_P[(x,t), S^*] < 1)\,,$$

where d_P was defined in (1.18) and S^* is the convex hull of S and the set $\{(x^\lambda, t) : (x,t) \in S\}$. Next, boundedness of u and standard local parabolic estimates give

$$|\nabla u(x,t)| \le C_\lambda \qquad ((x,t) \in S^*)\,.$$

Furthermore,

$$u(\tilde{x}^{\lambda^*}, \tilde{t}) \ge u(\tilde{x}, \tilde{t}) \ge u(\tilde{x}^\lambda, \tilde{t}) \tag{2.15}$$

and

$$|\tilde{x}^{\lambda^*} - \tilde{x}^\lambda| = |\tilde{x}_1^{\lambda^*} - \tilde{x}_1^\lambda| = 2(\lambda - \lambda^*) \le 2\varepsilon\,.$$

Also, by (f1) and $h(\tilde{x} \cdot v) \ge 0$

$$\begin{aligned} 0 > Lw_\lambda(\tilde{x}, \tilde{t}) &= h(\tilde{x}^\lambda \cdot v)f(u(\tilde{x}^\lambda, \tilde{t})) - h(\tilde{x} \cdot v)f(u(\tilde{x}, \tilde{t})) \\ &\ge h(\tilde{x}^\lambda \cdot v)f(u(\tilde{x}^\lambda, \tilde{t})) - h(\tilde{x} \cdot v)f(u(\tilde{x}^{\lambda^*}, \tilde{t})) \\ &= h(\tilde{x}^\lambda \cdot v)[f(u(\tilde{x}^\lambda, \tilde{t})) - f(u(\tilde{x}^{\lambda^*}, \tilde{t}))] + [h(\tilde{x}^\lambda \cdot v) - h(\tilde{x} \cdot v)]f(u(\tilde{x}^{\lambda^*}, \tilde{t}))\,. \end{aligned} \tag{2.16}$$

Let us estimate each term separately. Since the segment connecting $\tilde{x}$ and $\tilde{x}^{\lambda^*}$ belongs to S^*, one has by (2.14), (2.15), and the definition of C_f and C_λ

$$
\begin{aligned}
(2.17)\quad h(\tilde{x}^\lambda \cdot v)[f(u(\tilde{x}^\lambda, \tilde{t})) - f(u(\tilde{x}^{\lambda^*}, \tilde{t}))]& \\
&\geq h(2(\lambda^* + d^*) + 1)C_f(u(\tilde{x}^\lambda, \tilde{t}) - u(\tilde{x}^{\lambda^*}, \tilde{t})) \\
&\geq -2h(2(\lambda^* + d^*) + 1)C_f C_\lambda \varepsilon\,.
\end{aligned}
$$

To estimate the second term, notice that $\tilde{x}_1 \leq \lambda^* - \delta$ implies

$$\tilde{x}^\lambda \cdot v - \tilde{x} \cdot v = 2(\lambda - \tilde{x}_1)v_1 \geq 2(\lambda - \lambda^* + \delta)v_1 \geq 2\delta v_1\,.$$

Thus by the monotonicity of h and (2.13) we have

$$h(\tilde{x}^\lambda \cdot v) - h(\tilde{x} \cdot v) \geq \inf_{y \in [0, \lambda^* + d^* + 1]} (h(y + 2\delta v_1) - h(y)) > 0\,. \tag{2.18}$$

A substitution of (2.17) and (2.18) into (2.16) yields

$$0 > -2h(2(\lambda^* + d^*) + 1)C_f C_\lambda \varepsilon + [\inf_{y \in [0, \lambda^* + d^* + 1]} (h(y + 2\delta v_1) - h(y))]f(u(\tilde{x}^{\lambda^*}, \tilde{t}))\,,$$

or equivalently

$$f(u(\tilde{x}^{\lambda^*}, \tilde{t})) < \frac{2h(2(\lambda^* + d^*) + 1)C_f C_\lambda}{\inf_{y \in [0, \lambda^* + d^* + 1]}(h(y + 2\delta v_1) - h(y))}\varepsilon\,.$$

Hence, by (f1) it follows that for sufficiently small $\varepsilon > 0$ one has $u(\tilde{x}^{\lambda^*}, \tilde{t}) \leq \eta$, and for such ε, (2.12) holds true for any $z \in [0, u(\tilde{x}^{\lambda^*}, \tilde{t})]$. Then (2.12), (2.13) and (2.15) imply

$$
\begin{aligned}
Lw_\lambda(\tilde{x}, \tilde{t}) &\geq h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda, \tilde{t})) - f(u(\tilde{x}, \tilde{t}))] \\
&\geq h(\lambda^* + d^* + 1)\frac{\delta}{h(\lambda^* + d^* + 1)(\lambda^* + 1 + \frac{d^*}{v_1})^3} w_\lambda(\tilde{x}, \tilde{t}) \\
&= \frac{\delta}{(\lambda^* + 1 + \frac{d^*}{v_1})^3} w_\lambda(\tilde{x}, \tilde{t})\,.
\end{aligned}
$$

Easy calculations show that

$$\frac{\delta}{(\lambda^* + 1 + \frac{d^*}{v_1})^3} \leq \frac{\delta}{(\delta + \lambda - y)^3} = -\frac{g''(y)}{2} \leq -\frac{g''(y)}{g(y)} \qquad \left(y \in \left[\frac{-d^*}{v_1}, \lambda^*\right]\right)\,,$$

and since $\tilde{x}_1 \geq \frac{-d^*}{v_1}$,

$$Lw_\lambda(\tilde{x}, \tilde{t}) \geq \frac{\delta}{(\lambda^* + 1 + \frac{d^*}{v_1})^3} w_\lambda(\tilde{x}, \tilde{t}) \geq -\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)} w_\lambda(\tilde{x}, \tilde{t}) = -g''(\tilde{x}_1)\bar{w}_\lambda(\tilde{x}, \tilde{t})\,.$$

Consequently, (2.8) implies

$$L\bar{w}_\lambda(\tilde{x}, \tilde{t}) \geq 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x}, \tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}\,.$$

Case 3. Consider $(\tilde{x}, \tilde{t}) \in S$ with $\tilde{x}_1 \in [\lambda^* - \delta, \lambda]$. Then by (d2)

$$\tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leq \lambda v_1 + d^* \leq \lambda^* v_1 + d^* + 1\,,$$

and therefore for already fixed b_2 and C_f we have

$$\begin{aligned}Lw_\lambda(\tilde{x},\tilde{t}) &\geq h(\tilde{x}\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t}))-f(u(\tilde{x},\tilde{t}))] \geq h(\lambda^* v_1+d^*+1)C_f w_\lambda(\tilde{x},\tilde{t})\\ &= b_2 w_\lambda(\tilde{x},\tilde{t})\,.\end{aligned}$$

Moreover, (2.11) implies

$$-g''(y)=\frac{2\delta}{(\delta+\lambda-y)^3}\geq 2b_2+1\geq g(y)b_2+1 \qquad (y\in[\lambda^*-\delta,\lambda])\,.$$

After a substitution into the previous estimate and then into (2.8), we obtain

$$L\bar{w}_\lambda(\tilde{x},\tilde{t})\geq 2(\partial_{x_1}\bar{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}-\frac{\bar{w}_\lambda(\tilde{x},\tilde{t})}{g(\tilde{x}_1)}\,.$$

The rest of the proof uses comparison principle similarly as in Lemma 2.7, for more details see [23, Proof of Claim 4]. □

Proof of Theorem 2.3. We proceed by contradiction, that is, we assume $M := \|u\|_{L^\infty(\Omega\times\mathbb{R})} > 0$. Then by the continuity of u, there are $t_0\in\mathbb{R}$ and a smooth bounded domain $K_0\subset\Omega$ with $|K_0|\leq 1$ (here $|K_0|$ denotes the Lebesgue measure of K_0) such that $u(x,t_0)>0$ for all $x\in K_0$. Define

$$K_\sigma := \{x+\sigma e_1 : x\in K_0\} \qquad (\sigma\geq 0)\,.$$

Since Ω is convex and unbounded in x_1, one has $K_\sigma\subset\Omega$ for all $\sigma\geq 0$. Let $\mu>0$ be the first eigenvalue of the problem

$$\begin{aligned}-\Delta\phi_0 &= \mu\phi_0, && x\in K_0\,,\\ \phi_0 &= 0, && x\in\partial K_0\,,\end{aligned}$$

where the eigenfunction ϕ_0 is normalized such that $\max_{K_0}\phi_0 = 1$. Set

$$\phi_\sigma(x) := \phi_0(x_1-\sigma,x') \qquad (x=(x_1,x')\in K_\sigma)$$

and

$$\psi_\sigma(t) := \int_{K_\sigma} u(x,t)\phi_\sigma(x)\,dx \qquad (t\in\mathbb{R})\,.$$

Since by Lemma 2.9 u is nondecreasing in x_1 and $u>0$ in $K_0\times\{t_0\}$,

$$\psi_\sigma(t_0)\geq\psi_0(t_0) =: c_0 > 0\,.$$

Denote

$$K^*_\sigma(t) := \{x\in K_\sigma : u(x,t)\phi_\sigma(x)\geq c_0/2\} \qquad (t\geq t_0)\,.$$

If $\psi_\sigma(t^*)\geq c_0$ for some $t^*\geq t_0$, then (using $|K_\sigma|\leq 1$)

$$c_0\leq\int_{K_\sigma}u(x,t^*)\phi_\sigma(x)\,dx\leq |K^*_\sigma(t^*)|\cdot M+\frac{c_0}{2}|K_\sigma|\leq |K^*_\sigma(t^*)|\cdot M+\frac{c_0}{2}\,.$$

Consequently, $|K^*_\sigma(t^*)|\geq\xi := c_0/(2M) > 0$. Next,

$$\begin{aligned}\int_{K^*_\sigma(t^*)}u(x,t^*)\phi_\sigma(x)\,dx &\geq \xi\frac{c_0}{2}\geq\xi\int_{K_\sigma\setminus K^*_\sigma(t^*)}u(x,t^*)\phi_\sigma(x)\,dx\\ &= \xi\int_{K_\sigma}u(x,t^*)\phi_\sigma(x)\,dx-\xi\int_{K^*_\sigma(t^*)}u(x,t^*)\phi_\sigma(x)\,dx\,.\end{aligned}$$

It follows that
$$\int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx \geq \frac{\xi}{1+\xi}\int_{K_\sigma} u(x,t^*)\phi_\sigma(x)\,dx = \frac{c_0}{2M+c_0}\psi_\sigma(t^*)\,.$$
Since K is bounded, we can choose R such that K is a subset of the ball of radius R centered at the origin. Then for sufficiently large $\sigma \geq 0$
$$\begin{aligned} x\cdot v &= x_1v_1 + x_2v_2 \geq -|x_1-\sigma|v_1 + v_1\sigma - R|v_2| \geq R(-v_1-|v_2|) + v_1\sigma \\ &\geq \frac{1}{2}v_1\sigma \qquad (x\in K_\sigma)\,. \end{aligned}$$
Hence, for sufficiently large $\sigma \geq 0$ and (h2) one has
$$\begin{aligned} \frac{d}{dt}\psi_\sigma(t^*) &= \int_{K_\sigma}\Delta u(x,t^*)\phi_\sigma(x)\,dx + \int_{K_\sigma} h(x\cdot v)f(u(x,t^*))\phi_\sigma(x)\,dx \\ &\geq \int_{K_\sigma} u(x,t^*)\Delta\phi_\sigma(x)\,dx + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma} f(u(x,t^*))\phi_\sigma(x)\,dx \\ &\geq \int_{K_\sigma} u(x,t^*)\Delta\phi_\sigma(x)\,dx + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma^*(t^*)}\frac{f(u(x,t^*))}{M}u(x,t^*)\phi_\sigma(x)\,dx \\ &\geq -\mu\psi_\sigma(t^*) + h\left(\frac{1}{2}v_1\sigma\right)f\left(\frac{c_0}{2}\right)\frac{1}{M}\int_{K_\sigma^*(t^*)} u(x,t^*)\phi_\sigma(x)\,dx \\ &\geq \psi_\sigma(t^*)\left[-\mu + h\left(\frac{1}{2}v_1\sigma\right)f\left(\frac{c_0}{2}\right)\frac{1}{M}\frac{c_0}{2M+c_0}\right] \\ &\geq \psi_\sigma(t^*)\,. \end{aligned}$$
Thus, if $\psi_\sigma(t^*) \geq c_0$, then $\psi'_\sigma(t^*) \geq 0$, and consequently $\psi'_\sigma(t) \geq \psi_\sigma(t) \geq c_0$ for each $t \geq t^*$. Since $\psi_\sigma(t_0) \geq c_0$, one has $\psi'_\sigma(t) \geq c_0 > 0$ for each $t > t_0$. Therefore $\psi_\sigma(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of u. ☐

3. Proofs of main results

In this section, we use the notation introduced in the previous sections. Especially, recall the definitions of $\mathbb{R}^N_\lambda$ (see (2.1)), H_λ (see (2.2)), x^λ (see (2.4)), and d_p (see (1.18)).

Our main technical tools are the following doubling lemmas.

Lemma 3.1. *Let (X,d) be a compact metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Theta := \Sigma \setminus D$. Also, let $M : D \to (0,\infty)$ be a bounded function on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*
$$M(y)d(y,\Theta) > 2k\,,$$
then there exists $x \in D$ such that
$$M(x)d(x,\Theta) > 2k, \quad M(x) \geq M(y)\,,$$
and
$$M(z) \leq 2M(x) \qquad (z \in D \cap B^*(x, kM^{-1}(x)))\,, \tag{3.1}$$
where $B^*(y,R) := \{x \in X : d^*(x,y) \leq R\}$ and $d^*(x,y) = |d(x,\Theta) - d(y,\Theta)|$.

Lemma 3.2. *The statement of Lemma 3.1 holds true if (X, d) is a complete metric space and $B^*(x, kM^{-1}(x))$ in (3.1) is replaced by $B(x, kM^{-1}(x))$, where $B(x, R) := \{x \in X : d(x, y) \le R\}$.*

Lemma 3.2 was proved in [25, Lemma 5.1]. The proof of Lemma 3.1 is analogous to the proof of [25, Lemma 5.1]. One only replaces every d by d^* and uses compactness of X, when passing to the limit.

Proof of Theorem 1.1. This proof was partly inspired by the proofs of the corresponding results in [7, 26, 36]. We use the equivalent formulation introduced in Remark 1.3. If (1.17) fails, then there exist $(T_k)_{k\in\mathbb{N}} \subset (0, \infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1) with T replaced by T_k, and $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_k)$ such that

$$M_k(y_k, s_k) := u_k^{\frac{p-1}{3}}(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)) \qquad (k \in \mathbb{N}),$$

where $d_k(s) := \min\{s, T_k - s\}^{\frac{1}{2}}$. Now, for each $k \in \mathbb{N}$, Lemma 3.2 with $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $d = d_P$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \Omega \times \{0, T_k\}$ implies the existence of $(x_k, t_k) \in \bar{\Omega} \times (0, T_k)$ with

$$\begin{aligned} M_k(x_k, t_k) &\ge M_k(y_k, s_k) > 2kd_k^{-1}(t_k) \\ M_k(x_k, t_k) &\ge M_k(y_k, s_k) > 2k \\ 2M_k(x_k, t_k) &\ge M_k(x, t) \qquad ((x, t) \in G_k), \end{aligned} \tag{3.2}$$

where

$$G_k := \{(x, t) \in \Omega \times (0, T_k) : d_P((x, t), (x_k, t_k)) < k\lambda_k\},$$

and

$$\lambda_k := M_k^{-1}(x_k, t_k) \to 0 \qquad \text{as} \quad k \to \infty.$$

Here we used that $d_P((x, t), \Theta_k) = d_k(t)$ for each $(x, t) \in \Sigma_k$. By (3.2)

$$|t - t_k| < k^2\lambda_k^2 < \frac{d_k^2(t_k)}{4} = \frac{1}{4}\min\{t_k, T_k - t_k\} \qquad ((x, t) \in G_k),$$

and therefore

$$\left\{x \in \Omega : |x - x_k| < \frac{k\lambda_k}{2}\right\} \times \left(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}\right) \subset G_k.$$

Since the function a is bounded, we can, after passing to a subsequence, assume that $\mathcal{A} := \lim_{k\to\infty} a(x_k)$ exists.

Case (1). First assume $\mathcal{A} \ne 0$. We define a sequence $(v_k)_{k\in\mathbb{N}}$, of rescaled copies of u as

$$v_k(x, t) := \lambda_k^{\frac{3}{(p-1)}} u(x_k + \lambda_k^{\frac{3}{2}} x, t_k + \lambda_k^3 t) \qquad ((x, t) \in D_k),$$

where

$$D_k := \left\{x \in \lambda_k^{-\frac{3}{2}}(\Omega - x_k) : |x| < \frac{k}{2\lambda_k^{\frac{1}{2}}}\right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k}\right). \tag{3.3}$$

Then $v_k(0,0) = 1$ and, by (3.2), $0 \le v_k(x,t) \le 2$ for each $(x,t) \in D_k$. Moreover, v_k satisfies

$$(3.4) \quad (v_k)_t = \Delta v_k + a(x_k + \lambda_k^{\frac{3}{2}} x) v_k^p, \qquad (x,t) \in D_k\,,$$

$$(3.5) \qquad v_k = 0, \qquad (x,t) \in \left\{ y \in \lambda_k^{-\frac{3}{2}}(\partial\Omega - x_k) : |y| < \frac{k}{2\lambda_k^{\frac{1}{2}}} \right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k} \right).$$

By passing to a suitable subsequence we may assume either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to \infty \quad \text{or} \quad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to c^* \ge 0\,.$$

If (i) holds, then (3.4), L^p estimates, and the Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\text{loc}}(\mathbb{R}^N \times \mathbb{R})$, $\sigma \in (0,1)$ to a function v_∞ satisfying

$$(v_\infty)_t = \Delta v_\infty + \mathcal{A} v_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}\,.$$

Moreover, $v_\infty(0,0) = 1$ and $v_\infty \le 2$. However, if $\mathcal{A} > 0$ and $p < p_B(N)$ (for the definition of $p_B(N)$ see (1.10)) this contradicts [5, Remark 2.6]. If $\mathcal{A} < 0$ and $p > 1$ we have a contradiction to Lemma 2.1

If (ii) holds, then after an application of a suitable orthogonal change of coordinates, the L^p estimates and the Schauder's estimates, yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\text{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function v_∞ satisfying

$$\begin{aligned} (v_\infty)_t &= \Delta v_\infty + \mathcal{A} v_\infty^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ v_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,, \end{aligned}$$

with $v_\infty(0,0) = 1$ and $v_\infty \le 2$. However, if $\mathcal{A} > 0$ and $p < p_S(N) \le p_B(N-1)$ this contradicts [26, Theorem 2.1]. If $\mathcal{A} < 0$ and $p > 1$ we have a contradiction to Lemma 2.2.

Case (2). Assume $\mathcal{A} = 0$. Since a is bounded in $C^2(\bar{\Omega})$, we can assume, after passing to a subsequence, that there exists a vector $\mathcal{B} := \lim_{k\to\infty} \nabla a(x_k) \in \mathbb{R}^N$. Then (1.3) implies $\mathcal{B} \ne 0$.

If $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence, we can, after appropriate restriction, assume the existence of $x_\infty := \lim_{k\to\infty} x_k$. Then $\mathcal{A} = a(x_\infty) = 0$. Set $\tilde{z}_k := x_\infty$ and $V_k := \mathcal{V} := \Omega$ for each $k \in \mathbb{N}$

If $(x_k)_{k\in\mathbb{N}}$ does not have a convergent subsequence, we can assume $|x_k - x_l| \ge 3$ for each $k \ne l$. Let V_k be the connected component of $B_1(x_k) \cap \Omega$ containing x_k, where $B_1(y)$ is the unit ball centered at y. By [16, Lemma 6.37], there exists an extension of $a \in C^2(\bar{V}_k)$ to $C^2(\bar{B}_1(x_k))$, which we denote again by a. Since $V_k \cap V_l = \emptyset$ for $k \ne l$, the function a is well defined on $\mathcal{V} := \cup_{k\in\mathbb{N}} \bar{B}_1(x_k)$.

Denote $\tilde{\Gamma} := \{x \in \bar{\mathcal{V}} : a(x) = 0\}$. Since $a \in C^2(\mathcal{V})$, $\mathcal{A} = 0$, and $\mathcal{B} \ne 0$, there is $(\tilde{z}_k)_{k\in\mathbb{N}} \subset \tilde{\Gamma}$ with $|x_k - \tilde{z}_k| \to 0$ as $k \to \infty$. Define δ_k and $(z_k)_{k\in\mathbb{N}} \subset \tilde{\Gamma}$ such that

$$\delta_k := |z_k - x_k| = \operatorname{dist}(x_k, \tilde{\Gamma}) \le |x_k - \tilde{z}_k| \to 0\,.$$

Then $a \in C^2(\mathcal{V})$ yields $\lim_{k\to\infty} \nabla a(z_k) = \lim_{k\to\infty} \nabla a(x_k) \neq 0$. Thus we may assume $|\nabla a(z_k)| \neq 0$, and therefore

$$\delta_k = \frac{|\nabla a(z_k)(x_k - z_k)|}{|\nabla a(z_k)|} \qquad (k \in \mathbb{N}) .$$

Using that $z_k \in \tilde{\Gamma}$, that is, $a(z_k) = 0$, we obtain

$$a(x_k + \lambda_k x) = \nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2) . \tag{3.6}$$

We define a sequence $(w_k)_{k\in\mathbb{N}}$, of rescaled copies of u as

$$w_k(x,t) := \lambda_k^{\frac{3}{(p-1)}} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in \tilde{D}_k) ,$$

where

$$\tilde{D}_k := \left\{ x \in \lambda_k^{-1}(V_k - x_k) : |x| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right) .$$

Then, $w_k(0,0) = 1$ and $0 \le w_k(x,t) \le 2$ for each $(x,t) \in \tilde{D}_k$, and w_k satisfies

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} a(x_k + \lambda_k x) w_k^p, \qquad (x,t) \in \tilde{D}_k , \tag{3.7}$$

$$w_k = 0, \qquad (x,t) \in \left\{ y \in \lambda_k^{-1}(\partial\Omega - x_k) : |y| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right) . \tag{3.8}$$

Hence, by (3.6)

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} \left[\nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2) \right] w_k^p, \qquad (x,t) \in \tilde{D}_k . \tag{3.9}$$

Case (2a). Assume that there is a suitable subsequence of $(x_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} =: d^* \in \mathbb{R} .$$

By passing to a yet another subsequence we may assume that either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to \infty \quad \text{or} \quad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to c^* \ge 0 .$$

If (i) holds, then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty + (d^* + \mathcal{B} \cdot x) w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R} ,$$

satisfying $w_\infty(0,0) = 1$, $w_\infty \le 2$. Standard regularity theory implies that w_∞ is in fact a classical solution. After a suitable orthogonal transformation and translation, we obtain a nontrivial nonnegative bounded solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm |\mathcal{B}| x_n w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R} ,$$

a contradiction to [23, Theorem 1.1] for any $p > 1$.

If (ii) holds, then $\operatorname{dist}(x_k, \partial\Omega) \to 0$ as $k \to \infty$. After a suitable rotation we have $\nu_\Omega(x_k) \to -e_1$ as $k \to \infty$. Then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\rm loc}(\mathbb{R}^N_{c^*}\times\mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*}\times\mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned}(w_\infty)_t &= \Delta w_\infty + (d^* + \mathcal{B}\cdot x)w_\infty^p, && (x,t)\in\mathbb{R}^N_{c^*}\times\mathbb{R}\,,\\ w_\infty &= 0, && (x,t)\in\partial\mathbb{R}^N_{c^*}\times\mathbb{R}\,,\end{aligned}$$

with $w_\infty(0,0)=1$ and $w_\infty\le 2$. Standard regularity theory yields that w_∞ is in fact a classical solution. Also $a\in C^2(\bar\Omega)$, $\mathrm{dist}(x_k,\partial\Omega)\to 0$ and (1.13) imply

$$0<\frac{\tilde c}{2}\le \liminf_{k\to\infty}\left|\frac{\nabla a(x_k)}{|\nabla a(x_k)|}+e_1\right| = \left|\frac{\mathcal{B}}{|\mathcal{B}|}+e_1\right|\,.$$

Thus, $\mathcal{B}$ is not a multiple of $-e_1$. Now, after a suitable translation, we obtain a contradiction, to Corollary 2.4 for any $p>1$.

Case (2b). After passing to a subsequence, we may assume that

$$\lim_{k\to\infty}\frac{\nabla a(z_k)(x_k-z_k)}{\lambda_k} = \pm|\mathcal{B}|\lim_{k\to\infty}\frac{\delta_k}{\lambda_k}=\pm\infty\,.$$

Setting

$$y=\frac{x}{\alpha_k}\,,\quad s=\frac{t}{\alpha_k^2}\,,$$

where

$$\alpha_k := \left(\frac{\lambda_k}{\delta_k|\nabla a(z_k)|}\right)^{\frac12} = \left(\frac{\lambda_k}{|\nabla a(z_k)(x_k-z_k)|}\right)^{\frac12}\to 0$$

we transform (3.9) to

$$\begin{aligned}(w_k)_s &= \Delta_y w_k + \frac{\alpha_k^2}{\lambda_k}a(x_k+\lambda_k\alpha_k y)w_k^p\\ &= \Delta_y w_k + \frac{\nabla a(z_k)(x_k-z_k+\lambda_k x)+O(\delta_k^2+\lambda_k^2|x|^2)}{|\nabla a(z_k)(x_k-z_k)|}w_k^p\\ &= \Delta_y w_k + [\pm 1 + \alpha_k^3\nabla a(z_k)y + O(\delta_k+\alpha_k^4\lambda_k|y|^2)]w_k^p\,, \qquad (y,s)\in\hat D_k\,,\end{aligned}$$

where

$$\hat D_k := \left\{y\in(\lambda_k\alpha_k)^{-1}(\Omega-x_k) : |y|<\frac{k}{2\alpha_k}\right\}\times\left(-\frac{k^2}{4\alpha_k^2},\frac{k^2}{4\alpha_k^2}\right)\,.$$

Moreover, by (3.8)

$$w_k = 0$$

$$\left((y,s)\in\left\{y\in(\lambda_k\alpha_k)^{-1}(\partial\Omega-x_k) : |y|<\frac{k}{2\alpha_k}\right\}\times\left(-\frac{k^2}{4\alpha_k^2},\frac{k^2}{4\alpha_k^2}\right)\right)\,.$$

By passing to a yet another subsequence, we may assume either

$$\text{(i)}\quad \frac{\mathrm{dist}(x_k,\partial\Omega)}{\lambda_k\alpha_k}\to\infty \quad\text{or}\quad \text{(ii)}\quad \frac{\mathrm{dist}(x_k,\partial\Omega)}{\lambda_k\alpha_k}\to c^*\ge 0\,.$$

If (i) holds, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\rm loc}(\mathbb{R}^N\times\mathbb{R})$ to a function $w_\infty\in C(\mathbb{R}^N\times\mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm w_\infty^p, \qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}\,,$$

and $w_\infty(0,0) = 1$, $w_\infty \le 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [5] (with "+" sign) for any $1 < p < p_B(N)$ and Lemma 2.1 (with "-" sign) for any $p > 1$.

If (ii) holds, then after a suitable orthogonal change of coordinates and a translation, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty \pm w_\infty^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}\,, \\ w_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}\,, \end{aligned}$$

and $w_\infty(0,0) = 1$, $w_\infty \le 2$. Standard regularity theory implies that w_∞ is a classical solution. However this contradicts [26, Theorem 2.1] (with "+" sign) for any $1 < p < p_S(N) \le p_B(N-1)$ and Lemma 2.2 (with "-" sign) for any $p > 1$. □

Let us formulate a sufficient condition that guarantees (1.20).

Lemma 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $1 < p < p_B(N)$, and assume that $a \in C^2(\bar\Omega)$. For a nonnegative classical solution u of* (1.1), (1.2) *define $x^* : (0,T) \to \Omega$ such that*

$$u(x^*(t), t) = \sup_{x\in\Omega} u(x,t) \qquad (t \in (0,T))\,.$$

If there exist $\varepsilon^* > 0$ and $t_0 \in [0,T]$ such that $\mathrm{dist}(x^*(t), \Gamma) \ge \varepsilon^*$ for each $t \in [t_0, T]$, then (1.20) *holds with C depending on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$, ε^* and t_0.*

Proof. As in the proof of Theorem 1.1, we use the equivalent formulation introduced in Remark 1.3. Assume that (1.20) fails. Then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1), and a sequence $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_k)$ such that

$$\tilde M_k(y_k, s_k) > 2k(1 + d_k^{-1}(s_k))\,,$$

where

$$\tilde M_k := u_k^{\frac{p-1}{2}}\,, \qquad d_k(t) = \min\{t, T_k - t\}^{\frac12}\,.$$

Now, Lemma 3.1 with compact $X_k = \Sigma_k = \bar\Omega \times [0, T_k]$, $D_k = \bar\Omega \times (0, T_k)$ and $\Theta_k = \bar\Omega \times \{0, T_k\}$ implies the existence of a sequence $(x'_k, t_k) \in \Omega \times (0, T_k)$ with

$$\begin{aligned} &\tilde M_k(x'_k, t_k) \ge \tilde M_k(y_k, s_k) > 2k d_k^{-1}(t_k) \\ &\tilde M_k(x'_k, t_k) \ge \tilde M_k(y_k, s_k) > 2k \\ &2\tilde M_k(x'_k, t_k) \ge \tilde M_k(x,t) \qquad ((x,t) \in G'_k)\,, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} G'_k &:= \{(x,t) \in \Omega \times (0,T) : d^*_k((x,t),(x'_k,t_k)) < k\lambda'_k\}\,, \\ d^*_k((x,t),(y,s)) &:= |d_k(t) - d_k(s)| \qquad ((x,t),(y,s) \in X_k)\,, \end{aligned}$$

and

$$\lambda'_k := \tilde M^{-1}(x'_k, t_k) \to 0 \quad \text{as} \quad k \to \infty\,.$$

Observe that d_k^* does not depend on x, and therefore (3.10) remains true if we replace x'_k by $x_k := x^*(t_k)$ and G'_k by

$$G_k := \{(x,t) \in \Omega \times (0,T) : d_k^*((x,t),(x_k,t_k)) < k\lambda_k\} \subset G'_k \,,$$

where

$$\lambda_k := \tilde{M}^{-1}(x_k,t_k) \to 0 \,.$$

By our assumptions $\lim_{k\to\infty} a(x_k) \neq 0$. The rest of the proof is now the same as Case (1) in the proof of Theorem 1.1 (see also [26, Theorem 4.1]) with v_k replaced by

$$v_k(x,t) := \lambda^{\frac{2}{p-1}} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in D_k) \,,$$

and D_k by

$$D_k := \left\{ (x,t) \in \lambda_k^{-1}(\Omega - x_k) : |x| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{2}, \frac{k^2}{2} \right) . \qquad \square$$

Proof of Proposition 1.5. In the proof we implicitly assume that all constants depend on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$ and T. Fix any $\xi \in \partial\Omega$ with $a(\xi) = 0$. Since Ω is convex, we can, after a suitable rotation, assume

$$\xi_1 = \sup_{x \in \Omega} x_1, \quad \text{and therofore} \quad \nu_\Omega(\xi) = e_1 \,.$$

Since ξ is a local minimizer of a in $\bar{\Omega}$, all tangential derivatives of a vanish at ξ. Then (1.7) implies $\partial_{x_1} a(\xi) < 0$. Denote

$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\} \,.$$

Assume $u \not\equiv 0$, otherwise the statement is trivial. Observe, that u satisfies

$$u_t = \Delta u + \alpha(x,t) u, \qquad (x,t) \in \Omega \times (0,T) \,,$$

where $\alpha(x,t) = a(x) u^{p-1}$. By Theorem 1.1, α is bounded on $\Omega \times (0, T/2)$ and the bound depends only on the implicitly assumed constants. Next, Hopf boundary lemma (see [19, Lemma 2.6]) implies $\partial_{e_1} u(\xi, \frac{T}{2}) < 0$. By the convexity of Ω, we can choose $\lambda < \xi_1$, sufficiently close to ξ_1 such that

$$w_\lambda(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is well defined. Since $\partial_{x_1} u(\xi, \frac{T}{2}) < 0$ and $\partial_{x_1} a(\xi) < 0$, we can increase $\lambda < \xi_1$ such that

$$w_\lambda(x, \frac{T}{2}) > 0, \quad \text{and} \quad a(x^\lambda) > a(x) \qquad (x \in \Omega_\lambda) \,.$$

Observe that $\xi_1 - \lambda \geq c_1 > 0$, where c_1 is independent of ξ. Since $a(x^\lambda) > a(x)$ for $x \in \Omega_\lambda$, w_λ satisfies

$$(w_\lambda)_t \geq \Delta w_\lambda + \alpha^*(x,t) w_\lambda, \qquad (x,t) \in \Omega_\lambda \times (0,T) \,,$$

where

$$\alpha^*(x,t) := a(x) \frac{u^p(x^\lambda,t) - u^p(x,t)}{u(x^\lambda,t) - u(x,t)} \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is bounded on compact subintervals of $(0,T)$. Similarly as in (2.5)

$$w_\lambda(x,t) \geq 0 \qquad ((x,t) \in \partial\Omega_\lambda \times (0,T)) \,.$$

Now, the maximum principle implies $w_\lambda > 0$ in $\Omega_\lambda \times \left(\frac{T}{2}, T\right)$. Therefore $|x^*(t) - \xi| \geq c_0$ for each $t \in \left(\frac{T}{2}, T\right)$. Since c_0 is independent of ξ and $\Gamma \subset \partial\Omega$, one has

$$\operatorname{dist}(x^*(t), \Gamma) \geq \operatorname{dist}(x^*(t), \partial\Omega) \geq c_0 > 0 \qquad \left(t \in \left(\frac{T}{2}, T\right)\right),$$

and the statement of the proposition follows from Lemma 3.3. □

Lemma 3.4. *Let $N = 1$, $\Omega = (0,1)$ and fix $\mu \in [0, \frac{1}{2})$. Assume $a \in C^2([0,1])$ has exactly one nondegenerate zero $\mu \in [0, 2\mu]$. Also assume $a(x) < 0$ for $x \in [0, \mu)$ and*

$$u_0(x) \leq u_0(x^\mu) \qquad (x \in (0, \mu)). \tag{3.11}$$

If $u \not\equiv 0$ is a nonnegative solution of the problem (1.1), (1.2), *then $|x^*(t) - \mu| \geq c_0 > 0$ and c_0 depends on N, p, a, $\|u_0\|_{L^\infty((0,1))}$, T.*

Proof. For each $\lambda \in (0, \frac{1}{2})$, define $w_\lambda : (0, \lambda) \times (0, \infty) \to \mathbb{R}$ as $w_\lambda(x,t) := u(x^\lambda, t) - u(x,t)$. Since $a(x^\mu) \geq 0 \geq a(x)$ for each $x \in [0, \mu]$,

$$a(x^\mu)u^p(x^\mu, t) - a(x)u^p(x,t) \geq 0 \qquad ((x,t) \in [0,\mu] \times (0,T)).$$

Thus,

$$(w_\mu)_t - (w_\mu)_{xx} \geq 0 \qquad ((x,t) \in (0,\mu) \times (0,T)).$$

By (3.11)

$$w_\mu(x, 0) = u_0(x^\mu) - u_0(x) \geq 0 \qquad (x \in (0,\mu)).$$

Since $u \not\equiv 0$, the maximum principle implies $u > 0$ in $(0,1) \times (0,T)$. Then similarly as in (2.5)

$$w_\mu(0,t) > 0 \quad \text{and} \quad w_\mu(\mu, t) = 0 \qquad (t \in (0,T)).$$

Then, the maximum principle $w_\mu > 0$ in $(0,\mu) \times (0,T)$ and $\partial_x w_\mu(\mu, t) < 0$ for $t \in (0,T)$. Hence, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$w_\lambda(x, T/2) \geq 0 \qquad (x \in (0,\lambda), \lambda \in [\mu, \mu + \varepsilon_0)).$$

As above one can show

$$w_\lambda(0,t) > 0 \quad \text{and} \quad w_\lambda(\lambda, t) = 0 \qquad (t \in (T/2, T)).$$

Since $a'(\mu) > 0$, we can decrease $\varepsilon_0 > 0$ to obtain $a(x^\lambda) \geq a(x)$ for each $x \in (0, \lambda)$ and each $\lambda \in [\mu, \mu + \varepsilon_0)$. Then

$$(w_\lambda)_t - \Delta w_\lambda \geq a(x)[u^p(x^\lambda, t) - u^p(x,t)] = c(x,t)w_\lambda \qquad ((x,t) \in (0,\lambda) \times (t_0, T)),$$

where $c(x,t)$ is a continuous function on $[0,\lambda] \times [t_0, T)$ (possibly unbounded as $t \to T$) The maximum principle implies $w_\lambda(x,t) > 0$ for each $(x,t) \in (0,\lambda) \times (t_0, T)$. In particular $x^*(t) \geq \lambda > \mu$, and therefore $|x^*(t) - \mu| \geq c_0 > 0$ for each $t \in (t_0, T)$. □

Proof of Proposition 1.7. Lemma 3.4 with $\mu = \mu_1$ implies $|x^*(t) - \mu_1| > \varepsilon^* > 0$. If we replace x by $1 - x$ and use Lemma 3.4 with $\mu = 1 - \mu_2$ again, we obtain $|x^*(t) - \mu_2| > \varepsilon^* > 0$. Now, the proposition follows from Lemma 3.3. □

Proof of Proposition 1.6. Without loss of generality assume $a(0) \le 0$, otherwise replace x by $1-x$. If $\mu < \frac{1}{2}$, then the proposition follows from Lemma 3.4 and Lemma 3.3. Assume $\mu \in [\frac{1}{2}, 1]$. Similarly as in the proof of Lemma 3.4, we can show that $w_\mu(x,t) := u(x^\mu, t) - u(x,t)$ is well defined on $[\mu, 1]$ and satisfies

$$w_\mu(x,t) < 0 \qquad ((x,t) \in (\mu,1) \times (0,T)) \quad \text{and} \quad w'_\mu(\mu,t) < 0 \qquad (t \in (0,T)).$$

Hence, for $\lambda > \mu$ sufficiently close to μ we have $w_\lambda(x, \frac{T}{2}) < 0$ for any $x \in (\lambda, 1)$. Similarly as in Lemma 3.4 (using the maximum principle) we prove $w_\lambda(x,t) < 0$ for any $(x,t) \in (\lambda, 1) \times (\frac{T}{2}, T)$. Consequently, $|x^*(t) - \mu| > \lambda - \mu > 0$ for all $t \in (\frac{T}{2}, T)$ and the proposition follows from Lemma 3.3. □

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