Qualitative properties of solutions to mixed-diffusion bistable equations

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Abstract

We consider a fourth-order extension of the Allen-Cahn model with mixed-diffusion and Navier boundary conditions. Using variational and bifurcation methods, we prove results on existence, uniqueness, positivity, stability, a priori estimates, and symmetry of solutions. As an application, we construct a nontrivial bounded saddle solution in the plane.

Keywords: higher-order equations, bilaplacian, extended Fisher-Kolmogorov equation

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1 Introduction

We study the following fourth-order equation with Navier boundary conditions

$$\begin{aligned} \Delta^2 u - \beta \Delta u &= u - u^3 && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{1.1}$$

These boundary conditions are relevant in many physical contexts [18] and they permit to rewrite (1.1) as a second order elliptic system with Dirichlet boundary conditions. In our best knowledge this model was analyzed only in one-dimension, see [33] and references therein. We discuss some of these results later in this introduction. In this paper we present results on existence, uniqueness, positivity, stability, a priori estimates, and symmetries of solutions in higher-dimensional domains when $\beta \geq \sqrt{8}$. The case $\beta < \sqrt{8}$ requires different approaches and techniques and we only prove partial results in this case.

The problem (1.1) is a stationary version of

$$\partial_t u + \gamma \Delta^2 u - \Delta u = u - u^3, \qquad \gamma > 0, \tag{1.2}$$

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which was first proposed in 1988 by Dee and van Saarloos [14] as a higher-order model for some physical, chemical, and biological systems. The right-hand side of (1.2) is of bistable type, meaning it has two constant stable states $u \equiv \pm 1$ separated by a third unstable state $u \equiv 0$, see [14]. The distinctive feature of this model is that the structure of equilibria is richer than its second order counterpart

$$\partial_t u - \Delta u = u - u^3, \tag{1.3}$$

giving rise to more complicated patterns and dynamics. The equation (1.3) is related to the Fisher-KPP equation (*Fisher-Kolmogorov-Petrovskii-Piscunov* or sometimes simply called *Fisher-Kolmogorov* equation)[1] proposed by Fisher [16] to model the spreading of an advantageous gene into a 1-dimensional population and mathematically analyzed by Kolmogorov, Petrovskii, and Piscunov [26].

The equilibria of (1.3) satisfy the well-known *Allen-Cahn* or *real Ginzburg-Landau* equation

$$-\Delta u = u - u^3, \tag{1.4}$$

which is a classical phase-transition model. Its associated energy functional

$$\frac{1}{2}\int |\nabla u|^2\, dx + \frac{1}{4}\int (|u|^2 - 1)^2\, dx \tag{1.5}$$

is used to describe the pattern and the separation of the (stable) phases ± 1 of a substance or a material within the van der Waals-Cahn-Hilliard gradient theory of phase transitions [9]. For instance, it has important physical applications in the study of interfaces in both gases and solids, e.g. for binary metallic alloys [1] or bi-phase separation in fluids [37]. In these models the function u describes the pointwise state of the material or the fluid. The constant equilibria corresponding to the global minimum points ± 1 of the potential $\frac{1}{4}(|u|^2 - 1)^2$ are called the pure phases, whereas other configurations u represent mixed states, and orbits connecting ± 1 describe phase transitions.

To understand the formation of more complex patterns in layering phenomena—observed for instance in concentrated soap solutions and metallic alloys—some nonlinear models for materials include second order derivatives in the energy functional. The basic model can be seen as an extension of (1.5), namely

$$\int [(\nabla^2 u)^2 + g(u)|\nabla u|^2 + W(u)]\, dx,$$

where $\nabla^2 u$ denotes the Hessian matrix of u. It appears as a simplification of a nonlocal model [23] analyzed in one-dimension in [7, 11, 30, 32] and in higher dimensions in [10, 17, 22]. In [22], the Hessian $\nabla^2 u$ is replaced by Δu as a simplification of the model and it was also proposed as model for phase-field theory of edges in anisotropic crystals in [38]. Finally, we also mention the study of amphiphilic films [29] and the description of the phase separations in ternary mixtures containing oil, water, and amphiphile, see [20], where the scalar order parameter u is related to the local difference of concentrations of water and oil.

[1]in the original model, the nonlinearity u^3 is replaced by u^2

These models of phase transition or phase separation motivate the study of the stationary solutions of (1.2). After scaling, equilibria of (1.2) considered in $\mathbb{R}^N$ solve

$$\Delta^2 u - \beta \Delta u = u - u^3. \tag{1.6}$$

We refer to (1.6) as the *Extended-Fisher-Kolmogorov* equation (EFK). This fourth-order model has been mostly investigated in the one-dimensional setting:

$$u'''' - \beta u'' + u^3 - u = 0. \tag{1.7}$$

When $\beta \in [\sqrt{8}, \infty)$, there is a full classification of bounded solutions of (1.7), which mirrors that of the second order equation. Specifically, each bounded solution is either constant, a unique kink (up to translations and reflection), or a periodic solution indexed by the first integral, whereas there are no pulses.

For $\beta \in [0, \sqrt{8})$, infinitely many kinks, pulses, and chaotic solutions appear. The structure of some of these solutions can be quantified by defining homotopy classes, but a full classification is not available. The threshold $\sqrt{8}$ is related to a change in stability of constant states $u = \pm 1$. The proof of these results rely on purely one-dimensional techniques, for instance, stretching arguments, phase space analysis, shooting methods, first integrals, etc. For more details on the one-dimensional EFK we refer to [6, 33] and the references therein.

For $N \geq 2$ let us mention [5], where the authors prove the analog of the Gibbon's conjecture and some Liouville-type results.

To prove our main results we treat (1.1) by a variational approach: fix the functional space $H := H^2(\Omega) \cap H^1_0(\Omega)$ associated with Navier boundary conditions (see [18] for a survey on Navier and other boundary conditions), where $H^2(\Omega)$ and $H^1_0(\Omega)$ denote the usual Sobolev spaces, and let $J_\beta : H \to \mathbb{R}$ be the energy functional given by

$$J_\beta(u) := \int_\Omega \left(\frac{|\Delta u|^2}{2} + \beta \frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2} \right) dx \qquad \text{for } u \in H. \tag{1.8}$$

It is standard to prove that a critical point u of J_β is a weak solution of (1.1), that is, u satisfies

$$\int_\Omega \Delta u \Delta v + \beta \nabla u \nabla v + (u^3 - u) v \, dx = 0 \qquad \text{for all } v \in H.$$

Extracting qualitative information of global minimizers is far from trivial, since many important tools used in second order problems for the analysis of global minimizers are no longer available. For example, one cannot use arguments involving the positive part $u^+ := \max\{u, 0\}$, absolute value, or rearrangements of functions since they do not belong to $H^2(\Omega)$ in general. Furthermore the validity of maximum principles (or more generally positivity preserving properties) is a delicate issue in fourth-order problems.

As a first step, we prove the existence, uniqueness, and bounds of positive solutions of (1.1). For the rest of the paper, $\lambda_1(\Omega) = \lambda_1 > 0$ denotes the first

Dirichlet eigenvalue of $-\Delta$ in Ω and *hyperrectangle* refers to a product of N bounded nonempty open intervals. We say that a solution $u \in H$ of (1.1) is *stable* if

$$J''_\beta(u)[v,v] = \int_\Omega |\Delta v|^2 + \beta|\nabla v|^2 + (3u^2-1)v^2\,dx \geq 0 \quad \text{for all } v \in H.$$

We say that u is *strictly stable* if the inequality is strict for any $v \not\equiv 0$.

Theorem 1.1. *Let $\beta > 0$ and $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a smooth bounded domain or a hyperrectangle. If $\lambda_1^2 + \beta\lambda_1 \geq 1$, then $u \equiv 0$ is the unique weak solution of* (1.1)*. If*

$$\lambda_1^2 + \beta\lambda_1 < 1\,, \tag{1.9}$$

then

1. *There is $\varepsilon > 0$ such that* (1.1) *admits for each $\beta \in (\bar\beta - \varepsilon, \bar\beta)$ a positive classical solution u, where $\bar\beta = \frac{1-\lambda_1^2}{\lambda_1}$.*
2. *For each $\beta \geq \frac{\sqrt{8}}{(\sqrt{27}-2)^{1/2}}$ there is a positive classical solution u of* (1.1) *such that $\|u\|_{L^\infty(\Omega)} \leq \frac{1}{\beta^2}(\frac{4+\beta^2}{3})^{\frac{3}{2}}$ and $\Delta u < \frac{\beta}{2}u$ in Ω.*
3. *For every $\beta \geq \sqrt{8}$ there exists a unique positive solution u of* (1.1)*. Moreover, this solution is strictly stable and satisfies $\|u\|_{L^\infty(\Omega)} \leq 1$.*

Our assumptions on Ω are needed for higher-order elliptic regularity results. We single out hyperrectangles to use them in the construction of saddle solutions and patterns. Indeed, by reflexion, positive solutions of (1.1) in regular polygons that tile the plane give rise to periodic planar patterns.

Observe that (1.9) holds for all big enough domains. As mentioned above, the threshold $\sqrt{8}$ is related to a change in the stability of constant states $u = \pm 1$. For $\beta \geq \sqrt{8}$ the states are saddle-node type whereas for $\beta < \sqrt{8}$ they are saddle-focus type. Hence in the latter case we can not expect u being bounded by 1. One can prove oscillations around one for radial global minimizers arguing as in [4, proof of Theorem 6]. Intuitively, for $\beta \geq \sqrt{8}$ the Laplacian is the leading term and the equation inherits the dynamics of the second order Allen-Cahn equation, while for $\beta \in (0, \sqrt{8})$ the bilaplacian increases its influence resulting in a much richer and complex set of solutions. We present numerical approximations[2] of positive solutions using minimization techniques in Figure 1 below.

Note that the first claim in Theorem 1.1 holds for any $\beta > 0$, but only for appropriate values of λ_1.

Theorem 1.1 follows directly from Theorem 3.3 in Section 3 and Theorem 11.1 in Section 11. The proof is based on variational and bifurcation techniques. For the variational part (Theorem 3.3), we minimize an auxiliary problem modified in a way to guarantee the sign and L^∞ bounds of global minimizers. Next, we prove that global minimizers of the auxiliary problem are local minimizers of

[2]Computed with FreeFem++ [21] and Mathematica 10.0, Wolfram Research Inc., 2014.

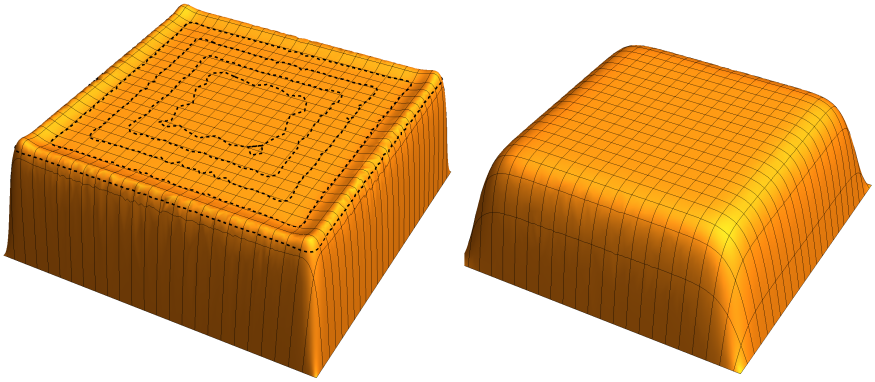

Figure 1: Numerical approximation of the global minimizer of (1.8) for $\Omega = [0, 50]^2$ with $\beta = 0.1$ (left) and $\beta = 4$ (right). The dotted lines represent the level set $\{u = 1\}$.

(1.8). The uniqueness is more involved, and is proved using stability, maximum principles, and bifurcation from a simple eigenvalue (Theorem 11.1).

We depict a numerical approximation[3] of the bifurcation branch in Figure 2. This branch can be continued even for $\beta < 0$; in this case, (1.1) is called the *Swift-Hohenberg equation*. See Section 13 for an example of such a branch and we refer again to [33] for a survey on (1.1) for $\beta < 0$, see also Remark 11.2 for a brief discussion on the explicit values of the bifurcation points.

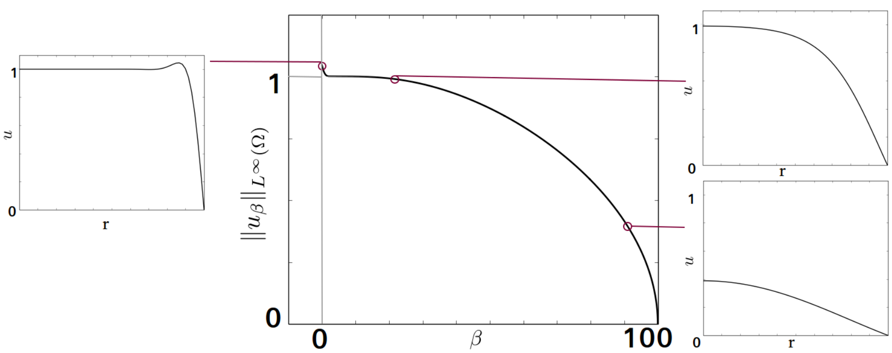

Figure 2: Numerical approximation of the bifurcation branch and some radial solutions. Here Ω is a ball in $\mathbb{R}^2$ of radius 240.483.

We now use the solution given by Theorem 1.1 to construct a saddle solution for (1.1). We call u a *saddle solution* if $u \not\equiv 0$ and $u(x, y)xy \geq 0$ for all $(x, y) \in \mathbb{R}^2$. See Figure 3 below.

Theorem 1.2. *For* $\beta \geq \sqrt{\frac{8}{\sqrt{27}-2}}$ *the problem* $\Delta^2 u - \beta \Delta u = u - u^3$ *in* $\mathbb{R}^2$ *has a saddle solution.*

[3]Computed with AUTO-07P [15].

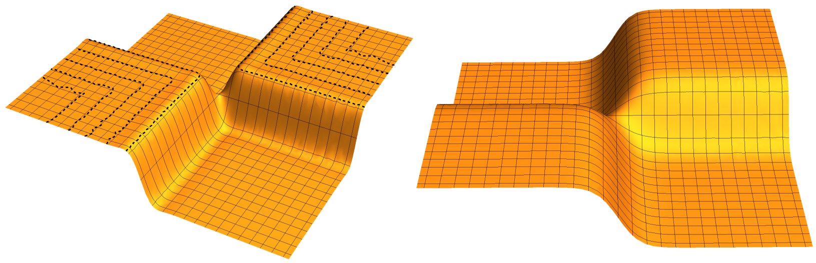

Figure 3: Saddle solutions for $\beta < \sqrt{8}$ (left) and $\beta \geq \sqrt{8}$ (right). The dotted lines are the level set $\{u = 1\}$.

We refer to [8, 13] for more information on saddle solutions for second order bistable equations.

In the following we explore properties of positive solutions with a special focus on stability and symmetry properties. The next result states that positive solutions of (1.1) are strictly stable if $\beta \geq \sqrt{8}$. This again shows that for $\beta \geq \sqrt{8}$, (1.1) recovers properties of the Allen-Cahn equation (1.4).

Theorem 1.3. *Let $\partial\Omega$ be of class $C^{1,1}$ and $\beta \geq \sqrt{8}$. Then any positive solution of* (1.1) *is strictly stable.*

The assumption $\beta \geq \sqrt{8}$ allows us to analyze stability properties of positive solutions using results on spectral properties of linear cooperative systems from [35] and [31].

For radial domains we can link stability and symmetry.

Theorem 1.4. *Let Ω be a ball or an annulus and let u be a stable solution of* (1.1) *with $\beta > \sqrt{12 - 2\lambda_1}$ such that $\|u\|_{L^\infty(\Omega)} \leq 1$. Then u is a radial function.*

Note that Theorem 1.4 does not assume positivity of solutions. We believe that the restriction on β is of a technical nature, but it is needed in our approach, see also Remark 7.1.

More generally, for reflectionally symmetric domains we have the following. We say that a domain is *convex and symmetric in the e_1-direction* if for every $x = (x_1, x_2, \ldots, x_N) \in \Omega$ we have $\{(tx_1, x_2, \ldots, x_N) : t \in [-1, 1]\} \subset \Omega$.

Proposition 1.5. *Let $\beta \geq \sqrt{8}$ and let $\Omega \subset \mathbb{R}^N$ be a hyperrectangle or a bounded smooth domain which is convex and symmetric in the e_1-direction. Then, any positive solution of* (1.1) *satisfies*

$$u(x_1, x_2, \ldots, x_N) = u(-x_1, x_2, \ldots, x_N) \qquad \text{for all } x = (x_1, \ldots, x_N) \in \Omega,$$
$$\partial_{x_1} u(x) < 0 \qquad \text{for all } x \in \Omega \text{ such that } x_1 > 0.$$

The proof of Proposition 1.5 follows a moving-plane argument for systems, see Figure 1 (left) for the described symmetry. Note that for $\beta \in (0, \sqrt{8})$ the solution oscillates when close to 1 in big enough domains, in particular it is not monotone, although it may still be symmetric. For positive solutions on balls, Proposition 1.5 implies Theorem 1.4 without the stability assumption.

Next, we focus on positivity of global minimizers under additional assumptions, however we conjecture that the positivity holds in general, even for $\beta < \sqrt{8}$.

The next theorem states positivity of global radial minimizers, that is, functions $u \in H_r := \{v \in H : v \text{ is radial in } \Omega\}$ such that $J_\beta(u) \leq J_\beta(v)$ for all $v \in H_r$.

Theorem 1.6. *Let Ω be a ball or an annulus,* (1.9) *hold, and let $u \in H$ be a global radial minimizer of* (1.8) *with $\|u\|_{L^\infty(\Omega)} \leq 1$. Then $\partial_r u$ does not change sign if Ω is a ball and $\partial_r u$ changes sign exactly once if Ω is an annulus.*

The global minimizer satisfies the uniform bound for $\beta \geq \sqrt{8}$, see Proposition 5.1 below. The proof of Theorem 1.6 relies on a new *flipping technique* that preserves differentiability while diminishing the energy and it is therefore well suited for variational fourth-order problems. Theorem 1.6 clearly implies that global radial minimizers do not change sign. For β large we can relax the assumptions on the solution, as stated in the following.

Corollary 1.7. *Let Ω be a ball or an annulus, $\beta > \sqrt{12} - 2\lambda_1$,* (1.9) *hold, and let u be a global minimizer of* (1.8) *in H. Then u is radial and does not change sign in Ω. Moreover, $\partial_r u$ does not change sign if Ω is a ball while $\partial_r u$ changes sign exactly once if Ω is an annulus.*

Proof. From $\beta > \sqrt{12} - 2\lambda_1$ and (1.9) follows $\beta \geq \sqrt{8}$ and from Proposition 5.1, we deduce that $\|u\|_{L^\infty(\Omega)} \leq 1$. Theorem 1.4 therefore implies that u is radial. Then u is in particular the global minimizer in H_r and the corollary follows from Theorem 1.6. □

As last theorem we present a uniqueness and convergence result for

$$\begin{aligned} \gamma\Delta^2 u - \Delta u &= u - u^3 && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{1.10}$$

when $\gamma \to 0$.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with the first Dirichlet eigenvalue $\lambda_1 < 1$. Let $\gamma \geq 0$ and u_γ be a global minimizer in H of*

$$\int_\Omega \left(\gamma \frac{|\Delta u|^2}{2} + \frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2} \right) dx. \tag{1.11}$$

There is an open neighborhood of 0 such that, for all $\gamma \in I \cap [0, 1]$: u_γ is the unique global minimizer in H and $u_\gamma > 0$ in Ω. Moreover, the function $I \to C^4(\Omega)$; $\gamma \mapsto u_\gamma$ is continuous and u_0 is the global minimizer of (1.11) *in $H^1_0(\Omega)$ with $\gamma = 0$.*

The paper is organized as follows. In Section 2 we prove a crucial auxiliary lemma for obtaining a priori estimates. Section 3 contains the proof of Theorem 3.3, the variational part in the proof of Theorem 1.1. In Section 4 and 5 we prove a priori estimates of solutions. Our study of the stability of solutions Theorem 1.3 is contained in Section 6. Section 7 is devoted to the study of radial symmetry of stable solutions Theorem 1.4. In Section 8 we present our flipping method and prove Theorem 1.6. Proposition 1.5 can be found in Section 9 and the saddle solution is constructed in Section 10. The bifurcation result Theorem 11.1 involved in the proof of Theorem 1.1 is contained in Section 11. The convergence to the second order equation, Theorem 1.8, is proved in Section 12. And finally, in Section 13 we include two numerical approximations of bifurcation branches.

To close this Introduction, let us mention that the case $\beta < \sqrt{8}$ requires a different approach due to possible oscillations around 1. In particular, in this case strict stability, uniqueness, and symmetry properties of positive solutions and global minimizers are not known.

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2 Auxiliary lemma

We prove a very helpful lemma that allows us to obtain a priori bounds on solutions. For the rest of the paper denote $C_0(\bar{\Omega})$ the space of continuous functions vanishing on $\partial\Omega$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta > 0$, $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$, and let $u \in C^4(\Omega) \cap C_0(\bar{\Omega})$ be a solution of $\Delta^2 u - \beta\Delta u = f(u)$ in Ω such that $\Delta u \in C_0(\bar{\Omega})$. Set $\overline{u} := \max_{\overline{\Omega}} u$, $\underline{u} := \min_{\overline{\Omega}} u$, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(s) := \frac{4}{\beta^2} f(s) + s$. Then*

$$\overline{u} \le \max_{[\underline{u},\overline{u}]} g \qquad \text{and} \qquad \underline{u} \ge \min_{[\underline{u},\overline{u}]} g. \tag{2.1}$$

Moreover,

1. *If $\overline{u} \le \max_{[0,\overline{u}]} g$ and $f < 0$ in $(1, \infty)$, then $\overline{u} \le \max_{[0,1]} g$.*
2. *If $\underline{u} \ge \min_{[\underline{u},0]} g$ and $f > 0$ in $(-\infty, -1)$, then $\underline{u} \ge \min_{[-1,0]} g$.*

Proof. Let $w \in C^2(\Omega) \cap C_0(\bar{\Omega})$ be given by $w(x) := -\Delta u(x) + \frac{\beta}{2} u(x)$. We prove only the second inequality in (2.1) as the first one follows similarly. Fix $x_0, \xi_0 \in \overline{\Omega}$ such that $u(x_0) = \underline{u}$ and $w(\xi_0) = \min_{\overline{\Omega}} w$. If $x_0 \in \partial\Omega$, then

$$\underline{u} = u(x_0) = 0 = g(u(x_0)) \ge \min_{[\underline{u},\overline{u}]} g$$

and the second inequality in (2.1) follows. If $\xi_0 \in \partial\Omega$, then $-\Delta u + \frac{\beta}{2}u \geq 0$ in Ω and the maximum principle implies that $\underline{u} = 0$ and the second inequality in (2.1) follows. If $x_0, \xi_0 \in \Omega$, then $w(\xi_0) \leq w(x_0) = -\Delta u(x_0) + \frac{\beta}{2}u(x_0) \leq \frac{\beta}{2}u(x_0)$. Since $-\Delta w(\xi_0) \leq 0$,

$$
\begin{aligned}
f(u(\xi_0)) + \frac{\beta^2}{4}u(\xi_0) &= \Delta^2 u(\xi_0) - \beta\Delta u(\xi_0) + \frac{\beta^2}{4}u(\xi_0) \\
&= -\Delta w(\xi_0) + \frac{\beta}{2}w(\xi_0) \leq \frac{\beta^2}{4}u(x_0),
\end{aligned}
$$

that implies $\underline{u} \geq \min_{[\underline{u}, \overline{u}]} g$.

We now prove the claim 2. only as claim 1. is analogous. Assume $\underline{u} \leq -1$, otherwise the statement is trivial. Since $f > 0$ in $(-\infty, -1)$, then $g(s) > s$ in $(-\infty, -1)$, and therefore $\min_{[\underline{u}, -1]} g \geq \min_{[\underline{u}, -1]} s = \underline{u}$. Thus, if $\underline{u} \geq \min_{[\underline{u}, 0]} g$ then $\underline{u} \geq \min_{[-1, 0]} g$ as claimed. ☐

2.1 Regularity

In this section we prove two technical regularity results. The first one is rather standard application of known arguments in the fourth order setting and we include details for reader's convenience. The second lemma is more subtle and the crucial point is that the constants in bounds are independent of parameter γ. As such it forms the cornerstone of the proof of Theorem 1.8.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a smooth bounded domain or hyperrectangle and fix $\beta, \gamma > 0$, and $f \in L^\infty(\Omega)$. Let $u \in H$ be a weak solution of $\gamma\Delta^2 u - \beta\Delta u = f$, that is,*

$$
\int_\Omega \gamma\Delta u\Delta\phi + \beta\nabla u\nabla\phi - f\phi\, dx = 0 \qquad \forall\phi \in H. \tag{2.2}
$$

Then for each $p > 1$ one has $u \in W^{4,p}(\Omega) \cap C^3(\overline{\Omega})$ with $u = \Delta u = 0$ on $\partial\Omega$ and

$$
\|u\|_{W^{4,p}(\Omega)} \leq C\|f\|_{L^\infty(\Omega)},
$$

where $C = C(\beta, \gamma, p, \Omega)$. In addition, if $f \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, then $u \in C^{4,\alpha}(\overline{\Omega})$ with $\Delta u \in C_0(\overline{\Omega})$ and

$$
\|u\|_{C^{4,\alpha}(\Omega)} \leq \tilde{C}\|f\|_{C^\alpha(\Omega)}, \tag{2.3}
$$

where $\tilde{C} = \tilde{C}(\beta, \gamma, p, \Omega)$.

Proof. Assume first that $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain. By Riesz representation theorem there are weak solutions $\bar{u}, \bar{v} \in H^1_0(\Omega)$ of the equations

$$
-\Delta\bar{v} + \frac{\beta}{\gamma}\bar{v} = f \quad \text{and} \quad -\gamma\Delta\bar{u} = \bar{v} \qquad \text{in } \Omega. \tag{2.4}
$$

Then, by [19, Lemma 9.17] applied to $\bar{v}$ and $\bar{u}$, for every $0 < p < \infty$ there is $C_1 > 0$ such that $\|\bar{v}\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$ and [28, Ch 9 Sec 2 Thm 3] $\|\bar{u}\|_{W^{4,p}(\Omega)} \leq C\|\bar{v}\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$. By embedding theorems $\bar{u} \in C^3(\overline{\Omega})$ and $\bar{u} = 0 = \bar{v} = -\gamma\Delta\bar{u}$ on $\partial\Omega$.

By integration by parts, $\bar{u}$ satisfies (2.2) and $u \equiv \bar{u}$ in Ω, since weak solutions of (2.2) are unique. Finally, if $f \in C^\alpha(\Omega)$, by Schauder estimates [27, Thm. 6.3.2], $\bar{v} \in C^{2,\alpha}(\overline{\Omega})$, $u \in C^{4,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and (2.3) holds.

Now without loss of generality assume that Ω is a hyperrectangle of the form $\Omega = \prod_{i=1}^N [0, l_i]$ for some $l_i > 0$, $i = 1, \ldots, N$. Let $\bar{u}, \bar{v} \in H_0^1(\Omega)$ be weak solutions of (2.4). Using odd reflections and the Dirichlet boundary conditions we extend $\bar{u}$ and $\bar{v}$ to $\tilde{\Omega} = \prod_{i=1}^N [-l_i, 2l_i]$ and obtain weak solutions of (2.4) defined in $\tilde{\Omega}$. Then interior regularity [28, Theorem 1 Sec 4 Ch 9] implies that for any $\bar{\Omega}_1 \subset \Omega_2 \subset \tilde{\Omega}$

$$\|\bar{u}\|_{W^{4,p}(\Omega_1)} \leq C(\|f\|_{L^\infty(\Omega_2)} + \|\bar{u}\|_{L^2(\Omega_2)})\,. \tag{2.5}$$

Note that we replaced L^p by L^2 on the right hand side, which can be done due to $W^{4,p_1} \hookrightarrow L^p$ for appropriate $p_1 < p$ and iteration in (2.5). Also, $\|\bar{u}\|_{L^2(\tilde{\Omega})} \leq C\|f\|_{L^2(\tilde{\Omega})}$ by testing (2.4) by $\bar{v}$ and $\bar{u}$ respectively and using standard estimates.

As before, $u \equiv \bar{u}$ in Ω by integration by parts and uniqueness of weak solutions. Finally, (2.3) follows analogously from interior Schauder estimates [27, Thm. 7.11]. $\square$

Lemma 2.3. *Let Ω be a smooth domain, $\beta, \gamma > 0$, and let $u \in H \cap L^\infty(\Omega)$ be a weak solution of $\gamma\Delta^2 u - \beta\Delta u = u - u^3$ in H. There is $\gamma_0 = \gamma_0(\beta, p, \Omega) > 0$ and $C = C(\beta, p, \Omega)$ such that for every $p > 1$ and $\gamma \in (0, \gamma_0)$ one has $u \in W^{6,p}(\Omega)$ and*

$$\|u\|_{W^{6,p}(\Omega)} \leq C(1 + \|u\|^9_{L^\infty(\Omega)}). \tag{2.6}$$

Proof. In this proof, C denotes different positive constants which depend on Ω, β, and p, but are *independent* of γ.

Since $u \in L^\infty(\Omega)$ we have by bootstrap and Lemma 2.2 that $u \in C^{4,\alpha}(\bar{\Omega})$ with $\Delta u \in C^0(\overline{\Omega})$. Then (u, v) with $v = -\gamma\Delta u$ and $f = u - u^3$ solves (2.4) in the classical sense. By [28, Ch. 8. Sec. 5. Thm 6], there is $\gamma_0 = \gamma_0(\beta, p, \Omega) > 0$ such that, for every $\gamma \in (0, \gamma_0)$,

$$\|v\|_{W^{2,p}(\Omega)} \leq C(1 + \|u\|^3_{L^\infty(\Omega)})\,.$$

On the other hand, $-\beta\Delta u = u - u^3 - \gamma\Delta^2 u = u - u^3 + \Delta v$ and by [19, Lemma 9.17] for every $p \in (0, \infty)$ we have

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|u - u^3 + \Delta v\|_{L^p(\Omega)} \leq C(1 + \|u\|^3_{L^\infty(\Omega)})\,. \tag{2.7}$$

In particular, $\|\Delta u\|_{L^p(\Omega)} \leq C(1 + \|u\|^3_{L^\infty(\Omega)})$. Set $w = \Delta u$. Since $u \in C^{4,\alpha}(\Omega)$, we have that w is a weak solution of

$$\gamma\Delta^2 w - \beta\Delta w = f_1 := \Delta(u - u^3) = w - 6u|\nabla u|^2 - 3u^2 w \quad \text{in } \Omega$$

with $w = 0$ on $\partial\Omega$. Moreover, $\Delta w = \Delta^2 u = \frac{1}{\gamma}(u - u^3 + \beta\Delta u) = 0$ on $\partial\Omega$, since $u \in C^{4,\alpha}(\bar{\Omega})$ and $u = \Delta u = 0$. Observe that $\|f_1\|_{L^p(\Omega)} \leq C(1 + \|u\|^7_{L^\infty(\Omega)})$. Then we can repeat the argument for w and f_1 instead of u and f respectively to obtain that $w \in C^{4,\alpha}(\Omega)$ and

$$\|w\|_{W^{2,p}(\Omega)} \leq C(1 + \|u\|^7_{L^\infty(\Omega)})\,. \tag{2.8}$$

Finally, denote $\xi := \Delta w$. Then $\xi \in C^{2,\alpha}(\bar{\Omega})$, $\|\xi\|_{L^p(\Omega)} \leq C(1 + \|u\|^7_{L^\infty(\Omega)})$, and ξ is a weak solution of $\gamma\Delta^2\xi - \beta\Delta\xi = f_2 := \Delta f_1$ and $\xi = 0$ on $\partial\Omega$. Additionally, $\Delta\xi = \Delta^2 w = \frac{1}{\gamma}(f_1 + \beta\Delta w) = 0$ on $\partial\Omega$, where we used $u = \Delta u = w = \Delta w$ on $\partial\Omega$. Here we are fundamentally using that f_1 vanishes on $\partial\Omega$. Note that $\|f_2\|_{L^p(\Omega)} \leq C(1 + \|u\|^9_{L^\infty(\Omega)})$. Thus we can iterate the above procedure one more time with u and f replaced by ξ and f_2 respectively, to obtain

$$\|\xi\|_{W^{2,p}(\Omega)} \leq C(1 + \|u\|^9_{L^\infty(\Omega)})\,. \tag{2.9}$$

Note that we cannot iterate anymore since f_2 is not vanishing on $\partial\Omega$, and therefore we cannot obtain boundary conditions for higher derivatives. Finally, (2.6) follows by (2.7), (2.8), (2.9), and [28, Ch.9 Sec. 2 Thm 3]. $\square$

3 Existence of positive solutions

For any $\beta > 0$ we fix the following constants for the rest of the paper

$$C_\beta := \sqrt{\frac{4+\beta^2}{4}}, \quad M_\beta := \frac{1}{\beta^2}\left(\frac{4+\beta^2}{3}\right)^{\frac{3}{2}}, \quad K_0 := \sqrt{\frac{8}{\sqrt{27}-2}} \approx 1.58. \tag{3.1}$$

Remark 3.1. *i)* $C_\beta > 1$ *is the unique positive root of* $h(s) = \frac{4}{\beta^2}(s - s^3) + s$. *In particular,* $h > 0$ *in* $(0, C_\beta)$.

ii) $M_\beta \geq 1$ *is the maximum value of* h *in* $(0, \infty)$ *attained at* $\sqrt{\frac{\beta^2+4}{12}}$. *Note that if* $\beta \geq \sqrt{8}$ *then* h *is increasing on* $(0, 1)$.

iii) $M_\beta \leq C_\beta$ *if* $\beta \geq K_0$.

Lemma 3.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain,* $\beta > 0$, $f : \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(s) := \begin{cases} -\frac{\beta^2}{4}s & \text{if } s < 0, \\ s - s^3 & \text{if } s \in [0, C_\beta], \\ C_\beta - C_\beta^3 = -\frac{\beta^2}{4}C_\beta & \text{if } s > C_\beta, \end{cases} \tag{3.2}$$

and let u *be a classical solution of*

$$\begin{aligned} \Delta^2 u - \beta\Delta u &= f(u) && \text{in } \Omega, \\ u = \Delta u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Then $0 \leq u \leq M_\beta$. *Moreover, if* $\beta \geq \sqrt{8}$, *then* $u \leq 1$. *In particular, if* $\beta \geq K_0$, *then* u *is a classical solution of* (1.1).

Proof. By Lemma 2.1 (and using the same notation) $\underline{u} \geq \min_{\mathbb{R}} g \geq 0$ by the definition of C_β, that is, $u \geq 0$ in Ω. On the other hand, again by Lemma 2.1 we have that $\overline{u} \leq \max_{[0,\overline{u}]} g = \max_{[0,1]} g = M_\beta$.

If $\beta \geq \sqrt{8}$, then g is nondecreasing in $[0,1]$, and therefore $\overline{u} \leq \max_{[0,\overline{u}]} g$ implies by Lemma 2.1 that $\overline{u} \leq \max_{[0,1]} g = g(1) = 1$. Finally, if $\beta \geq K_0$, then $M_\beta \leq C_\beta$, and thus $f(u(x)) = u(x) - u(x)^3$ for all $x \in \Omega$, that is, u solves (1.1). □

Theorem 3.3. *Let $\beta > 0$ and $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ be a smooth bounded domain or a hyperrectangle. If $\lambda_1^2 + \beta\lambda_1 \geq 1$, then $u \equiv 0$ is the unique weak solution of* (1.1)*. If $\lambda_1^2 + \beta\lambda_1 < 1$, then for $\beta \geq \frac{\sqrt{8}}{(\sqrt{27}-2)^{1/2}}$ there is a positive classical solution u of* (1.1) *such that $\|u\|_{L^\infty(\Omega)} \leq \frac{1}{\beta^2}(\frac{4+\beta^2}{3})^{\frac{3}{2}}$ and $\Delta u < \frac{\beta}{2}u$ in Ω. Additionally, if $\beta \geq \sqrt{8}$ then $\|u\|_{L^\infty(\Omega)} \leq 1$ and u is the unique positive solution of* (1.1)*.*

Proof. Let u be a weak solution of (1.1) and $\lambda_1^2 + \beta\lambda_1 \geq 1$. Assume there is a nontrivial weak solution $u \in H$ of (1.1). Then, by testing equation (1.1) with u we get that

$$0 = \int_\Omega |\Delta u|^2 + \beta|\nabla u|^2 + u^4 - u^2\,dx > (\lambda_1^2 + \beta\lambda_1 - 1)\int_\Omega u^2\,dx \geq 0,$$

by the Poincaré inequality. This is a contradiction and therefore $u \equiv 0$ is the unique weak solution if $\lambda_1^2 + \beta\lambda_1 > 1$.

Now, assume $\lambda_1^2 + \beta\lambda_1 < 1$ and let f be as in (3.2), and

$$J : H \to \mathbb{R}; \qquad J(u) := \int_\Omega \frac{|\Delta u|^2}{2} + \beta\frac{|\nabla u|^2}{2} - F(u)\,dx, \tag{3.4}$$

where $F(s) := \int_0^s f(t)\,dt$. Note that $F(s) \leq \frac{1}{4}$ for all $s \in \mathbb{R}$. Thus $J(u) \geq -\frac{|\Omega|}{4}$ for all $u \in H$. A standard lower semicontinuity argument shows that J attains a nontrivial global minimizer u in H and that u is a weak solution of $\Delta^2 u - \beta\Delta u = f(u)$ in Ω with Navier boundary conditions. Observe that the minimizer is nontrivial, since $\lambda_1^2 + \beta\lambda_1 < 1$ implies $J(\delta\phi_1) < 0$ for sufficiently small $\delta > 0$, where ϕ_1 is the first Dirichlet eigenfunction of the Laplacian.

By Lemma 2.2, $u \in C^4(\Omega)\bigcap C(\overline{\Omega})$ and $\Delta u \in C_0(\Omega)$. By Lemma 3.2, u is a solution of (1.1) satisfying $0 \leq u \leq M_\beta$ if $\beta \geq K_0$; and $0 \leq u \leq 1$ if $\beta \geq \sqrt{8}$.

The strict positivity of u (recall $u \not\equiv 0$) and $-\Delta u + \frac{\beta}{2}u$ is a consequence of the maximum principle for second order equations and the following decomposition into a system

$$-\Delta u + \frac{\beta}{2}u = w, \quad -\Delta w + \frac{\beta}{2}w = (1 + \frac{\beta^2}{4})u - u^3 \quad \text{in } \Omega, \quad u = w = 0 \quad \text{on } \partial\Omega,$$

where $(1 + \frac{\beta^2}{4})u - u^3 \geq 0$ since $0 \leq u \leq M_\beta \leq C_\beta$ for $\beta \geq K_0$.

Finally, the uniqueness of the positive solution for $\beta \geq \sqrt{8}$ follows from Theorem 11.1. □

4 A priori bounds

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\beta > 0$, and let u be a classical solution of* (1.1).

i) If u is nonnegative in Ω, then $\|u\|_{L^\infty(\Omega)} \le M_\beta$.

ii) If $\beta \ge \sqrt{8}$ and u is nonnegative in Ω, then $\|u\|_{L^\infty(\Omega)} \le 1$.

iii) If $\beta \ge \sqrt{8}$ and $\|u\|_{L^\infty(\Omega)} < \sqrt{\frac{\beta^2}{2}+1}$, then $\|u\|_{L^\infty(\Omega)} \le 1$.

Proof. We use the notation of Lemma 2.1 with $f(s) = s - s^3$. We prove claim *iii*) first. Assume without loss of generality that $\|u\|_{L^\infty(\Omega)} = \overline{u} < \sqrt{\frac{\beta^2}{2}+1}$. By direct computation $g(s) < -s$ for $s \in (-\sqrt{\frac{\beta^2}{2}+1}, 0)$. Therefore

$$\max_{s\in[-\overline{u},0]} g(s) < \max_{s\in[-\overline{u},0]} -s = \overline{u}. \tag{4.1}$$

Moreover, by Lemma 2.1, $\overline{u} \le \max\limits_{[-\underline{u},\overline{u}]} g \le \max\limits_{[-\overline{u},\overline{u}]} g$. Then, in virtue of (4.1), we get $\overline{u} \le \max\limits_{[0,\overline{u}]} g$. It follows from Lemma 2.1 and $\beta \ge \sqrt{8}$ that $\overline{u} \le \max\limits_{[0,1]} g = 1$ as claimed.

Now, let $\beta > 0$ and assume $\underline{u} \ge 0$. By Lemma 2.1 and Remark 3.1, $\overline{u} \le \max\limits_{[0,\infty]} g = g\left(\sqrt{\frac{1}{3}+\frac{\beta^2}{12}}\right) = M_\beta$ and the first claim follows. If $\beta \ge \sqrt{8}$, then g is nondecreasing in $[0,1]$ and, by Lemma 2.1, $\overline{u} \le \max\limits_{[0,1]} g = g(1) = 1$. ☐

5 Bounds for the global minimizer

Similar a priori bounds were obtained in [5] for unbounded domains.

Proposition 5.1. *Let v be a global minimizer of* (1.1) *in H. Then*

$$\|v\|_{L^\infty(\Omega)} \le \frac{\sqrt{4+\beta^2}}{2} \quad \text{if } \beta \ge \sqrt{\frac{8}{\sqrt{27}-2}} \quad \text{and} \quad \|v\|_{L^\infty(\Omega)} \le 1 \quad \text{if } \beta \ge \sqrt{8}.$$

When $\beta \in (0, \sqrt{8})$ we expect that global minimizers are not bounded by one in big enough domains, see Figure 1 and Figure 2 below.

Proof. Assume $\beta \ge K_0$, let J_β be as in (1.8), and let v be its global minimizer in H, i.e. $J_\beta(v) \le J_\beta(w)$ for all $w \in H$. Further, for C_β as defined in (3.1) set $f : \mathbb{R} \to \mathbb{R}$,

$$f(s) := \begin{cases} s - s^3 & \text{if } s \in [-C_\beta, C_\beta], \\ \operatorname{sign}(s)(C_\beta - C_\beta^3) = -\frac{\beta^2}{4}\operatorname{sign}(s) C_\beta & \text{otherwise }, \end{cases}$$

and let J be as in (3.4) with this choice of f. Let u be the global minimizer of J in H. By Lemma 2.2, u is a classical solution of (3.3). Using Lemma 2.1 (and its notation), since $g \geq 0$ in $(0, \infty)$, we have $\underline{u} \geq \min_{[\underline{u}, \overline{u}]} g = \min_{[\underline{u}, 0]} g$, and consequently $\underline{u} \geq \min_{[-1,0]} g \geq -M_\beta$. If $\beta \geq \sqrt{8}$, we additionally have $\underline{u} \geq \min_{[-1,0]} g = -1$.

Replacing u by $-u$ and noting $\beta \geq K_0$ we conclude $\|u\|_{L^\infty(\Omega)} \leq M_\beta \leq C_\beta$ and $\|u\|_{L^\infty(\Omega)} \leq 1$ if $\beta \geq \sqrt{8}$. In particular $f(u(x)) = u(x) - u^3(x)$, and therefore

$$J(u) = J_\beta(u) \geq J_\beta(v) \geq J(v)\,,$$

where the last inequality is strict if $\|v\|_{L^\infty(\Omega)} > C_\beta$, a contradiction to the minimality of u. If $\beta \geq \sqrt{8}$ and $\|v\|_{L^\infty(\Omega)} > 1$, then, by Lemma 4.1 part *iii)*, $\|v\|_{L^\infty(\Omega)} \geq \sqrt{\frac{\beta^2}{2} + 1} > C_\beta$ and we obtain a contradiction as above. □

6 Stability of positive solutions

The proof of Theorem 1.3 is an easy consequence of the following.

Proposition 6.1. *Assume that $\beta \geq \sqrt{8}$, and let u be a positive solution of* (1.1)*. Then,*

$$\mu_1 = \inf_{v \in H} \frac{\int_\Omega |\Delta v|^2 + \beta |\nabla v|^2 + (u^2 - 1) v^2 \, dx}{\int_\Omega v^2 \, dx} = 0\,.$$

Indeed, assume for a moment that Proposition 6.1 holds. Then we have.

Proof of Theorem 1.3. Let $\mathcal{H} := \{v \in H \ : \ \|v\|_{L^2(\Omega)} = 1\}$. By Proposition 6.1,

$$\inf_{v \in \mathcal{H}} \int_\Omega |\Delta v|^2 + \beta |\nabla v|^2 + (3u^2 - 1) v^2 \, dx = \mu_1 + 2 \inf_{v \in \mathcal{H}} \int_\Omega u^2 v^2 \, dx > 0$$

and Theorem 1.3 follows. □

The proof of Proposition 6.1 uses the following result adjusted to our situation.

Theorem 6.2. *Let* $L := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$ *and M be a 2×2 matrix with real entries such that*

1. *$-M$ is* essentially positive*, that is, $-M_{12} \geq 0$ and $-M_{21} \geq 0$ in Ω.*
2. *M is* fully coupled*, that is, $M_{1,2} \not\equiv 0$ and $M_{2,1} \not\equiv 0$.*
3. *There is a* positive strict supersolution *ϕ of $L + M$, i.e., a function $\phi > 0$ such that $(L + M)\phi > 0$ in Ω,*

Then there are $\widetilde{v}, \widetilde{w} \in W^{2,N}_{loc}(\Omega) \bigcap C_0(\Omega)$ *unique (up to normalization) positive functions such that*

$$(L + M) \begin{pmatrix} \widetilde{v} \\ \widetilde{w} \end{pmatrix} = \lambda_0 \begin{pmatrix} \widetilde{v} \\ \widetilde{w} \end{pmatrix}, \tag{6.1}$$

where $\lambda_0 > 0$ *is the smallest eigenvalue (smallest real part) of* (6.1).

Moreover, there are unique (up to normalization) positive functions $v, w \in W^{2,N}_{loc}(\Omega) \bigcap C_0(\Omega)$, *such that*

$$(L + M) \begin{pmatrix} v \\ w \end{pmatrix} = \lambda_B\, B \begin{pmatrix} v \\ w \end{pmatrix}, \tag{6.2}$$

where $\lambda_B > 0$ *is the smallest eigenvalue (smallest real part) of* (6.2) *and* $B \not\equiv 0$ *is a matrix with* $B_{ij} \in C(\bar{\Omega})$ *and* $B_{ij} \geq 0$.

Theorem 6.2 is a particular case of [35, Theorem 1.1] and [31, Theorem 5.1]. In [31, Theorem 5.1] the result is formulated for matrices $B \in C_0(\bar{\Omega})$, but the same proof applies for $B \in C(\bar{\Omega})$.

Proof of Proposition 6.1. Let

$$L := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}, \qquad \widetilde{M} := \begin{pmatrix} \frac{\beta}{2} & -1 \\ u^2 - 1 - \frac{\beta^2}{4} & \frac{\beta}{2} \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6.3}$$

and

$$M := \widetilde{M} + \frac{\beta^2}{4} B = \begin{pmatrix} \frac{\beta}{2} & -1 \\ u^2 - 1 & \frac{\beta}{2} \end{pmatrix}.$$

Let $\varphi_1 > 0$ be the first Dirichlet eigenfunction of the Laplacian in Ω, that is,

$$\begin{aligned} -\Delta\varphi_1 &= \lambda_1 \varphi_1 && \text{in } \Omega, \\ \varphi_1 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{6.4}$$

Note that $-M$ is fully coupled, for $\beta \geq \sqrt{8}$ it is essentially positive by Lemma 4.1, and the function $\phi = \begin{pmatrix} \varphi_1 \\ \varphi_1 \end{pmatrix}$ is a positive strict supersolution of $L + M$, because $\lambda_1 + \frac{\beta}{2} - 1 > 0$ and $\lambda_1 + u^2 - 1 + \frac{\beta}{2} > 0$. Then, by Theorem 6.2, there are unique (up to normalization) positive functions $v, w \in W^{2,N}_{loc}(\Omega) \bigcap C_0(\Omega))$ such that

$$(L + \widetilde{M}) \begin{pmatrix} v \\ w \end{pmatrix} = (\lambda_B - \frac{\beta^2}{4})\, B \begin{pmatrix} v \\ w \end{pmatrix}.$$

Since $\partial\Omega$ is $C^{1,1}$ standard regularity arguments imply that $v \in C^4(\Omega) \cap C_0(\bar{\Omega})$, $\Delta v \in C_0(\bar{\Omega})$ and then

$$\Delta^2 v - \beta \Delta v + (u^2 - 1)v = (\lambda_B - \frac{\beta^2}{4})v =: \mu_1 v \qquad \text{in } \Omega,$$

where μ_1 is the first eigenvalue. By multiplying this equation by $u > 0$ and integrating by parts we get that

$$0 = \mu_1 \int_\Omega v\, u\, dx$$

since u is a solution of (1.1). Therefore $\mu_1 = 0$, because u and v are positive. This ends the proof by the variational characterization of μ_1. $\square$

Remark 6.3. *Without the assumption $\beta \geq \sqrt{8}$ the result still holds for any solution u such that $0 < u \leq 1$ in Ω. This is required for the essential positivity of $-M$.*

7 Radial symmetry of stable solutions

Proof of Theorem 1.4. Let L and B be as in (6.3) and let

$$\widetilde{Q} := \begin{pmatrix} \frac{\beta}{2} & -1 \\ 3u^2 - 1 - \frac{\beta^2}{4} & \frac{\beta}{2} \end{pmatrix}.$$

Note that

$$Q := \widetilde{Q} + \left(\frac{\beta^2}{4} - 2\right) B = \begin{pmatrix} \frac{\beta}{2} & -1 \\ 3(u^2 - 1) & \frac{\beta}{2} \end{pmatrix}$$

is a fully coupled matrix and $-Q$ is essentially positive. Moreover, since $\beta > \sqrt{12} - 2\lambda_1$ there is $\delta > 0$ such that $1/(\lambda_1 + \frac{\beta}{2}) < \delta < (\lambda_1 + \frac{\beta}{2})/3$. Then, if φ_1 is as in (6.4), we have

$$(L + Q) \begin{pmatrix} \delta\varphi_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} (\delta\lambda_1 + \frac{\beta}{2}\delta - 1)\varphi_1 \\ (\lambda_1 + 3(u^2 - 1)\delta + \frac{\beta}{2})\varphi_1 \end{pmatrix} > 0\,,$$

and therefore $L + Q$ has a positive strict supersolution. Therefore, by Theorem 6.2 there is $\lambda_B > 0$ and positive functions $v, w \in W^{2,N}_{loc}(\Omega) \bigcap C_0(\Omega)$ such that

$$(L + \widetilde{Q}) \begin{pmatrix} v \\ w \end{pmatrix} = \mu\, B \begin{pmatrix} v \\ w \end{pmatrix}, \qquad \text{with } \mu = \lambda_B - \frac{\beta^2}{4} + 2.$$

By standard regularity arguments, $v \in C^4(\Omega) \bigcap C_0(\Omega)$ and v solves

$$\Delta^2 v - \beta\Delta v + (3u^2 - 1)v = \mu v \qquad \text{in } \Omega,$$

that is, v is the first eigenfunction and μ is a simple (first) eigenvalue. Stability of u implies $\mu \geq 0$. Now let u_θ denote an angular derivative in which u is not radial. Then, in particular, u_θ must change sign in Ω and

$$\Delta^2 u_\theta - \beta\Delta u_\theta + (3u^2 - 1)u_\theta = 0 \qquad \text{in } \Omega,$$

since ∂_θ and $-\Delta$ commute and u is a solution of (1.1). Note that $u \in C^5(\Omega)$ by Lemma 2.3. This implies that u_θ is a sign-changing eigenfunction associated to the zero eigenvalue, but this contradicts the fact that $\mu \geq 0$ is simple. Therefore $u_\theta \equiv 0$ in Ω and u must be radial. $\square$

Remark 7.1. *Note that Theorem 1.4 holds true for any λ_1 if $\beta \geq \sqrt{12}$. On the other hand, if u is a nontrivial solution, then, by Theorem 3.3, we have that $\lambda_1^2 + \beta\lambda_1 < 1$. Combined with $\beta > \sqrt{12} - 2\lambda_1$, we obtain that the infimum of β's satisfying these inequalities is $\sqrt{8}$ with corresponding $\lambda_1 = \sqrt{3} - \sqrt{2}$.*

8 Positivity of global radial minimizers

Before we prove Theorem 1.6, we introduce some notation. Let

$$I := \begin{cases} [0, R) & \text{if } \Omega = B_R, \\ (R_0, R) & \text{if } \Omega = B_R \backslash B_{R_0}, \end{cases}$$

for some $R > R_0 > 0$, where B_r denotes open ball of radius r centered at the origin. To simplify the presentation we abuse a little bit the notation and we write u also to denote $u(r) = u(|x|)$. Recall that $H_r = \{u \in H : u \text{ is radially symmetric}\}$.

Proof of Theorem 1.6. Suppose first that $\Omega = B_R(0)$ and for a contradiction, assume that u changes sign. Then either u or $-u$ has a positive local maximum in $(0, R)$. Without loss of generality, assume there is $\eta \in (0, R)$ such that $1 \geq M := u(\eta) = \max\limits_{[0,R]} u > 0$. Let $v : \overline{I} \to \mathbb{R}$ be given by

$$v := \begin{cases} \frac{1-M}{1+M}(M - u) + M & \text{in } [0, \eta], \\ u & \text{in } (\eta, R]\,. \end{cases}$$

Note that v is just a rescaled reflection of u with respect to $u = M$ in $[0, \eta]$. Since $v'(\eta) = u'(\eta) = v'(0) = u'(0) = 0$ we have that $v \in C^1(I)$. Also

$$\begin{gathered} |v'| \leq |u'| \quad \text{in } (0, R) \quad \text{and} \quad |v''| \leq |u''| \quad \text{in } (0, R) \setminus \{\eta\}, \\ \text{with strict inequalities in } (0, \eta), \text{ whenever } u' \neq 0 \text{ and } u'' \neq 0. \end{gathered} \tag{8.1}$$

Furthermore in $(0, \eta)$ one has $0 \leq v \leq \frac{1-M}{1+M}(M + 1) + M = 1$ and

$$\begin{aligned} v - u &= \left(\frac{1 - M}{1 + M} + 1 \right) (M - u) \geq 0\,, \\ v + u &= \left(-\frac{1 - M}{1 + M} + 1 \right) u + \left(\frac{1 - M}{1 + M} + 1 \right) M = \left(\frac{2M}{1 + M} \right) (u + 1) \geq 0\,. \end{aligned}$$

Then $v^2 - u^2 = (v - u)(v + u) \geq 0$, thus $|1 - v^2| = 1 - v^2 \leq 1 - u^2 = |1 - u^2|$ in $[0, R]$, and then

$$\int_I (v^2 - 1)^2 \, dx \leq \int_I (u^2 - 1)^2 \, dx. \tag{8.2}$$

By (8.1) and (8.2) one has $J_\beta(v) < J_\beta(u)$, a contradiction to the minimality of u and thus u does not change sign in Ω.

The proof for the annulus $\Omega = B_R(0)\backslash B_{R_0}(0)$ for some $R > R_0 > 0$ is similar. If u changes sign, then there are $\eta, \mu \in I$ such that

$$M := u(\eta) = \max_{[0,R]} u > 0 \qquad \text{and} \qquad m := -u(\mu) = -\min_{[0,R]} u > 0.$$

Without loss of generality, assume that $\eta < \mu$. Let $v : \overline{I} \to \mathbb{R}$ be given by

$$v := \begin{cases} u & \text{in } [R_0, \eta], \\ \frac{m-M}{m+M}(M-u) + M & \text{in } (\eta, \mu), \\ -u & \text{in } [\mu, R]\,. \end{cases}$$

Then in (η, μ)

$$\begin{aligned} v &\leq \frac{m-M}{m+M}(M+m) + M = m \leq 1 \qquad \text{if } m > M, \\ v &= \frac{M-m}{m+M}(u-M) + M \leq M \leq 1 \qquad \text{if } m \leq M, \\ v - u &= \left(\frac{M-m}{m+M} - 1\right)u + \left(\frac{m-M}{m+M} + 1\right)M = \frac{2m}{m+M}(-u+M) \geq 0, \\ v + u &= \left(\frac{M-m}{m+M} + 1\right)u + \left(\frac{m-M}{m+M} + 1\right)M = \frac{2M}{m+M}(u+m) \geq 0\,. \end{aligned}$$

Therefore (8.1) (with strict inequality on (η, μ)) and (8.2) also hold in this case. Then $J_\beta(v) < J_\beta(u)$, which contradicts the minimality of u and thus u does not change sign in Ω..

It only remains to prove that assertions about $\partial_r u$. In the case of ball we set $I = (-R, R)$ and extend u to the whole interval by even reflection. Note that such extension is C^1 on the whole interval and C^2 except the origin.

Thus in for both ball and annulus we prove that $\partial_r u$ changes sign only once in I. Without loss of generality assume $u \geq 0$ in Ω. Now, suppose by contradiction that u' has more than one change of sign. Then u' has at least three changes of sign: two local maxima and one local minimum. Let $\eta_1 \in I$ be such that $u(\eta_1) = \max_I u =: M_1$ and $\tilde{\eta} \in I$ another local maximum. We only prove the case $\eta_1 < \tilde{\eta}$, the other one is analogous. Let $\mu \in (\eta_1, \tilde{\eta})$ be such that $u(\mu) = \min_{[\eta_1, \tilde{\eta}]} u =: m$ and $\eta_2 \in (\mu, \tilde{\eta}]$ such that $u(\eta_2) = \max_{[\mu, \tilde{\eta}]} u =: M_2$. Clearly $m < M_2 \leq M_1$. Define $v : I \to [0, 1]$ by

$$v := \begin{cases} u & \text{in } [R_0, \eta_1] \cup [\eta_2, R], \\ \frac{M_1 - M_2}{M_1 - m}(u - M_1) + M_1 & \text{in } (\eta_1, \mu), \\ M_2 & \text{in } [\mu, \eta_2), \end{cases}$$

By similar calculations as above it is easy to see that $v \in C^1(I)$, and that $J_\beta(v) < J_\beta(u)$, which contradicts the minimality of u and thus u' only changes sign once in I. Note that in the case of ball, $\partial_r v(0) = 0$, but v is not necessarily even. However either restriction of v to positive or negative interval has smaller energy than the corresponding u. $\square$

9 Symmetry of positive solutions

Proof of Proposition 1.5. We can write (1.1) as the following system.

$$\begin{aligned} -\Delta u + \frac{\beta}{2}u &= w && \text{in } \Omega, \\ -\Delta w + \frac{\beta}{2}w &= u - u^3 + \frac{\beta^2}{4}u && \text{in } \Omega, \\ u = w &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{9.1}$$

By Lemma 4.1 and $\beta \geq \sqrt{8}$ one has $\|u\|_{L^\infty(\Omega)} \leq 1$ and it is easy to check that a standard moving plane method as in [36] can be applied to (9.1). The regularity of the boundary can be relaxed up to Lipschitz regularity if one uses maximum principles for small domains, see [3]. □

10 Saddle solution

Proof of Theorem 1.2. Let $\beta \geq K_0$ (see (3.1)). By odd reflection, it suffices to find a positive $u \in C^4(\bar{\mathbb{R}}^2_+)$ solving

$$\begin{aligned} \Delta^2 u - \beta\Delta u &= u - u^3 && \text{in } \mathbb{R}^2_+, \\ \Delta u = u &= 0 && \text{on } \partial\mathbb{R}^2_+, \end{aligned} \tag{10.1}$$

where $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 > 0, x_2 > 0\}$. Indeed, if we extend u to $\mathbb{R}^2$ by odd reflections we obtain that $u \in C^1(\mathbb{R}^2)$. Since $u = 0$ on $H_1 := \{x : x_2 = 0\}$ one has $u_{x_1} = u_{x_1x_1} = u_{x_1x_1x_1} = 0$ on H_1. Also $\Delta u = 0$ on H_1 implies $u_{x_2x_2} = 0$, and consequently $u_{x_2x_2x_1} = 0$. However, u_{x_1}, $u_{x_1x_1}$, $u_{x_2x_2}$, $u_{x_1x_1x_1}$, $u_{x_2x_2x_1}$ are odd, and therefore continuous on H_1. Since all other partial derivatives (up to third order) are even, they are continuous on H_1 as well. The same procedure applies to $H_2 := \{x : x_1 = 0\}$ and we have $u \in C^3(\mathbb{R}^2)$.

From the equation (10.1) we also obtain continuity of $\Delta^2 u$ and $\Delta^2 u = 0$ on $\partial\mathbb{R}^2_+$. By a similar reasoning as above we obtain that the extension is of class $C^4(\mathbb{R}^2)$ and it is a classical solution of (1.1) with $\Omega = \mathbb{R}^2$, and consequently it is a saddle solution.

We find solution of (10.1) by a limiting procedure using the positive solution given by Theorem 3.3 over $\Omega_R := (0, R)^2$ and sending $R \to \infty$. Note that for R big enough, the solution given by Theorem 3.3 satisfies that $0 < u \leq M_\beta$ in Ω_R, and therefore, by Lemma 2.2 there is some $C > 0$ independent of R such that

$$\|u_R\|_{C^{4,\alpha}(\bar{\Omega}_R)} < C \quad \text{for all } R > 0. \tag{10.2}$$

By the Arzela-Ascoli theorem there is a sequence $R_N \to \infty$ such that $u_{R_N} \to u$ in $C^4(\bar{\mathbb{R}}^2_+)$ and u satisfies (10.1). We now prove that $u \not\equiv 0$. Indeed, fix

$$r > 64(1+\beta)\widetilde{C}^2 + 32, \tag{10.3}$$

where $\widetilde{C} > 0$ is a constant independent of R specified below. We show that

$$\|u_R\|_{L^\infty([0,r+1]^2)} \geq \frac{1}{\sqrt{2}} \quad \text{for all } R > r+3. \tag{10.4}$$

Assume by contradiction that there is $R > r+3$ such that

$$\|u_R\|_{L^\infty([0,r+1]^2)} \leq \frac{1}{\sqrt{2}}. \tag{10.5}$$

We define the following sets

$$\begin{aligned}
\omega_1 &:= \{x \in \Omega_R \,:\, \text{dist}(x, \partial\Omega_R) \leq 1,\ x_1 \leq r+1, x_2 \leq r+1\},\\
\omega_2 &:= \{x \in \Omega_R \,:\, \text{dist}(x, \partial\Omega_R) \geq 1,\ x_1 \leq r+1, x_2 \leq r+1\},\\
\omega_3 &:= \{x \in \Omega_R \,:\, \min\{x_1, x_2\} \geq r+1,\ \max\{x_1, x_2\} \leq r+2\},\\
\omega_4 &:= \{x \in \Omega_R \,:\, x_1 \geq r+2, \text{ or } x_2 \geq r+2\}.
\end{aligned}$$

Note that $\Omega_R = \bigcup_{i=1}^4 \omega_i$. Now, let $\phi_1 \in C^2(\bar{\Omega}_R) \bigcap C_0(\bar{\Omega}_R)$, $\phi_2 \in C^2(\bar{\Omega}_R)$ such that $0 \leq \phi_1 \leq 1$, $0 \leq \phi_2 \leq 1$, $\|\phi_i\|_{C^2(\Omega_R)} \leq K$ for $i = 1, 2$ and some $K > 0$ independent of R, and

$$\phi_1 \equiv 1 \quad \text{in } \omega_2, \qquad \phi_1 \equiv 0 \quad \text{in } \omega_4, \qquad \phi_2 \equiv 0 \quad \text{in } \omega_2, \qquad \phi_2 \equiv 1 \quad \text{in } \omega_4.$$

Further, let $\psi \in C^4(\Omega_R) \bigcap C_0(\Omega_R)$ be given by $\psi := \phi_1 + \phi_2 u_R$. Then $\psi \equiv 1$ in ω_2, $\psi \equiv u_R$ in ω_4, and there is some $\widetilde{C} > 0$, depending only on C (from (10.2)) and K, such that $\|\psi\|_{C^2(\Omega_R)} \leq \widetilde{C}$. For $i = 1, \ldots, 4$ let

$$J_i(v) := \int_{\omega_i} \frac{|\Delta v|^2}{2} + \beta \frac{|\nabla v|^2}{2} + \frac{1}{4}(v^2 - 1)^2 \, dx \quad \text{ for } v \in H^2(\Omega_R) \bigcap H_0^1(\Omega_R).$$

Note that $\sum_{i=1}^4 J_i(v) = J_\beta(v) + \frac{1}{4}|\Omega_R|$ for $v \in C^2(\Omega_R)$, here J_β is as in (1.8) for $\Omega = \Omega_R$. Then $\sum_{i=1}^4 J_i(\psi) \leq [(\frac{1}{2} + \frac{\beta}{2})\widetilde{C}^2 + \frac{1}{4}](|\omega_1| + |\omega_3|) + J_4(u_R)$, and by (10.5), $\sum_{i=1}^4 J_i(u_R) \geq \frac{|\omega_2|}{16} + J_4(u_R)$. Therefore

$$J_\beta(u_R) - J_\beta(\psi) \geq 2r\left(\frac{r}{32} - 2(1+\beta)\widetilde{C}^2 - 1\right) > 0,$$

by (10.3) a contradiction to the minimality of u_R. Therefore (10.4) holds and the maximum principle yields that $u > 0$ in $\mathbb{R}^2_+$ is a solution of (10.1). □

11 Bifurcation from a simple eigenvalue

Theorem 11.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a smooth bounded domain or a hyper-rectangle. If the first Dirichlet eigenvalue $\lambda_1 < 1$, then there is $\varepsilon > 0$ such that*

(1.1) *admits a positive solution* $u_\beta \in C^{4,\alpha}(\Omega)$ *for all* $\beta \in (\bar\beta - \varepsilon, \bar\beta)$*, where*

$$\bar\beta = \frac{1-\lambda_1^2}{\lambda_1}.$$

Additionally, if $\bar\beta \geq \sqrt{8}$ *and* Ω *is of class* C^1*, then* (1.1) *admits a unique positive solution* u_β *such that* $\|u_\beta\|_{L^\infty(\Omega)} \leq 1$ *for all* $\beta \in [\sqrt{8}, \bar\beta)$*.*

Proof. Let $X = \{u \in C^{4,\alpha}(\Omega) \cap C^2(\bar\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$ and $Y = C^{0,\alpha}(\Omega)$. Consider the operator

$$G : \mathbb{R} \times X \to Y; \quad G(\beta, u) := \Delta^2 u - \beta \Delta u - u + u^3.$$

Then we have $G(\beta, 0) = 0$ for all β. Moreover $u \in X$ solves (1.1) if and only if $G(\beta, u) = 0$. We consider the partial derivative

$$\partial_u G : (0,\infty) \times X \to \mathcal{L}(X,Y), \quad \partial_u G(\gamma, u)[v] = \Delta^2 v - \beta \Delta v - v + 3u^2 v.$$

For $\beta > 0$ we set

$$A_\beta := \partial_u G(\beta, 0); \quad A_\beta v = \Delta^2 v - \beta \Delta v - v.$$

Let $N(A_\beta)$ and $R(A_\beta)$ denote the kernel and the range of A_β respectively. Note that $v \in N(A_\beta)$ if and only if $\Delta^2 v - \beta \Delta v = v$ in Ω. Let φ_1 be the first eigenfunction of the Laplacian in Ω, see (6.4) with $\|\varphi_1\|_{L^2(\Omega)} = 1$.

By the definition of $\bar\beta > 0$, one has $\varphi_1 \in N(A_{\bar\beta})$. Moreover, by the Krein-Rutman Theorem $N(A_{\bar\beta}) = \{\alpha\varphi_1 : \alpha \in \mathbb{R}\}$. Further, since $A_{\bar\beta}$ is self adjoint, by the Fredholm Theory $R(A_{\bar\beta}) = \{v \in Y : \int_\Omega \varphi_1 v\, dx = 0\}$. In particular, since $\frac{d}{d\beta} A_\beta v = -\Delta v$ for $v \in X$. We find that $\frac{d}{d\beta} A_\beta \varphi_1 \mid_{\beta=\bar\beta} = \lambda_1 \varphi_1 \notin R(A_{\bar\gamma})$. Hence all the assumptions of [12, Lemma 1.1] are satisfied, and thus there is $\varepsilon > 0$ and C_1-functions $\beta : (-\varepsilon, \varepsilon) \to (0, \infty)$ and $u : (-\varepsilon, \varepsilon) \to X$ such that $\beta(0) = \bar\beta$ and $G(\beta(t), u(t)) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Since $\partial_{uu} G(\bar\beta, 0)[\varphi_1, \varphi_1] = 0$ and $\int_\Omega \varphi_1 \partial_{uuu} G(\bar\beta, 0)[\varphi_1, \varphi_1, \varphi_1]\, dx = 6 \int_\Omega \varphi_1^4\, dx > 0$ we have a subcritical bifurcation, and therefore

$$u = \pm c(\bar\beta - \beta)^{\frac{1}{2}} \varphi_1 + o(t^{\frac{1}{2}})$$

for some constant $c > 0$ and $t \in (0, \varepsilon)$. This proves the first claim.

For the second claim, assume $\bar\beta \geq \sqrt{8}$ and that Ω being C^1. Let $(0, T)$ be the maximal interval of existence for the curve u with $T \in (0, \infty]$. By this we mean that we can uniquely extend the curve, in particular the curve ceases to exist if it intersects another curve e.g. $(\gamma, 0)$ or it bifurcates. If $\beta(t) = \sqrt{8}$ for some $t \in (0, T)$, then the existence part of the claim follows, so assume that $\beta(t) > \sqrt{8}$ for all $t \in (0, T)$.

By the C^1- continuity of the curve $\mathcal{C} := \{u(t) : t \in (0, T)\}$ we have that $0 < u_\beta < 1$ for all $t \in (0, T)$ sufficiently close to zero. By Lemma 4.1 and the continuity of $\mathcal{C}$ it follows that $\|u(t)\|_{L^\infty(\Omega)} \leq 1$ for all $t \in (0, T)$. To show that $u(t) > 0$ for all $t \in (0, T)$ we argue by contradiction. Assume that $\bar t = \sup\{t \in$

$(0,T) : u(s) > 0$ in Ω for all $s \in (0,t]\} < T$. Observe that $\bar{t} > 0$ by Theorem 1.3 and the implicit function theorem.

Let $\bar{u} = u(\bar{t}) \in C^{4,\alpha}(\Omega)$. Note that $\bar{u}$ satisfies the system

$$\begin{aligned} -\Delta \bar{u} + \beta \bar{u} &= w && \text{in } \Omega, \\ -\Delta w &= \bar{u} - \bar{u}^3 && \text{in } \Omega, \\ w = u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

for some $\beta > 0$, and $\bar{u} - \bar{u}^3 \geq 0$ since $0 \leq \bar{u} \leq 1$. Then, by the maximum principle and the Hopf Lemma, either $\bar{u} \equiv 0$ or $\bar{u} > 0$ in Ω and $\partial_\nu \bar{u} < 0$ on $\partial\Omega$, where ν denotes the exterior normal vector to $\partial\Omega$. The former case contradicts $T > \bar{t}$ whereas the latter one contradicts the C^4-continuity of $\mathcal{C}$ and the definition of $\bar{u}$.

Therefore $u(t) > 0$ for all $t \in (0,T)$. Since $\|u(t)\|_{L^\infty(\Omega)} \leq 1$ for all $t \in (0,T)$, standard elliptic regularity theory yields that $\|u(t)\|_{C^{4,\alpha}(\Omega)} \leq C$ for all $t \in (0,T)$ and for some $C > 0$. Moreover, since the positivity is preserved along the curve, $u(T) \not\equiv 0$. Indeed, $(\beta(t), u(t))$ cannot return to a neighborhood of $(\bar{\beta}, 0)$ due to uniqueness. Also, it cannot intersect $(\beta, 0)$ as any other branch bifurcating from $(\beta, 0)$ $(\beta < \bar{\beta})$ consists locally of sign changing solutions because the corresponding eigenfunction direction is sign changing (perpendicular to the principal eigenfunction).

By the first part of Theorem 3.3, we know that $\beta(t) < \bar{\beta}$ for all $t \in (0,T)$. Then, bifurcation theory (see [25, [Theorem II.3.3]) implies that necessarily $(\sqrt{8}, \bar{\beta}) \subset \{\beta(t) \, : \, t \in (0,T)\}$.

We now prove that u_β is the unique positive solution of (1.1) for $\beta \in [\sqrt{8}, \bar{\beta})$. Indeed, let u_0 be a positive solution of (1.1) for some $\beta_0 \in [\sqrt{8}, \bar{\beta})$. By Theorem 1.3, u_0 is a strictly stable solution, and therefore $D_u G(u_0, \beta_0)$ is an invertible operator. Then, by the implicit function theorem there exists $\varepsilon > 0$ and a smooth curve $\gamma : (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \to X$ such that $G(\beta, \gamma(\beta)) = 0$ and for any solution of $G(\beta, u) = 0$ sufficiently close to (β_0, u_0) one has $u \in \gamma$.

By the same argument, we can extend γ to a maximal interval (β_1, β_2) with γ containing only positive solutions. Then the strict stability in Theorem 1.3 guarantees that γ is not bifurcating and it does not have turning points because it is parametrized by β. Since the only solution for $\beta \geq \bar{\beta}$ is zero, by the first part of Theorem 3.3, we have that $\beta_2 \leq \bar{\beta}$ and $\gamma(\beta_2)$ is a non-negative function. Arguing as before, one obtains that necessarily $\gamma(\beta_2) \equiv 0$ and then $\beta_2 = \bar{\beta}$. Here, as above, we have used that all other bifurcation points of the form $(\beta, 0)$ must have sign changing branches The strict stability of the branch bifurcating from $\bar{\beta}$ and the uniqueness of the branch close to the bifurcation point $(\bar{\beta}, 0)$ yield that necessarily $u_0 = u_{\beta_0}$, as desired.

If Ω is a hyperrectangle, one proves the positivity along the curve using Serrin's boundary point Lemma [34, Lemma 1] at corners and the rest of the proof remains unchanged.

□

Remark 11.2. *If Ω is a ball of radius R one can explicitly calculate the radius R for which the bifurcation occurs in the following way: By using polar coordinates one can solve in terms of Bessel functions the Dirichlet eigenvalue problem for λ_1, in particular $\lambda_1(B_R(0)) = \frac{j_{N/2-1,1}}{R^2}$, where $B_R(0)$ is a ball of radius $R > 0$ and $j_{N/2-1,1}$ is the first positive zero of the Bessel function $J_{N/2-1}$, see for instance [24, Section 4.1]. Then, for $\beta = \sqrt{8}$, the bifurcation occurs at balls of radius $R_N := \frac{j_{N/2-1,1}}{\sqrt{\sqrt{3}-\sqrt{2}}}$. For example, $R_2 :\approx 4.26$, $R_3 :\approx 5.57$, $R_4 :\approx 6.79$, $R_{10} :\approx 13.46$, etc... And more generally, for $\beta > 0$, the bifurcation occurs at balls of radius $R_{\beta,N} := \frac{\sqrt{2}j_{N/2-1,1}}{\sqrt{-\beta+\sqrt{\beta^2+4}}}$.*

12 Convergence to the second order equation

Proof of Theorem 1.8. Let $\gamma \in (0, 1/64)$, u_γ be the global minimizer of (1.11) and $\mu = \gamma^{\frac{-1}{4}}$. Note that $w : \mu\Omega \to \mathbb{R}$ given by $w(x) := u_\gamma(\mu^{-1}x)$ is a weak solution in H of $\Delta^2 w - \mu^2 \Delta w = w - w^3$ in $\mu\Omega$. Also note that $\mu \geq \sqrt{8}$ if $\gamma \leq 64$. By Proposition 5.1 we have that $\|u_\gamma\|_{L^\infty} = \|v\|_{L^\infty} \leq 1$ and, by Lemma 2.3, $\|u_\gamma\|_{C^{5,\alpha}(\Omega)} \leq C$ for some $C > 0$ independent of γ.

Let $u^* \in H_0^1(\Omega)$ be a global minimizer of (1.11) in $H_0^1(\Omega)$ with $\gamma = 0$. It is well known that u^* is a unique global minimizer, strictly stable, and it does not change sign (see, for example, [2]).

Now, since u_γ is bounded in $C^{5,\alpha}$ independently of γ it is easy to see that $u_\gamma \to u^*$ in C^4 as $\gamma \to 0$, by the uniqueness of solutions of the limit problem.

Let $G \in C^1(\mathbb{R} \times C^4(\Omega))$ be given by $G(\gamma, u) = \gamma\Delta^2 u - \Delta u - u + u^3$. Notice that $\partial_u G(\gamma, u) \in \mathcal{L}(C^4(\Omega), \mathbb{R})$ and $\partial_u G(0, u^*) \neq 0$ by the strict stability of u^*. Therefore, by the implicit function theorem (see for example [25, Theorem I.1.1]), there is a neighborhood $I \times V \subset \mathbb{R} \times C^4(\Omega)$ of $(0, u^*)$ and a continuous function $\lambda : I \to V$ with $\lambda(0) = u^*$ such that $G(\gamma, \lambda(\gamma)) = 0$ for all $\gamma \in I$ and every solution of $G(\gamma, u) = 0$ in $I \times V$ is of the form $(\gamma, \lambda(\gamma))$ for some $\gamma \in I$. Since $u_\gamma \to u^*$ in C^4 as $\gamma \to 0^+$ and u_γ is arbitrary global minimizer, we obtain that u_γ is the unique global minimizer for all $\gamma \in I$. Finally, if the first Dirichlet eigenvalue $\lambda_1(\Omega) < 1$, then $u^* \not\equiv 0$ and by the strong maximum principle $u^* > 0$ in Ω, by the Hopf Lemma $\partial_\nu u^* < 0$ in $\partial\Omega$, where ν denotes the exterior normal vector, and therefore $u_\gamma > 0$ in Ω for all $\gamma \in I$, by making I a smaller neighborhood of 0 if necessary. □

Remark 12.1. *Note that the proof of Theorem 1.8 also proves the existence of solutions in $C^{5,\alpha}(\Omega)$ for equation (1.10) with $\gamma \in [-\gamma_0, 0]$. This equation is known as the 4-NLS equation, see [33]. Also note that, if $\lambda_1(\Omega) \geq 1$, there are no non-trivial solutions by the Poincaré inequality.*

13 Numerical approximations

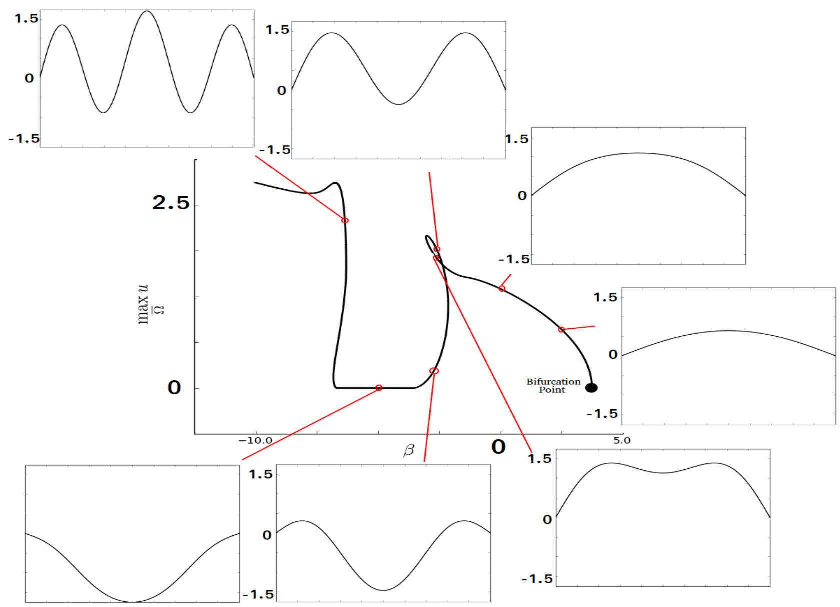

Figure 4: Bifurcation branch from $\bar{\beta} = \frac{1-\lambda_1^2}{\lambda_1}$ with $\Omega = (0, 2\pi)$ and some sols. to (1.1). Along the branch the positivity is lost and even negative solutions appear. Here the branch does not return to the region $\beta > 0$ since the complexity of solutions is constrained by $|\Omega|$. Indeed, $|u| < \sqrt{2}$ for $\beta > 0$ (see the introduction) implies $\int_\Omega (u'')^2 + \beta(u')^2 \, dx = \int_\Omega u^2 - u^4 \, dx \leq 6|\Omega|$.

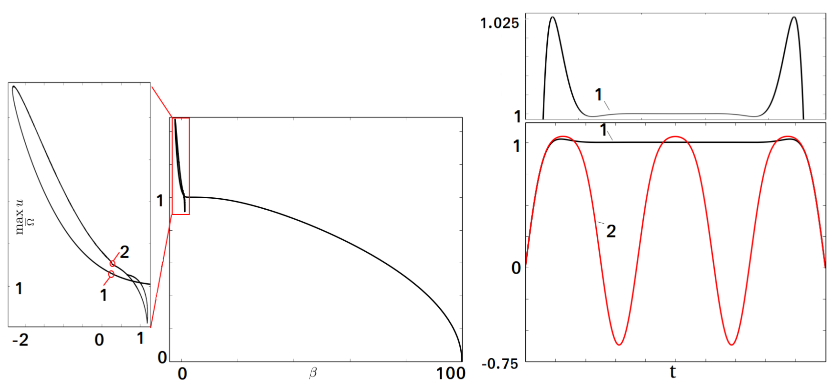

Figure 5: Bifurcation branch from $\bar{\beta} = \frac{1-\lambda_1^2}{\lambda_1}$ with $\Omega = (0, 10\pi)$ (left). On the right down we have two solutions of (1.1) corresponding to $\beta = 0.35$. Sol. 1 belongs to the lower part of the branch, sol. 2 is in the upper part. On the right up we have a close-up showing that sol. 1 has its maximum above 1 and present oscillations.

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