Large Prandtl Number Asymptotics in Randomly Forced Turbulent Convection

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Abstract

We establish the convergence of statistically invariant states of the three dimensional Boussinesq Equations in the infinite Prandtl number limit. The equations are subject to a temperature gradient on the boundary and to internal heating in the bulk driven by a stochastic, white in time, gaussian forcing. For the active scalar equations given by the infinite Prandtl system we show that the associated invariant measures are unique and that moreover the corresponding Markovian dynamics evolving probability distributions are contractive in an appropriate Kantorovich-Wasserstein metric. This contractive property of the limit system allows us to reduce the question of the convergence of statistically stationary states to one of finite time asymptotics and certain Prandtl uniform moment bounds.

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1 Introduction

Buoyancy driven convection plays a central role in a wide variety of physical processes: from earth's climate system to the internal dynamics of stars. As such it is of fundamental importance to identify and predict robust statistical quantities in these complex flows and to connect such statistics with the basic equations governing their dynamics, for example the Boussinesq equations. In particular characterizing pattern formation, mean heat transport, and small scale dynamics as a function of physical parameters and boundary conditions remains a topic of intensive research theoretically, numerically, and experimentally; see e.g. [BPA00, Man06, AGL09, LX10] for a broad overview of recent developments.

It has long been understood that statistically invariant states of the nonlinear partial differential equations of fluid dynamics provide mathematical objects which are expected to contain various robust statistical quantities found in turbulent fluid flows. An ongoing challenge is therefore to address the existence, uniqueness, ergodicity, and dependence of these measures on parameters in a variety of specific contexts. While one may certainly pose these questions for deterministic equations (cf. [FMRT01]) the stochastic setting can be more tractable given the regularizing effect of noise on the associated probability distribution functions. Moreover stochastic terms can be physically reasonable in various settings. In convection problems energy may be supplied to the system through both boundary and volumetric heat fluxes, which for instance models radioactive decay processes in the earths mantle; see [Rob67, TZ67, LDB04, WD11, GS12, BN12]. Both sources can therefore have an essentially stochastic character in situations of physical interest.

In this and a companion work, [FGHR], we study statistically invariant states of the stochastically driven Boussinesq equations

$$\frac{1}{Pr}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \Delta \mathbf{u} = \nabla p + Ra\hat{\mathbf{k}}T, \quad \nabla \cdot \mathbf{u} = 0, \tag{1.1}$$

$$dT + \mathbf{u} \cdot \nabla T dt = \Delta T dt + \sum_{k=1}^{N} \sigma_k dW^k, \tag{1.2}$$

for the (non-dimensionalized) velocity field $\mathbf{u} = (u_1, u_2, u_3)$, pressure p, and temperature T of a buoyancy driven fluid. The system (1.1)–(1.2) evolves in a three dimensional domain $\mathcal{D} = [0, L]^2 \times [0, 1]$ and is supplemented with the boundary conditions

$$\mathbf{u}_{|z=0} = \mathbf{u}_{|z=1} = 0, \quad T_{|z=0} = \tilde{Ra}, \ T_{|z=1} = 0, \quad \mathbf{u}, T \text{ are periodic in } \mathbf{x} = (x_1, x_2). \tag{1.3}$$

The dimensionless physical parameters in the problem are the *Prandtl* number Pr and *Rayleigh* numbers Ra and $\tilde{Ra}$; see Remark 3.3 below for further details. The driving noise is given by a collection of independent white noise processes $dW^k = dW^k(t)$ acting in spatial directions $\sigma_k = \sigma_k(x)$ which form a complete orthogonal basis of eigenfunctions (ordered with respect to eigenvalues) of the Laplace operator on $\mathcal{D}$ supplemented with homogeneous Dirichlet boundary conditions for $z = 0, 1$ and periodic in $\mathbf{x}$. The stochastic terms in (1.2) have been normalized so that

$$\sum_{k=1}^{N} \|\sigma_k\|^2_{L^2(\mathcal{D})} = 1\,,$$

with the strength of the body forcing expressed in the physical parameters Ra and $\tilde{Ra}$; see (3.7) below.

Our principal aim here will be to establish convergence properties of statistically invariant states of (1.1)–(1.3) in the *Large Prandtl number limit* when the physical parameter Pr in (1.1) diverges to ∞. In other words we will establish the convergence to invariant measures of the active scalar equation

$$-\Delta \mathbf{u} = \nabla p + Ra\hat{\mathbf{k}}T, \quad \nabla \cdot \mathbf{u} = 0, \tag{1.4}$$

$$dT + \mathbf{u} \cdot \nabla T dt = \Delta T dt + \sum_{k=1}^{N} \sigma_k dW^k \tag{1.5}$$

complemented with boundary conditions for T and $\mathbf{u}$ as in (1.3). The analysis of convection in the large Prandtl number limit is of basic interest in a variety of physical contexts, most notably in modeling certain portions of the earth's mantle and for convection in gasses under high pressure, where the Prandtl number can reach the order of 10^{24}, see [CD99, DC01, OS11]. It is worth emphasizing that the system (1.4)–(1.5) has very complex dynamics even without stochastic forcing when the Rayleigh number(s) are sufficiently large; see [CD99, BPA00, DC01, Wan04a, Par06, AGL09, LX10, OS11].

Let us now present a heuristic version of our main results to provide intuition. For the precise formulation see Theorem 3.1 below.

Theorem 1.1. *Fix any $Ra, \tilde{Ra} > 0$ and consider (1.1)–(1.3) and (1.4)–(1.5) with $N \geq N(Ra, \tilde{Ra})$ independently forced directions in the temperature equation. Then (1.4)–(1.5) possesses a unique ergodic invariant measure μ_∞. Let $\{\mu_{Pr}\}_{Pr \in (0,\infty)}$ be a sequence of statistically invariant states associated to (1.1)–(1.3).*[1] *Then μ_{Pr} converges to μ_∞ in a suitable metric. In particular, for any sufficiently regular observable ϕ on the phase space of T,*

$$\left| \int \phi(T) d\mu_{Pr} - \int \phi(T) d\mu_\infty \right| \leq C(Pr)^{-q}, \tag{1.6}$$

where $C = C(\phi, Ra, \tilde{Ra}), q = q(Ra, \tilde{Ra}) > 0$ are independent of Pr and q is independent of ϕ.

Our method of proof can be simplified to yield analogous results for the 2D version of the stochastically forced Boussinesq equation (1.1)–(1.3). Theorem 1.1 may also be seen as a complement to a series of recent works [Wan04a, Wan04b, Wan05, Wan07, Wan08a, Wan08b] which addresses large parameter limits for flows driven only through the boundary and in a deterministic framework. Here we show that the addition of stochastic terms to the Boussinesq system allows for stronger convergence results, but the proofs require a different framework.

The starting point of our analysis is to establish a strict contraction for the semigroup $\{P^0_t\}_{t\geq 0}$ evolving probability distributions associated to (1.4)–(1.5). We will show that there exists $\kappa \in (0,1)$ such that for $t_* > 0$ sufficiently large

$$\rho(\mu P^0_{t_*}, \tilde{\mu} P^0_{t_*}) \leq \kappa \rho(\mu, \tilde{\mu}) \tag{1.7}$$

for any measures $\mu, \tilde{\mu}$ on this phase space associated with T. Here ρ is a Kantorovich-Wasserstein metric defined by a carefully chosen distance on the phase space; see (2.9) below. In a series of recent works, [HM08, HM10, HMS11], a general scheme for proving a strict contraction, as in (1.7), was developed.

Once an estimate of the form (1.7) is established, the convergence of statistically invariant states, as in (1.6), can be translated to the convergence of solutions on finite time intervals. More specifically, since ρ is a metric defined by Lipschitz norms of observables, when $\nu_{Pr}(t)$, $T^{Pr}(t)$ and $\nu_\infty(t)$, $T^\infty(t)$ represent the probability laws and associated solutions of (1.1)–(1.2) and (1.4)–(1.5) respectively, one may show that

$$\rho(\nu_{Pr}(t), \nu_\infty(t)) \leq C\left(\mathbb{E}\exp(\eta\|T^{Pr}(t)\|^2) \cdot \mathbb{E}\exp(\eta\|T^\infty(t)\|^2) \cdot \mathbb{E}\|T^{Pr}(t) - T^\infty(t)\|^q\right)^{1/4} \tag{1.8}$$

for suitable choices of $C, \eta, q > 0$ independent of $Pr, t > 0$. By then combining (1.7) with (1.8) we find that

$$\rho(\Pi\mu_{Pr}, \mu_0) \leq C\left(\mathbb{E}\exp(\eta\|T^{Pr}_S(t_*)\|^2) \cdot \mathbb{E}\exp(\eta\|T^{\infty,Pr}_S(t_*)\|^2) \cdot \mathbb{E}\|T^{Pr}_S(t_*) - T^{\infty,Pr}_S(t_*)\|^q\right)^{1/4} \tag{1.9}$$

where T^{Pr}_S are stationary solutions of (1.1)–(1.2) and $T^{\infty,Pr}_S$ are solutions of the infinite Prandtl number system starting from initial conditions given by $T^{Pr}_S(0)$. Here Π is the projection onto the T portion of the phase space and the time t_* in (1.9) comes from (1.7). For specific details on the relationship between (1.7)–(1.9) see Section 2.2 below.

In summary through (1.9) the strict contraction property (1.7) reduces the question of convergence of stationary states to establishing some Pr-independent exponential moment bounds, and proving the convergence of solutions of the finite Prandtl system to those of the infinite Prandtl system at a *fixed positive finite time*. Moreover (1.9) provides a means for establishing the rate of convergence for $\rho(\Pi\mu_{Pr}, \mu_0) \to 0$

[1] In the precise statement of this result, Theorem 3.1, we will impose an additional technical assumption on μ_{Pr}. This condition is roughly analogous to a finite energy criteria for weak solutions of the 3D Navier-Stokes equations. In [FGHR] we have demonstrated the existence of such states μ_{Pr}.

as $Pr \to \infty$. It is also worth noting here that the approach leading to the bound (1.9) does not rely on uniqueness of statistically invariant states for finite values of $Pr > 0$.[2]

We therefore face two challenges. The first is to establish the strict contraction (1.7) for (1.4)–(1.5). Guided by the classical Doob-Khasminskii Theorem [Doo48, Km60, DPZ96] and as encompassed by the more recent developments in [HM08, HM10, HMS11] one would expect a contraction of the type (1.7) when the Markov semigroup is smoothing, suitable moment bounds hold, and there is some form of irreducibility in the dynamics. The second major challenge is to address to the finite-time convergence desired in (1.9). We stress here that this convergence to zero does not follow from continuous dependence on parameters. Indeed, the system (1.1)–(1.3) with large Pr is a singular perturbation of (1.4)–(1.5). For the infinite Prandtl system (1.4)–(1.5), $\mathbf{u}$ is determined from T at each time step by solving a Stokes problem and thus $\mathbf{u}$ does not satisfy an independent evolution equation. On the other hand one prescribes in (1.1)–(1.3) an initial condition for $\mathbf{u}$ which is absent in the limit $Pr \to \infty$.

Regarding the first challenge, the question of smoothing for the Markov semigroup can be translated to a control problem; see (B.10) below. In our setting, when the number of forced directions $N = N(Ra, \tilde{Ra})$ is sufficiently large, an appropriate control can be found through Foias-Prodi type considerations [FP67]. Since (1.4)–(1.5) may be seen as an advection diffusion system with $\mathbf{u}$ being two derivatives smoother than T, such a strategy largely repeats the approach used in previous works on the 2D stochastic Navier-Stokes equations [EMS01, HM06, HM08, KS12]. On the other hand establishing suitable moment bounds is more delicate due to the non-homogenous boundary conditions imposed in (1.3) and requires a careful use of the maximum principle along with exponential martingale estimates. These bounds have been carried out in the companion work [FGHR].

Thus the main obstacle to demonstrating (1.7) is to establish irreducibility, which is not so transparent from the path set out in previous works e.g. [EM01, HM06, CGHV13, FGHRT13]. This is because the system (1.4)–(1.5) with its stochastic terms removed can have highly non-trivial dynamics; [CD99, DC01, Wan04a, Par06, OS11]. We proceed to show that despite this observation every invariant measure of (1.4)–(1.5) contains zero as a common point of support. Indeed we establish with the use of another Foias-Prodi bound that a Girsanov shift of (1.4)–(1.5) has almost trivial dynamics with positive probability. We then employ moment estimates and stopping time arguments to translate this non-zero probability back to the original system (1.4)–(1.5) yielding the desired irreducibility

We turn to the second major challenge: the convergence of solutions at a fixed finite time as $\mathrm{Pr} \to \infty$, as desired in (1.9). Here we need to develop a suitable asymptotic analysis for (1.1)–(1.2). This is a non-trivial task since the small parameter lies in front of the time derivative terms in (1.1). Similarly to [Wan04a, Wan04b, Wan05, Wan07, Wan08a, Wan08b] we derive an 'intermediate system' which is close to the finite Prandtl system over bound time intervals and which converges to the infinite Prandtl system after an $O(\varepsilon)$ time transient. However, in our case time regularity properties crucially used in previous works is missing due to the stochastic forcing. As a substitute we derive a stochastic evolution equation for the velocity component and use martingale properties of associated Itō integrals. Our analysis then takes advantage of some previously unobserved cancellations in the error terms and delicate stopping time arguments.

The manuscript is organized as follows. We begin with a 'warm-up problem' by carrying out an asymptotic analysis for a system of stochastic ordinary differential equations which model some of the difficulties which are encountered in the analysis of the full system. This preliminary step gives us the occasion to introduce the formalities of the Kantorovich-Wasserstein metric in Section 2.2. In Section 3 we introduce the rigorous mathematical setting of Boussinesq equations, (1.1)–(1.3), which will serve as a foundation for the rest of the analysis. Section 4 is devoted to establishing the contraction (1.7) for the infinite Prandtl system (1.4)–(1.5). In Section 5 we carry out the finite time convergence analysis which follows ideas from the formal asymptotic procedure derived for the toy model. An appendix recalls various elements essentially contained in previous works that we have used in our analysis. Section A is devoted to recalling various moment estimates from [FGHR] for a class of drift-diffusion equations and for (1.4)–(1.5). In Section B we detail gradient estimates on the Markov semigroup given by (1.4)–(1.5) which are carried out in a similar fashion to e.g. [HM06].

[2] In fact we do not even require (1.1)–(1.3) to define a Markovian semigroup, since we only use the existence of statistically invariant states as in [FG95] which are established in [FGHR] via Galerkin truncation in the velocity equation in (1.1)–(1.2).

2 A Simple Stochastic Model for the Large Prandtl Number Asymptotic

To understand some of the main issues involved in the large Prandtl number limit we begin by considering the simplified setting of a class finite dimensional toy models. Both the usage of contraction properties for the Markov semigroup defined by the limit system and the formal asymptotics employed to derive finite time convergence results will guide our analysis of the full system (1.1)–(1.3) in Sections 3–5 below. The reader anxious to see the analysis of the Boussinesq equations can skip immediately to Section 3.

2.1 The Equations and General Problem Set-up

Fix $M_1, M_2, N > 0$ and let $W = (W_1, \ldots, W_N)$ be a standard N-dimensional Brownian motion, relative to some ambient filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For each $\varepsilon > 0$, consider the following coupled systems of stochastic ordinary differential equations

$$\varepsilon\left(\frac{du^\varepsilon}{dt} + b_1(u^\varepsilon, u^\varepsilon)\right) + a_1(u^\varepsilon) = Ra\cdot e(\theta^\varepsilon), \quad u^\varepsilon(0) = u_0, \tag{2.1}$$

$$d\theta^\varepsilon + (b_2(u^\varepsilon, \theta^\varepsilon) + a_2(\theta^\varepsilon))\,dt = \sum_{k=1}^N \sigma_k dW^k, \quad \theta^\varepsilon(0) = \theta_0, \tag{2.2}$$

evolving on $\mathbb{R}^{M_1+M_2}$. Here $u^\varepsilon : \Omega \times [0,\infty) \to \mathbb{R}^{M_1}$, $\theta^\varepsilon : \Omega \times [0,\infty) \to \mathbb{R}^{M_2}$ are adapted stochastic processes and σ_k are fixed elements in $\mathbb{R}^{M_2}$. We denote $|\sigma|^2 = \sum_{k=1}^N |\sigma_k|^2$ that is $|\cdot|$ the Hilbert-Schmidt norm on $N \times M_1$ matrices. The operators $a_i : \mathbb{R}^{M_i} \to \mathbb{R}^{M_i}$, $i = 1, 2$ and $e : \mathbb{R}^{M_2} \to \mathbb{R}^{M_1}$ are linear and we suppose that a_1, a_2 are strictly positive definite with

$$\langle a_1(u), u\rangle \geq |u|^2, \quad \langle a_2(\theta), \theta\rangle \geq |\theta|^2, \tag{2.3}$$

for all $u \in \mathbb{R}^{M_1}$, $\theta \in \mathbb{R}^{M_2}$. We assume that $b_1 : \mathbb{R}^{2M_1} \to \mathbb{R}^{M_1}$, $b_2 : \mathbb{R}^{M_1+M_2} \to \mathbb{R}^{M_2}$ are bilinear and satisfy the cancellation properties

$$\langle b_2(\tilde{u}, u), u\rangle = 0, \quad \langle b_2(u, \theta), \theta\rangle = 0, \tag{2.4}$$

for any $\tilde{u}, u \in \mathbb{R}^{M_1}$, $\theta \in \mathbb{R}^{M_2}$. We denote the operator norm of a multilinear operator Q by $|Q|$ and note that although (2.3) imply $|a_i^{-1}| \leq 1$, we indicate below the dependence of constants on $|a^{-1}|$. For any fixed value of the parameter $Ra > 0$ we our goal is to study the behavior of statistically invariant states $U_S^\varepsilon = (u_S^\varepsilon, \theta_S^\varepsilon)$ of (2.1)–(2.2) in the limit as $\varepsilon \to 0$ and its relationship to solutions of

$$a_1(u^0) = Ra\cdot e(\theta^0), \tag{2.5}$$

$$d\theta^0 + \left(b_2(u^0, \theta^0) + a_2(\theta^0)\right)dt = \sum_{k=1}^N \sigma_k dW^k, \quad \theta^0(0) = \theta_0, \tag{2.6}$$

the formal limit system.

Under the given assumptions it is not hard to see that, for any fixed values of $\varepsilon \geq 0, Ra > 0$ the problems (2.1)–(2.2) and (2.5)–(2.6), complemented with the initial conditions $U_0 = (u_0, \theta_0) \in \mathbb{R}^{M_1+M_2}$ and $\theta_0 \in \mathbb{R}^{M_2}$ respectively, posses unique, probabilistically strong solutions depending continuously on the initial conditions; see e.g. [AFS08]. Having fixed $Ra > 0$, we denote $\{P_t^\varepsilon\}_{t\geq 0}$, $\{P_t^0\}_{t\geq 0}$ as the associated (Feller) Markov semigroups; see (3.11), (3.12) below. Denote $\Pi_\theta : \mathbb{R}^{M_1+M_2} \to \mathbb{R}^{M_2}$ as the projection onto the θ component of (u, θ), that is, $\Pi_\theta(u, \theta) = \theta$.

The existence of invariant measures for $\{P_t^\varepsilon\}_{t\geq 0}$ and $\{P_t^0\}_{t\geq 0}$ easily follows from a standard Krylov-Bogoliubov procedure [KB37] and energy type estimates for (2.1)–(2.2) and (2.5)–(2.6). These, possibly non-unique, invariant measures are denoted μ_ε and μ_0 respectively. If $U_{0,S}^\varepsilon = (u_{0,S}^\varepsilon, \theta_{0,S}^\varepsilon)$ is an $\mathcal{F}_0$-measurable random variable distributed as μ_ε, then we denote the correspond solution of (2.1)–(2.2) as $(u_S^\varepsilon, \theta_S^\varepsilon)$ and call this a *statistically steady state* of the system. Denote by $\Pi_\theta\mu_\varepsilon$ the push forward of μ_ε under Π_θ.[3]

[3] Recall that the push forward is defined as

$$\Pi_\theta\mu_\varepsilon(A) = \mu_\varepsilon(\Pi_\theta^{-1}A) \quad \text{for any } A \in \mathcal{B}(\mathbb{R}^{M_2}). \tag{2.7}$$

2.2 Contraction in the Wasserstein Metric and Its Consequences

With our toy model now in hand we next recall the general setting of the Kantorovich-Wasserstein distance in which we will establish the convergence of statistically invariant states as $\varepsilon \to 0$. We then give the precise details of the strategy outlined in (1.7)–(1.9) in the introduction which, under the additional assumptions on (2.5)–(2.6), reduces $\Pi_\theta \mu^\varepsilon \to \mu^0$ to the question of the closeness of solutions of (2.1)–(2.2) to (2.5)–(2.6) on finite time intervals.

We make use of the Kantorovich-Wasserstein distance as follows: let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and for $\eta > 0$ we consider a weighted metric on $\mathbb{X}$ as

$$\rho_\eta(\theta, \tilde{\theta}) = \inf_{\substack{\gamma \in C^1([0,1];\mathbb{X}) \\ \gamma(0)=\theta, \gamma(1)=\tilde{\theta}}} \int_0^1 \exp(\eta \|\gamma\|^2) \|\gamma'(s)\| ds, \qquad \text{for any } \theta, \tilde{\theta} \in \mathbb{X}.$$

Notice that

$$\|\theta - \tilde{\theta}\| \leq \rho_\eta(\theta, \tilde{\theta}) \leq \exp(2\eta(\|\theta\|^2 + \|\tilde{\theta}\|^2)) \|\theta - \tilde{\theta}\| . \tag{2.8}$$

Let $\mathrm{Pr}_1(\mathbb{X})$ be the set of Borel probability measures μ on $\mathbb{X}$ with $\int \rho_\eta(0, U) d\mu(U) < \infty$. On $\mathrm{Pr}_1(\mathbb{X})$ we define the Kantorovich-Wasserstein metric, relative to ρ_η, equivalently as

$$\rho_\eta(\mu, \tilde{\mu}) := \sup_{\|\phi\|_{Lip,\eta} \leq 1} \left| \int_{\mathbb{X}} \phi(\theta) d\mu(\theta) - \int_{\mathbb{X}} \phi(\theta) d\tilde{\mu}(\theta) \right| := \inf_{\Gamma \in \mathcal{C}(\mu, \tilde{\mu})} \int_{\mathbb{X} \times \mathbb{X}} \rho_\eta(\theta, \tilde{\theta}) d\Gamma(\theta, \tilde{\theta}) \tag{2.9}$$

where $\|\phi\|_{Lip,\eta} := \sup_{\theta \neq \tilde{\theta}} \frac{|\phi(\theta) - \phi(\tilde{\theta})|}{\rho_\eta(\theta, \tilde{\theta})}$ for $\phi : \mathbb{X} \to \mathbb{R}$. The set $\mathcal{C}(\mu, \tilde{\mu})$ is the collection of couplings that is Borel probability measures Γ in $Pr(\mathbb{X} \times \mathbb{X})$ with $\mu, \tilde{\mu}$ as its marginals. This mean that the last term in (2.9) is equivalent to

$$\rho_\eta(\mu, \tilde{\mu}) = \inf \mathbb{E} \rho_\eta(X, Y) , \tag{2.10}$$

where the infimum is taken over all $\mathbb{X}$-valued random variables X, Y distributed as $\mu, \tilde{\mu}$ respectively. See e.g. [Vil08] or [Dud02] for a more detailed treatment and further background on these metrics.

For any $\eta > 0$ we define the set of observables

$$V_\eta(\mathbb{X}) := \left\{ \phi \in C^1(\mathbb{X}) : \|\phi\|_\eta := \sup_{\theta \in \mathbb{X}} \frac{\|\phi(\theta)\| + \|\nabla \phi(\theta)\|}{\exp(\eta \|\theta\|^2)} < \infty \right\} . \tag{2.11}$$

It is shown in [HM08, Proposition 4.1] that for any $\phi \in C^1(\mathbb{X})$,

$$C \|\phi\|_{2\eta} \leq \|\phi\|_{Lip,\eta} \leq C \|\phi\|_\eta .$$

These bounds are useful for translating the convergence of measures in ρ_η to convergence of observables; see (2.14) below.

For simplicity we will now take as an assumption that the semigroup $\{P_t^0\}_{t \geq 0}$ associated to (2.5)–(2.6) is contractive in ρ_η for some $\eta > 0$ depending only on fixed constants in the problem, $\mathrm{Ra}, |\sigma|$. Specifically we assume that there exist $t_* > 0$ and $\kappa \in (0, 1)$ such that

$$\rho_\eta(\mu P_{t_*}^0, \tilde{\mu} P_{t_*}^0) \leq \kappa \rho_\eta(\mu, \tilde{\mu}), \tag{2.12}$$

for any $\mu, \tilde{\mu} \in \mathrm{Pr}_1(\mathbb{R}^{M_2})$.

Remark 2.1. *In view of results from [HM08, HM10, HMS11], the condition* (2.12) *can be verified in many interesting situations in finite and infinite dimensions. In particular one may establish this condition* (2.12) *for our toy* (2.5)–(2.6) *if the Hörmander bracket condition is satisfied. On the other hand we verify that* (2.12) *for the Boussinesq system with* $\mathrm{Pr} = \infty$, (1.4)–(1.5) *in Section 4 below.*

Under (2.12) the invariant measure μ_0 for (2.5)–(2.6) is clearly unique and hence ergodic. Observe that for any invariant measure μ_ε of $\{P_t^\varepsilon\}_{t\geq 0}$, $\varepsilon > 0$, we have that

$$\begin{aligned}\rho_\eta(\Pi\mu_\varepsilon, \mu_0) &= \rho_\eta(\Pi(\mu_\varepsilon P^\varepsilon_{t_*}), \mu_0 P^0_{t_*}) \leq \rho_\eta(\Pi(\mu_\varepsilon P^\varepsilon_{t_*}), (\Pi\mu_\varepsilon)P^0_{t_*}) + \rho_\eta((\Pi\mu_\varepsilon)P^0_{t_*}, \mu_0 P^0_{t_*})\\ &\leq \rho_\eta(\Pi(\mu_\varepsilon P^\varepsilon_{t_*}), (\Pi\mu_\varepsilon)P^0_{t_*}) + \kappa\rho_\eta(\Pi\mu_\varepsilon, \mu_0).\end{aligned}$$

Let $\theta^\varepsilon_S(t_*) = \theta^\varepsilon(t_*, (u^\varepsilon_{0,S}, \theta^\varepsilon_{0,S}))$ and $\theta^{0,\varepsilon}_S(t_*) = \theta^0(t_*, \theta^\varepsilon_{0,S})$ denote respectively solutions of (2.1)–(2.2) and (2.5)–(2.6) at time t_* with initial conditions distributed as μ_ε and $\Pi_\theta\mu_\varepsilon$. Clearly the joint distribution of $\theta^\varepsilon_S(t_*), \theta^{0,\varepsilon}_S(t_*)$ has marginals $\Pi(\mu_\varepsilon P^\varepsilon_{t_*}), (\Pi\mu_\varepsilon)P^0_{t_*}$.

By (2.10) and (2.8) we have that for any $\eta, q > 0$

$$\begin{aligned}\rho_\eta(\Pi(\mu_\varepsilon P^\varepsilon_{t_*}), (\Pi\mu_\varepsilon)P^0_{t_*}) &\leq \mathbb{E}\rho_\eta(\theta^\varepsilon_S(t_*), \theta^{0,\varepsilon}_S(t_*)) \leq \mathbb{E}\left(\exp(2\eta(|\theta^\varepsilon_S(t_*)|^2 + |\theta^{0,\varepsilon}_S(t_*)|^2))|\theta^\varepsilon_S(t_*) - \theta^{0,\varepsilon}_S(t_*)|\right)\\ &\leq C\mathbb{E}\left(\exp(3\eta(|\theta^\varepsilon_S(t_*)|^2 + |\theta^{0,\varepsilon}_S(t_*)|^2))|\theta^\varepsilon_S(t_*) - \theta^{0,\varepsilon}_S(t_*)|^{q/2}\right)\\ &\leq C\left(\mathbb{E}\exp(12\eta|\theta^\varepsilon_S(t_*)|^2)\cdot\mathbb{E}\exp(12\eta|\theta^{0,\varepsilon}_S(t_*)|^2))\right)^{1/4}\left(\mathbb{E}|\theta^\varepsilon_S(t_*) - \theta^{0,\varepsilon}_S(t_*)|^q\right)^{1/2},\end{aligned}$$

where $C > 0$ depends only on q and η. Combining these two observations and rearranging we infer for any $\eta, q > 0$

$$\rho_\eta(\Pi\mu_\varepsilon, \mu_0) \leq C\left(\mathbb{E}\exp(12\eta|\theta^\varepsilon_S(t_*)|^2)\cdot\mathbb{E}\exp(12\eta|\theta^{0,\varepsilon}_S(t_*)|^2))\right)^{1/4}\left(\mathbb{E}|\theta^\varepsilon_S(t_*) - \theta^{0,\varepsilon}_S(t_*)|^q\right)^{1/2}, \tag{2.13}$$

where C depends only on q, η and κ. Thus to prove (2.12) it suffices to establishing uniform in $\varepsilon > 0$ moment bounds and mean convergence properties on finite time intervals.

In summary working from (2.13) we now prove.

Proposition 2.1. *Fix any $Ra > 0$ and consider the systems (2.1)–(2.2) and (2.5)–(2.6) satisfying (2.3), (2.4). Assume furthermore that (2.5)–(2.6) satisfies the contraction condition (2.12) for some fixed $t_* > 0$, $\kappa \in (0,1)$, and $\eta \leq \frac{1}{6|\sigma|^2}$. Let μ_0 be the unique invariant measure associated to (2.5)–(2.6). Then, for any collection $\{\mu_\varepsilon\}_{\varepsilon>0}$ of invariant measures corresponding to (2.1)–(2.2) one has*

$$\rho_\eta(\Pi_\theta\mu_\varepsilon, \mu_0) \leq C\varepsilon^q, \tag{2.14}$$

where $C = C(|\sigma|, Ra, t_*, \kappa)$ and $q = q(|\sigma|, Ra, t_*, \kappa) > 0$ are both independent of the (possibly non-unique) choice of elements $\{\mu_\varepsilon\}_{\varepsilon>0}$. This implies that for any collection of statically stationary solutions $(u^\varepsilon_S, \theta^\varepsilon_S)$ of (2.1)–(2.2) and the unique stationary solution θ^0_S of (2.5)–(2.6),

$$|\mathbb{E}(\phi(\theta^\varepsilon_S) - \phi(\theta^0_S))| \leq C\|\phi\|_{2\eta}\varepsilon^q \tag{2.15}$$

for any observable $\phi \in V_{2\eta}(\mathbb{R}^{M_2})$ and $C = C(Ra, |\sigma|, t_*, \kappa)$ independent of ϕ and $\varepsilon > 0$.

2.3 Uniform Moment Bounds

The first step in the proof of Proposition 2.1 is to establish uniform moment bounds for solutions of (2.2) independently of $\varepsilon \geq 0$. Such bounds which essentially rely on energy estimates and exponential martingale bounds should be expected in view of (2.3)–(2.4). Specifically we prove

Lemma 2.1. *Fix any $\mathcal{F}_t$-adapted process $v \in C([0,\infty); \mathbb{R}^{M_1})$ and $\xi_0 \in \mathcal{F}_0$ and consider any adapted $\xi \in C([0,\infty); \mathbb{R}^{M_1})$ solving*

$$d\xi + (b_2(v,\xi) + a_2(\xi))dt = \sum_{k=1}^N \sigma_k dW^k, \quad \xi(0) = \xi_0, \tag{2.16}$$

where a_2 and b_2 are subject to (2.3), (2.4). Then:

(i) For any $\eta \leq \frac{1}{2|\sigma|^2}$, any $\varepsilon, K > 0$ and any solution ξ of (2.16) one has

$$\mathbb{P}\left(\sup_{t\geq 0}\left(|\xi(t)|^2 + \int_0^t |\xi(s)|^2 ds - |\sigma|^2 t - |\xi_0|^2\right) > K \middle| \mathcal{F}_0\right) \leq e^{-\eta K}. \tag{2.17}$$

Consequently, for any $t \geq 0$,

$$\mathbb{E}\exp\left(\eta\left(\sup_{s\in[0,t]}|\xi(s)|^2 + \int_0^t |\xi(s)|^2 ds\right)\right) \leq C\exp(\eta t|\sigma|^2)\mathbb{E}\exp(\eta|\xi_0|^2), \tag{2.18}$$

where $C = C(\eta)$ is independent of v.

(ii) For any $t \geq 0$ and $\eta \leq \frac{1}{2|\sigma|^2}$

$$\mathbb{P}\left(|\xi(t)|^2 - |\sigma|^2 - e^{-t}|\xi_0|^2 > K \middle| \mathcal{F}_0\right) \leq e^{-\eta K}, \tag{2.19}$$

and hence for every $t \geq 0$

$$\mathbb{E}\exp\left(\eta|\xi(t)|^2\right) \leq C\exp(\eta|\sigma|^2)\mathbb{E}\exp(\eta e^{-t}|\xi_0|^2), \tag{2.20}$$

where $C = C(\eta)$ is independent of t and v.

(iii) For any stationary solution ξ_S of (2.16) and any $\eta \leq \frac{1}{2|\sigma|^2}$

$$\mathbb{E}\exp(\eta|\xi_S|^2) \leq C < \infty, \tag{2.21}$$

where $C = C(\eta)$ is independent of v.

(iv) Finally if $(\theta_S^\varepsilon, u_S^\varepsilon)$ is a stationary solution of (2.1)–(2.2) then for any $\tilde{\eta} < \frac{1}{4Ra^2|e|^2|\sigma|^2}$

$$\mathbb{E}\exp(\tilde{\eta}|u_S^\varepsilon|^2) \leq C < \infty. \tag{2.22}$$

where $C = C(\tilde{\eta})$ is independent of $\varepsilon > 0$.

Proof. By applying Itō's lemma to (2.16) and using (2.3), (2.4) we obtain $d|\xi|^2 + 2|\xi|^2 dt \leq |\sigma|^2 dt + 2\langle\xi, \sigma\rangle dW$. Hence with the use of exponential martingale bounds and standard estimates we establish (2.17)–(2.21) as in e.g. [HM06, HM08, KS12, GH14]; see also Appendix A below.

It remains to prove (2.22). To estimate $(u_S^\varepsilon, \theta_S^\varepsilon)$ we multiply (2.1) with u_S^ε. Due to (2.3) and (2.4) we have $\varepsilon\frac{d}{dt}|u_S^\varepsilon|^2 + |u_S^\varepsilon|^2 \leq Ra^2|e|^2|\theta_S^\varepsilon|^2$ implying that

$$|u_S^\varepsilon(t)|^2 \leq \exp(-\varepsilon^{-1}t)|u_{S,0}^\varepsilon|^2 + \frac{Ra^2|e|^2}{\varepsilon}\int_0^t \exp(\varepsilon^{-1}(s-t))|\theta_S^\varepsilon|^2 ds \leq \exp(-\varepsilon^{-1}t)|u_{S,0}^\varepsilon|^2 + Ra^2|e|^2 \sup_{s\in[0,t]}|\theta_S^\varepsilon(s)|^2.$$

On the other hand θ_S^ε satisfies a system of the form (2.16). Thus by (2.18) and (2.21) for any $\tilde{\eta} \leq \frac{1}{4Ra^2|e|^2|\sigma|^2}$ and any $M > 0$

$$\mathbb{E}\exp(\tilde{\eta}|u_S^\varepsilon(t)|^2 \wedge M) \leq C\exp(t)\left(\mathbb{E}\exp\left(2\tilde{\eta}(\exp(-\varepsilon^{-1}t)|u_{S,0}^\varepsilon|^2 \wedge M\right)\right)^{1/2}, \tag{2.23}$$

where $C = C(\tilde{\eta})$ is independent of $\varepsilon > 0$ and $M > 0$. Since the distribution of $u_S^\varepsilon(t)$ is independent of t, one obtains for sufficiently large t such that $\exp(-\varepsilon^{-1}t) \leq \frac{1}{2}$ that $\mathbb{E}\exp(\tilde{\eta}|u_S^\varepsilon|^2 \wedge M) \leq C$ for a constant C independent of $M > 0, \varepsilon > 0$. The bound (2.22) now follows from the monotone convergence theorem, completing the proof. □

2.4 Convergence on Finite Time Intervals

With the ε-independent bounds (2.18)–(2.22) in hand, we next establish convergence on finite time intervals to obtain the desired decay in (2.13). Throughout this section for any $\varepsilon > 0$ we denote u_S^ε, θ_S^ε and $u_S^{0,\varepsilon}$, $\theta_S^{0,\varepsilon}$ respectively as the solutions of (2.1)–(2.2) and (2.5)–(2.6) supplemented with initial conditions $(u_{0,S}^\varepsilon, \theta_{0,S}^\varepsilon)$ and $\theta_{0,S}^\varepsilon$ distributed as a μ_ε and $\Pi\mu_\varepsilon$ where μ_ε are the invariant measures of (2.1)–(2.2).

Remark 2.2. *For clarity of presentation we have restricted our analysis to the initial conditions for (2.1)–(2.2) and (2.5)–(2.6) suitable for establishing (2.13). However, straightforward modifications of forthcoming methods show that for any $t > 0$ and any collection $\{(u_0^\varepsilon, \theta_0^\varepsilon)\}_{\varepsilon>0} \subset \mathbb{R}^{M_1+M_2}$ with $\{u_0^\varepsilon\}_{\varepsilon>0}$ bounded and $\theta_0^\varepsilon \to \theta_0$ we have*

$$\mathbb{E}|\theta^\varepsilon(t,(u_0^\varepsilon,\theta_0^\varepsilon)) - \theta^0(t,\theta_0)|^q \leq C(\varepsilon^q + |\theta_0^\varepsilon - \theta_0|^q),$$

from some suitable $C = C(Ra, t, |\sigma|, \sup_{\varepsilon>0}|u_0^\varepsilon|) > 0$ and $q = q(Ra, t, |\sigma|) > 0$, both independent of $\varepsilon > 0$.

2.4.1 Formal Asymptotics: The Corrector

Our first step is to derive an intermediate system of equations that is close to both the systems (2.1)–(2.2) and (2.5)–(2.6) but only after an $O(\varepsilon)$ time transient for the limit system.

To this end we observe that (2.1)–(2.2) is formally approximated by

$$\frac{d}{dt}\tilde{u}^\varepsilon + \varepsilon^{-1}a_1(\tilde{u}^\varepsilon) = \varepsilon^{-1}Ra \cdot e(\tilde{\theta}^\varepsilon), \quad d\tilde{\theta}^\varepsilon + \left(b_2(\tilde{u}^\varepsilon, \tilde{\theta}^\varepsilon) + a_2(\tilde{\theta}^\varepsilon)\right)dt = \sum_{k=1}^N \sigma_k dW^k.$$

Thus, under the (as yet unjustified) supposition that there is a clear separation of time scales between the motion of $\tilde{u}^\varepsilon$ and $\tilde{\theta}^\varepsilon$ we therefore propose the effective dynamics for (2.1)–(2.2) as

$$\tilde{u}^\varepsilon(t) = \exp(-\varepsilon^{-1}a_1 t)u_{0,S}^\varepsilon + a_1^{-1}(Ra \cdot e(\tilde{\theta}^\varepsilon)) - \exp(-\varepsilon^{-1}a_1 t)a_1^{-1}(Ra \cdot e(\theta_{0,S}^\varepsilon)), \tag{2.24}$$

$$d\tilde{\theta}^\varepsilon + \left(b_2(\tilde{u}^\varepsilon, \tilde{\theta}^\varepsilon) + a_2(\tilde{\theta}^\varepsilon)\right)dt = \sum_{k=1}^N \sigma_k dW^k, \quad \tilde{\theta}^\varepsilon(0) = \theta_{0,S}^\varepsilon. \tag{2.25}$$

In simple terms we are making use of the fact that solutions of systems of the form $\frac{d}{dt}u + \varepsilon^{-1}Au = \varepsilon^{-1}g$, $u(0) = u_0$, where A is a matrix and g is a constant, admits the Duhamel representation $u(t) = \exp(-\varepsilon^{-1}At)u_0 + (A^{-1}g - \exp(-\varepsilon^{-1}At)A^{-1}g)$.

2.4.2 Convergence of the Corrector to the Limit System

We first estimate the difference between corrector (2.24)–(2.25) and the formal limit system (2.5)–(2.6). We will prove the following

Lemma 2.2. *For every $\varepsilon > 0$ let $\tilde{\theta}^\varepsilon$ and $\theta_S^{0,\varepsilon}$ correspond to solutions of (2.24)–(2.25) and of (2.5)–(2.6) respectively both starting from initial conditions $\theta_{0,S}^\varepsilon$ distributed as any $\Pi_\theta\mu_\varepsilon$, where μ_ε is an invariant measures of the system (2.1)–(2.2). Then, for any $t > 0$, and any $\varepsilon > 0$,*

$$\mathbb{E}|\tilde{\theta}^\varepsilon(t) - \theta_S^{0,\varepsilon}(t)|^2 \leq C\varepsilon\,, \tag{2.26}$$

where $C = C(Ra, t, |\sigma|)$ is independent of $\varepsilon > 0$ and (the possibly non-unique) μ_ε.

Proof. Set $v^\varepsilon := \tilde{u}^\varepsilon - u_S^{0,\varepsilon}$, $\psi^\varepsilon := \tilde{\theta}^\varepsilon - \theta_S^{0,\varepsilon}$. Comparing (2.24)–(2.25) to (2.5)–(2.6) we see that v^ε and ψ^ε satisfy

$$v^\varepsilon(t) = a_1^{-1}(Ra \cdot e(\psi^\varepsilon)) + \exp(-\varepsilon^{-1}a_1 t)u_{0,S}^\varepsilon - \exp(-\varepsilon^{-1}a_1 t)a_1^{-1}(Ra \cdot e(\theta_{0,S}^\varepsilon)), \tag{2.27}$$

$$\frac{d\psi^\varepsilon}{dt} + a_2(\psi^\varepsilon) = b_2(u_S^{0,\varepsilon}, \theta_S^{0,\varepsilon}) - b_2(\tilde{u}^\varepsilon, \tilde{\theta}^\varepsilon) = -b_2(v^\varepsilon, \theta_S^{0,\varepsilon}) - b_2(\tilde{u}^\varepsilon, \psi^\varepsilon)\,, \qquad \psi^\varepsilon(0) = 0\,. \tag{2.28}$$

First, from (2.27) using (2.3) we obtain

$$|v^\varepsilon(t)|^2 \le 3Ra^2|e|^2|a_1^{-1}|^2|\psi^\varepsilon(t)|^2 + 3\exp(-4\varepsilon^{-1}t)(|u^\varepsilon_{0,S}|^2 + Ra^2|e|^2|a_1^{-1}|^2|\theta^\varepsilon_{0,S}|^2). \tag{2.29}$$

On the other hand from (2.28) with (2.3), (2.4), we obtain $\frac{d}{dt}|\psi^\varepsilon|^2 + 2|\psi^\varepsilon|^2 \le -2\langle b_2(v^\varepsilon, \theta_S^{0,\varepsilon}), \psi^\varepsilon\rangle$ and hence with (2.29) we have, for any $\eta > 0$,

$$\frac{d}{dt}|\psi^\varepsilon|^2 \le C|v^\varepsilon|^2 + \frac{\eta}{2}|\theta_S^{0,\varepsilon}|^2|\psi^\varepsilon|^2 \le \left(CRa^2 + \frac{\eta}{2}|\theta_S^{0,\varepsilon}|^2\right)|\psi^\varepsilon|^2 + C\exp(-4\varepsilon^{-1}t)(|u^\varepsilon_{0,S}|^2 + Ra^2|\theta^\varepsilon_{0,S}|^2),$$

where $C = C(\eta, |a_1^{-1}|, |e|, |b_2|)$ is independent of $\varepsilon > 0$. Using Grönwall's inequality, $\int_0^t \exp(\varepsilon^{-1}(s-t))\, ds \le \varepsilon$ and $\psi^\varepsilon(0) = 0$ we have

$$|\psi^\varepsilon(t)|^2 \le \varepsilon C \exp\left(CRa^2 t + \frac{\eta}{2}\int_0^t |\theta_S^{0,\varepsilon}|^2 ds\right)(|u^\varepsilon_{0,S}|^2 + Ra^2|\theta^\varepsilon_{0,S}|^2),$$

for a constant $C = C(\eta, |a_1^{-1}|, |e|, |b_2|)$ independent of $\epsilon > 0$. We now take expected values and apply Hölder's inequality. By choosing $\eta > 0$ sufficiently small in order to make use of (2.18) for $\theta_S^{0,\varepsilon}$ and then applying (2.21) and (2.22) to bound terms involving $\theta^\varepsilon_{0,S}$ and $u^\varepsilon_{0,S}$ we obtain

$$\begin{aligned}\mathbb{E}|\psi^\varepsilon(t)|^2 &\le \varepsilon C(1+Ra^2)\exp\left(tC(Ra^2 + |\sigma|^2)\right)\cdot\left(\mathbb{E}\exp(\eta|\theta^\varepsilon_{0,S}|^2)\cdot\mathbb{E}(|u^\varepsilon_{0,S}|^4 + |\theta^\varepsilon_{0,S}|^4)\right)^{1/2}\\ &\le \varepsilon C(1+Ra^2)\exp\left(tC(Ra^2+|\sigma|^2)\right)\end{aligned}$$

for a constant $C = C(|a_1^{-1}|, |e|, |b_2|)$ which independent of $\varepsilon > 0$. The proof is now complete. □

2.4.3 Estimates between the Corrector and Small Parameter Solutions

We next compare solutions of the small parameter system (2.1)–(2.2) to the corrector (2.24)–(2.25) and prove

Lemma 2.3. *For any $\varepsilon > 0$ let θ^ε_S and $\tilde\theta^\varepsilon$ be solutions of (2.1)–(2.2) and of (2.24)–(2.25) respectively both with same initial conditions distributed as $\Pi_\theta\mu_\varepsilon$, where μ_ε is any invariant measure μ_ε of (2.1)–(2.2). Then, for every $t > 0$, there exist $C = C(Ra, |\sigma|, t) > 0$, $\delta = \delta(Ra, |\sigma|, t) > 0$, both independent of $\varepsilon > 0$ and the particular choice of μ_ε such that*

$$\mathbb{E}\left(\sup_{s\in[0,t]}|\theta^\varepsilon_S(s) - \tilde\theta^\varepsilon(s)|^{2\delta}\right) \le C\varepsilon^\delta, \tag{2.30}$$

for every $\varepsilon > 0$.

Proof. Set $\tilde v^\varepsilon = u^\varepsilon_S - \tilde u^\varepsilon$, $\tilde\psi^\varepsilon = \theta^\varepsilon_S - \tilde\theta^\varepsilon$. Similarly to (2.28), $\tilde\psi^\varepsilon$ satisfies

$$\frac{d\tilde\psi^\varepsilon}{dt} + a_2(\tilde\psi^\varepsilon) = -b_2(\tilde v^\varepsilon, \theta^\varepsilon_S) - b_2(\tilde u^\varepsilon, \tilde\psi^\varepsilon), \qquad \tilde\psi^\varepsilon(0) = 0\,,$$

and hence with (2.3)–(2.4)

$$\frac{d}{dt}|\tilde\psi^\varepsilon|^2 + 2|\tilde\psi^\varepsilon|^2 \le 2|\langle b_2(\tilde v^\varepsilon, \theta^\varepsilon_S), \tilde\psi^\varepsilon\rangle|. \tag{2.31}$$

Given that $\tilde\psi^\varepsilon(0) = 0$, and $|\langle b_2(\tilde v^\varepsilon, \theta^\varepsilon_S), \tilde\psi^\varepsilon\rangle| \le |\tilde\psi^\varepsilon|^2 + C|\theta^\varepsilon_S|^2|\tilde v^\varepsilon|^2$ we have that

$$\sup_{s\in[0,t]}|\tilde\psi^\varepsilon(s\wedge\tau)|^2 \le C\sup_{s\in[0,t]}|\theta^\varepsilon_S(s\wedge\tau)|^2\int_0^{t\wedge\tau}|\tilde v^\varepsilon(t')|^2 dt' \tag{2.32}$$

for any $t \ge 0$ and any stopping time τ, where $C = C(|b_2|)$.

In order to obtain bounds for $\tilde v^\varepsilon$ we rewrite (2.24) as

$$a_1(\tilde u^\varepsilon(t)) = Ra\cdot e(\tilde\theta^\varepsilon) + \exp(-\varepsilon^{-1}a_1 t)(a_1(u^\varepsilon_{0,S}) - Ra\cdot e(\theta^\varepsilon_{0,S})).$$

As such, comparing with (2.1),

$$\varepsilon\left(\frac{du^\varepsilon_S}{dt}+b_1(u^\varepsilon_S,u^\varepsilon_S)\right)+a_1(\tilde{v}^\varepsilon)=Ra\cdot e(\tilde{\psi}^\varepsilon)-\exp(-\varepsilon^{-1}a_1t)(a_1(u^\varepsilon_{0,S})-Ra\cdot e(\theta^\varepsilon_{0,S})). \tag{2.33}$$

Our next step is to derive a stochastic equation for $\tilde{u}^\varepsilon$. Returning to (2.24)–(2.25) we obtain

$$\begin{aligned}d\tilde{u}^\varepsilon &= a_1^{-1}(Ra\cdot e(d\tilde{\theta}^\varepsilon))-\varepsilon^{-1}\exp(-\varepsilon^{-1}a_1t)\big[a_1(u^\varepsilon_{0,S})-Ra\cdot e(\theta^\varepsilon_{0,S})\big]dt\\ &=-a_1^{-1}(Ra\cdot e(b_2(\tilde{u}^\varepsilon,\tilde{\theta}^\varepsilon)+a_2(\tilde{\theta}^\varepsilon)))dt+\sum_{k=1}^N a_1^{-1}(Ra\cdot e(\sigma_k))dW^k\\ &\quad-\varepsilon^{-1}\exp(-\varepsilon^{-1}a_1t)\left(a_1(u^\varepsilon_{0,S})-Ra\cdot e(\theta^\varepsilon_{0,S})\right)dt.\end{aligned} \tag{2.34}$$

Combining (2.33) and (2.34) we infer

$$\begin{aligned}d\tilde{v}^\varepsilon+\frac{1}{\varepsilon}a_1(\tilde{v}^\varepsilon)dt=&\frac{Ra}{\varepsilon}\cdot e(\tilde{\psi}^\varepsilon)dt\\ &-\Big(b_1(u^\varepsilon_S,u^\varepsilon_S)-a_1^{-1}(Ra\cdot e(b_2(\tilde{u}^\varepsilon,\tilde{\theta}^\varepsilon)+a_2(\tilde{\theta}^\varepsilon)))\Big)dt-\sum_{k=1}^N a_1^{-1}(Ra\cdot e(\sigma_k))dW^k\,,\end{aligned}$$

and hence by the Itō lemma

$$\begin{aligned}d|\tilde{v}^\varepsilon|^2+\frac{2}{\varepsilon}\langle a_1(\tilde{v}^\varepsilon),\tilde{v}^\varepsilon\rangle dt=&\frac{2Ra}{\varepsilon}\langle e(\tilde{\psi}^\varepsilon),\tilde{v}^\varepsilon\rangle dt+\sum_{k=1}^N|a_1^{-1}(Ra\cdot e(\sigma_k))|^2dt\\ &-2\Big\langle b_1(u^\varepsilon_S,u^\varepsilon_S)-a_1^{-1}(Ra\cdot e(b_2(\tilde{u}^\varepsilon,\tilde{\theta}^\varepsilon)+a_2(\tilde{\theta}^\varepsilon))),\tilde{v}^\varepsilon\Big\rangle dt\\ &-2\sum_{k=1}^N\langle a_1^{-1}(Ra\cdot e(\sigma_k)),\tilde{v}^\varepsilon\rangle dW^k.\end{aligned} \tag{2.35}$$

Next, with (2.3) and routine bounds we infer

$$\begin{aligned}d|\tilde{v}^\varepsilon|^2+\frac{1}{\varepsilon}|\tilde{v}^\varepsilon|^2dt\le&\frac{C\mathrm{Ra}^2}{\varepsilon}|\tilde{\psi}^\varepsilon|^2dt+CRa^2|\sigma|^2dt+C\varepsilon\Big(|u^\varepsilon_S|^4+Ra^2|\tilde{u}^\varepsilon|^2|\tilde{\theta}^\varepsilon|^2+|\tilde{\theta}^\varepsilon|^2\Big)dt\\ &-2\sum_{k=1}^N\langle a_1^{-1}(Ra\cdot e(\sigma_k)),\tilde{v}^\varepsilon\rangle dW^k\,,\end{aligned}$$

where $C=C(|a_i|,|a_i^{-1}|,|e|,b_i)$ is again independent of $\varepsilon>0$. Integrating in time, multiplying by ε and recalling that $\tilde{v}^\varepsilon(0)=0$ we obtain

$$\begin{aligned}\int_0^{t\wedge\tau}|\tilde{v}^\varepsilon|^2dt'\le&C\mathrm{Ra}^2\int_0^{t\wedge\tau}(|\tilde{\psi}^\varepsilon|^2dt'+\varepsilon|\sigma|^2)dt'+C\varepsilon^2\int_0^{t\wedge\tau}\Big(|u^\varepsilon_S|^4+Ra^2|\tilde{u}^\varepsilon|^2|\tilde{\theta}^\varepsilon|^2+|\tilde{\theta}^\varepsilon|^2\Big)dt'\\ &-2\varepsilon\sum_{k=1}^N\int_0^{t\wedge\tau}\langle a_1^{-1}(Ra\cdot e(\sigma_k)),\tilde{v}^\varepsilon\rangle dW^k\,,\end{aligned} \tag{2.36}$$

for any $t\ge0$ and any stopping time τ.

We now combine (2.32) and (2.36) as follows. Consider the stopping times

$$\tau_\kappa:=\inf_{t\ge0}\{|\theta^\varepsilon_S(t)|^2\ge\kappa\}, \tag{2.37}$$

for each $\kappa\ge0$. Applying (2.32) with these stopping times, making use of (2.36) and then taking expected values yields

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}|\tilde{\psi}^\varepsilon(s\wedge\tau_\kappa)|^2\le&\kappa Ra^2C\int_0^t\left(\mathbb{E}\sup_{s\in[0,r]}|\tilde{\psi}^\varepsilon(s\wedge\tau_\kappa)|^2+\varepsilon|\sigma|^2\right)dr\\ &+\varepsilon^2\kappa C\int_0^t\mathbb{E}\left(|u^\varepsilon_S|^4+Ra^2|\tilde{u}^\varepsilon|^2|\tilde{\theta}^\varepsilon|^2+|\tilde{\theta}^\varepsilon|^2\right)ds,\end{aligned} \tag{2.38}$$

where $C > 0$ is independent of Ra, $\kappa > 0$ and $\varepsilon > 0$. In order the bound the final term in (2.38) we estimate $\tilde{u}^\varepsilon$ defined in (2.24) by a triangle inequality. Next, we use that $\tilde{\theta}^\varepsilon$ satisfies drift diffusion equation (2.16), and consequently we apply (2.18), (2.21)–(2.22) to infer

$$\mathbb{E}\left(|u_S^\varepsilon|^4 + Ra^2|\tilde{u}^\varepsilon|^2|\tilde{\theta}^\varepsilon|^2 + |\tilde{\theta}^\varepsilon|^2\right) \leq C(1 + Ra^4),$$

where $C = C(|\sigma|, |a_1|, |a_1^{-1}|)$ is independent of $\varepsilon > 0$ and Ra. Hence,

$$\mathbb{E} \sup_{s\in[0,t]} |\tilde{\psi}^\varepsilon(s \wedge \tau_\kappa)|^2 \leq \kappa(1 + Ra^4)C \int_0^t \left(\mathbb{E} \sup_{s\in[0,r]} |\tilde{\psi}^\varepsilon(s \wedge \tau_\kappa)|^2 + t\varepsilon\right) dr,$$

and hence, with the Grönwall inequality we find

$$\mathbb{E}\left(\sup_{s\in[0,t]} |\tilde{\psi}^\varepsilon(s)|^2 \mathbb{1}_{\tau_\kappa > t}\right) \leq \mathbb{E} \sup_{s\in[0,t]} |\tilde{\psi}^\varepsilon(s \wedge \tau_\kappa)|^2 \leq \varepsilon t \exp(\kappa C_1) \tag{2.39}$$

which holds for a constant C_1 independent of $\varepsilon > 0$ and $\kappa > 0$.

Set $X_\varepsilon(t) := \sup_{s\in[0,t]} |\tilde{\psi}^\varepsilon(s)|^2$ and for each $t \geq 0$, $\kappa > 0$, $\varepsilon > 0$ define the sets

$$E_{t,\kappa,\varepsilon} := \left\{\sup_{s\in[0,t]} |\theta_S^\varepsilon|^2 \geq \kappa\right\} = \{\tau_\kappa \leq t\}.$$

For each $t > 0$ and each sufficiently small $\eta = \eta(|\sigma|) > 0$ one finds by Markov inequality, (2.18), and (2.21) that for small $\eta > 0$

$$\mathbb{P}(E_{t,\kappa,\varepsilon}) \leq e^{-\eta\kappa}\mathbb{E}\exp\left(\eta \sup_{s\in[0,t]} |\theta_S^\varepsilon(s)|^2\right) \leq e^{-\eta\kappa} C \exp(\eta t|\sigma|^2), \tag{2.40}$$

where $C > 0$ is independent of $\varepsilon > 0$ and $t, \kappa > 0$. On the other hand, by (2.39) for any $\delta > 0$,

$$\begin{aligned}
\mathbb{E}X_\varepsilon(t)^\delta &= \sum_{k=0}^\infty \mathbb{E}\left(X_\varepsilon(t)^\delta \mathbb{1}_{k \leq \left(\sup_{s\in[0,t]} |\theta_S^\varepsilon|^2\right) < k+1}\right) = \sum_{k=0}^\infty \mathbb{E}\left(X_\varepsilon(t)^\delta \mathbb{1}_{\tau_k \leq t}\mathbb{1}_{\tau_{k+1} > t}\right) \\
&\leq \sum_{k=0}^\infty \left(\mathbb{E}(X_\varepsilon(t)\mathbb{1}_{\tau_{k+1}>t})\right)^\delta \left(\mathbb{P}(\tau_k \leq t)\right)^{1-\delta} \\
&\leq \varepsilon^\delta t^\delta \exp(\eta t|\sigma|^2) C \sum_{k=0}^\infty \exp(\delta C_1(k+1) - (1-\delta)\eta k),
\end{aligned} \tag{2.41}$$

where we have used (2.39), (2.40) for the final bound. Here C is independent ε and C_1 is the constant appearing in (2.39). Thus when $\delta < \frac{\eta}{C_1 + \eta}$ the series in (2.41) converges and the desired bound (2.30) follows for any such value of δ. The proof is thus complete. □

3 The Boussinesq Equations and Their Mathematical Setting

We begin our analysis of the stochastic Boussinesq Equations by recalling some details of the mathematical setting. The section concludes with a mathematically precise restatement of Theorem 1.1. Here and below we implicitly assume that C, c, C_0 etc. are constants depending on the domain $\mathcal{D}$, any other dependence is indicated explicitly.

For the forthcoming analysis it is convenient to consider an equivalent, homogenous, form of the stochastic Boussinesq Equations. Introducing the 'small parameter' $\varepsilon = Pr^{-1} > 0$ and making the change of variable

$\theta^\varepsilon = T - \tilde{Ra}(1-z)$ we can rewrite (1.1)–(1.2) as

$$\varepsilon(\partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) - \Delta \mathbf{u}^\varepsilon = \nabla \tilde{p}^\varepsilon + Ra\hat{\mathbf{k}}\theta^\varepsilon, \quad \nabla \cdot \mathbf{u}^\varepsilon = 0, \tag{3.1}$$

$$d\theta^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla\theta^\varepsilon dt = \tilde{Ra} \cdot u_d^\varepsilon dt + \Delta\theta^\varepsilon dt + \sum_{k=1}^{N} \sigma_k dW^k, \tag{3.2}$$

supplemented with the homogenous boundary conditions

$$\mathbf{u}^\varepsilon_{|z=0} = \mathbf{u}^\varepsilon_{|z=1} = 0, \quad \theta^\varepsilon_{|z=0} = \theta^\varepsilon_{|z=1} = 0, \quad \mathbf{u}^\varepsilon, \theta^\varepsilon \text{ are periodic in } \mathbf{x} = (x_1, x_2). \tag{3.3}$$

Note here that we have implicitly modified the pressure in (3.1) by $Ra(z - \frac{1}{2}z^2)$ since $(1-z)\hat{\mathbf{k}} = \nabla(z - \frac{1}{2}z^2)$. The corresponding infinite Prandtl system ($\varepsilon = 0$) is given by

$$-\Delta \mathbf{u}^0 = \nabla \tilde{p} + Ra\hat{\mathbf{k}}\theta^0, \quad \nabla \cdot \mathbf{u}^0 = 0, \tag{3.4}$$

$$d\theta^0 + \mathbf{u}^0 \cdot \nabla\theta^0 dt = \tilde{Ra} \cdot u_d^0 dt + \Delta\theta^0 dt + \sum_{k=1}^{N} \sigma_k dW^k, \tag{3.5}$$

again with initial conditions $\theta^0(0) = \theta^0_0$ and boundary conditions as in (3.3).

Remark 3.1. *Notice that we do not prescribe an initial condition for* $\mathbf{u}^0$ *as this component of* (3.4)-(3.5) *does not satisfy an independent evolution equation. Indeed* (3.5) *can be rewritten as*

$$d\theta^0 + (L\theta^0) \cdot \nabla\theta^0 dt = \tilde{Ra}(L\theta^0)_d dt + \Delta\theta^0 dt + \sum_{k=1}^{N} \sigma_k dW^k, \tag{3.6}$$

where $L = RaA^{-1}P\hat{k}$ *for* A *the Stokes operator and* P *the Leray projector, that is* $L\theta$ *is the solution of* $-\Delta\mathbf{u} = \nabla\tilde{p} + Ra\hat{\mathbf{k}}\theta$, $\nabla \cdot \mathbf{u} = 0$*; see* (5.2) *below.*

Remark 3.2. *The systems* (3.1)–(3.2) *or* (3.4)-(3.5) *can be reformulated in terms of* $T = \theta^\varepsilon + \tilde{Ra}(1-z)$*, which satisfies* (1.1)–(1.2) *or* (1.4)–(1.5) *and have boundary conditions given as in* (1.3)*. Our analysis will make use of both of these formulations.*

Remark 3.3. *As noted above, parameters in the problem are the Prandtl* ($Pr = \varepsilon^{-1}$) *and Rayleigh numbers* (Ra, $\tilde{Ra}$)*, which are unit-less. In terms of basic physical quantities of interest we have that*

$$\varepsilon^{-1} = Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{g\alpha\gamma h^{5/2}}{\nu\kappa^{3/2}}, \quad \tilde{Ra} = \frac{\sqrt{\kappa h}T_1}{\gamma}, \tag{3.7}$$

where ν *is the kinematic viscosity,* κ *the thermal diffusivity,* g *the gravitational constant,* α *the coefficient of thermal expansion,* h *the distance between the confining plates,* $T_b - T_t$ *the temperature differential, and* $\gamma = \mathcal{H}/\rho c$ *the intensity* $\mathcal{H}$ *of the volumetric heat flux*[4] *normalized by the density* ρ *and specific heat* c *of the fluid. We refer the interested reader to [FGHR] where the dimensionless form of the stochastically driven Boussinesq equations,* (1.1)–(1.2) *or equivalently* (3.1)–(3.3)*, are derived.*

3.1 The Functional Setting

We next define the phase space for the Boussinesq equations. Our setting is very close to the classical framework for the Navier-Stokes equations; see e.g. [CF88, Tem01] for further details.

For every $\varepsilon > 0$ define $H := H_1 \times H_2$ as the phase space for (3.1)–(3.3), where

$$H_1 := \{\mathbf{u} \in (L^2(\mathcal{D}))^3 : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}_{|z=0,1} = 0, \mathbf{u} \text{ is periodic in } \mathbf{x}\},$$
$$H_2 := \{\theta \in L^2(\mathcal{D}) : \theta \text{ is periodic in } \mathbf{x}\}$$

[4]In our setting $\mathcal{H}$ has units of power/$\sqrt{\text{volume*time}}$.

and we abuse notation setting by $H = H_2$ for (3.4)–(3.5) when $\varepsilon = 0$. As usual H is endowed with the standard L^2-norm which we denote as $\|\cdot\|$. All other norms are written as $\|\cdot\|_X$ below for a given space X. We define H^1 type spaces as

$$V_1 := \{\mathbf{u} \in (H^1(\mathcal{D}))^3 : \nabla \cdot \mathbf{u} = 0, \mathbf{u}_{|z=0,1} = 0, \mathbf{u} \text{ is periodic in } \mathbf{x}\},$$
$$V_2 := \{\theta \in H^1(\mathcal{D}) : \theta_{|z=0,1} = 0, \theta \text{ is periodic in } \mathbf{x}\}.$$

Let $V = V_1 \times V_2$ if $\varepsilon > 0$ and $V = V_2$ if $\varepsilon = 0$. Finally set

$$\Pi_\theta : H \to H_2 \text{ to be the projection onto the } \theta \text{ component of } H. \tag{3.8}$$

As above in (2.7), for any Borel measure $\mu \in \Pr(H)$, we take $\Pi_\theta \mu$ to be the push-forward of μ by Π_θ.

We have the following general results concerning the existence and uniqueness of solutions of (3.1)–(3.3) and (3.4)–(3.5):

Proposition 3.1 (Existence, Uniqueness and Continuous Dependence of Solutions on Data)**.**

(i) *For every $\epsilon > 0$ and any given $\mu^0 \in Pr(H)$ there exists a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ upon which is defined an H-valued stochastic process $(\mathbf{u}^\varepsilon, \theta^\varepsilon)$ with the regularity*

$$(\mathbf{u}^\varepsilon, \theta^\varepsilon) \in L^2(\Omega; L^2_{loc}([0,\infty); V) \cap L^\infty_{loc}([0,\infty); H)),$$

which is weakly continuous in H, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, satisfies (3.1)–(3.2) weakly and such that $(\mathbf{u}^\varepsilon(0), \theta^\varepsilon(0))$ is distributed as μ^0. We say that such a pair $(\mathcal{S}, (\mathbf{u}^\varepsilon, \theta^\varepsilon))$ is a weak-martingale solution of (3.1)–(3.3).

(ii) *Additionally, for any $\varepsilon > 0$, there exists a martingale solution $(\mathcal{S}, (\mathbf{u}^\varepsilon_S, \theta^\varepsilon_S))$ which is stationary in time. These stationary solutions $(\mathcal{S}, (\mathbf{u}^\varepsilon_S, \theta^\varepsilon_S))$ may be chosen in such a way that, for any $p \geq 2$ there is an $\eta = \eta(p) > 0$,*

$$\sup_{\varepsilon>0} \int_H \exp(\eta(\|\mathbf{u}\|^2 + \|\theta\|^2_{L^p})) d\mu_\varepsilon(\mathbf{u}, \theta) < \infty\,, \tag{3.9}$$

where $\mu_\varepsilon(\cdot) = \mathbb{P}((\mathbf{u}^\varepsilon_S, \theta^\varepsilon_S) \in \cdot)$.

(iii) *Now consider the case when $\varepsilon = 0$. Fix a stochastic basis $\mathcal{S}$ and any $\mathcal{F}_0$ measurable random variable $\theta_0 \in L^2(\Omega, H)$. Then there exists a unique process θ^0*

$$\theta^0 \in L^2(\Omega; L^2_{loc}([0,\infty); V) \cap C([0,\infty); H)) \tag{3.10}$$

which is $\mathcal{F}_t$-adapted, weakly solves (3.4)–(3.5) and satisfies the initial condition $\theta^0(0) = \theta_0$.

(iv) *For a given stochastic basis $\mathcal{S}$ and each $\theta_0 \in H$ denote $\theta^0(\cdot, \theta_0, W)$ as the unique corresponding stochastic process satisfying (3.4)–(3.5) with (3.10). We have that $\theta_0 \mapsto \theta^0(t, \theta_0, W)$ is Fréchet differentiable in $\theta_0 \in H$ for any $t \geq 0$ and any fixed realization $W(\cdot) = W(\cdot, \omega)$. On the other hand $W \mapsto \theta^0(t, \theta_0, W)$ is Fréchet differentiable in W from $C_0([0,t], \mathbb{R}^N)$ to H for each fixed $\theta_0 \in H$ and $t > 0$.*

These results are standard for a system like (3.1)–(3.3); see e.g. [DPZ92, FG95, DGHT11]. The only novel difficulty in view of existing methods is the uniform moment bound (3.9). The existence of such a collection of solutions is established using the maximum principle and exponential moment bounds in the companion work [FGHR]; cf. Appendix A below.

The Markovian framework for (3.4)–(3.5) is defined as above for the toy model (2.5)–(2.6). The transition functions are given by

$$P^0_t(\theta_0, A) := \mathbb{P}(\theta^0(t, \theta_0) \in A), \quad t \geq 0,\ \theta_0 \in H, A \in \mathcal{B}(H) \tag{3.11}$$

and associated semigroup by

$$P^0_t \phi(\theta_0) := \mathbb{E}\phi(\theta^0(t, \theta_0)), \quad t \geq 0,\ \phi \in M_b(H)\,, \tag{3.12}$$

where $M_b(H)$ is the set of bounded measurable functions on H. This semigroup acts on Borel probability measures μ according to

$$\mu P_t^0(A) = \int_H P_t^0(\theta, A) d\mu(\theta), \quad A \in \mathcal{B}(H).$$

In view of the continuous dependence on initial conditions the semigroup $\{P_t^0\}_{t\geq 0}$ is Feller, that is, it maps the set of continuous bounded functions on H, $C_b(H)$ to itself.

3.2 Statement of the Main Results

We now precisely formulate the main result of the work (cf. Theorem 1.1).

Theorem 3.1. *Let $\{P_t^0\}_{t\geq 0}$ be the Markov semigroup associated to (3.4)–(3.5) defined in (3.12). There exists $N_0 > 0$ and $\eta_0 > 0$ depending only on Ra and $\tilde{Ra}$ such that*

(i) if $N \geq N_0$, that is, if we directly force the first N eigenfunctions in (3.5), then

$$\rho_\eta(\mu P_t^0, \tilde{\mu} P_t^0) \leq C \exp(-\gamma t) \rho_\eta(\mu, \tilde{\mu}), \tag{3.13}$$

for any $\mu, \tilde{\mu} \in Pr_1(H)$, $\eta \in (0, \eta_0)$ and every $t \geq 0$, where ρ_η is defined in (2.9). In particular there exists a unique ergodic invariant measure $\mu \in Pr_1(H)$ of (3.4)–(3.5).

(ii) There exists a collection of measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ corresponding to statistically invariant states of (3.1)–(3.3) which satisfy the uniform moment condition (3.9) for some sufficiently small $\eta > 0$ and $p \geq 3$. For any such collection, if $N \geq N_0$ we have that

$$\rho_\eta(\Pi_\theta \mu_\varepsilon, \mu_0) \leq C \varepsilon^q \,, \tag{3.14}$$

where $C = C(Ra, \tilde{Ra})$ and $q = q(Ra, \tilde{Ra})$ are both independent of $\varepsilon > 0$. As such,

$$\mathbb{E}(\phi(\theta_S^\varepsilon) - \phi(\theta_S^0)) \leq C \|\phi\|_{2\eta} \varepsilon^q \,, \tag{3.15}$$

for any element $\phi \in V_{2\eta}(H)$ as defined in (2.11). Here $\{\theta_S^\varepsilon\}_{\varepsilon>0}$ and θ_S^0 are stationary solutions of (3.1)–(3.3) and (3.4)–(3.5) respectively corresponding to $\{\mu_\varepsilon\}_{\varepsilon>0}$ and μ_0.[5]

The proofs of (i) and (ii) are carried out in Section 4 and 5 respectively. We conclude this section by making several important remarks.

Remark 3.4.

(i) Using the approach detailed here, the infinite Prandtl limit analogous to (3.14), (3.15) may also be established for a two dimensional version of (3.1)–(3.3). Here one can also show that (3.1)–(3.3) has a well defined Markov semigroup. Thus, any statistically invariant state corresponds to an invariant measure of the associated semigroup. This additional ingredient of Markovianity in two dimensions allows us to show in [FGHR] that the ε-independent exponential moment bounds in (3.9) hold for all invariant measures.

(ii) In 3D the existence of a sequence of statistically invariant states of (3.1)–(3.3) satisfying the uniform moment bound (3.9) is established in the companion work [FGHR]. Here by contrast to the 2D case we have not been able to show that every sequence of statistically invariant states of (3.1)–(3.3) have such (uniform) exponential moments.

(iii) An interesting outstanding issue is to establish the convergence of the velocity fields in the large Prandtl number limit. This convergence of the extended measures μ_ε as $\varepsilon \to 0$ as compared to (3.14) will be addressed in future work.

[5]It is worth noting here that θ_S^ε are only stationary Martingale solutions. As such we cannot suppose that the collection of these solution $\{\theta_S^\varepsilon\}_\varepsilon$ are all defined relative to the same stochastic basis. This subtlety will not cause us any trouble in what follows and we shall essential suppress this technical point in order to avoid notational confusion.

4 Contraction in the Wasserstein Distance for the Infinite Prandtl System

In this section we establish some properties of the infinite Prandtl system, (3.4)–(3.5). These properties provides a sufficient condition for the contraction bound (3.13) as a consequence of a general result in [HM08, Theorem 3.1].

Proposition 4.1. *There exist $\eta_0 > 0$ and N_0, depending only on $Ra, \tilde{R}a$, such that whenever the number of forced modes N in (3.4)–(3.5) exceeds N_0 and for any $0 < \eta < \eta_0$ we have*

(a) **Lyapunov structure:** *For all $t^* > 0$, there exists $C_1 = C_1(t^*, \eta)$ such that*

$$\mathbb{E}\left(\exp(\eta\|\theta^0(t,\theta_0^0)\|^2)(1+\|\mathcal{J}_{0,t}\|)\right) \leq C_1 \exp(\eta(1+4Ra\tilde{R}a)e^{-t/2}\|\theta_0^0\|^2)\,, \tag{4.1}$$

for each $\theta_0^0 \in H$ and every $t \in [0,t^*]$. Here the operator $\mathcal{J}_{0,t}$ is the Fréchet derivative of $\theta^0(t,\theta_0)$ with respect to initial condition θ_0^0; see (B.1) and (B.8) below.

(b) **Gradient Bound for Markov semigroup:** *for any $\phi \in C_b^1(H)$, and every $t \geq 0$, $\theta \in H$*

$$\|\nabla P_t^0\phi(\theta)\| \leq C\exp(\eta\|\theta\|^2)\left(\sqrt{P_t^0(|\phi(\theta)|^2)} + \delta(t)\sqrt{P_t^0(\|\nabla\phi(\theta)\|^2)}\right), \tag{4.2}$$

where $\delta(t) \to 0$ as $t \to \infty$. Here again δ and $C > 0$ depend only on $Ra, \tilde{R}a$, and η.

(c) **Irreducibility condition:** *for any $M > 0$, $\delta > 0$ there is $t_* = t_*(M,\delta,\eta)$ such that for each $t \geq t_*$*

$$\inf_{\|\theta_0\|,\|\tilde{\theta}_0\| \leq M}\ \sup_{\Gamma \in \mathcal{C}(\delta_{\theta_0}P_t^0,\delta_{\tilde{\theta}_0}P_t^0)} \Gamma\{(\theta,\tilde{\theta}) \in H \times H : \rho_\eta(\theta,\tilde{\theta}) < \delta\} > 0, \tag{4.3}$$

where, as above in (2.9), $\mathcal{C}(\delta_{\theta_0}P_t^0, \delta_{\tilde{\theta}_0}P_t^0)$ denotes the collection of all couplings of the measures $\delta_{\theta_0}P_t^0$ and $\delta_{\tilde{\theta}_0}P_t^0$

Proving the first condition, (a), essentially reduces to establishing a moment bound which follows from estimates found in [FGHR], and which we recall below in Appendix A (see Proposition A.2). The second condition, (4.2), can be translated to a control problem though the use of Malliavin calculus which in our setting amounts to proving a relatively straightforward Foias-Prodi type estimate. Since both (a) and (b) can be established by methods essentially contained in previous works we relegate further details to appendices (see Sections A and B below). As already mentioned above, the principal novel challenge here is to prove the irreducibility condition (c) which we turn to next.

4.1 Irreducibility

In previous related works the proof of irreducibility essentially relies on the fact that the governing equations without the stochastic forcing have a trivial attractor which is stable under small force perturbations; see e.g. [EM01, HM06, CGHV13, FGHRT13]. Here for (1.1)–(1.2) the dynamics without body forces can be highly non-trivial. Our approach shows that we can reduce (4.3) to a control problem though the Girsanov theorem and careful stopping time arguments. We believe that this strategy may be applicable to other problems.

As a preliminary step we show that (4.3) may be reduced to proving a slightly simpler bound.

Lemma 4.1. *For a given $N \geq 0$ consider (3.4)–(3.5) with N independently forced directions. If for every $M, \delta > 0$ there is a $t_* = t_*(M,\delta) > 0$ such that*

$$\inf_{\|\theta_0\| \leq M} \mathbb{P}(\|\theta^0(t,\theta_0)\| < \delta) > 0,\ \textit{for each } t \geq t_*\,, \tag{4.4}$$

then Proposition 4.1 (c), that is, (4.3), holds for such an N and any $\eta > 0$.

Proof. For any $\theta_0, \tilde{\theta}_0 \in H$ consider the element $\tilde{\Gamma} \in \mathcal{C}(\delta_{\theta_0} P_t^0, \delta_{\tilde{\theta}_0} P_t^0)$ defined on cylinder sets as

$$\tilde{\Gamma}(A \times B) = P_t(\theta_0, A) \times P_t(\tilde{\theta}_0, B), \quad A, B \in \mathcal{B}(H).$$

For each $t > 0$ and any $M, \eta, \gamma > 0$ one has

$$\begin{aligned}
\inf_{\|\theta_0\|,\|\tilde{\theta}_0\| \le M} \ \sup_{\Gamma \in \mathcal{C}(\delta_{\theta_0} P_t, \delta_{\tilde{\theta}_0} P_t)} & \Gamma\{(\theta, \tilde{\theta}) \in H \times H : \rho_\eta(\theta, \tilde{\theta}) < \gamma\} \\
& \ge \inf_{\|\theta_0\|,\|\tilde{\theta}_0\| \le M} \tilde{\Gamma}\left\{(\theta, \tilde{\theta}) \in B_1 \times B_1 : \|\theta\| + \|\tilde{\theta}\| < \gamma \exp(-2\eta)\right\} \\
& \ge \left(\inf_{\|\theta_0\| \le M} P_t\left(\theta_0, \left\{\theta \in H : \|\theta\| < \min\{\gamma/2 \cdot \exp(-2\eta), 1\}\right\}\right)\right)^2 \\
& = \left(\inf_{\|\theta_0\| \le M} \mathbb{P}(\|\theta(t, \theta_0)\| < \min\{\gamma/2 \cdot \exp(-2\eta), 1\})\right)^2,
\end{aligned}$$

where we have used (2.8) for the first inequality. By now applying (4.4) with $\delta = \min\{\gamma/2 \cdot \exp(-2\eta), 1\})$ and the given $M > 0$ the desired result now follows. $\square$

In order to establish (4.3) the rest of the section is therefore devoted to showing that

Proposition 4.2. *There exists an $N_0 = N_0(Ra, \tilde{Ra})$ sufficiently large such that, for any $N \ge N_0$ and every $M, \delta > 0$, there is a $t_* = t_*(M, \delta) > 0$ such that (4.4) is satisfied.*

Proof of Proposition 4.2. We first consider the modified system

$$\Delta \bar{\mathbf{u}} = \nabla \bar{p} + Ra \hat{\mathbf{k}} \bar{\theta}, \quad \nabla \cdot \bar{\mathbf{u}} = 0, \tag{4.5}$$

$$d\bar{\theta} + \bar{\mathbf{u}} \cdot \nabla \bar{\theta} dt = (\tilde{Ra} \cdot \bar{u}_d + \Delta \bar{\theta} - \lambda_N P_N \bar{\theta}) dt + \sum_{k=1}^N \sigma_k dW^k, \quad \bar{\theta}(0) = \theta_0, \tag{4.6}$$

and establish the analogue of (4.4) for $\bar{\theta}$ when N is sufficiently large.[6] For this purpose we consider the change of variable $\psi := \bar{\theta} - \sum_{k=1}^N \sigma_k W^k = \bar{\theta} - \sigma W$ which satisfies

$$\partial_t \psi + \bar{\mathbf{u}} \cdot \nabla \psi = \tilde{Ra} \cdot \bar{u}_d + \Delta \psi - \lambda_N P_N \psi + (\Delta \sigma W - \lambda_N P_N \sigma W - \bar{\mathbf{u}} \cdot \nabla(\sigma W)), \qquad \psi(0) = \theta_0.$$

Taking an inner product with ψ, using that $\bar{\mathbf{u}}$ is divergence free, the inverse Poincaré inequality (see (B.14) below) and the bound

$$\|\nabla \bar{\mathbf{u}}\| \le Ra \|\bar{\theta}\| \le Ra(\|\psi\| + \|\sigma W\|) \tag{4.7}$$

which follows from (4.5) we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\psi\|^2 + \lambda_N \|\psi\|^2 &\le (\tilde{Ra} \|\bar{\mathbf{u}}\| + \|\Delta \sigma W\| + \lambda_N \|\sigma W\| + \|\bar{\mathbf{u}}\| \|\nabla \sigma W\|_{L^\infty}) \|\psi\| \\
&\le C(\tilde{Ra} Ra(\|\psi\| + \|\sigma W\|) + \|\Delta \sigma W\| + \lambda_N \|\sigma W\| + Ra(\|\psi\| + \|\sigma W\|) \|\nabla \sigma W\|_{L^\infty}) \|\psi\|.
\end{aligned}$$

Rearranging we conclude

$$\frac{d}{dt} \|\psi\| + (\lambda_N - Ra(\tilde{Ra} + \|\nabla \sigma W\|_{L^\infty})) \|\psi\| \le (Ra \tilde{Ra} + \lambda_N + Ra \|\nabla \sigma W\|_{L^\infty}) \|\sigma W\| + \|\Delta \sigma W\|. \tag{4.8}$$

Next, we use the fact that, with positive probability, each of $\|\sigma W\|, \|\nabla \sigma W\|, \|\Delta \sigma W\|$ stays close to zero over finite time intervals. For $\gamma > 0$, $t > 0$, $N > 0$ consider the sets

$$\mathcal{X}_{\gamma,t,N} := \left\{ \sup_{s \in [0,t]} \|\nabla \sigma W\|_{L^\infty} \le 1, \sup_{s \in [0,t]} \|\Delta \sigma W\| \le \frac{\gamma}{2}, \sup_{s \in [0,t]} \|\sigma W\| \le \gamma \left(\frac{1}{2(Ra \tilde{Ra} + \lambda_N + Ra)} \wedge 1 \right) \right\}.$$

[6] Here recall that P_N denotes the projection onto the first N modes of the Stokes problem (with boundary conditions as in (3.3)) and λ_N is the corresponding largest eigenvalue in this collection.

Since σ is spatially smooth we infer from standard properties of Brownian motion that $\mathbb{P}(\mathcal{X}_{\gamma,t,N}) > 0$ for any $\gamma > 0$, $t > 0$ and $N > 0$. On the other hand, on $\mathcal{X}_{\gamma,t,N}$ the differential inequality

$$\frac{d}{dt}\|\psi\| + (\lambda_N - Ra(\tilde{Ra} + 1))\|\psi\| \leq \gamma$$

holds over the interval $[0, t]$. Hence, choosing N sufficiently large so that

$$\lambda_N \geq \max\{2Ra(\tilde{Ra} + 1), 1\} \tag{4.9}$$

we infer that

$$\|\bar{\theta}(t, \theta_0)\| \leq \|\psi(t)\| + \|\sigma W\| \leq 2\gamma + e^{-\lambda_N t/2}\|\theta_0\|$$

on $\mathcal{X}_{\gamma,t,N}$. Note that (4.9) sets the condition on N_0 in the statement of Proposition 4.2. In any case, for a given $M > 0$, $\delta > 0$ by choosing $t_* = t_*(M, \delta)$ such that $e^{-\lambda_N t_*} M \leq \frac{\delta}{2}$ we have for any $t \geq t_*$

$$\inf_{\|\theta_0\| \leq M} \mathbb{P}(\|\bar{\theta}(t, \theta_0)\| < \delta) \geq \mathbb{P}(\mathcal{X}_{\delta/4,t,N}) > 0\,. \tag{4.10}$$

In order to now infer (4.4) from (4.10) we apply the Girsanov theorem to a slightly modified version of (4.5)–(4.6) in conjunction with further bounds on (4.5)–(4.6). For $K > 0$ and $\theta_0 \in H$ define $\tilde{\theta}_K = \tilde{\theta}_K(\cdot, \theta_0)$ as the solution of (4.5)–(4.6) with the term $-\lambda_N P_N \bar{\theta}$ is replaced with $-\lambda_N P_N \tilde{\theta}_K \chi_K(\|P_N \tilde{\theta}_K\|)$. Here χ_K is a smooth, non-negative cut-off function which is one for $|x| \leq K$ and zero for $|x| \geq K + 1$. Consider the stopping times

$$\tau_K(\theta_0) = \inf_{t \geq 0}\left\{\|P_N \tilde{\theta}_K(t, \theta_0)\| \geq K\right\},$$

for any $K > 0$ and any $\theta_0 \in H$. It is not hard show that, for any $K > 0$ and any $\theta_0 \in H$

$$\mathbb{P}\left(\bar{\theta}(t \wedge \tau_K(\theta_0), \theta_0) = \tilde{\theta}_K(t \wedge \tau_K(\theta_0), \theta_0),\ \text{for every } t \geq 0\right) = 1. \tag{4.11}$$

On the other hand, for any $\theta_0 \in H$ and $K > 0$, $\tilde{\theta}_K(\cdot, \theta_0)$ is absolutely continuous with respect to the processes $\theta^0(\cdot, \theta_0)$ solving (3.4)–(3.5). Indeed for $\theta_0 \in H$ and $K > 0$ define

$$\mathcal{M}_{\theta_0,K}(t) = \exp\left(-\int_0^t \alpha_{\theta_0,K}\, dW - \frac{1}{2}\int_0^t |\alpha_{\theta_0,K}|^2 ds\right), \tag{4.12}$$

where

$$\alpha_{\theta_0,K} = -\lambda_N \sigma^{-1} P_N \tilde{\theta}_K(t, \theta_0)\chi_K(\|P_N \tilde{\theta}_K(t, \theta_0)\|)$$

and define

$$d\mathbb{Q}_{\theta_0,K} := \mathcal{M}_{\theta_0,K}\, d\mathbb{P}\,.$$

Notice that, since $|\sigma^{-1} P_N \tilde{\theta}_K(s, \theta_0)\chi_K(\|P_N \tilde{\theta}_K\|)| \leq \|\sigma^{-1}\| \cdot (K + 1)$, the Novikov condition is satisfied and Girsanov theorem applies to $\tilde{\theta}_K(\cdot, \theta_0)$ and $\mathcal{M}_{\theta_0,K}$ for any $K > 0$ and any $\theta_0 \in H$.

Now, according to the Girsanov theorem, for any $\delta > 0$ and $t \geq 0$, we have

$$\mathbb{P}(\|\theta(t, \theta_0)\| < \delta) = \mathbb{Q}_{\theta_0,K}(\|\tilde{\theta}_K(t, \theta_0)\| < \delta) = \mathbb{E}\left(\mathcal{M}_{\theta_0,K}(t)\mathbb{1}_{\|\tilde{\theta}_K(t,\theta_0)\| < \delta}\right),$$

and hence for any $\delta > 0$, $\theta_0 \in H$ and for any $\daleth, K, t > 0$, Markov inequality implies

$$\mathbb{P}(\|\theta(t, \theta_0)\| < \delta) \geq \daleth\mathbb{P}\left(\|\tilde{\theta}_K(t, \theta_0)\| < \delta, \mathcal{M}_{\theta_0,K}(t) \geq \daleth\right) \geq \daleth\mathbb{P}\left(\|\bar{\theta}(t, \theta_0)\| < \delta, \mathcal{M}_{\theta_0,K}(t) \geq \daleth, \tau_K(\theta_0) > t\right),$$

where we used (4.11) for the final inequality. On the other hand

$$\begin{aligned}\mathbb{P}\left(\|\bar\theta(t,\theta_0)\|<\delta\right)&\le\mathbb{P}\left(\|\bar\theta(t,\theta_0)\|<\delta,\mathcal{M}_{\theta_0}(t)\ge\beth\right)+\mathbb{P}\left(\mathcal{M}_{\theta_0}(t)<\beth\right)\\&\le\mathbb{P}\left(\|\bar\theta(t,\theta_0)\|<\delta,\mathcal{M}_{\theta_0,K}(t)\ge\beth,\tau_K(\theta_0)>t\right)+\mathbb{P}\left(\mathcal{M}_{K,\theta_0}(t)<\beth\right)+\mathbb{P}(\tau_K(\theta_0)<t)\,.\end{aligned}$$

These two bounds yield

$$\begin{aligned}&\inf_{\|\theta_0\|\le M}\mathbb{P}(\|\theta(t,\theta_0)\|<\delta)\\&\qquad\ge\beth\inf_{\|\theta_0\|\le M}\mathbb{P}\left(\|\bar\theta(t,\theta_0)\|<\delta\right)-\beth\sup_{\|\theta_0\|\le M}\Big(\mathbb{P}\left(\mathcal{M}_{K,\theta_0}(t)<\beth\right)+\mathbb{P}(\tau_K(\theta_0)<t)\Big),\end{aligned}\tag{4.13}$$

for any $M,\delta,t>0$ and for any $K,\beth>0$.

Since the first term on the left hand side of (4.13) is independent of $K>0$ and has the same dependence on $\beth>0$ as the second term we finish the argument by showing that for every fixed $M,K,t>0$

$$\sup_{\|\theta_0\|\le M}\mathbb{P}\left(\mathcal{M}_{\theta_0,K}(t)<\beth\right)\to0,\qquad\text{as }\beth\to0,\tag{4.14}$$

and for every given $M,t>0$

$$\sup_{\|\theta_0\|\le M}\mathbb{P}(\tau_K(\theta_0)<t)\to0,\qquad\text{as }K\to\infty.\tag{4.15}$$

For the first bound (4.14), we have from (4.12) and Itō isometry

$$\begin{aligned}\mathbb{P}\left(M_{\theta_0,K}(t)<\beth\right)=&\mathbb{P}\left(\int_0^t\alpha_{\theta_0,K}dW+\frac12\int_0^t|\alpha_{\theta_0,K}|^2ds>\log(\beth^{-1})\right)\\\le&\frac{1}{\log(\beth^{-1})}\mathbb{E}\left(\left|\int_0^t\alpha_{\theta_0,K}dW\right|+\frac12\int_0^t|\alpha_{\theta_0,K}|^2ds\right)\\\le&\frac{2}{\log(\beth^{-1})}\mathbb{E}\left(1+\lambda_N^2\|\sigma^{-1}P_N\|^2\int_0^t\|P_N\tilde\theta(t,\theta_0)\|^2\chi_K(\|P_N\tilde\theta_K(t,\theta_0)\|)ds\right),\\\le&\frac{2\left(1+\lambda_N^2\|\sigma^{-1}P_N\|^2(K+1)^2t\right)}{\log(\beth^{-1})},\end{aligned}$$

valid for any $\beth\in(0,1)$, $K>0$, and any $\theta_0\in H$. For the second bound observe that, in view of (4.11),

$$\mathbb{P}(\tau_K(\theta_0)<t)\le\mathbb{P}\left(\sup_{s\in[0,t]}\|P_N\bar\theta(s,\theta_0)\|\ge K\right)\le\frac{1}{K^2}\mathbb{E}\left(\sup_{s\in[0,t]}\|\bar\theta(s,\theta_0)\|^2\right).\tag{4.16}$$

From the Itō formula follows

$$d\|\bar\theta\|+2\lambda_N\|P_N\bar\theta\|^2dt+2\|\nabla\bar\theta\|^2dt=\left(2\tilde Ra\langle\tilde u_d,\bar\theta\rangle+1\right)dt+\langle\sigma,\bar\theta\rangle dW.$$

Integrating in time and using (4.5), inverse Poincaré inequality (see (B.14)), and (4.7) we infer for any $s\ge0$

$$\|\bar\theta\|^2+2\lambda_N\mathbb{E}\int_0^s\|\bar\theta\|^2dr\le\|\theta_0\|^2+2Ra\tilde Ra\mathbb{E}\int_0^s\|\bar\theta\|^2dr+s+2\sup_{r\in[0,s]}\left|\int_0^r\langle\sigma,\bar\theta\rangle dW\right|.$$

Using the assumption (4.9) and the Birkholder-Davis-Gundy inequality we infer

$$\mathbb{E}\left(\sup_{s\in[0,t]}\|\bar\theta(s,\theta_0)\|^2\right)\le\|\theta_0\|^2+17t.\tag{4.17}$$

Combining (4.16) and (4.17) thus yields the second bound (4.15).

With (4.10), (4.13), (4.14), and (4.15) now in hand we now conclude the proof by arguing as follows. Given any $\delta > 0$ and any $M > 0$ choose t_* as above so that $e^{-\lambda_N t_*} M \leq \frac{\delta}{2}$. Fix any $t \geq t_*$ and by (4.10) $a = a(M, \delta, t) := \inf_{\|\theta_0\| \leq M} \mathbb{P}(\|\bar{\theta}(t, \theta_0) < \delta) > 0$. Now by (4.15) we can pick K sufficiently large so that $\sup_{\|\theta_0\| \leq M} \mathbb{P}(\tau_K(\theta_0) < t) \leq a/4$. With K, M, t fixed we choose $\daleth > 0$ small enough so that $\sup_{\|\theta_0\| \leq M} \mathbb{P}(\mathcal{M}_{K,\theta_0}(t) < \daleth) \leq a/4$. Finally by combining these choices with (4.13) we finally obtain that

$$\inf_{\|\theta_0\| \leq M} \mathbb{P}(\|\theta(t, \theta_0)\| < \delta) \geq \frac{\daleth a}{2} > 0\,.$$

The proof of Proposition 4.2 is thus complete. □

5 Convergence on Finite Time Intervals

Having established the contraction condition (3.13), we now prove the second part of Theorem 3.1. Observe that (3.13) implies an analogue of (2.13) for (3.4)–(3.5), where $|\cdot|$ is replaced by the L^2 norms $\|\cdot\|$. In view of the uniform bound (3.9), Theorem 3.1, (ii) is thus an immediate consequence of the following proposition.

Proposition 5.1. *For each $\varepsilon > 0$ let θ_S^ε be the second component of a statistically stationary solution of (3.1)–(3.2) satisfying the uniform bound (3.9) and let $\theta_S^{0,\varepsilon}$ be solution of (3.4)–(3.5) with the initial condition $\theta_S^{0,\varepsilon}(0) := \theta_{S,0}^\varepsilon$. Then, for any $\varepsilon > 0$ and any $t \geq 0$,*

$$\mathbb{E}\|\theta_S^\varepsilon(t) - \theta_S^{0,\varepsilon}(t)\|^\delta \leq C\varepsilon^\delta \tag{5.1}$$

for constants $C = C(Ra, \tilde{Ra}, t) > 0$, $\delta = \delta(Ra, \tilde{Ra}, t) > 0$ independent of $\varepsilon > 0$.[7]

5.1 Defining the corrector

In order to proceed as in Section 2.4.1 and define the 'corrector' we first recall some properties of the steady Stokes problem

$$-\Delta \mathbf{u} = \nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \tag{5.2}$$

and the associated linear evolution given as

$$\varepsilon \partial_t \mathbf{u} - \Delta \mathbf{u} = \nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{5.3}$$

for any $\varepsilon > 0$ and relative to the (sufficiently regular) data $\mathbf{f}$, $\mathbf{u}_0$. Both (5.2) and (5.3) are supplemented with same mixed periodic-Dirichlet boundary conditions as for $\mathbf{u}$ in (3.3).

Classically, the equations (5.2) and (5.3) can be understood in terms of the functional framework given in Section 3.1; see e.g. [Tem01] for a systematic treatment. Recall that the Stokes operator is given by $A = -P\Delta$ which is self-adjoint, positive and unbounded on H_1. Here P is the Leray projection on divergence free vector fields $(L^2(\mathcal{D}))^3 \to H_1$.[8] As such we may rewrite (5.2), as $A\mathbf{u} = P\mathbf{f}$. Along the lines of the classical elliptic theory one may show that for any $\mathbf{f} \in (L^2(\mathcal{D}))^3$ there exists a unique $\mathbf{u} \in D(A) = V_1 \cap (H^2(\mathcal{D}))^3$ denoted by $\mathbf{u} = A^{-1}P\mathbf{f}$ which satisfies

$$\|A^{-1}P\mathbf{f}\|_{H^2} \leq C\|\mathbf{f}\|. \tag{5.4}$$

Turning to (5.3) we observe that for any $\mathbf{f} \in L^2_{loc}([0, \infty); H_1)$ and $\mathbf{u}_0 \in H_1$ there exists a unique solution $\mathbf{u}$ of (5.3) relative to this data with $\mathbf{u} \in L^2_{loc}([0, \infty); D(A)) \cap C([0, \infty); H_1)$. The flow associated with (5.3), i.e. (5.3) with $\mathbf{f} \equiv 0$, defines an analytic semigroup which we denote as $\exp(-\varepsilon^{-1}At)$.

[7] Indeed, given θ_S^ε and its associated basis $\mathcal{S}^\varepsilon$, we can obtain (a unique) pathwise solution of $\theta_S^{0,\varepsilon}$ relative to this basis according to Proposition 3.1, (ii). While the forthcoming estimates may therefore take place in a different stochastic basis for different values of $\varepsilon > 0$ the constants C and δ will be shown to be independent of particular sequence of basses.

[8] Equivalently $A\mathbf{u} = -\Delta \mathbf{u} - \nabla p$, where $p = p(\mathbf{u})$ the 'pressure' is the unique H^1 function satisfying $\Delta p = \mathrm{div}(\Delta u)$ in the weak sense.

With these preliminaries in hand, essentially following (2.24)-(2.25), we define the corrector system.

$$\begin{aligned}\tilde{\mathbf{u}}^\varepsilon(t) &= \exp(-\varepsilon^{-1}At)P_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0} + A^{-1}(P(Ra\cdot\hat{\mathbf{k}}\tilde{\theta}^\varepsilon)) - \exp(-\varepsilon^{-1}At)A^{-1}(P(Ra\hat{k}\theta^\varepsilon_{S,0}))\\ &= A^{-1}(P(Ra\cdot\hat{\mathbf{k}}\tilde{\theta}^\varepsilon)) + \mathbf{w}^\varepsilon(t)\,, \end{aligned} \tag{5.5}$$

$$d\tilde{\theta}^\varepsilon + \left(\tilde{\mathbf{u}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon - \Delta\tilde{\theta}^\varepsilon\right)dt = \tilde{Ra}\cdot\tilde{u}^\varepsilon_d dt + \sum_{k=1}^N \sigma_k dW^k, \quad \tilde{\theta}^\varepsilon(0) = \theta^\varepsilon_{S,0}\,, \tag{5.6}$$

where $\mathbf{w}^\varepsilon(t) = \exp(-\varepsilon^{-1}At)(P_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0} - A^{-1}(P(Ra\cdot\hat{\mathbf{k}}\theta^\varepsilon_{S,0})))$ so that under this definition $\tilde{\mathbf{u}}^\varepsilon(0) = P_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0}$. Here, for technical reasons, we slightly modify the initial condition on $\tilde{\mathbf{u}}^\varepsilon$ in (5.5) compared to (2.24) by taking N^ε such that $\varepsilon\lambda^2_{N^\varepsilon}\sim 1$, where P_{N^ε} is the projection onto the first N^ε modes of the Stokes operator A. Observe that for $t\geq 0$, $\tilde{\mathbf{u}}^\varepsilon$ solves

$$-\Delta\tilde{\mathbf{u}}^\varepsilon = \nabla p^\varepsilon + Ra\cdot\hat{\mathbf{k}}\tilde{\theta}^\varepsilon + \Delta\mathbf{w}^\varepsilon(t), \quad \nabla\cdot\tilde{\mathbf{u}}^\varepsilon = 0\,, \tag{5.7}$$

with $\mathbf{w}^\varepsilon$ solving

$$\begin{aligned}&\varepsilon\partial_t\mathbf{w}^\varepsilon - \Delta\mathbf{w}^\varepsilon = \nabla q^\varepsilon, \quad \nabla\cdot\mathbf{w}^\varepsilon = 0,\\ &\mathbf{w}^\varepsilon(0) = P_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0} - \mathbf{y}^\varepsilon_{S,0}, \quad \text{where } \mathbf{y}^\varepsilon_{S,0} \text{ solves} \quad \Delta\mathbf{y}^\varepsilon_{S,0} + \nabla q = Ra\cdot\hat{\mathbf{k}}\theta^\varepsilon_{S,0}, \quad \nabla\cdot\mathbf{y}^\varepsilon_{S,0} = 0\,.\end{aligned} \tag{5.8}$$

and supplemented with the same boundary conditions as $\mathbf{u}$ in (3.3).

5.2 Comparing the corrector to the infinite Prandtl System

Having defined the corrector $(\tilde{\mathbf{u}}^\varepsilon, \tilde{\theta}^\varepsilon)$ we proceed as above to prove (5.1) in two steps. We begin by estimating difference between the corrector and the infinite Prandtl system. In comparison to Lemma 2.2 we have

Lemma 5.1. *For $\varepsilon > 0$, let $(\tilde{\mathbf{u}}^\varepsilon, \tilde{\theta}^\varepsilon)$ be the solution of (5.5)–(5.6) and let $(\mathbf{u}^{0,\varepsilon}_S, \theta^{0,\varepsilon}_S)$ to be the solution of (3.4)–(3.5) with initial condition $\theta^\varepsilon_{S,0}$ being the second component of stationary solutions of (3.1)–(3.2) satisfying (3.9). Then for any $\varepsilon > 0$ and any $t > 0$ there exist $C = C(\tilde{Ra}, Ra, t)$ and $\delta = \delta(\tilde{Ra}, Ra, t)$ both independent of $\varepsilon > 0$ such that*

$$\mathbb{E}\|\tilde{\theta}^\varepsilon(t) - \theta^{0,\varepsilon}_S(t)\|^\delta \leq C\varepsilon^\delta. \tag{5.9}$$

Proof. Let $\mathbf{v}^\varepsilon = \tilde{\mathbf{u}}^\varepsilon - \mathbf{u}^{0,\varepsilon}_S$ and $\phi^\varepsilon = \tilde{\theta}^\varepsilon - \theta^{0,\varepsilon}_S$. Using that $\tilde{\theta}^\varepsilon$ and $\theta^{0,\varepsilon}_S$ share the same initial condition we see that $\mathbf{v}^\varepsilon$ and ϕ^ε satisfy

$$-\Delta\mathbf{v}^\varepsilon = \nabla p^\varepsilon + Ra\hat{k}\phi^\varepsilon - \Delta\mathbf{w}^\varepsilon, \qquad \nabla\cdot\mathbf{v}^\varepsilon = 0\,, \tag{5.10}$$

$$\partial_t\phi^\varepsilon - \Delta\phi^\varepsilon = \tilde{Ra}\cdot v^\varepsilon_d - \mathbf{v}^\varepsilon\cdot\nabla\theta^{0,\varepsilon}_S - \tilde{\mathbf{u}}^\varepsilon\cdot\nabla\phi^\varepsilon, \qquad \phi^\varepsilon(0) = 0, \tag{5.11}$$

where $\mathbf{w}^\varepsilon$ obeys (5.8).

Starting from (5.11), using that $\tilde{\mathbf{u}}^\varepsilon$, $\mathbf{v}^\varepsilon$ are divergence free along with the Poincaré inequality and standard Sobolev embeddings we estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi^\varepsilon\|^2 + \|\nabla\phi^\varepsilon\|^2 &= \int\left(\tilde{Ra}\cdot\phi^\varepsilon\tilde{v}^\varepsilon_d - \mathbf{v}^\varepsilon\cdot\nabla\theta^{0,\varepsilon}_S\phi^\varepsilon\right)dx \leq \tilde{Ra}\|\phi^\varepsilon\|\|\mathbf{v}^\varepsilon\| + \|\mathbf{v}^\varepsilon\|_{L^6}\|\nabla\phi^\varepsilon\|\|\theta^{0,\varepsilon}_S\|_{L^3}\\ &\leq \|\nabla\phi^\varepsilon\|^2 + C\|\nabla\mathbf{v}^\varepsilon\|^2\left(\tilde{Ra}^2 + \|\theta^{0,\varepsilon}_S\|^2_{L^3}\right)\,. \end{aligned}\tag{5.12}$$

Next, testing (5.10) with $\mathbf{v}^\varepsilon$ and using Hölder and Poincaré inequalities, we find

$$\|\nabla\mathbf{v}^\varepsilon\|^2 \leq 2Ra^2\|\phi^\varepsilon\|^2 + 2\|\nabla\mathbf{w}^\varepsilon\|^2. \tag{5.13}$$

To bound $\mathbf{w}^\varepsilon$ we infer directly from (5.8) that

$$\frac{\varepsilon}{2}\|\mathbf{w}^\varepsilon(t)\|^2 + \int_0^t\|\nabla\mathbf{w}^\varepsilon(s)\|^2 ds \leq \frac{\varepsilon}{2}\|\mathbf{w}^\varepsilon_0\|^2 \leq \frac{\varepsilon}{2}(\|\mathbf{u}^\varepsilon_{S,0}\| + Ra\|\theta^\varepsilon_{S,0}\|)^2, \tag{5.14}$$

for every $t \geq 0$.

Rearranging in (5.12), applying (5.13), (5.14), and recalling that $\phi^\varepsilon(0) = 0$ we infer

$$\sup_{s\in[0,t]} \|\phi^\varepsilon(s)\|^2 \leq C \sup_{s\in[0,t]} \left(\tilde{Ra}^2 + \|\theta_S^{0,\varepsilon}(s)\|^2_{L^3}\right) \left(\int_0^t Ra^2 \sup_{t'\in[0,r]} \|\phi^\varepsilon(t')\|^2 dr + \varepsilon(\|\mathbf{u}^\varepsilon_{S,0}\| + Ra\|\theta^\varepsilon_{S,0}\|)^2\right),$$

and hence with Grönwall's inequality we have

$$\sup_{s\in[0,t]} \|\phi^\varepsilon(s)\|^2 \leq \varepsilon(\|\mathbf{u}^\varepsilon_{S,0}\| + Ra\|\theta^\varepsilon_{S,0}\|)^2 \exp\left(C(Ra^2 t + 1) \sup_{s\in[0,t]} \left(\tilde{Ra}^2 + \|\theta_S^{0,\varepsilon}(s)\|^2_{L^3}\right)\right).$$

Choosing $\delta = \delta(Ra, \tilde{Ra}, t)$ sufficiently small we infer the desired bound, (5.9) by applying Proposition A.1 and using the condition (3.9). □

5.3 Comparing the Corrector to the Large Prandtl System

The harder case is to compare the corrector to the large Prandtl system. The reader may refer back to Lemma 2.3 for the for the analogous bounds for the toy model.

Lemma 5.2. *For each $\varepsilon > 0$ suppose $(\mathbf{u}^\varepsilon_S, \theta^\varepsilon_S)$ is a stationary solutions of (3.1)–(3.3) satisfying the uniform bound condition (3.9). Take $(\tilde{\mathbf{u}}^\varepsilon, \tilde{\theta}^\varepsilon)$ to be the solution of (5.5)–(5.6) with the initial condition $\theta^\varepsilon_{S,0} = \theta^\varepsilon_S(0)$. Then for any $\varepsilon > 0$ and any $t > 0$ there exist $C = C(\tilde{Ra}, Ra, t)$ and $\delta = \delta(\tilde{Ra}, Ra, t)$ both independent of $\varepsilon > 0$ such that*

$$\mathbb{E}\|\tilde{\theta}^\varepsilon(t) - \theta_S^{0,\varepsilon}(t)\|^{2\delta} \leq C\varepsilon^\delta. \tag{5.15}$$

Remark 5.1. *Note that $C(\tilde{Ra}, Ra, t) \to \infty$ as $t \to 0^+$. This is due to the fact that unlike $\mathbf{u}^\varepsilon_S$, $\tilde{\mathbf{u}}^\varepsilon$ does not satisfy an evolution equation and we do not prescribe an initial condition for $\tilde{\mathbf{u}}^\varepsilon$ in (5.5).*

Proof. Define $\tilde{\mathbf{v}}^\varepsilon = \mathbf{u}^\varepsilon_S - \tilde{\mathbf{u}}^\varepsilon$ and $\tilde{\phi}^\varepsilon = \theta^\varepsilon_S - \tilde{\theta}^\varepsilon$. Referring to (5.5)–(5.6) and to (3.1)–(3.3) we see that $\tilde{\phi}^\varepsilon$ satisfies

$$\partial_t\tilde{\phi}^\varepsilon - \Delta\tilde{\phi}^\varepsilon = \tilde{Ra}\cdot\tilde{v}^\varepsilon_d - \tilde{\mathbf{v}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon - \mathbf{u}^\varepsilon_S\cdot\nabla\tilde{\phi}^\varepsilon, \qquad \tilde{\phi}^\varepsilon(0) = 0,$$

and therefore testing with $\tilde{\phi}^\varepsilon$ and using that $\nabla\cdot\tilde{\mathbf{v}}^\varepsilon = 0$ we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}^\varepsilon\|^2 + \|\nabla\tilde{\phi}^\varepsilon\|^2 = \int(\tilde{Ra}\cdot\tilde{v}^\varepsilon_d - \tilde{\mathbf{v}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon)\tilde{\phi}^\varepsilon dx \leq \tilde{Ra}\|\tilde{\mathbf{v}}^\varepsilon\|\|\tilde{\phi}^\varepsilon\| + \|\tilde{\mathbf{v}}^\varepsilon\|_{L^6}\|\nabla\tilde{\phi}^\varepsilon\|\|\tilde{\theta}^\varepsilon\|_{L^3}.$$

Hence from standard Sobolev embeddings and the Poincaré inequality

$$\frac{d}{dt}\|\tilde{\phi}^\varepsilon\|^2 \leq C\left(\|\tilde{\theta}^\varepsilon\|^2_{L^3} + \tilde{Ra}^2\right)\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2.$$

Integrating in time we infer that

$$\sup_{s\in[0,t]}\|\tilde{\phi}^\varepsilon(s\wedge\tau)\|^2 \leq \sup_{s\in[0,t\wedge\tau]}\left(\|\tilde{\theta}^\varepsilon(s)\|^2_{L^3} + \tilde{Ra}^2\right)\int_0^{t\wedge\tau}\|\nabla\tilde{\mathbf{v}}^\varepsilon(t')\|^2 dt' \tag{5.16}$$

for any $t > 0$ and any stopping time $\tau \geq 0$.

We now turn to derive an evolution equation for $\mathbf{v}^\varepsilon$. Recalling that $\tilde{\mathbf{u}}^\varepsilon$ satisfies (5.7) and comparing this equation with (3.1) satisfied by $\mathbf{u}^\varepsilon_S$ we find (cf. (2.33))

$$\varepsilon(\partial_t\mathbf{u}^\varepsilon_S + \mathbf{u}^\varepsilon_S\cdot\nabla\mathbf{u}^\varepsilon_S) - \Delta\tilde{\mathbf{v}}^\varepsilon = \nabla q^\varepsilon + Ra\hat{\mathbf{k}}\tilde{\phi}^\varepsilon - \Delta\mathbf{w}^\varepsilon, \tag{5.17}$$

where, as above, $\mathbf{w}^\varepsilon$ maintains (5.8). From (5.5), (5.6), and (5.8), we find that $\tilde{\mathbf{u}}^\varepsilon$ satisfies

$$\begin{aligned} d\tilde{\mathbf{u}}^\varepsilon &= \partial_t \mathbf{w}^\varepsilon + RaA^{-1}(P(\hat{\mathbf{k}}d\tilde{\theta}^\varepsilon)) \\ &= \frac{1}{\varepsilon}(-\Delta \mathbf{w}^\varepsilon + \nabla q^\varepsilon) - RaA^{-1}\Big(P\Big(\hat{\mathbf{k}}\Big(\tilde{\mathbf{u}}^\varepsilon \cdot \nabla\tilde{\theta}^\varepsilon - \Delta\tilde{\theta}^\varepsilon - \tilde{Ra}\cdot \tilde{u}_d^\varepsilon\Big)\Big)\Big)dt + Ra\sum_{k=1}^N A^{-1}(P(\hat{\mathbf{k}}\sigma_k))dW^k . \end{aligned} \tag{5.18}$$

Multiplying (5.17) by ε^{-1}, subtracting the resulting system from (5.18) and rearranging we obtain

$$\begin{aligned} d\tilde{\mathbf{v}}^\varepsilon - \frac{1}{\varepsilon}\Delta\tilde{\mathbf{v}}^\varepsilon dt &= \frac{1}{\varepsilon}\left(\nabla\tilde{q}^\varepsilon + Ra\hat{\mathbf{k}}\tilde{\phi}^\varepsilon\right)dt \\ &+ Ra\left(A^{-1}\Big(P\Big(\hat{\mathbf{k}}\Big(\tilde{\mathbf{u}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon - \Delta\tilde{\theta}^\varepsilon + \tilde{Ra}\cdot\tilde{u}_d^\varepsilon\Big)\Big)\Big) - \mathbf{u}_S^\varepsilon\cdot\nabla\mathbf{u}_S^\varepsilon\right)dt - Ra\sum_{k=1}^N A^{-1}(P(\hat{\mathbf{k}}\sigma_k))dW^k , \end{aligned} \tag{5.19}$$

with $\nabla\cdot\tilde{\mathbf{v}}^\varepsilon = 0$. Using (5.19) we now estimate $\tilde{\mathbf{v}}^\varepsilon$ as follows. The Itō formula and (5.20) reveals that

$$\begin{aligned} d\|\tilde{\mathbf{v}}^\varepsilon\|^2 + \frac{2}{\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 dt =& \frac{2}{\varepsilon}Ra\langle\tilde{\phi}^\varepsilon, \tilde{v}_d^\varepsilon\rangle dt + 2Ra\left\langle A^{-1}\left(P\left(\hat{\mathbf{k}}\left(\tilde{\mathbf{u}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon - \Delta\tilde{\theta}^\varepsilon - \tilde{Ra}\cdot\tilde{u}_d^\varepsilon\right)\right)\right) - \mathbf{u}_S^\varepsilon\cdot\nabla\mathbf{u}_S^\varepsilon, \tilde{\mathbf{v}}^\varepsilon\right\rangle dt \\ &+ Ra^2\sum_{k=1}^N |A^{-1}(P(\hat{\mathbf{k}}\sigma_k))|^2 dt - 2Ra\sum_{k=1}^N\langle A^{-1}(P(\hat{\mathbf{k}}\sigma_k)), \tilde{\mathbf{v}}^\varepsilon\rangle dW^k \\ :=& (T_1 + T_2 + T_3 + T_4 + T_5 + T_6)dt + SdW. \end{aligned} \tag{5.20}$$

With the Young and Poincaré inequalities we have

$$|T_1| \le \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + \frac{C\mathrm{Ra}^2}{\varepsilon}\|\tilde{\phi}^\varepsilon\|^2. \tag{5.21}$$

For T_2 we use that A^{-1} is self-adjoint on H, $D(A)\subset H$ and that $\tilde{\mathbf{u}}^\varepsilon, \tilde{\mathbf{v}}^\varepsilon$ are divergence free, to obtain

$$|T_2| = 2Ra\left|\int \tilde{\mathbf{u}}^\varepsilon\cdot\nabla\tilde{\theta}^\varepsilon(A^{-1}\tilde{\mathbf{v}}^\varepsilon)_d dx\right| = 2Ra\left|\int \tilde{\mathbf{u}}^\varepsilon\cdot\nabla(A^{-1}\tilde{\mathbf{v}}^\varepsilon)_d\,\tilde{\theta}^\varepsilon dx\right| ,$$

where $(A^{-1}\tilde{\mathbf{v}}^\varepsilon)_d$ represents the third component of the vector field $A^{-1}\tilde{\mathbf{v}}^\varepsilon$. Hence (5.4) and the imbedding $H^2 \hookrightarrow L^\infty$ imply

$$|T_2| \le 2Ra\|\tilde{\mathbf{u}}^\varepsilon\|\|\tilde{\theta}^\varepsilon\|\|\nabla(A^{-1}\tilde{\mathbf{v}}^\varepsilon)\|_{L^\infty} \le CRa\|\tilde{\mathbf{u}}^\varepsilon\|\|\tilde{\theta}^\varepsilon\|\|\nabla\tilde{\mathbf{v}}^\varepsilon\| \le \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + \varepsilon C(Ra^4\|\tilde{\theta}^\varepsilon\|^4 + \|\tilde{\mathbf{u}}^\varepsilon\|^4).$$

Using (5.5), the triangle inequality, and (5.14) we have

$$\|\tilde{\mathbf{u}}^\varepsilon\| \le CRa\|\tilde{\theta}^\varepsilon\| + \|\mathbf{w}^\varepsilon(t)\| \le \|\mathbf{u}_{S,0}^\varepsilon\| + CRa(\|\theta_{S,0}^\varepsilon\| + \|\tilde{\theta}^\varepsilon\|). \tag{5.22}$$

Combining this observation with the previous bound we infer

$$|T_2| \le \frac{1}{4\varepsilon}\|\nabla\mathbf{v}^\varepsilon\|^2 + \varepsilon C(Ra^4(\|\tilde{\theta}^\varepsilon\|^4 + \|\theta_{S,0}^\varepsilon\|^4) + \|\mathbf{u}_{S,0}^\varepsilon\|^4) . \tag{5.23}$$

For the terms T_3 and T_4 we use the regularity of Stokes operator and (5.22) to obtain

$$\begin{aligned} |T_3| + |T_4| &\le \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + 4\varepsilon Ra^2(\|\tilde{\theta}^\varepsilon\|^2 + \tilde{Ra}^2\|\tilde{\mathbf{u}}^\varepsilon\|^2) \\ &\le \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + \varepsilon Ra^2(\tilde{Ra}^2 + 1)C(\|\tilde{\theta}^\varepsilon\|^2 + \|\theta_{S,0}^\varepsilon\|^2 + \|\mathbf{u}_{S,0}^\varepsilon\|^2) . \end{aligned} \tag{5.24}$$

To address T_5 we take advantage of an additional cancellation. Since $\mathbf{u}_S^\varepsilon = \tilde{\mathbf{v}}^\varepsilon + \tilde{\mathbf{u}}^\varepsilon$ we find

$$|T_5| = 2Ra|\langle\mathbf{u}_S^\varepsilon\cdot\nabla\tilde{\mathbf{u}}^\varepsilon, \tilde{\mathbf{v}}^\varepsilon\rangle| = 2Ra|\langle\mathbf{u}_S^\varepsilon\cdot\nabla\tilde{\mathbf{v}}^\varepsilon, \tilde{\mathbf{u}}^\varepsilon\rangle| \le \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + 4\varepsilon Ra^2\|\tilde{\mathbf{u}}^\varepsilon\|_{L^\infty}^2\|\mathbf{u}_S^\varepsilon\|^2 .$$

Next, using (5.5), (5.4), the imbedding $H^2 \hookrightarrow L^\infty$, and standard properties of analytic semigroups we obtain

$$\begin{aligned}\|\tilde{\mathbf{u}}^\varepsilon\|_{L^\infty} &\leq CRa\|\tilde{\theta}^\varepsilon\| + \|\mathbf{w}^\varepsilon(t)\|_{L^\infty} \leq CRa\|\tilde{\theta}^\varepsilon\| + \|P_{N_\varepsilon}\mathbf{u}^\varepsilon_{S,0}\|_{H^2} + \|\mathbf{y}^\varepsilon_{S,0}\|_{H^2}\\ &\leq CRa\|\tilde{\theta}^\varepsilon\| + \|P_{N_\varepsilon}\mathbf{u}^\varepsilon_{S,0}\|_{H^2} + Ra\|\theta^\varepsilon_{S,0}\|\end{aligned}$$

Recalling that N^ε is chosen such that $\varepsilon\lambda^2_{N^\varepsilon} \sim 1$ we conclude with the generalized Poincaré inequality and further standard manipulations that

$$\begin{aligned}|T_5| &\leq \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + C\varepsilon Ra^2(Ra^2(\|\tilde{\theta}^\varepsilon\|^2 + \|\theta^\varepsilon_{S,0}\|^2) + \|P_{N_\varepsilon}\mathbf{u}^\varepsilon_{S,0}\|^2_{H^2})\|\mathbf{u}^\varepsilon_S\|^2\\ &\leq \frac{1}{4\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 + CRa^4(\|\tilde{\theta}^\varepsilon\|^4 + \|\theta^\varepsilon_{S,0}\|^4 + \|\mathbf{u}^\varepsilon_{S,0}\|^4 + \|\mathbf{u}^\varepsilon_S\|^4). \end{aligned}\tag{5.25}$$

Finally we observe $|T_6| \leq CRa^2$. Combining the bounds (5.21)–(5.25) and rearranging in (5.20) we find

$$\begin{aligned}d\|\tilde{\mathbf{v}}^\varepsilon\|^2 + \frac{1}{\varepsilon}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 dt &\leq \frac{4Ra^2}{\varepsilon}\|\tilde{\phi}^\varepsilon\|^2 + \mathrm{Ra}^2\|\sigma\|^2 + C(1+Ra^4)(1+\tilde{Ra}^2)(\|\tilde{\theta}^\varepsilon\|^4 + \|\theta^\varepsilon_{S,0}\|^4 + \|\mathbf{u}^\varepsilon_{S,0}\|^4 + \|\mathbf{u}^\varepsilon_S\|^4 + 1)\\ &\quad - 2Ra\sum_{k=1}^N \langle A^{-1}(P(\hat{\mathbf{k}}\sigma_k)), \tilde{\mathbf{v}}^\varepsilon\rangle dW^k\,,\end{aligned}$$

where the constant $C > 0$ is independent of $Ra, \tilde{Ra}$ and $\varepsilon > 0$. By (5.5) we have that $\tilde{\mathbf{v}}^\varepsilon(0) = Q_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0}$, where $Q_{N^\varepsilon} := I - P_{N^\varepsilon}$ and P_{N^ε} is the projection onto the first N^ε eigenfunctions of the Stokes operator. Consequently for any $t \geq 0$, any stopping time τ

$$\begin{aligned}\int_0^{t\wedge\tau}\|\nabla\tilde{\mathbf{v}}^\varepsilon\|^2 dt \leq& \varepsilon\|Q_{N^\varepsilon}\mathbf{u}^\varepsilon_{S,0}\|^2 + 4Ra^2\int_0^{t\wedge\tau}(\|\tilde{\phi}^\varepsilon\|^2 + \varepsilon\|\sigma\|^2)dt'\\ &+ \varepsilon C(\mathrm{Ra}^4+1)(\tilde{Ra}^2+1)\int_0^{t\wedge\tau}\left[\|\tilde{\theta}^\varepsilon\|^4 + \|\theta^\varepsilon_{S,0}\|^4 + \|\mathbf{u}^\varepsilon_S\|^4 + \|\mathbf{u}^\varepsilon_{S,0}\|^4 + 1\right]dt'\\ &- \varepsilon Ra\sum_{k=1}^N\int_0^{t\wedge\tau}\langle A^{-1}(P(\hat{\mathbf{k}}\sigma_k)), \tilde{\mathbf{v}}^\varepsilon\rangle dW^k,\end{aligned}\tag{5.26}$$

where C is independent of $\varepsilon > 0$, $Ra, \tilde{Ra}$.

Next for any $\kappa > 0$ define the stopping times

$$\tau_\kappa := \inf_{t\geq 0}\left\{\|\tilde{\theta}^\varepsilon(t)\|^2_{L^3} \geq \kappa\right\}\,. \tag{5.27}$$

From this definition, (5.16), (5.26) we now infer

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\|\tilde{\phi}^\varepsilon(s\wedge\tau)\|^2\\ &\quad\leq 4Ra^2\left(\kappa + \tilde{Ra}^2\right)\int_0^t\mathbb{E}\sup_{s\in[0,t']}\|\tilde{\phi}^\varepsilon(s\wedge\tau)\|^2 dt' + \varepsilon\left(\kappa + \tilde{Ra}^2\right)(\mathbb{E}\|\mathbf{u}^\varepsilon_{S,0}\|^2 + Ra^2\|\sigma\|^2 t)\\ &\qquad + \varepsilon C(\kappa + \tilde{Ra}^4 + 1)(\mathrm{Ra}^4+1)\int_0^t\mathbb{E}\left[\|\mathbf{u}^\varepsilon_S\|^4 + \|\mathbf{u}^\varepsilon_{S,0}\|^4 + \|\tilde{\theta}^\varepsilon\|^4 + \|\theta^\varepsilon_{S,0}\|^4 + 1\right]dt,\end{aligned}$$

which implies with the Gronwall inequality that

$$\mathbb{E}\sup_{s\in[0,t]}\|\tilde{\phi}^\varepsilon(s\wedge\tau)\|^2 \leq \varepsilon\exp\left(C(Ra^4+1)\left(\kappa + \tilde{Ra}^4\right)t + 1\right)\mathcal{M}_\varepsilon(t)\,, \tag{5.28}$$

where

$$\mathcal{M}_\varepsilon(t) := \mathbb{E}\|\mathbf{u}^\varepsilon_{S,0}\|^2 + \int_0^t\mathbb{E}\left[\|\mathbf{u}^\varepsilon_S\|^4 + \|\mathbf{u}^\varepsilon_{S,0}\|^4 + \|\tilde{\theta}^\varepsilon\|^4 + \|\theta^\varepsilon_{S,0}\|^4 + \|\sigma\|^2 + 1\right]ds,$$

and the constant C is independent of κ, ε, Ra, and $\tilde{Ra}$.

Now in view of (3.9) and making another usage of Proposition A.1, we observe that $\mathcal{M}_\varepsilon$ is bounded independently of $\varepsilon > 0$. Therefore (5.28) is of the form (2.39). As above we define

$$E_{t,\kappa,\varepsilon} := \left\{ \sup_{s\in[0,t]} \|\tilde{\theta}^\varepsilon\|_{L^3}^2 \geq \kappa \right\} = \{\tau_\kappa \leq t\}$$

and as in (2.40) we infer from (A.3) that

$$\mathbb{P}(E_{t,\kappa,\varepsilon}) \leq Ce^{-\kappa\eta} \tag{5.29}$$

for a constant $C = C(\tilde{Ra}, t)$ independent of $\varepsilon > 0$, $\kappa > 0$. Thus, applying (5.29) to (5.28) as in (2.41) we now infer (5.15) completing the proof of Lemma 5.2. □

A Appendix: Moment Bounds For Stochastic Drift-Diffusion Equations

In this appendix we collect some moment bounds proved in [FGHR] which have been used extensively in the analysis above.

As in [FGHR] we consider the following class of stochastic divergence-free drift diffusion systems

$$d\xi + \mathbf{v}\cdot\nabla\xi dt = (\tilde{Ra}\cdot v_3 + \Delta\xi)dt + \sum_{k=1}^{N}\sigma_k dW^k, \quad \xi(0) = \xi_0 \tag{A.1}$$

evolving on the three dimensional domain $\mathcal{D} = [0, L]^2 \times [0, 1]$. Here $\tilde{Ra} > 0$ is a fixed parameter and $\mathbf{v} = (v_1, v_2, v_3)$ any sufficiently regular and adapted, divergence free vector field. Both $\mathbf{v}$ and ξ are supposed to satisfies the same mixed Dirichlet-Periodic boundary condition as $\mathbf{u}^\varepsilon, \theta^\varepsilon$ in (3.3). Recall that by the change of variable $T = \xi + \tilde{Ra}(1 - z)$ we may reformulate (A.1) as

$$dT + \mathbf{v}\cdot\nabla T dt = \Delta T dt + \sum_{k=1}^{N}\sigma_k dW^k, \quad T(0) = T_0 = \xi_0 + \tilde{Ra}(1 - z)\,, \tag{A.2}$$

where $\mathbf{v}$ and T satisfy boundary conditions as in (1.3). As such, bounds for ξ solving (A.1) immediately translate to bounds for T.

In [FGHR] we prove:

Proposition A.1. *Suppose that $\mathbf{v} \in L^2_{loc}([0,\infty); V_1 \cap (H^2(\mathcal{D}))^3) \cap C([0,\infty); H_1)$ a.s. and is $\mathcal{F}_t$-adapted. Fix any $p \geq 2$ and any initial condition $\xi_0 \in H \cap L^p(\mathcal{D})$ which is $\mathcal{F}_0$ measurable with*

$$\mathbb{E}\exp(\eta\|\xi_0\|_{L^p}^2) < \infty,$$

for some $\eta > 0$. Then there exists $\eta_0 = \eta_0(\sigma, \tilde{Ra}, p) > 0$ such that for any $t \geq 0$ and any positive $\eta \leq \eta_0$,

$$\mathbb{E}\exp\left(\frac{\eta}{2^{p/2+2}}\sup_{s\in[0,t]}\|\xi\|_{L^p}^2\right) \leq C_1\mathbb{E}\exp\left(\eta\|\xi_0\|_{L^p}^2 + \eta pt(\|\sigma\|_{L^p}^2 + 2^{p/2}(4\tilde{Ra}^2 + 1))\right) \tag{A.3}$$

for a constant $C = C(\tilde{Ra}, p)$ independent of t, η, ξ_0, and $\mathbf{v}$. Furthermore,

$$\mathbb{E}\exp\left(\frac{\eta}{2^{p/2+2}}\|\xi(t)\|_{L^p}^2\right) \leq C\mathbb{E}\exp\left(\eta(e^{-\kappa t}\|\xi_0\|^2)\right)\,, \tag{A.4}$$

where again $C = C(\tilde{Ra}, p, \|\sigma\|_{L^p}, \mathcal{D})$ and $\kappa = \kappa(\tilde{Ra}, \mathcal{D}) > 0$ are independent of t, η, ξ_0, and $\mathbf{v}$.

We now return to the infinite Prandtl system (3.4)–(3.5) and recall a bound analogous to (A.4) but which uses more of the specific structure for the advecting velocity field.

Proposition A.2. *Fix an initial condition $\theta_0^0 \in H$ which is $\mathcal{F}_0$ measurable, and let $\theta^0 = \theta^0(t, \theta_0^0)$ denote the corresponding solution to (3.4)–(3.5). There is a universal constant $\eta^* > 0$ such that for any $t > 0$ and $\eta \in (0, \eta^*]$, there exists $C = C(Ra, \tilde{Ra}) > 0$ such that*

$$\mathbb{E}\left(\exp\left(\eta\|\theta^0\|^2 + \frac{\eta e^{-t/4}}{4}\int_0^t \|\nabla\theta^0\|^2 ds\right)\right) \leq C\exp\left(\eta(1 + 4Ra\tilde{Ra})e^{-t/2}\|\theta_0^0\|^2\right).$$

The proof of Proposition A.2 can be found in [FGHR].

Remark A.1. *Using Proposition A.2 and (B.17) below we can easily establish the Lyapunov bound (4.1) with*

$$C_1 = \exp\left(\frac{CRa^4 e^{t^*/2}}{\eta^2} + Ra\tilde{Ra}\right).$$

B Gradient Estimates On the Markov Semigroup

In this section we establish the gradient bound for the Markov semigroup generated by (3.4)–(3.5) in order to prove (4.2). For this purpose we begin by briefly recalling how (4.2) is translated to a control problem through the use of Malliavin calculus. We refer to e.g. [Nua09] or [NP12] for further general background on this subject and to [HM06, HM11, FGHRT13] for the application of this formalism in a setting close to ours.

Define the random operators

$$\mathcal{J}_{0,t}\xi := \lim_{\delta\to 0}\frac{\theta^0(t, \theta_0 + \delta\xi, W) - \theta^0(t, \theta_0, W)}{\delta} \tag{B.1}$$

for any $\xi \in H$ and

$$\mathcal{A}_{0,t}w := \lim_{\delta\to 0}\frac{\theta^0(t, \theta_0, W + \delta\int_0^{\cdot} w) - \theta^0(t, \theta_0, W)}{\delta} \tag{B.2}$$

for any $w \in L^2(\Omega; L^2([0,t]; \mathbb{R}^N))$. Here $\mathcal{A}_{0,t}w = \langle \mathfrak{D}\theta^0, w\rangle$, where the unbounded operator $\mathfrak{D} : L^2(\Omega; H) \mapsto L^2(\Omega; L^2(0, t, \mathbb{R}^N) \otimes H)$ is the Malliavin derivative and w is any element in the domain of the dual operator δ of $\mathfrak{D}$.

For our purposes it is sufficient to recall that any $\mathcal{F}_t$-adapted process in $\in L^2(\Omega; L^2([0,t]; \mathbb{R}^N))$ belongs to the domain of δ and $\delta(w)$ corresponds to the Itō integral of w so that

$$\mathbb{E}\langle \mathfrak{D}X, w\rangle = \mathbb{E}\left(X\int_0^t w dW\right) \tag{B.3}$$

for any $X \in \mathrm{Dom}(\mathfrak{D})$ and any $\mathcal{F}_t$-adapted w. This is a special case of the *Malliavin integration by parts formula*. We furthermore recall that $\mathfrak{D}$ satisfies a chain rule namely that if $\phi \in C^1(H)$ and $\theta \in \mathrm{Dom}(\mathfrak{D})$ then $\phi(\theta) \in \mathrm{Dom}(\mathfrak{D})$ and

$$\mathfrak{D}\phi(\theta) = \nabla\phi(\theta)\mathfrak{D}\theta. \tag{B.4}$$

Combining (B.3)–(B.4) and making use of the Itō isometry we infer that,

$$\begin{aligned}\nabla P_t^0\phi(\theta_0)\xi =& \mathbb{E}\left(\nabla\phi(\theta^0(t,\theta_0))\mathcal{J}_{0,t}\xi\right) = \mathbb{E}\left(\phi(\theta^0(t,\theta_0))\int_0^t wdW\right) + \mathbb{E}\left(\nabla\phi(\theta^0(t,\theta_0))\left(\mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}w\right)\right)\\ \leq& \sqrt{P_t^0(|\phi(\theta)|^2)}\left(\mathbb{E}\int_0^t |w|^2 dt\right)^{1/2} + \sqrt{P_t^0(\|\nabla\phi(\theta)\|^2)}\left(\mathbb{E}\|\mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}w\|^2\right)^{1/2}\end{aligned} \tag{B.5}$$

for any $\phi \in C_b^1(H)$, $\theta_0 \in H$ and any (adapted) $w \in L^2(\Omega; L^2([0,t]; \mathbb{R}^N))$.

Our desired bound (4.2) follows from (B.5) if, for every $\xi \in H$ with $\|\xi\| = 1$ there is (adapted) $w = w(\xi) \in L^2([0,\infty);\mathbb{R}^N)$ such that

$$\mathbb{E}\|\mathcal{J}_{0,t}\xi - \mathcal{A}_{0,t}w(\xi)\|^2 \leq C\exp(2\eta\|\theta_0\|^2)\delta(t)\,, \tag{B.6}$$

$$\sup_{\|\xi\|=1}\mathbb{E}\int_0^\infty |w(\xi)|^2 dt \leq C\exp(2\eta\|\theta_0\|^2). \tag{B.7}$$

where $\delta(t) \to 0$ as $t \to \infty$ and C, η, and δ are independent of θ_0.

To solve the control problem (B.6)–(B.7) we observe that (B.1) and (B.2) admit explicit characterizations as linearizations of (3.4)–(3.5). For any $\xi \in H^0$ we let $\rho(t) = \rho(t,\xi) := \mathcal{J}_{0,t}\xi$, which satisfies

$$\partial_t\rho + \mathbf{u}^0\cdot\nabla\rho + \mathbf{v}^0\cdot\nabla\theta^0 = \tilde{Ra}\cdot v_d^0 + \Delta\rho, \quad -\Delta\mathbf{v}^0 = \nabla p + \mathrm{Ra}\hat{\mathbf{k}}\rho, \quad \nabla\cdot\mathbf{v}^0 = 0\,, \quad \rho(0) = \xi, \tag{B.8}$$

supplemented by boundary conditions as in (3.3).[9] On the other hand, setting $\tilde{\rho} := \mathcal{A}_{0,t}w$ for any $w \in L^2([0,t],\mathbb{R}^N)$ we have

$$\partial_t\tilde\rho + \mathbf{u}^0\cdot\nabla\tilde\rho + \tilde{\mathbf{v}}^0\cdot\nabla\theta^0 = \tilde{Ra}\cdot\tilde v_d^0 + \Delta\tilde\rho + \sum_{k=1}^N \sigma_k w_k,\ -\Delta\tilde{\mathbf{v}}^0 = \nabla p + \mathrm{Ra}\hat{\mathbf{k}}\tilde\rho,\ \nabla\cdot\tilde{\mathbf{v}}^0 = 0\,,\ \tilde\rho(0) = 0, \tag{B.9}$$

again with boundary conditions as in (3.3).

Denote $\bar\rho(t) = \bar\rho(t,\xi,w) = \rho - \tilde\rho$ and $\bar{\mathbf{v}} := \mathbf{v} - \tilde{\mathbf{v}}$ for any $w \in L^2([0,\infty);\mathbb{R}^N$ and $\xi \in H$. We now choose w as a function of ξ as follows. Let P_N be the projection on the first N eigenfunctions of the Laplacian with boundary conditions as in (3.3). Set $w(t) := \sigma^{-1}\lambda P_N\bar\rho$, where $\lambda > 0$ and N will be selected below.[10] Relative to this choice of $w = w(\xi)$, $\bar\rho$ satisfies

$$\partial_t\bar\rho + \mathbf{u}^0\cdot\nabla\bar\rho + \bar{\mathbf{v}}^0\cdot\nabla\theta^0 = \tilde{Ra}\cdot\bar v_d^0 + \Delta\bar\rho - \lambda P_N\bar\rho, \quad \Delta\bar{\mathbf{v}}^0 = \nabla p + \mathrm{Ra}\hat{\mathbf{k}}\bar\rho, \quad \nabla\cdot\bar{\mathbf{v}}^0 = 0, \quad \bar\rho(0) = \xi. \tag{B.10}$$

Testing (B.10) with $\bar\rho$ and $\bar{\mathbf{v}}^0$ respectively, and using that both $\mathbf{u}^0$ and $\bar{\mathbf{v}}^0$ are divergence free vector fields, we obtain

$$\frac{d}{dt}\|\bar\rho\|^2 + 2\|\nabla\bar\rho\|^2 + 2\lambda\|P_N\bar\rho\|^2 = 2\int_{\mathcal{D}}\left(\tilde{Ra}\bar v_d^0 - \bar{\mathbf{v}}^0\cdot\nabla\theta^0\right)\bar\rho dx \tag{B.11}$$

and

$$\|\nabla\bar{\mathbf{v}}^0\| \leq \mathrm{Ra}\|\bar\rho\|. \tag{B.12}$$

With standard Sobolev embeddings and (B.12) we have, for any $\eta > 0$,

$$\begin{aligned}\left|\int_{\mathcal{D}}\left(\tilde{Ra}\bar v_d^0 - \bar{\mathbf{v}}^0\cdot\nabla\theta^0\right)\bar\rho dx\right| \leq& \|\bar{\mathbf{v}}^0\|_{L^6}\|\nabla\theta^0\|\|\bar\rho\|_{L^3} + \tilde{Ra}\|\bar{\mathbf{v}}^0\|\|\bar\rho\|\\ \leq& C\|\nabla\bar{\mathbf{v}}^0\|\|\nabla\theta^0\|\|\bar\rho\|^{1/2}\|\nabla\bar\rho\|^{1/2} + \tilde{Ra}\|\nabla\bar{\mathbf{v}}^0\|\|\bar\rho\|\\ \leq& CRa\|\nabla\theta^0\|\|\bar\rho\|^{3/2}\|\nabla\bar\rho\|^{1/2} + Ra\tilde{Ra}\|\bar\rho\|^2\\ \leq& \|\nabla\bar\rho\|^2 + (C(Ra)^{4/3}\|\nabla\theta^0\|^{4/3} + Ra\tilde{Ra})\|\bar\rho\|^2\\ \leq& \|\nabla\bar\rho\|^2 + (\eta\|\nabla\theta^0\|^2 + C)\|\bar\rho\|^2\,,\end{aligned} \tag{B.13}$$

where $C = C(\mathrm{Ra},\tilde{Ra},\eta) = \frac{\tilde{C}Ra^4}{\eta^2} + Ra\tilde{Ra}$ and $\tilde{C}$ is a universal constant. Also since P_N and $-\Delta$ commute we have for $Q_N := I - P_N$

$$\|\nabla\bar\rho\|^2 = -\langle P_N\bar\rho,\Delta P_N\bar\rho\rangle - \langle Q_N\bar\rho,\Delta Q_N\bar\rho\rangle = \|\nabla P_N\bar\rho\|^2 + \|\nabla Q_N\bar\rho\|^2 \geq \|\nabla Q_N\bar\rho\|^2 \geq \lambda_N\|Q_N\bar\rho\|^2\,, \tag{B.14}$$

[9] Notice that (B.8) can also be written as

$$\partial_t\rho + (L\theta^0)\cdot\nabla\rho + (L\rho)\cdot\nabla\theta^0 = \tilde{Ra}(L\rho) + \Delta\rho, \quad \rho(0) = \xi.$$

where $L = RaA^{-1}P\hat{k}$ and A is the stokes operator, P; cf. (5.2) and (3.6) above. Similar formulations can also be given for (B.9), (B.10).

[10] Of course the choice of N will determine the number of modes subject to stochastic perturbation. Observe that w is well defined as $\{\sigma_k\}_{k=1}^N$ is the set of the first N (nonzero) eigenvectors of the Laplacian.

where the last inequality follows from the generalized Poincarè inequality. Choose $2\lambda = \lambda_N$ (with N to be chosen below) and combine (B.11) and (B.13) to infer

$$\frac{d}{dt}\|\bar{\rho}\|^2 + (\lambda_N - (\eta_0\|\nabla\theta^0\|^2 + C))\|\bar{\rho}\|^2 \leq 0,$$

and hence, since $\bar{\rho}(0) = \xi$,

$$\|\bar{\rho}(t)\|^2 \leq \|\xi\|^2 \exp\left(\eta_0 \int_0^t \|\nabla\theta^0\|^2 \, dr + (C - \lambda_N)t\right) . \tag{B.15}$$

Applying Proposition A.1 we conclude that, for any $\theta_0^0 \in H$, and $\eta \in (0, \eta_0]$,

$$\mathbb{E}\|\bar{\rho}(t)\|^2 \leq C\|\xi\|^2 \exp\left(\eta\|\theta_0^0\|^2 + (C + \eta - \lambda_N)t\right) ,$$

where $C = C(\mathrm{Ra}, \tilde{Ra})$ is independent of ξ and θ_0^0 and $t \geq 0$. By now choosing N large enough such that $\lambda_N > 2(C + \eta\|\sigma\|^2)$ we obtain

$$\mathbb{E}\|\bar{\rho}(t)\|^2 \leq C\|\xi\|^2 \exp\left(\eta\|\theta_0^0\|^2 - \frac{\lambda_N}{2}t\right) , \tag{B.16}$$

where $C = C(\mathrm{Ra}, \tilde{Ra})$ is independent of ξ and θ_0^0 and $t \geq 0$. This yields the first bound (B.6).

To obtain the second desired bound, (B.7), we use (B.16) to estimate

$$\mathbb{E}\int_0^\infty |w(\xi)|^2 dt = \|\sigma^{-1}\|^2\lambda_N^2\mathbb{E}\int_0^\infty \|P_N\bar{\rho}\|^2 \, dt \leq C\exp(\eta\|\theta_0^0\|^2) ,$$

where $C = C(\lambda_N, Ra, \tilde{Ra})$ is independent of θ_0^0 yielding (B.7). The bound (4.2) now follows.

Remark B.1. *We can use the same argument leading to* (B.15) *to show that*

$$\|\rho(t)\|^2 \leq \|\xi\|^2 \exp\left(\eta \int_0^t \|\nabla\theta^0\|^2 \, dr + Ct\right) .$$

That is, for any $\eta > 0$*,*

$$\|\mathcal{J}_{0,t}\| \leq \exp\left(\eta \int_0^t \|\nabla\theta^0\|^2 \, dr + Ct\right) , \tag{B.17}$$

where, as above, $C = C(Ra, \tilde{Ra}) = \frac{\tilde{C}Ra^4}{\eta^2} + Ra\tilde{Ra}$*.*

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