Counter-example for Liouville theorems for indefinite problems on half spaces

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Abstract

We show that nonlinear Liouville theorems does not hold in general for indefinite problems on half spaces. Thus, in order to use blow-up method to obtain a priori estimates of indefinite elliptic equations, one has to impose assumptions on the nodal set of nonlinearity. The counter example is constructed by shooting method in one-dimensional case and then extended to higher dimensions.

1 Introduction

This paper is motivated by studies of the indefinite elliptic problems of the form

$$\begin{aligned} -\Delta u &= m(x)|u|^{p-1}u, && x \in \Omega \\ u &= 0 && x \in \partial\Omega\,, \end{aligned} \tag{1}$$

and the parabolic counterparts. In this context the indefinite problem means that the function m is changing sign in $\bar{\Omega}$. Here, and below we assume that $\Omega \subset \mathbb{R}^N$ is a smooth domain (of class $C^{2,\alpha}$ for some $\alpha > 0$) and the problem is superlinear and subcritical, that is, $1 < p < p_S$, where $p_s := \infty$ for $N = 1, 2$ and $p_S := (N+2)/(N-2)$ for $N \geq 3$. The assumptions on the function m will be specified below.

Indefinite elliptic problems attracted a lot of attention during recent decades see e.g [1, 2, 5, 6, 7, 16] and references therein. In order to investigate their qualitative properties it is important to obtain a priori bounds for solutions. By a priori estimates we mean estimates of the form

$$\|u\|_X \leq C(N, p, \Omega, m)\,, \tag{2}$$

where $X := L^\infty(\bar{\Omega})$. We remark that analogous problem for parabolic problems investigates blow-up rates of solutions see e.g. [9, 14, 17] and references therein.

In order to obtain a priori estimates, one can use various strategies (see [15]). In this paper we focus on the scaling method, which often yields optimal results with respect to exponent p, if precise asymptotics of the nonlinearity is known, in our case u^p, for u close to infinity.

Let us briefly explain how the scaling method connects a priori estimates and Liouville theorems. Detailed exposition for elliptic and parabolic problems can be found for example in [7, 9, 13]. We are not going to discuss the optimality of assumptions, especially assumptions on the exponent p. An interested reader can find a detailed analysis in [15], see also references therein.

In this paper the term Liouville theorem refers to the following statement. Any bounded, non-negative solution of a given problem is trivial, that is, the solution is equal to zero everywhere. Equivalently, there is no non-trivial, non-negative, bounded solution of a given problem.

Before we proceed, we need the following notation:

$$\mathbb{R}^N_c := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > c\} \qquad (c \in \mathbb{R}),$$

and

$$\Omega^+ := \{x \in \Omega : m(x) > 0\}, \quad \Omega^- := \{x \in \Omega : m(x) < 0\}, \quad \Omega^0 := \{x \in \Omega : m(x) = 0\}.$$

Assume that m is a continuous function and there are positive continuous functions α_1, α_2 defined on the small neighborhood of Ω_0 in Ω and $\gamma_1, \gamma_2 > 0$ such that

$$m(x) = \begin{cases} \alpha_1(x)[\mathrm{dist}(x, \Omega_0)]^{\gamma_1} & x \in \Omega^+, \\ \alpha_2(x)[\mathrm{dist}(x, \Omega_0)]^{\gamma_2} & x \in \Omega^-. \end{cases}$$

We assume that (2) fails, that is, we assume that for each $k \in \mathbb{N}$ there exist a solution u_k of the problem (1) and $x_k \in \Omega$ such that

$$u_k(x_k) \geq 2k \qquad (k \in \mathbb{N}).$$

After an application of doubling lemma (see [13, Lemma 5.1]), appropriate scaling, and elliptic regularity we can distinguish following cases.

If there is a subsequence of $(x_k)_{k\in\mathbb{N}}$ (denoted again (x_k)) such that $x_k \to x_0$ with $x_0 \in \bar{\Omega}$ and $x_0 \notin \Omega^0$, then there must exist a bounded nonnegative function v with $v(0,0) = 1$ that solves

$$0 = \Delta v + \kappa v^p, \qquad x \in \mathbb{R}^N, \tag{3}$$

or

$$\begin{aligned} 0 &= \Delta v + \kappa v^p, && x \in \mathbb{R}^N_{c^*}, \\ v &= 0, && x \in \partial\mathbb{R}^N_{c^*}, \end{aligned} \tag{4}$$

for some $c^* \in \mathbb{R}$, where $\kappa \in \{-1, 1\}$. However, by the results of Gidas and Spruck [10] if $\kappa = 1$ and [4, 8] if $\kappa = -1$, the Liouville theorem holds for problem (3) and (4) with $1 < p < p_S$. This means $v \equiv 0$, which contradicts $v(0,0) = 1$.

If $x_0 \in \Omega^0$, then the problem is more involved and it was discussed in [7], see also references therein, under the assumption $\bar{\Omega}^0 \subset \Omega$, that is, m does not vanish on $\partial\Omega$. Then v with $v(0,0) = 1$ can in addition to (3) and (4) solve

$$0 = \Delta v + h(x_1)v^p, \qquad x \in \mathbb{R}^N, \tag{5}$$

where $h(x) = x^{\gamma_1}$ for $x > 0$ and $h(x) = -|x|^{\gamma_2}$ for $x < 0$. However, by [7, 12] the problem (5) satisfies the Liouville theorem for any continuous, nondecrasing function h, such that

$$h(0) = 0, \quad h \text{ is strictly increasing for } x > 0, \quad \lim_{x\to\infty} h(x) = \infty\,. \tag{6}$$

Hence $v \equiv 0$, a contradiction to $v(0,0) = 1$. We remark that we can allow h to depend on x_1 only, due to translational and rotational invariance of the problem (5).

The situation in the remaining case is more interesting. If we allow $\bar{\Omega}_0 \cap \partial\Omega \neq \emptyset$, then v with $v(0,0) = 1$, can, in addition to the cases above, solve

$$\begin{aligned} 0 &= \Delta v + h(x \cdot b)v^p, && x \in \mathbb{R}^N_{c^*}\,, \\ v &= 0, && x \in \partial\mathbb{R}^N_{c^*}\,, \end{aligned} \tag{7}$$

where b is a unit vector, $c^* \in \mathbb{R}$, and $h(x) = x^{\gamma_1}$ for $x > 0$ and $h(x) = -|x|^{\gamma_2}$ for $x < 0$. Notice that we cannot choose $b = (1, 0, \cdots, 0)$ since the problem is defined on the half space, and therefore it is not rotationally invariant. In order to obtain a contradiction as above, one has to prove Liouville theorem for (7). It follows from the following Corollary that was proved in [9].

Corollary 1. *Assume $b \neq -e_1$ and $c^* \in \mathbb{R}$, or $b = -e_1$ and $c^* \geq 0$. If $h : \mathbb{R} \to \mathbb{R}$ is continuous, non-decreasing function such that* (6) *holds, then there is no non-negative, non-trivial, bounded solution v of* (7)*.*

We remark that the result in [9] treats more general nonlinearities. In the case $b = -e_1$ and $c^* \geq 0$, Lioville theorem holds under more general assumptions on h (see [9, 17]). One might expect that the Liouville theorem will continue to be true when $b = -e_1$ and $c^* < 0$.

However, the main result of this paper (see Proposition 1 below), shows that such Liouville theorem does not hold. More precisely, if $b = -e_1$, then for each $c^* < 0$ there exists a bounded, positive solution of (7). The construction of the solution in one dimensional case ($N = 1$) is based on the shooting method in two directions. A counter-example in higher dimensions is obtained by an extension of the one dimensional solution by a constant. Similarly, one can obtain a counter-example to parabolic Liouville theorems.

This counter-example shows that the blow-up method based on scaling needs additional assumptions on m, if $m(x_0) = 0$ for some $x_0 \in \partial\Omega$. For example we need to assume, as in [9], that Ω_0 intersects $\partial\Omega$ transversally.

Since one might consider more general functions m, or one might be interested in ordinary differential equations, we consider more general problems than required for our counter-examples.

More specifically, we consider a function $h \in C(\mathbb{R})$ such that:

$$h(x) > 0 \quad \text{for} \quad x > 0, \qquad h(x) < 0 \quad \text{for} \quad x < 0, \tag{8}$$

$$\int_{-\infty}^{0} h(x)\,dx = -\infty\,, \qquad \int_{0}^{\infty} h(x)\,dx = \infty\,, \tag{9}$$

$$\text{there exists } \varepsilon^* > 0 \text{ such that } h \text{ is non-decreasing on } (-\varepsilon^*, 0). \tag{10}$$

The main result of the paper is the following proposition.

Proposition 1. *Let $p > 1$ and assume that a continuous function h satisfies* (8)–(10)*. Then for each $a > 0$ there exists a bounded, non-negative, nontrivial solution u of the problem*

$$\begin{aligned} u'' &= h(x)|u|^{p-1}u, \qquad && x \in (-a, \infty)\,, \\ u(-a) &= 0\,. \end{aligned} \tag{11}$$

Moreover, $u'(x) < 0$ for $x \geq 0$ and $\lim_{x\to\infty} u(x) = 0$.

Remark 1. The nonlinearity $|u|^{p-1}u$ can be replaced by a Lipschitz function $f : [0, \infty) \to \mathbb{R}$, such that $f(0) = 0$, $f(u) > 0$ for $u > 0$, f is non-decreasing for $u > 0$, and

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty, \qquad \lim_{u\to 0^+} \frac{f(u)}{u} = 0\,.$$

In that case we extend f as a Lipschitz function to whole $\mathbb{R}$ such that $f(u) < 0$ for $u < 0$.

If the assumption

$$\int_0^\infty h(x)\,dx = \infty$$

is removed, Proposition 1 still holds true without the statement $\lim_{x\to\infty} u(x) = 0$.

If the problem is scale invariant, then the proof can be simplified and we can also address the question of uniqueness.

Proposition 2. *If $h(x) = sign(x)|x|^\alpha$ for some $\alpha > 0$, then the solution in Proposition 1 is unique.*

The following corollary states a counter-example to Liouville theorem for indefinite problems on half spaces. It shows that Corollary 1 cannot be improved. A counterexample is given by a function $v(x_1, \cdots, x_N) = u(x_1)$, where u is a function from Proposition 1.

Corollary 2. *If $b = -e_1$, $c^* < 0$, and h satisfies* (8)–(10)*, then the problem* (7) *possess a bounded, nonnegative solution.*

2 Proof of Proposition 1 and Proposition 2

Let us prove Proposition 1 first. Fix $\xi \in (0, \infty)$. Let $u_k : (\tau_k, T_k) \to \mathbb{R}$ be the solution of the initial value problem

$$\begin{aligned} u_k'' &= h(x)|u_k|^{p-1}u_k\,, \qquad && x \in (\tau_k, T_k)\,, \\ u_k(0) &= \xi\,, \qquad u_k'(0) = k\,, && \end{aligned} \tag{12}$$

where (τ_k, T_k) is the maximal existence interval of u_k. By a standard theory, $-\infty \leq \tau_k < 0 < T_k \leq \infty$.

Remark 2. *Decay at infinity.* If $h(x) = |x|^\alpha$ for some $\alpha > 0$, then one can proceed as in [11, Theorem 2.1] and obtain that for $1 < p < p_S$, every solution u satisfies $u(x) \leq C|x|^{-\frac{2+\alpha}{p-1}}$ and $|u'| \leq C|x|^{-\frac{p+1+\alpha}{p-1}}$ for each $|x| > 1$. Observe that [11] discusses problem with $h(x) = -|x|^\alpha$, but one can easily modify the proof of [11, Lemma 2.1] by replacing Liouville theorem of Gidas and Spruck [10] by e.g. [4, 8].

Variational approach. For each $\xi > 0$ the existence of a unique $k(\xi)$ such that there exists global ($T_k = \infty$), positive solution, follows from variational approach. Specifically, let X be the Banach space of functions with finite norm

$$\|u\|_X := \left(\frac{1}{2}\int_0^\infty (u'(x))^2\,dx\right)^{\frac{1}{2}} + \left(\frac{1}{p+1}\int_0^\infty h(x)|u(x)|^{p+1}\,dx\right)^{\frac{1}{p+1}}.$$

Then it is easy to check that the functional

$$F[u] := \frac{1}{2}\int_0^\infty (u'(x))^2\,dx + \frac{1}{p+1}\int_0^\infty h(x)|u(x)|^{p+1}\,dx$$

is coercive, strictly convex, and continuous. Moreover, the set $M := \{u \in X : u(0) = \xi\}$ is convex and closed (therefore weakly closed) so there exists a unique global minimizer of F on M. The minimizer satisfies Euler-Lagrange equation (12) for some k. Since $F[u] = F[|u|]$, the minimizer is non-negative. Also, as $u \equiv 0$ is an equilibrium of (12), every non-negative, non-trivial solution is positive. Notice that this method also implies decay rate of the minimizer at infinity.

However, the variational approach requires the solution to be in the space X, which we cannot guarantee a priori. Also, it gives merely existential result and it does not specify how k depends on ξ.

Fowler transformation. If $h(x) = |x|^\alpha$, one can proceed as in [3] and transform the problem by Fowler transformation $X(t) := -xu'u^{-1}$, $Z(t) := x^{1+\alpha}u^p(u')^{-1}$, and $x = e^t$. Then X and Z satisfies

$$\begin{aligned} X' &= X[X + Z + 1] \\ Z' &= Z[(1+\alpha) - pX - Z]\,. \end{aligned} \tag{13}$$

The existence of solutions of (12) is equivalent to the existence of heteroclinic trajectories connecting equilibria $(0,0)$, $(\frac{1+\alpha}{p-1}, -\frac{p+1+\alpha}{p-1})$ of the system (13). This approach yields very precise asymptotic $-XZ = x^{2+\alpha}u^{p-1} \to \frac{(1+\alpha)(p+1+\alpha)}{(p-1)^2}$ as $x \to \infty$ (and analogous expression for u').
However, since this method does not apply readily to general h and the proof of the existence of heteroclinic orbits is not elementary, we rather use other approach.

We prove the existence of solutions for (12) by shooting method. Notice that this method applies to general h and no decay of u is required. Moreover, it allows us to derive more precise information on dependence of k on ξ.

Claim 1. *If $u_k'(x_0) \geq 0$ and $u_k(x_0) > 0$ for some $x_0 > 0$, then $u_k'(x) > 0$ for each $x > x_0$ and $\lim_{x\to T_k} u_k(x) = \infty$.*

Proof of Claim 1. By (8), $u_k''(x_0) = h(x_0)u_k^p(x_0) > 0$, and therefore $u_k'(x) > u_k'(x_1) > u_k'(x_0) \geq 0$, for each $x > x_1 > x_0$ sufficiently close to x_0. If $u_k'(x) > u_k'(x_1) > 0$ for each $x > x_1$, Claim 1 follows.

Otherwise, there exists the smallest $x_2 > x_1$ with $u_k'(x_2) = u_k'(x_1)$. Then $u_k'(x) > 0$ on $[x_1, x_2]$, and consequently $u_k(x) > 0$ on $[x_1, x_2]$. Moreover, for each $x \in [x_1, x_2]$ one has $u_k''(x) = h(x)u_k^p(x) > 0$, that is, u_k is strictly convex on $[x_1, x_2]$, a contradiction to $u_k'(x_2) = u_k'(x_1)$. $\square$

Claim 2. *If $u_k(x_0) \leq 0$ for some $x_0 > 0$, then $u_k(x) < 0$ for each $x > x_0$ and $\lim_{x\to T_k} u_k(x) = -\infty$.*

Proof of Claim 2. Let $x^* := \inf\{x > 0 : u_k(x) = 0\}$. Since $u_k(0) = \xi > 0$, x^* is well defined and $x^* > 0$. Suppose that there is $x_1 > x^*$ such that $u_k(x_1) \geq 0$. Then either $u \geq 0$ on $[x^*, x_1]$, or u has a negative minimum at $x_2 \in [x^*, x_1]$. In the first case x^* is a local minimizer of u. By the uniqueness of solutions of initial value problems one has $u \equiv 0$, a contradiction to $u(0) = \xi > 0$. In the second case $u_k''(x_2) = h(x_2)|u_k|^{p-1}u_k(x_2) < 0$, a contradiction. Hence, $x_0 = x^*$ and $u < 0$ on (x_0, ∞).

Finally, since $u_k'' = h(x)|u|^{p-1}u(x) < 0$ for each $x \in (x_0, \infty)$, u_k is concave on (x_0, ∞) and the second statement follows. $\square$

Denote

$$\mathcal{K}_0 := \{k : u_k(x) \leq 0 \text{ for some } x \geq 0\}\,,$$
$$\mathcal{K}_2 := \{k : u_k(x) \geq 2 \text{ for some } x \geq 0\}\,.$$

Claim 3. *The sets $\mathcal{K}_0$ and $\mathcal{K}_2$ are non-empty, open, and disjoint. Moreover $(-\infty, -2\xi - H\xi^p) \subset \mathcal{K}_0$, where $H := \sup_{x\in[0,1]} h(x)$.*

Proof of Claim 3. From Claim 2 it follows that $(0, \infty) \subset \mathcal{K}_2 \neq \emptyset$. If $k \in \mathcal{K}_0$, then $\lim_{x\to T_k} u_k(x) \to -\infty$ and if $k \in \mathcal{K}_2$, then $\lim_{x\to T_k} u_k(x) \to \infty$. Thus $\mathcal{K}_0 \cap \mathcal{K}_2 = \emptyset$.

If $k_0 \in \mathcal{K}_0$ then, by Claim 2, there exists x_1 such that $u_{k_0}(x_1) < -1$. The continuous dependence of solutions on initial data implies $u_k(x_1) < -1$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_0$ is open.

Analogously if $k_0 \in \mathcal{K}_2$, then by Claim 1 there is x_0 such that $u_{k_0}(x_0) > 3$. Then the continuous dependence of solutions on the initial data yields $u_k(x_0) > 3$ for any k sufficiently close to k_0. Thus, $\mathcal{K}_2$ is open as well.

Finally, we show the second statement, which also implies $\mathcal{K}_0 \neq \emptyset$. Fix $k < -2\xi - H\xi^p$ and suppose that there is the smallest $x_0 \in [0, 1]$ with $u_k'(x_0) = -\xi$. Then $u_k(x) \leq u_k(0) = \xi$ on $(0, x_0)$. However,

$$u_k'(x_0) = u_k'(0) + \int_0^{x_0} u_k''(x)\,dx = k + \int_0^{x_0} h(x)u_k^p(x)\,dx \leq k + H\xi^p < -\xi\,,$$

a contradiction.

Hence, $u_k'(x) < -\xi$ on $[0, 1]$, and therefore $u_k(x) \leq 0$ for some $x \in [0, 1]$. $\square$

Denote

$$M := \mathbb{R} \setminus (\mathcal{K}_0 \cap \mathcal{K}_2)\,.$$

By Claim 3, $M \neq \emptyset$ and since u_k ($k \in M$) is bounded, $T_k = \infty$. Also, by Claim 1, $u_k' < 0$ in $[0, \infty)$ for each $k \in M$, and therefore $\lim_{x\to\infty} u_k(x) =: L \geq 0$ exists. If $L > 0$, then

$$0 \geq u'(x) = u'(0) + \int_0^x u''(t)\,dt = k + \int_0^x h(t)u^p(t)\,dt \geq k + L^p \int_0^x h(t)\,dt\,,$$

a contradiction to (9) for sufficiently large x. This implies $L = 0$.

Claim 4. $M = \{k^*\}$.

Proof of Claim 4. Suppose that there are $k_1, k_2 \in M$ with $k_1 > k_2$. Then for $x_0 > 0$ sufficiently small, one has

$$u_{k_1}(x_0) - u_{k_2}(x_0) > 0 \quad \text{and} \quad (u_{k_1} - u_{k_2})'(x) > 0 \qquad (x \in [0, x_0])\,.$$

Since $\lim_{x\to\infty} u_{k_1}(x) - u_{k_2}(x) = 0$, there exists the smallest $x_1 > x_0$ with $u_{k_1}'(x_1) = u_{k_2}'(x_1)$. Then $u_{k_1}(x) > u_{k_2}(x)$ for $x \in (x_0, x_1)$; however,

$$\begin{aligned} u_{k_1}'(x_1) &= u_{k_1}'(x_0) + \int_{x_0}^{x_1} u_{k_1}''(x)\,dx = u_{k_1}'(x_0) + \int_{x_0}^{x_1} h(x)u_{k_1}^p(x)\,dx \\ &> u_{k_2}'(x_0) + \int_{x_0}^{x_1} h(x)u_{k_2}^p(x)\,dx = u_{k_1}'(x_0) + \int_{x_0}^{x_1} u_{k_2}''(x)\,dx = u_{k_2}'(x_1)\,, \end{aligned}$$

a contradiction. □

Define the function $k : (0, \infty) \to (-\infty, 0)$ such that $k(\xi)$ is the unique k for which the problem (12) has bounded positive solution on $(0, \infty)$. Let u_ξ be the solution of such problem:

$$\begin{aligned} u_\xi'' &= h(x)|u_\xi|^{p-1}u_\xi\,, & & x \in (\tau_\xi, \infty)\,, \\ u_\xi(0) &= \xi\,, & u_\xi'(0) &= k(\xi)\,, \end{aligned} \tag{14}$$

where τ_ξ defines, as above, the existence time of u_ξ. Recall that u_ξ is decreasing and decays to 0 as $x \to \infty$. Notice that the subscript now indicates the value of $u_\xi(0)$ rather than $u_\xi'(0)$.

Claim 5. *The function* $k : (0, \infty) \to (-\infty, 0)$ *is continuous,* $\lim_{\xi\to\infty} k(\xi) = -\infty$*, and* $\lim_{\xi\to 0^+} k(\xi) = 0$*.*

Proof. First, let us prove continuity. For a contradiction suppose that there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \xi_n = \xi_0 \in (0, \infty)$ such that $k(\xi_0) \neq \lim_{n\to\infty} k(\xi_n) =: M$. Let u be the solution of the problem (12) with $u'(0) = k$ replaced by $u'(0) = M$. Since $M \neq k(\xi_0)$, the solution is either not bounded or not positive. Thus, by Claim 1 and Claim 2 there exists x_0 such that either $u(x_0) < -2$ or $u(x_0) > 3$. The continuous dependence of solution on initial conditions yields that $u_{\xi_n}(x_0) < -2$ or $u_{\xi_n}(x_0) > 3$ for sufficiently large n. This contradicts the definition of $k(\xi_n)$, and proves that k is continuous.

From Claim 3 and negativity of k it follows that $0 > k(\xi) \geq -2\xi - H\xi^p$, and the statement $\lim_{\xi\to 0^+} k(\xi) = 0$ follows.

We finish the proof by showing that $k(\xi) \leq -\frac{\xi}{2}$ for large ξ. Otherwise, there exists large ξ such that $k(\xi) > -\frac{\xi}{2}$ and the convexity of u_ξ yields that $u'_\xi(x) > -\frac{\xi}{2}$ for each $x \in [0,1]$. Hence, $u_\xi(x) > \frac{\xi}{2}$ for each $x \in [0,1]$. Since u_ξ is a nonincreasing function

$$\begin{aligned} 0 \geq u'_\xi(1) &= u'_\xi(0) + \int_0^1 u''_\xi(t)\,dt \\ &= k(\xi) + \int_0^1 h(t)u^p_\xi(t)\,dt \geq -\frac{\xi}{2} + \left(\frac{\xi}{2}\right)^p \int_0^1 h(t)\,dt\,, \end{aligned}$$

a contradiction for sufficiently large ξ. □

Claim 6. *For each $\xi > 0$, there exists $x^* < 0$ such that $u_\xi(x^*) = 0$.*

Proof of Claim 6. For a contradiction assume $u_\xi(x) > 0$ for each $x \in (\tau_\xi, 0)$. Since $u''_\xi(x) = h(x)u^p_\xi < 0$, u_ξ is concave on $(\tau_\xi, 0)$. Therefore, $0 \leq u_\xi(x) \leq 1 + u'_\xi(0)x$ for each $x \in (\tau_\xi, 0)$, and in particular $\tau_\xi = -\infty$.

Next, we show that $u'_\xi(x_0) > 0$ for some $x_0 < 0$. If not, then u_ξ decreases on $(-\infty, 0)$ and $u_\xi(x) \geq u_\xi(0) = \xi$ for all $x < 0$. However,

$$\begin{aligned} 0 \geq u'_\xi(x) = u'_\xi(0) - \int_x^0 u''_\xi(s)\,ds &= k(\xi) - \int_x^0 h(s)u^p_\xi(s)\,ds \\ &\geq k(\xi) - \xi^p \int_x^0 h(s)\,ds\,, \end{aligned}$$

a contradiction to (9) for large negative x.

Thus $u'_\xi(x_0) > 0$ for some $x_0 < 0$, and since u_ξ is concave, $u'_\xi(x) \geq u'_\xi(x_0) > 0$ for each $x < x_0$. Hence, $u_\xi(x^*) = 0$ for some $x^* < 0$, a contradiction. □

Denote $a(\xi) := \sup\{x < 0 : u_\xi(x) = 0\}$. By Claim 6, a is well defined and negative for each ξ. Also, continuous dependence of k on ξ implies the continuity of a.

Claim 7. *The range of a is $(-\infty, 0)$, that is, $\mathcal{R} := \{a(\xi) : \xi \in (0,\infty)\} = (-\infty, 0)$.*

Proof. By the continuity of a is suffices to prove $\sup \mathcal{R} = 0$ and $\inf \mathcal{R} = -\infty$.

First, for a contradiction assume $\max\{\sup \mathcal{R}, -\varepsilon^*\} =: -\varepsilon < 0$, where ε^* was defined in (10). We show that for a sufficiently large ξ, $u'_\xi(x) = 0$ for some $x \in [-\frac{\varepsilon}{4}, 0]$. For a contradiction suppose $u'_\xi(x) < 0$ for each $x \in [-\frac{\varepsilon}{4}, 0]$. Then, u_ξ decreases on $[-\frac{\varepsilon}{4}, 0]$, and by (10), $u''_\xi = h(x)u^p_\xi$ increases on $[-\frac{\varepsilon}{4}, 0]$.

If $u'_\xi(x) \geq \frac{k(\xi)}{2}$ for some $x \in (-\frac{\varepsilon}{8}, 0)$, then the increasing second derivative of u_ξ yields $u'_\xi(x) = 0$ for some $x \in [-\frac{\varepsilon}{4}, 0]$, a contradiction. Otherwise $u'_\xi(x) <$

$\frac{k(\xi)}{2}$ for all $x \in (-\frac{\varepsilon}{8}, 0)$. Then, since u_ξ decreases on $[-\frac{\varepsilon}{4}, 0]$, one has $u_\xi(x) \geq u_\xi(-\frac{\varepsilon}{8}) \geq -\frac{\varepsilon}{16} k(\xi)$ for each $x \in (-\frac{\varepsilon}{4}, -\frac{\varepsilon}{8})$. Moreover,

$$\begin{aligned}
0 > u_\xi'\left(-\frac{\varepsilon}{4}\right) &= u_\xi'(0) - \int_{-\frac{\varepsilon}{4}}^{0} u_\xi''(t)\, dt = k(\xi) - \int_{-\frac{\varepsilon}{4}}^{0} h(t) u_\xi^p(t)\, dt \\
&\geq k(\xi) - \int_{-\frac{\varepsilon}{4}}^{-\frac{\varepsilon}{8}} h(t) \left(-\frac{\varepsilon k(\xi)}{16}\right)^p dt = k(\xi) - c_\varepsilon |k(\xi)|^p \,,
\end{aligned}$$

where $c_\varepsilon > 0$, a contradiction for any sufficiently large $k(\xi)$ (and by Claim 5, for sufficiently large ξ).

Let $b_\xi := \sup\{x < 0 : u_\xi'(x) = 0\}$. We showed that $b_\xi > -\frac{\varepsilon}{4}$ for any sufficiently large ξ. Let $U_\xi := u_\xi(b_\xi)$, then $U_\xi \geq \xi$ since u_ξ decreases on $(b_\xi, 0)$. Assume that there exists $x \in (-\frac{\varepsilon}{2}, b_\xi)$ such that $u_\xi(x) < U_\xi/2$. Then the concavity of u_ξ yields that $u_\xi(x) < 0$ for some $x \in (-\varepsilon - b_\xi, b_\xi)$, a contradiction to the definition of ε. Hence, $u_\xi(x) > U_\xi/2$ for each $x \in (-\frac{\varepsilon}{2}, b_\xi)$. However,

$$\begin{aligned}
0 < u_\xi\left(-\frac{\varepsilon}{2}\right) &= u_\xi(b_\xi) + u_\xi'(b_\xi)(-\frac{\varepsilon}{2} - b(\varepsilon)) - \frac{1}{2}\int_{-\frac{\varepsilon}{2}}^{b_\xi} (-\frac{\varepsilon}{2} - t) u_\xi''(t)\, dt \\
&= U_\xi + \frac{1}{2}\int_{-\frac{\varepsilon}{2}}^{b_\xi} (\frac{\varepsilon}{2} + t) h(t) u_\xi^p(t)\, dt \leq U_\xi + \frac{U_\xi^p}{2}\int_{-\frac{\varepsilon}{2}}^{-\frac{\varepsilon}{4}} (\frac{\varepsilon}{2} + t) h(t)\, dt \\
&= U_\xi + c_\varepsilon U_\xi^p \,,
\end{aligned}$$

where $c_\varepsilon < 0$, a contradiction for sufficiently large U_ξ, and therefore ξ. We have showed $\sup \mathcal{R} = 0$.

Assume $M := \inf \mathcal{R} > -\infty$. First, we claim $\lim_{\xi \to 0^+} u_\xi(b_\xi) = 0$, where b_ξ was defined above. Otherwise, there is a sequence (ξ_n) converging to 0 such that $\lim_{n\to\infty} u_{\xi_n}(b_{\xi_n}) =: \delta > 0$. Since u_ξ is concave, $u_{\xi_n}(b_{\xi_n}) \leq \xi_n + k(\xi_n) b_{\xi_n}$, and therefore $b_{\xi_n} < (u_{\xi_n}(b_{\xi_n}) - \xi_n)/k(\xi_n)$ (recall $k(\xi) < 0$). By Claim 5, $k(\xi_n) \to 0^-$ as $n \to \infty$ and $u_{\xi_n}(b_{\xi_n}) - \xi_n \to \delta$. Thus $b_{\xi_n} \to -\infty$ as $n \to \infty$. Since u_ξ decreases on $(b_\xi, 0)$, it is positive there, and consequently $M \leq a(\xi_n) \leq b_\xi \to -\infty$, a contradiction. Therefore, $u_\xi(b_\xi) \to 0$ as $\xi \to 0^+$.

Since u_ξ is concave, u_ξ increases on $(a(\xi), b_\xi)$. Hence, $u_\xi(x) \leq u_\xi(b_\xi)$ for each $x \in (a(\xi), b_\xi)$. Then,

$$\begin{aligned}
0 = u_\xi(a(\xi)) &= u_\xi(b_\xi) + u_\xi'(b_\xi)(a(\xi) - b_\xi) - \frac{1}{2}\int_{a(\xi)}^{b_\xi} (a(\xi) - t) u_\xi''(t)\, dt \\
&= u_\xi(b_\xi) - \frac{1}{2}\int_{a(\xi)}^{b_\xi} (a(\xi) - t) h(t) u^p(t)\, dt \geq u_\xi(b_\xi) - \frac{u_\xi^p(b_\xi)}{2}\int_{a(\xi)}^{b_\xi} (a(\xi) - t) h(t)\, dt \\
&\geq u_\xi(b_\xi) - \frac{u_\xi^p(b_\xi)}{2}\int_{-M}^{0} (-M - t) h(t)\, dt = u_\xi(b_\xi) - c_M u_\xi^p(b_\xi) \,,
\end{aligned}$$

where $c_M > 0$, a contradiction for small $u_\xi(b_\xi)$ (that is, small ξ). $\square$

This finishes the proof of Proposition 1. $\square$

Proof of Proposition 2. It is trivial to check, that assumptions (8)–(10) are satisfied for $h(x) = \mathrm{sign}(x)|x|^\alpha$, and therefore all claims in the proof of Proposition 1 holds true. In particular, for each $a < 0$ there exists a solution of (11). Fix a and two bounded, positive, nontrivial, solutions u, v of (11). Notice, by the scale invariance, that $v_\lambda(x) = \lambda^{\frac{2+\alpha}{p-1}} v(\lambda x)$ satisfies the equation in (11) and v_λ is a positive bounded function.

Without loss of generality assume $u(0) \leq v(0)$. Then there exists $\lambda \in (0, 1]$ such that $v_\lambda(0) = u(0)$. Moreover, Claim 4 yields that $v'_\lambda(0) = u'(0)$, and consequently $u = v_\lambda$ by the uniqueness of the initial value problem. If $\lambda \neq 1$, then $0 = u(-a) = v_\lambda(-a) = \lambda^{\frac{2+\alpha}{p-1}} v(-\lambda a) > 0$, a contradiction. Thus, $\lambda = 1$ and $u = v$, the uniqueness follows. $\square$

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