

Symmetry of positive solutions of asymptotically symmetric parabolic problems on \mathbb{R}^N

Juraj Földes ¹

Abstract In this paper we investigate symmetry properties of positive solution of quasilinear parabolic problems in the whole space. As the main result, we prove that if the problem converges exponentially to a symmetric one, then the solution converges to the space of symmetric functions. We also show, that this result does not hold true, if the convergence is not exponential.

Keywords Asymptotic symmetry, Positive bounded solutions, Method of moving hyperplanes, Maximum principle, Harnack inequality.

1 Introduction

In this paper we study quasilinear parabolic equation

$$\partial_t u = A_{ij}(t, u, \nabla u)u_{x_i x_j} + F(t, u, \nabla u) + G(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (1.1)$$

where ∇g denotes the gradient of a function g . The functions A and F satisfy certain regularity, ellipticity, and symmetry assumptions as specified in the next section. The function G that decays to 0 as t approaches infinity, is considered to be a perturbation of the problem. In (1.1), and also in the rest of the paper, we use summation convention, that is, when an index appears twice in a single term, we are summing over all of its possible values, usually from 1 to N .

Our goal is to show that every positive, classical, global, bounded solution u of (1.1) is asymptotically symmetric. Before we make these statements precise, let us give a brief account of older results.

¹J. Földes

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
e-mail: juraj.foldes@vanderbilt.edu

The first results on reflectional symmetry were established by Gidas, Ni and Nirenberg [13] for positive solutions of elliptic equations on bounded domains. Specifically, if Ω is a bounded, smooth domain, convex in x_1 , and symmetric with respect to the hyperplane

$$H_0 := \{x \in \mathbb{R}^N : x_1 = 0\},$$

and f is a Lipschitz function, then a positive classical solution u of

$$\Delta u + f(u) = 0, \quad x \in \Omega, \quad (1.2)$$

$$u = 0, \quad x \in \partial\Omega, \quad (1.3)$$

is even in x_1 and nonincreasing in the set

$$\Omega_0 := \{x \in \Omega : x_1 > 0\}.$$

The used techniques included the maximum principle and the method of moving hyperplanes introduced by Alexandrov [2] and developed by Serrin [31], who used it for overdetermined elliptic problems. Later, the results of Gidas et al. were generalized by Li [19] to fully nonlinear problems, and Berestycki and Nirenberg [8] extended them to nonsmooth domains Ω . We refer the reader to the surveys [6, 24, 26] for more results, references, and generalizations.

In another paper, Gidas, Ni and Nirenberg [14] considered (1.2) with $\Omega = \mathbb{R}^N$ and a smooth nonlinearity f satisfying $f(0) = 0$, and certain hypothesis near 0. They proved that each positive solution, which decays to 0 at a suitable rate, is radially symmetric. Later, Li [20] showed that any decay of solution as $|x| \rightarrow \infty$ is sufficient for symmetry, provided $f(0) = 0$ and $f'(0) < 0$. The later condition was weakened by Li and Ni [21], who assumed that $f'(z) \leq 0$ for any z sufficiently close to 0. All these papers also treat fully nonlinear problems. The described results were extended in various directions such as cooperative systems of equations, more general unbounded domains, or more general equations. We again refer the reader to [6, 24, 26] for more references.

The situation is more complicated for parabolic problems, as one cannot expect the solution to be symmetric, unless the initial data are symmetric. However, one can prove that the solution approaches the space of symmetric functions as time approaches infinity. To make this concept precise, for any open $\Omega \subset \mathbb{R}^N$ we define ω -limit set of u to be

$$\omega(u) := \{z : z = \lim_{n \rightarrow \infty} u(\cdot, t_n) \text{ for some } t_n \rightarrow \infty\},$$

where the convergence is in the space $C_0(\Omega)$, the space of continuous functions on Ω that vanish on $\partial\Omega$ and decay to zero at infinity (if Ω is unbounded). The space $C_0(\bar{\Omega})$ is equipped with the supremum norm.

If Ω is a bounded domain, symmetric with respect to H_0 , we say that u is *asymptotically symmetric* if z is even in x_1 and decreasing in x_1 in Ω_0 for each $z \in \omega(u)$.

The first results on asymptotic symmetry appeared in [15], where Hess and Poláčik proved asymptotic symmetry for positive classical solutions of the problem

$$\begin{aligned} u_t &= \Delta u + f(t, u), & x \in \Omega, \\ u &= 0, & x \in \partial\Omega. \end{aligned}$$

Here, Ω is a smooth bounded domain convex in x_1 , symmetric with respect to H_0 and f is Hölder in t and Lipschitz in u . In an independent work Babin [3, 4] showed asymptotic symmetry for autonomous fully nonlinear problem and later, Babin and Sell [5] allowed nonlinearity to depend on t . However, these results require additional compactness and positivity assumptions compared to [15].

These drawbacks were removed in [29], where Poláčik proved the asymptotic symmetry for positive, classical solutions of a general fully nonlinear parabolic problem on bounded domains. The results required certain strong positivity assumptions that were further discussed in [12].

Unlike for elliptic equations, symmetric properties of solutions on \mathbb{R}^N are much less understood. The difficulties arise from the fact that the center of symmetry is not a priori fixed. Even if one is able to prove the symmetry of every function $z \in \omega(u)$ with respect to some hyperplane, it is not immediate to show that all functions in $\omega(u)$ are symmetric with respect to the same hyperplane. Having this in mind, we say that u defined on $\Omega = \mathbb{R}^N$ is *asymptotically symmetric*, if there is $\lambda_0 \in \mathbb{R}$ such that all functions $z \in \omega(u)$ are symmetric with respect to the same hyperplane

$$H_{\lambda_0} := \{x \in \mathbb{R}^N : x_1 = \lambda_0\},$$

and decreasing in the halfspace

$$\mathbb{R}_{\lambda_0}^N := \{x \in \mathbb{R}^N : x_1 > \lambda_0\}.$$

In [27], Poláčik proved that a nonnegative solution u of (1.1) is asymptotically symmetric, provided $G \equiv 0$ and assumptions (N1)–(N4), (2.4), (2.5) from the

next section are satisfied. In [28], Poláčik discussed entire solutions, that is, solutions defined for all times (positive and negative), and he showed that each nonnegative entire solution is symmetric at each time.

We were not able to locate any symmetry results in the literature if $G \neq 0$. However, these can be obtained if the problem (1.1) is asymptotically autonomous, that is, if F and A_{ij} are independent of t , and u converges to a solution of the elliptic problem

$$0 = A_{ij}(u, \nabla u)u_{x_i x_j} + F(u, \nabla u), \quad x \in \mathbb{R}^N. \quad (1.4)$$

Then by the symmetry results for elliptic problems [14], this equilibrium is symmetric, and therefore the solution of the parabolic problem is asymptotically symmetric.

The convergence to a nonnegative equilibrium was obtained for asymptotically autonomous problems, that is, for the problems that are approaching an autonomous one as $t \rightarrow \infty$. First, let us explain the existing results on the following model problem. Let u be a classical, global, nonnegative solution of the problem

$$\begin{aligned} u_t &= \Delta u + F(u) + G(x, t), & (x, t) &\in \Omega \times (0, \infty), \\ u(x, t) &= 0, & (x, t) &\in \partial\Omega \times (0, \infty). \end{aligned} \quad (1.5)$$

Huang and Takáč in [16] (see also [10]) proved that the solution u of (1.5) converges to a solution v of the problem

$$\begin{aligned} 0 &= \Delta v + F(v), & x &\in \Omega, \\ v &= 0, & x &\in \partial\Omega, \end{aligned} \quad (1.6)$$

provided Ω is a smooth bounded domain, F satisfies certain analyticity assumptions and

$$\sup_{t \in (0, \infty)} t^{1+\delta} \int_t^\infty \|G(\cdot, s)\|_{L^2(\Omega)} ds < \infty. \quad (1.7)$$

Huang and Takáč also treated more general gradient-like problems with self-adjoint differential operators.

Later, Chill and Jendoubi [11] considered the problem (1.5) with $\Omega = \mathbb{R}^N$ and

$$F(u) = \sum_{p \in P} c_p |u|^{p-1} u,$$

where P is a finite subset of $(1, \frac{N+2}{N-2})$ and $c_q > 0$ for $q = \max_{p \in P} p$. Moreover, they assumed that there exists a compact set $K \subset \mathbb{R}^N$ with $\text{supp } G(\cdot, t) \subset K$ for each $t > 0$. As a result, they proved that (1.7) implies the convergence of positive solutions u , with bounded $H^1(\mathbb{R}^N)$ norm, to a solution of (1.6).

In this paper we generalize symmetry results from [27] to nonnegative solutions of the problem (1.1) with $G \not\equiv 0$. Under the assumptions (N1)–(N4) listed in the next section, we prove that each positive solution of (1.1) is asymptotically symmetric, provided there exists $\mu > 0$ with

$$\|G\|_{X_{(t,\infty)}} \leq C e^{-\mu t} \quad (t \geq 0), \quad (1.8)$$

where

$$X_{(s,t)} := L^\infty(\mathbb{R}^N \times (s,t)) \oplus L^{N+1}(\mathbb{R}^N \times (s,t)) \quad (t, s \in (0, \infty], s < t) \quad (1.9)$$

is the space of functions f that can be written in the form $f = g + h$ with $g \in L^\infty(\mathbb{R}^N \times (s,t))$ and $h \in L^{N+1}(\mathbb{R}^N \times (s,t))$, equipped with the norm

$$\|f\|_{X_{(s,t)}} = \inf_{g+h=f} (\|g\|_{L^\infty(\mathbb{R}^N \times (s,t))} + \|h\|_{L^{N+1}(\mathbb{R}^N \times (s,t))}). \quad (1.10)$$

Notice that G is not assumed to be globally integrable in x . This generalization proves to be useful for perturbations that depends on the solution or derivatives of solution, since these are only assumed to be bounded. Indeed, if instead of $G : (x, t) \rightarrow \mathbb{R}$ we consider a function $\tilde{G} : (x, t, u, p, q) \in \Omega \times [0, \infty) \times \mathbb{R}^{1+N+N^2} \rightarrow \mathbb{R}$, then our results apply, if

$$\tilde{G} : (x, t) \mapsto \tilde{G}(x, t, u(x, t), Du(x, t), D^2u(x, t)) \quad ((x, t) \in \mathbb{R}^N \times [0, \infty))$$

satisfies (1.8). An example of such function \tilde{G} is

$$\tilde{G} : (x, t, u, Du, D^2u) \mapsto e^{-t} g(u, Du, D^2u), \quad (1.11)$$

where g is continuous. Notice that problem (1.1) with G replaced by \tilde{G} is fully nonlinear. Therefore, our symmetry results cover certain fully nonlinear problems that converge exponentially to quasilinear ones as $t \rightarrow \infty$. However, it is not known if the symmetry results hold for general fully nonlinear equations.

If we apply our results on reflectional symmetry in various directions, the standard arguments show that all functions in the ω -limit set are radially

symmetric with respect to the same origin. In a future paper we show how to apply our symmetry results in the study of the asymptotic behavior of solution of asymptotically autonomous problems, that is, when F and (A_{ij}) are independent of t .

The asymptotic symmetry of positive solutions does not hold true if we merely assume that G converges to 0 as $t \rightarrow \infty$. A counterexample is given in Example 2.3 below, with $\|G\|_{X(t,\infty)} \approx \frac{1}{t}$. However, it is not known if the exponential decay (as stated in (1.8)) is necessary. Especially, we leave as an open problem, whether the integrability of $t \mapsto \|G\|_{X(t,\infty)}$ is sufficient for asymptotic symmetry of solutions.

To prove the symmetry results, we extend linear estimates for parabolic equations such as Alexandrov-Krylov estimate and the Harnack inequality to more general inhomogeneities (right hand sides) on unbounded domains. Since these results might be of independent interest, especially for applications to unbounded domains, we devote them a separate section. Once the linear estimates are established, we follow the framework from [27] to prove the symmetry results. The application of methods from [27] is not completely straightforward and a special care should be taken when treating perturbations on unbounded sets, since various constants might depend on the diameter of the set or the length of the time interval. In that case, we restrict our arguments to bounded time intervals and use iterative methods.

The rest of the paper is organized as follows. In the next section we state our main results. Section 3 contains general linear estimates of parabolic problems, and in Section 4, we prove the symmetry results.

2 Main results

Consider parabolic problem (1.1). We assume that the real valued functions $(A_{ij})_{1 \leq i, j \leq N}, F : (t, u, p) \mapsto \mathbb{R}$ are defined on $[0, \infty) \times [0, \infty) \times \mathbb{R}^N$ and satisfy the following conditions.

- (N1) *Regularity.* The functions $(A_{ij})_{1 \leq i, j \leq N}, F$ are continuous on $[0, \infty) \times [0, \infty) \times \mathbb{R}^N$ and continuously differentiable with respect to u and $p = (p_1, \dots, p_N)$ uniformly in $t \in [0, \infty)$. This means, that if h stands for any of $\partial_u A_{ij}, \partial_u F, \partial_{p_k} A_{ij}$ or $\partial_{p_k} F$ for some $1 \leq i, j, k \leq N$, then for

each $M > 0$ one has

$$\limsup_{\substack{0 \leq u, v, |p|, |q| \leq M, t \geq 0 \\ |u-v| + |p-q| \rightarrow 0}} |h(t, u, p) - h(t, v, q)| = 0. \quad (2.1)$$

(N2) *Ellipticity.* There is a positive constant α_0 such that for each $\xi \in \mathbb{R}^N$

$$A_{ij}(t, u, p)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad ((t, u, p) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^N).$$

(N3) *Symmetry.* For each $(t, u, p) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^N$ and $1 \leq i, j \leq N$ one has

$$\begin{aligned} A_{ij}(t, u, p) &= A_{ij}(t, u, -p_1, p_2, \dots, p_N), \\ F(t, u, p) &= F(t, u, -p_1, p_2, \dots, p_N), \\ A_{1j} &= A_{j1} \equiv 0 \quad \text{if } j \neq 1. \end{aligned}$$

(N4) *Stability of 0.* $F(t, 0, 0) = 0$ and there is a constant $\gamma > 0$ such that

$$\partial_u F(t, 0, 0) < -2\gamma \quad (t \geq 0).$$

Remark 2.1. The assumption (N4) and uniform continuity of $\partial_u F$ in t imply the existence of $\varepsilon_\gamma^* > 0$ with

$$\partial_u F(t, u, p) < -\gamma \quad ((t, u, p) \in [0, \infty) \times [0, \varepsilon_\gamma^*] \times B_{\varepsilon_\gamma^*}),$$

where B_r is an open ball centered at the origin with the radius r .

The assumptions on G are as follows (recall that $X_{(s,t)}$ was defined in (1.9)).

(G1) $G \in X_{(t,t+1)}$ for each $t \in [0, \infty)$ and

$$\lim_{t \rightarrow \infty} \|G\|_{X_{(t,t+1)}} = 0. \quad (2.2)$$

Some results require exponential decay of G .

(G2) For each $t \in (0, \infty)$ one has $G \in X_{(t,\infty)}$. Moreover, there exist $\mu > 0$ and $C_\mu > 0$ such that

$$\|G\|_{X_{(t,\infty)}} \leq \frac{C_\mu}{2} e^{-\mu t} \quad (t > 0). \quad (2.3)$$

One can easily verify that, with possibly changed μ , (G2) is equivalent to the following statement. For each $\varepsilon > 0$, there exists $t_\varepsilon > 0$ with $\|\tilde{G}\|_{X(t_\varepsilon, \infty)} \leq \varepsilon$, where $\tilde{G}(x, t) = e^{\mu(t-t_\varepsilon)}G(x, t)$. Notice that if we replace $X(t, \infty)$ by $X(t, t+1)$ in (G2), we obtain an equivalent assumption. As explained in the introduction, the space X allows us to treat, possibly unbounded, perturbations depending on u , Du or D^2u .

We assume that u is a classical, nonnegative, global solution of (1.1), that is, $u \in C^{2,1}(\mathbb{R}^N \times (0, \infty))$ and u satisfies (1.1) everywhere. Moreover, we assume

$$S := \sup_{\substack{(x,t) \in \mathbb{R}^N \times [0, \infty) \\ 1 \leq i, j \leq N}} \{|u(x, t)|, |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)|\} < \infty, \quad (2.4)$$

and

$$\limsup_{|x| \rightarrow \infty, t \in [0, \infty)} \{|u(x, t)|, |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)|\} = 0 \quad (1 \leq i, j \leq N). \quad (2.5)$$

Observe that (N1) combined with (2.4) yields the existence of $\beta_0 > 0$ such that

$$\sup_{t \geq 0} |h(t, v, p) - h(t, w, q)| \leq \beta_0 |(v, p) - (w, q)| \\ (v, w \in [0, S], p, q \in \mathbb{R}^N, |p|, |q| \leq S), \quad (2.6)$$

where h stands for F or A_{ij} , and S was defined in (2.4). Although we suppose (N2) and (2.6) with fixed constant α_0 , we really need it to be true on the range of (u, Du, D^2u) for each considered solution u . Since u is bounded and has bounded derivatives, (N2) needs to hold true only for $u, |p| < S$.

By (2.5), there is ρ_γ^* such that $|u|, |\nabla u| < \varepsilon_\gamma^*$ in $(\mathbb{R}^N \setminus B_{\rho_\gamma^*}) \times [0, \infty)$, and therefore by Remark 2.1

$$\partial_u F(t, u(x, t), \nabla u(x, t)) < -\gamma \quad ((x, t) \in (\mathbb{R}^N \setminus B_{\rho_\gamma^*}) \times [0, \infty)). \quad (2.7)$$

Uniformity of the limit (2.5) in t is not technical. When omitted the symmetry results may fail even for $G \equiv 0$. For more details see [28] and references therein.

It is not sufficient to merely assume $\partial_u F(t, 0, 0) < 0$ in (N4). Indeed, for appropriate $p > 1$ Poláčik and Yanagida [30] constructed a positive solution of the problem

$$u_t = \Delta u + u^p, \quad (x, t) \in \mathbb{R}^N \times (0, \infty)$$

satisfying (2.4) and (2.5) that is not asymptotically symmetric. If we set $F(t, u, q) = u^p - e^{-t}u$ and $G(x, t) = e^{-t}u(x, t)$, then $\partial_u F(t, 0, 0) < 0$ and G satisfies (G2). However, u is not asymptotically symmetric.

The assumptions (2.4) and (2.5) guarantee that u is globally defined and $\{u(\cdot, t) : t \geq 0\}$ is relatively compact in $E := C_0^1(\mathbb{R}^N)$, which stands for the space of $C^1(\mathbb{R}^N)$ functions, bounded together with their first order derivatives, equipped with the standard C^1 norm. Define the ω -limit set of u as

$$\omega(u) = \{z : z = \lim_{n \rightarrow \infty} u(\cdot, t_n) \text{ for some } t_n \rightarrow \infty\}, \quad (2.8)$$

where the convergence is in the topology of the space $C_0^1(\mathbb{R}^N)$.

Then $\omega(u)$ is nonempty, compact set in E , and it attracts the solution in the following sense

$$\lim_{t \rightarrow \infty} \text{dist}_E(u(\cdot, t), \omega(u)) = 0. \quad (2.9)$$

We are ready to formulate our first symmetry result.

Theorem 2.2. *Assume (N1)–(N4), (G1), and let u be a global solution of (1.1) satisfying (2.4) and (2.5). Then either u converges to 0 in $L^\infty(\mathbb{R}^N)$ or there exist $\lambda \in \mathbb{R}$ and $\phi \in \omega(u)$ such that for each $x \in \mathbb{R}_\lambda^N$ one has*

$$\begin{aligned} \phi(2\lambda - x_1, x') &= \phi(x) & ((x_1, x') = x \in \mathbb{R}^N), \\ \partial_{x_1} \phi(x) &< 0 & (x \in \mathbb{R}_\lambda^N). \end{aligned} \quad (2.10)$$

If we in addition assume (G2), then either $\omega(u) = \{0\}$ or there is $\lambda \in \mathbb{R}$ such that (2.10) holds for all $\phi \in \omega(u)$.

The following example shows, that the last statement of Theorem 2.2 does not hold if we merely assume (G1). In particular it is not true that all functions in the ω -limit set are symmetric with respect to the same hyperplane.

Example 2.3. Let v be a positive function satisfying (2.4), (2.5), and

$$0 = \Delta v + g(v), \quad x \in \mathbb{R}^N, \quad (2.11)$$

for appropriate function g with $g'(0) < 0$. Such a function v exists for example for $g(u) = \lambda u + u^p$ (see e.g. [7] and references therein) with $\lambda < 0$, $1 < p < p_S$, where $p_S := \frac{N+2}{N-2}$ for $N \geq 3$ and $p_S := \infty$ for $N \leq 2$ is the critical Sobolev exponent. By [14], v is radially symmetric and radially

decreasing with center at a point $x_0 \in \mathbb{R}^N$. Let $\eta : [0, \infty) \rightarrow \mathbb{R}$ be a bounded differentiable function and define $u : \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}$ by $u(x, t) := v(x_1 + \eta(t), x')$ for any $(x, t) = ((x_1, x'), t) \in \mathbb{R}^N \times [0, \infty)$. Then u satisfies (2.4), (2.5), and

$$u_t = \Delta u + g(u) + G(x, t), \quad (x, t) \in \mathbb{R}^N \times [0, \infty),$$

where

$$G(x, t) := v_{x_1}(x_1 + \eta(t), x')\eta'(t) \quad ((x, t) = ((x_1, x'), t) \in \mathbb{R}^N \times [0, \infty)).$$

It is easy to see that we can choose η with the following properties. There are sequences $(s_k)_{k \in \mathbb{N}}, (t_k)_{k \in \mathbb{N}}$ with $s_k, t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\eta(t_k) = 1$, $\eta(s_k) = 0$, and there is $C > 0$ with $|\eta'(t)| \leq \frac{C}{t}$ for all $t > 0$. Since v_{x_1} is bounded,

$$\lim_{t \rightarrow \infty} \|G\|_{X_{(t, t+1)}} \leq \lim_{t \rightarrow \infty} \|G\|_{L^\infty(\mathbb{R}^N \times (t, t+1))} \leq \lim_{t \rightarrow \infty} \|v_{x_1}\|_{L^\infty(\mathbb{R}^N \times (0, \infty))} \frac{C}{t} = 0,$$

and in particular G satisfies (G1). However, $v(x_1 + s, x') \in \omega(u)$ for any $s \in [0, 1]$, and therefore the functions in $\omega(u)$ are not symmetric with respect to the same hyperplane.

Finally, we state the corollary of Theorem 2.2 on asymptotic radial symmetry. We omit the proof since it uses the same arguments as in the case $G \equiv 0$ (cf. [27]). The formulation of results on rotational symmetry, if the problem is rotationally symmetric, is left to the reader.

Corollary 2.4. *In addition to (N1)–(N4) and (G2), assume $A_{ij} \equiv 0$ if $i \neq j$ and*

$$A_{ii}(t, u, p) = A_{ii}(t, u, q), \quad F(t, u, p) = F(t, u, q) \quad \text{whenever } |p| = |q|.$$

Let u be a global solution of (1.1) satisfying (2.4) and (2.5). Then either u converges to 0 in $L^\infty(\mathbb{R}^N)$ or there exists $\xi \in \mathbb{R}$ such that for each $\phi \in \omega(u)$ there is $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\begin{aligned} \phi(x - \xi) &= \tilde{\phi}(|x|) & (x \in \mathbb{R}^N), \\ \partial_r \tilde{\phi}(r) &< 0 & (r = |x| > 0). \end{aligned}$$

Remark 2.5. Assume $f \in C^1([0, \infty))$, $f(0)$, and $f'(0) < 0$. Also assume $G \in C^\alpha((0, \infty), L^\infty(\mathbb{R}^N))$, ($\alpha > 0$) satisfies (G2). If u solves a semilinear problem

$$u_t = \Delta u + f(u) + G(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty),$$

with $\|u\|_{L^\infty(\mathbb{R}^N \times (0, \infty))} < \infty$ and

$$\limsup_{|x| \rightarrow \infty, t \in [0, \infty)} |u(x, t)| = 0, \quad (2.12)$$

then the statement of Corollary 2.4 holds true. Indeed, assumptions of Corollary 2.4 on A , F , and G are clearly satisfied. By standard regularity theory (cp. [23]), $\|u\|_{L^\infty(\mathbb{R}^N \times (0, \infty))} < \infty$ and $G \in C^\alpha((0, \infty), L^\infty(\mathbb{R}^N))$ imply (2.4). One can easily see that in the semilinear case only (2.12) is needed in the proof of main results.

3 Linear equations

This section is devoted to linear parabolic estimates as a preparation for the method of moving hyperplanes. The results that were already published are stated without proofs. However, at some places we have to extend existing results and for those we include proofs as well.

Recall the following standard notation. For an open set $Q \subset \mathbb{R}^N$ we denote by $\partial_P Q$ the parabolic boundary of Q (for precise definition see e.g. [17, 22]). We also define a time cut of Q to be

$$Q_M := \{(x, s) \in \bar{Q} : s \in M\} \quad (M \subset \mathbb{R}). \quad (3.1)$$

If $M = \{t\}$, we often write Q_t instead of $Q_{\{t\}}$.

For bounded sets U, U_1 in \mathbb{R}^N or \mathbb{R}^{N+1} , the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\text{diam } U$ stands for the diameter of U , and $|U|$ for its Lebesgue measure (if it is measurable). For any $\lambda \in (-\infty, \infty)$ we define an open half space: $\mathbb{R}_\lambda^N := \{x \in \mathbb{R}^N : x_1 > \lambda\}$, and for $\lambda = -\infty$ we set $\mathbb{R}_\lambda^N = \mathbb{R}^N$. The open ball in \mathbb{R}^N centered at x with radius r is denoted by $B(x, r)$ and if the ball is centered at the origin, that is, if $x = 0$, we also write $B_r := B(0, r)$. For any $\lambda \in \mathbb{R}$ and $R > 0$ we set $B_R^\lambda := B_R \cap \mathbb{R}_\lambda^N$. Symbols f^+ and f^- denote the positive and negative parts of a function f : $f^\pm := (|f| \pm f)/2 \geq 0$.

We consider time dependent elliptic operators L of the form

$$L(x, t) = a_{km}(x, t) \frac{\partial^2}{\partial x_k \partial x_m} + b_k(x, t) \frac{\partial}{\partial x_k}. \quad (3.2)$$

To simplify the notation we shall use the following definition.

Definition 3.1. Given an open set $Q \in \mathbb{R}^N \times (0, \infty)$ and positive numbers α_0, β_0 , we say that an operator L of the form (3.2) belongs to $E(\alpha_0, \beta_0, Q)$ if its coefficients a_{km}, b_k are measurable functions defined on Q and they satisfy

$$\begin{aligned} |a_{km}|, |b_k| &\leq \beta_0 & (k, m = 1, \dots, N), \\ a_{km}(x, t) \xi_k \xi_m &\geq \alpha_0 |\xi|^2 & ((x, t) \in Q, \xi \in \mathbb{R}^N). \end{aligned}$$

3.1 Nonlinear to linear

In this subsection we assume (N1)–(N4) and (G1). At some places, where explicitly stated, we also assume (G2). Fix a positive global solution u of (1.1) satisfying (2.4) and (2.5). We show, how symmetries of the problem give rise to linear equations from the nonlinear ones.

We say that a pair of functions (\tilde{u}, \tilde{G}) is *admissible*, if \tilde{u} satisfies (2.4), (2.5), \tilde{G} satisfies (G1), and \tilde{u} is a positive solution of (1.1) with G replaced by \tilde{G} . In particular (u, G) is an admissible pair.

Let (\tilde{u}, \tilde{G}) be an admissible pair different to (u, G) . If we denote $w := u - \tilde{u}$, then

$$\begin{aligned} w_t &= L(x, t)w + c(x, t)w + f(x, t), & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ \lim_{|x| \rightarrow \infty} \sup_{t \in (0, \infty)} |w(x, t)| &= 0, \end{aligned} \quad (3.3)$$

where L has the form (3.2) with

$$\begin{aligned}
a_{ij}(x, t) &= A_{ij}(t, u(x, t), \nabla u(x, t)), \\
b_i(x, t) &= \int_0^1 F_{p_i}(t, u(x, t), \nabla \tilde{u}(x, t) + s(\nabla u(x, t) - \nabla \tilde{u}(x, t))) ds \\
&\quad + \tilde{u}_{x_k x_\ell}(x, t) \int_0^1 A_{k\ell, p_i}(t, u(x, t), \nabla \tilde{u}(x, t) + s(\nabla u(x, t) - \nabla \tilde{u}(x, t))) ds, \\
c(x, t) &= \int_0^1 F_u(t, \tilde{u}(x, t) + s(u(x, t) - \tilde{u}(x, t)), \nabla \tilde{u}(x, t)) ds \\
&\quad + \tilde{u}_{x_k x_\ell}(x, t) \int_0^1 A_{k\ell, u}(t, \tilde{u}(x, t) + s(u(x, t) - \tilde{u}(x, t)), \nabla \tilde{u}(x, t)) ds.
\end{aligned} \tag{3.4}$$

Then

$$L \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (0, \infty)), \quad \|c\|_{L^\infty(\mathbb{R}^N \times (0, \infty))} \leq \beta_0, \tag{3.5}$$

and, by (N1), Remark 2.1

$$c(x, t) < -\gamma, \tag{3.6}$$

whenever $u(x, t)$, $\tilde{u}(x, t)$, $|\nabla \tilde{u}(x, t)|$ and $|D^2 \tilde{u}(x, t)|$ are smaller than ε_γ^* , where ε_γ^* was defined in Remark 2.1. Observe, that we do not impose any smallness assumptions on $|\nabla u(x, t)|$ or $|D^2 u(x, t)|$.

Moreover,

$$f := G - \tilde{G} \in X_{(t, t+1)}, \quad \lim_{t \rightarrow \infty} \|f\|_{X_{(t, t+1)}} = 0. \tag{3.7}$$

If we suppose that (G2) holds for G and \tilde{G} , then

$$\|f\|_{X_{(t, \infty)}} \leq C_\mu e^{-\mu t} \quad (t > 0). \tag{3.8}$$

Uniform continuity of derivatives of $(A_{ij})_{1 \leq i, j \leq N}$ and F in conjunction with (2.4) yields that (a_{ij}) , (b_i) , and c are continuous in x and t .

Example 3.2. By (N4), $\tilde{u} \equiv 0$ and $\tilde{G} \equiv 0$ is an admissible pair. Thus $w = u - 0 = u$ solves the equation (3.3) such that (3.5) and (3.7) hold true with $f = G$. Moreover,

$$c(x, t) < -\gamma \quad ((x, t) \in \mathbb{R}^N \times (0, \infty) : u(x, t) \leq \varepsilon_\gamma^*), \tag{3.9}$$

and by (2.7),

$$c(x, t) < -\gamma \quad ((x, t) \in \mathbb{R}^N \times (0, \infty), |x| \geq \rho_\gamma^*). \tag{3.10}$$

Example 3.3. For any $x_0 \in \mathbb{R}^N$ define

$$\tilde{u}(x, t) := u(x + x_0, t) \text{ and } \tilde{G}(x, t) := G(x + x_0, t) \quad ((x, t) \in \mathbb{R}^N \times (0, \infty)).$$

Since $(A_{ij})_{1 \leq i, j \leq N}$ and F are independent of x , the pair (\tilde{u}, \tilde{G}) is admissible. Therefore, $w(x, t) := u(x + x_0, t) - u(x, t)$ satisfies (3.3), such that (3.5) and (3.7) hold true. Moreover, by (3.6) and (2.7)

$$c(x, t) < -\gamma \quad ((x, t) \in \mathbb{R}^N \times (0, \infty), |x| \geq \rho_\gamma^* + |x_0|).$$

The next example is crucial for the method of moving hyperplanes. To simplify the notation denote $x^\lambda := (2\lambda - x_1, x')$, the reflection of $x = (x_1, x') \in \mathbb{R}^N$ with respect to the hyperplane H_λ . We indicate explicitly the dependence of functions and operators on λ .

Example 3.4. By (N3),

$$\tilde{u}(x, t) := u(x^\lambda, t) \text{ and } \tilde{G}(x, t) := G(x^\lambda, t) \quad ((x, t) \in \mathbb{R}^N \times (0, \infty))$$

form an admissible pair. Thus, $w^\lambda := \tilde{u} - u$ satisfies (3.3) such that (3.5) and (3.7) hold true. Moreover, $|x| > 2|\lambda| + \rho_\gamma^* \geq \rho_\gamma^*$ implies $|x^\lambda| > \rho_\gamma^*$, and therefore (3.6) and (2.7) yield

$$c^\lambda(x, t) < -\gamma \quad ((x, t) \in \mathbb{R}^N \times (0, \infty), |x| \geq \rho_\gamma^* + 2|\lambda|). \quad (3.11)$$

By (N1), (2.6) (and (G2), if assumed), the constants α_0, β_0 , (and also C_μ, μ) are independent of λ . Notice that $w^\lambda(x, t) = 0$ for any $(x, t) \in H_\lambda \times [0, \infty)$. Hence, w^λ satisfies

$$\begin{aligned} w_t^\lambda &= L^\lambda(x, t)w^\lambda + c^\lambda(x, t)w^\lambda + f^\lambda(x, t), & (x, t) \in \mathbb{R}_\lambda^N \times (0, \infty), \\ w^\lambda &= 0, & (x, t) \in H_\lambda \times (0, \infty), \end{aligned} \quad (3.12)$$

$$\limsup_{|x| \rightarrow \infty, t > 0} w^\lambda(x, t) = 0.$$

Also, if G satisfies (G2), then \tilde{G} satisfies (G2) as well. Consequently (3.8) holds with f replaced by f^λ . Notice that (a_{ij}) in (3.4) are independent of λ .

3.2 Estimates of solutions

The results in this subsection might be of independent interest, therefore we state them in more general setting, than required for the proofs of our symmetry results.

Let Q be a domain in \mathbb{R}^{N+1} (bounded or unbounded), and let α_0, β_0 be positive constants. Consider a general linear parabolic equation

$$v_t = L(x, t)v + c(x, t)v + f(x, t), \quad (x, t) \in Q. \quad (3.13)$$

For any $s < t$ denote $X_{(s,t)}(Q)$ the space of functions $f : Q \rightarrow \mathbb{R}$ such that their extension by 0 to \mathbb{R}^{N+1} belongs to $X_{(s,t)}$ (cf. (1.10)). We denote $C_{loc}(\bar{Q})$ the space of continuous functions equipped with the topology induced by the locally uniform convergence.

First, we formulate Alexandrov – Krylov estimate, proved by Alexandrov [1] in the elliptic case, and later extended by Krylov [17] to the parabolic setting. In the literature, one can find many generalizations of these results. Here, we extend Cabré’s result [9] to functions f belonging to $X_{(s,t)}(Q)$. If $f \equiv 0$, we refer to the next theorem as the maximum or comparison principle.

Theorem 3.5. *Given $\tau < T$, fix an open set $Q \subset \mathbb{R}^N \times (\tau, T)$. If $v \in C_{loc}(\bar{Q}) \cap W_{N+1,loc}^{2,1}(Q)$ is a bounded supersolution of (3.13) (it satisfies (3.13) with “ = ” replaced by “ \geq ”) with $L \in E(\alpha_0, \beta_0, Q)$, a measurable function $c \leq 0$, and $f \in X_{(\tau,T)}(Q)$, then*

$$\sup_Q v^- \leq \sup_{\partial_P Q} v^- + C \|f^-\|_{X_{(\tau,T)}(Q)}, \quad (3.14)$$

where C depends on $N, \alpha_0, \beta_0, T - \tau$.

Proof. Fix arbitrary $\varepsilon > 0$ and choose f_1, f_2 such that $f_1^- + f_2^- = f^-$ and

$$\|f^-\|_{X_{(\tau,T)}(Q)} + \varepsilon \geq \|f_1^-\|_{L^{N+1}(Q_{(\tau,T)})} + \|f_2^-\|_{L^\infty(Q_{(\tau,T)})}.$$

Since $c \leq 0$, the bounded function $w : Q \rightarrow \mathbb{R}$

$$w(x, t) := v(x, t) + \sup_{\partial_P Q} v^- + (t - \tau) \|f_2^-\|_{L^\infty(Q_{(\tau,T)})} \quad ((x, t) \in Q),$$

satisfies

$$\begin{aligned} w_t &\geq L(x, t)w + c(x, t)w - f_1^-(x, t), & (x, t) \in Q, \\ w &\geq 0, & (x, t) \in \partial_P Q. \end{aligned} \quad (3.15)$$

Consequently, by [9, Corollary 1.16]

$$\sup_Q w^- \leq C \|f_1^-\|_{L^{N+1}(Q)}, \quad (3.16)$$

where C depends on $N, \alpha_0, \beta_0, T - \tau$. Then,

$$\begin{aligned} \sup_Q v^- &\leq \sup_Q w^- + \sup_{\partial_P Q} v^- + (t - \tau) \|f_2^-\|_{L^\infty(Q_{(\tau, T)})} \\ &\leq \sup_{\partial_P Q} v^- + C (\|f_2^-\|_{L^\infty(Q)} + \|f_1^-\|_{L^{N+1}(Q)}) \\ &\leq \sup_{\partial_P Q} v^- + C (\|f^-\|_{X_{(\tau, T)}(Q)} + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (3.14) follows. \square

Corollary 3.6. *If the assumption $c \leq 0$ of the previous theorem is replaced by $c \leq k$ for some $k \in \mathbb{R}$, and all other assumptions are retained, then*

a) if $k \geq 0$

$$\sup_{Q_{[\tau, T]}} v^- \leq e^{k(T-\tau)} \left(\sup_{\partial_P(Q_{[\tau, T]})} v^- + C \|f^-\|_{X_{(\tau, T)}(Q)} \right),$$

where C depends on $N, \alpha_0, \beta_0, T - \tau$.

b) if $k < 0$

$$\begin{aligned} \sup_{Q_T} v^- &\leq \max\{e^{k(T-\tau)} \|v^-\|_{L^\infty(Q_\tau)}, \sup_{(\partial_P Q_{[\tau, T]}) \setminus Q_\tau} v^-\} \\ &\quad + \frac{C}{1 - e^k} \sup_{t \in [\tau, T-1]} \|f^-\|_{X_{(t, t+1)}(Q)}, \end{aligned}$$

where C depends on N, α_0, β_0 . Notice that C is independent of $T - \tau$.

Proof of Corollary 3.6. The function $\tilde{v} := e^{-kt}v$ is a supersolution of (3.13) with c and f replaced by $c - k$ and \tilde{f} respectively, where $\tilde{f}(x, t) = e^{-kt}f(x, t)$. Since $c - k \leq 0$, Theorem 3.5 implies

$$\begin{aligned} e^{-kt_2} \sup_{Q_{t_2}} v^- &= \sup_{Q_{t_2}} \tilde{v}^- \leq \sup_{Q_{[t_1, t_2]}} \tilde{v}^- \\ &\leq \max\{\sup_{Q_{t_1}} \tilde{v}^-, \sup_{(\partial_P Q_{[t_1, t_2]}) \setminus Q_{t_1}} \tilde{v}^-\} + C \|\tilde{f}^-\|_{X_{(t_1, t_2)}(Q)} \\ &\quad (\tau \leq t_1 < t_2 \leq T), \quad (3.17) \end{aligned}$$

where C depends on $N, \alpha_0, \beta_0, t_2 - t_1$.

If $k \geq 0$, we set $t_1 = \tau$ and elementary manipulations imply

$$\sup_{Q_{t_2}} v^- \leq e^{k(t_2 - \tau)} \left(\sup_{\partial_P(Q_{[\tau, t_2]})} v^- + C \|f^-\|_{X(\tau, t_2)(Q)} \right).$$

Part a) follows, if we take supremum with respect to $t_2 \in [\tau, T]$.

Denote $\Gamma := \sup_{t \in [\tau, T-1]} \|f^-\|_{X(t, t+1)(Q)}$. If $k < 0$, then (3.17) with $t_2 = t_1 + 1$ yields

$$\sup_{Q_{t_1+1}} v^- \leq \max\{e^k \sup_{Q_{t_1}} v^-, \sup_{(\partial_P Q_{(t_1, t_1+1)}) \setminus Q_{t_1}} v^-\} + C\Gamma \quad (t_1 \in [\tau, T-1]),$$

where C depends on N, α_0, β_0 . Iterating the previous expression for $t_1 = \tau + j$, with $j \in \mathbb{N}$, $j \leq T - \tau - 1$, we obtain

$$\sup_{Q_{\tau+j}} v^- \leq \max\{e^{kj} \sup_{Q_\tau} v^-, \sup_{(\partial_P Q_{(\tau, \tau+j)}) \setminus Q_\tau} v^-\} + C\Gamma \sum_{i=0}^{j-1} e^{ki}. \quad (3.18)$$

Choose $j_0 \in \mathbb{N} \cup \{0\}$ such that $\tau + j_0 \leq T < \tau + j_0 + 1$. Then (3.17) with $t_1 = \tau + j_0$, $t_2 = T$ and (3.18) imply

$$\begin{aligned} \sup_{Q_T} v^- &\leq \max\{e^{k(T - (\tau + j_0))} \sup_{Q_{\tau+j_0}} v^-, \sup_{(\partial_P Q_{(\tau+j_0, T)}) \setminus Q_{\tau+j_0}} v^-\} + C\Gamma \\ &\leq \max\{e^{k(T - \tau)} \sup_{Q_\tau} v^-, \sup_{(\partial_P Q_{(\tau, T)}) \setminus Q_\tau} v^-\} + C\Gamma \sum_{i=0}^{\infty} e^{ki}, \end{aligned}$$

where C depends on N, α_0, β_0 and the part b) follows. \square

If $Q = \mathbb{R}_\lambda^N \times (\tau, T)$, $\lambda \in \mathbb{R}$, $\tau < T$, we can change variables such that c becomes negative in the neighborhood of H_λ and it does not change too much away from H_λ . Such results are usually obtained with an application of an appropriate supersolution. The observation that such procedure is possible for thin domains and domains of small measure was proved in [8]. In the next lemma we summarize properties of the supersolution constructed in [27, Lemma 2.5].

Lemma 3.7. *Given $\Theta, \varepsilon > 0$, there exist a function $g : [0, \infty) \rightarrow \mathbb{R}$ and a constant $\delta = \delta(N, \alpha_0, \beta_0, \Theta, \varepsilon) > 0$ with the following properties:*

$$\begin{aligned} g &\in C^1([0, \infty)) \cap C^2([0, \delta]) \cap C^2((\delta, \infty)), \\ \frac{1}{2} &\leq g \leq 2, \\ g''(\xi) + \Theta(|g'(\xi)| + g(\xi)) &\leq 0 \quad (\xi \in (0, \delta)), \\ g''(\xi) + \Theta|g'(\xi)| - \varepsilon g(\xi) &\leq 0 \quad (\xi \in (\delta, \infty)). \end{aligned}$$

Following [27, Remark 2.6], we obtain the following result.

Remark 3.8. Set $Q := \mathbb{R}_\lambda^N \times (\tau, T)$ for some $\lambda \in \mathbb{R}$ and $0 \leq \tau < T \leq \infty$. Let $v \in C_{loc}(\bar{Q}) \cap W_{N+1, loc}^{2,1}(Q)$ be a solution of (3.13) with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, and $f \in L^{N+1}(Q)$ satisfying

$$v = 0 \quad ((x, t) \in H_\lambda \times (\tau, T)) \quad \text{and} \quad \lim_{M \rightarrow \infty} \sup_{(x,t) \in Q, |x| \geq M} |v(x, t)| = 0.$$

For any $\gamma > 0$ set $\Theta = \frac{2\beta_0}{\gamma} + 1$, $\varepsilon = \frac{\gamma}{2}$ and let $\delta = \delta(N, \alpha_0, \beta_0, \gamma) > 0$ and g be as in Lemma 3.7. Then

$$w : (x, t) \mapsto \frac{v(x, t)}{g(x_1 - \lambda)} \quad ((x, t) \in Q)$$

is a solution of

$$\begin{aligned} w_t &= \hat{L}(x, t)w + \hat{c}(x, t)w + \hat{f}(x, t), & (x, t) \in \mathbb{R}_\lambda^N \times (\tau, T), \\ w &= 0, & (x, t) \in H_\lambda \times (\tau, T), \\ \lim_{M \rightarrow \infty} \sup_{(x,t) \in Q, |x| \geq M} |v(x, t)| &= 0 \end{aligned} \quad (3.19)$$

with $\hat{L}(x, t) \in E(\alpha_0, 5\beta_0, Q)$, $\|\hat{c}\|_{L^\infty(Q)} \leq 5\beta_0$ and

$$\|\hat{f}\|_{X_{(\tau, T)}(Q)} \leq 2\|f\|_{X_{(\tau, T)}(Q)}.$$

Moreover,

$$\hat{c}(x, t) \leq \begin{cases} -\frac{\gamma}{2} & (x, t) \in Q, x_1 \in [\lambda, \lambda + \delta), \\ c(x, t) + \frac{\gamma}{2} & (x, t) \in Q, x_1 \in [\lambda, \infty). \end{cases} \quad (3.20)$$

We conclude this section with a version of Krylov-Safonov Harnack inequality [18] (see also [22]) for sign changing solutions of nonhomogeneous problems. The statement is based on [29, Lemma 3.5], however it was modified to obtain the dependence of κ and κ_1 on $\text{diam } D$ instead of $\text{diam } U$.

Lemma 3.9. *Given numbers $d > 0$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, and τ with $\tau_1 - 2\theta \leq \tau \leq \tau_1 - \theta$, consider bounded domains $D, U \subset \mathbb{R}^N$ with*

$$D \subset\subset U, \quad \text{dist}(\bar{D}, \partial U) \geq d,$$

and denote $Q = U \times (\tau, \tau_4)$. Then there exist constants $\kappa, \kappa_1 > 0$ determined only by N , α_0 , β_0 , d , $\text{diam } D$, θ , $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_4 - \tau_3$ with the following property. If $v \in C_{loc}(\bar{Q}) \cap W_{N+1,loc}^{2,1}(Q)$ is a solution of (3.13), with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, and $f \in X_{(\tau, \tau_4)}(Q)$, then

$$\inf_{\bar{D} \times (\tau_3, \tau_4)} v \geq \kappa \|v^+\|_{L^\infty(D \times (\tau_1, \tau_2))} - \kappa_1 \|f\|_{X_{(\tau, \tau_4)}(Q)} - \sup_{\partial_F Q} e^{m(\tau_4 - \tau)} v^-,$$

where $m = \sup_Q c$.

Sketch of the proof. Since the proof closely follows [29, Proof of Lemma 3.5], we only outline differences (our statement includes a minor correction to [29, Lemma 3.5], as given in the addendum, see [25]). Instead of [29, Lemma 3.6] we employ the original Krylov-Safonov Harnack inequality for nonnegative solutions, [17, 18] where κ depends on N , $\text{diam } D$, α_0 , β_0 , θ , $\tau_2 - \tau_1$, $\tau_3 - \tau_2$ and $\tau_4 - \tau_3$, but not on $\text{diam } U$. Moreover, we use Theorem 3.5 instead of used Alexandrov-Krylov estimate to make κ_1 independent of $\text{diam } U$ and to replace L^{N+1} norm of f by X norm. The rest of the proof remains unchanged. \square

In the next corollary we formulate Harnack inequality for half spaces and the whole space. Based on the proof, one can easily formulate the results for other unbounded domains. If $\lambda = -\infty$ in the next corollary, we set $\mathbb{R}_\lambda^N := \mathbb{R}^N$ and $H_\lambda := \emptyset$.

Corollary 3.10. *Given numbers $d > 0$, $\lambda \in \mathbb{R} \cup \{-\infty\}$, $\theta > 0$, $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4$, and $\tau_1 - 2\theta \leq \tau \leq \tau_1 - \theta$, denote $Q := \mathbb{R}_\lambda^N \times (\tau, \tau_4)$. Fix a bounded domain $D \subset\subset \mathbb{R}_\lambda^N$ with $\text{dist}(\bar{D}, H_\lambda) \geq d$. If $v \in C_{loc}(\bar{Q}) \cap W_{N+1,loc}^{2,1}(Q)$ satisfies (3.13) with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, $f \in X_{(\tau, \tau_4)}(Q)$, and*

$$\lim_{M \rightarrow \infty} \sup_{(x,t) \in Q, |x| \geq M} |v(x, t)| = 0, \quad (3.21)$$

then there exist constants κ , κ_1 and p depending on N , α_0 , β_0 , d , $\text{diam}(D)$, θ , $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, and $\tau_4 - \tau_3$ such that

$$\inf_{\bar{D} \times (\tau_3, \tau_4)} v \geq \kappa \|v^+\|_{L^\infty(D \times (\tau_1, \tau_2))} - \sup_{\partial_P Q} e^{\beta_0(\tau_4 - \tau)} v^- - \kappa_1 \|f\|_{X_{(\tau, \tau_4)}(Q)}.$$

Proof. Choose large enough R such that $D \subset\subset B_R^\lambda$ and $\text{dist}(\partial B_R^\lambda, D) \geq \frac{d}{2}$. Then Lemma 3.9 applied with $U = B_R^\lambda$ implies

$$\begin{aligned} \inf_{\bar{D} \times (\tau_3, \tau_4)} v &\geq \kappa \|v^+\|_{L^\infty(D \times (\tau_1, \tau_2))} - \sup_{\partial_P Q} e^{\beta_0(\tau_4 - \tau)} v^- \\ &\quad - \sup_{|x|=R, t \in (\tau, \tau_4)} e^{\beta_0(\tau_4 - \tau)} v^-(x, t) - \kappa_1 \|f\|_{X_{(\tau, \tau_4)}(Q)}, \end{aligned}$$

where κ and κ_1 are as in Lemma 3.9. In particular they are independent of R . Passing $R \rightarrow \infty$ and using (3.21), we obtain the desired result. \square

We mostly use Corollary 3.10 with

$$\tau = \tau_1 - \vartheta \quad \text{and} \quad \tau_i = \tau + i\vartheta, \quad (i = 1, 2, 3, 4). \quad (3.22)$$

With this choice we obtain the following result.

Corollary 3.11. *For given $d > 0$, $\lambda \in \mathbb{R} \cup \{-\infty\}$, $\vartheta \in (0, 1)$ and $\tau > 1$. Denote $Q := \mathbb{R}_\lambda^N \times (\tau, \tau + 4\vartheta)$ and fix a bounded domain $D \subset\subset \mathbb{R}_\lambda^N$ with $\text{dist}(\bar{D}, H_\lambda) \geq d$. If $v \in C_{loc}(\bar{Q}) \cap W_{N+1, loc}^{2,1}(Q)$ satisfies (3.13) with $L \in E(\alpha_0, \beta_0, Q)$, $\|c\|_{L^\infty(Q)} \leq \beta_0$, $f \in X_{(\tau, \tau+4\vartheta)}(Q)$, and*

$$\lim_{M \rightarrow \infty} \sup_{(x,t) \in Q, |x| \geq M} |v(x, t)| = 0,$$

then there exist constants κ and κ_1 depending on N , α_0 , β_0 , d , $\text{diam}(D)$, ϑ such that

$$\inf_{\bar{D} \times (\tau+3\vartheta, \tau+4\vartheta)} v \geq \kappa \|v^+\|_{L^\infty(D \times (\tau+\vartheta, \tau+2\vartheta))} - \sup_{\partial_P Q} e^{4\beta_0\vartheta} v^- - \kappa_1 \|f\|_{X_{(\tau, \tau+4\vartheta)}(Q)}.$$

4 Proof of Theorem 2.2

In this section the notation and assumptions are as in Section 2. In particular, $(A_{ij})_{1 \leq i, j \leq N}$ and F satisfy (N1)–(N4) and G satisfies (G1). At some places,

where explicitly stated, we also assume (G2). Let u be a positive, global, classical solution of (1.1) satisfying (2.4) and (2.5).

In addition we denote

$$X_{(s,t)}^\lambda := X_{(s,t)}(\mathbb{R}_\lambda^N \times (s, t)) \quad (\lambda \in \mathbb{R}, s, t \in (0, \infty), s < t),$$

where $X_{(s,t)}(Q)$, for general $Q \subset \mathbb{R}^{N+1}$, was defined at the beginning of Subsection 3.2.

To start the proof, we assume

$$\limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} > 0, \quad (4.1)$$

otherwise $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ the the theorem follows.

Lemma 4.1. *Given any ball $B \subset \mathbb{R}^N$, there exists $k(B) > 0$ and $\tilde{T} > 0$ depending on N , α_0 , β_0 , and B such that*

$$u(x, t) \geq k(B) \quad ((x, t) \in \bar{B} \times (\tilde{T}, \infty)). \quad (4.2)$$

Proof. We claim that

$$\liminf_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} > 0. \quad (4.3)$$

Suppose not, that is, suppose

$$\liminf_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = 0. \quad (4.4)$$

We find a contradiction by showing that there exists $\tau > 0$ with

$$u(x, t) < 3\varepsilon := \frac{1}{2} \min \left\{ \limsup_{t \rightarrow \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}, \varepsilon_\gamma^* \right\} \\ ((x, t) \in \mathbb{R}^N \times (\tau, \infty)), \quad (4.5)$$

where ε_γ^* was defined in Remark 2.1.

According to Example 3.2, u satisfies (3.3) with $L \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (0, \infty))$ such that (3.5), (3.7), (3.9), and (3.10) hold. Let $C = C(\alpha_0, \beta_0, N)$ be a constant from Corollary 3.6 b). Then by (3.7) (or (G1)) and (4.4) there is τ with

$$\max \left\{ \frac{C}{1 - e^{-\gamma}} \sup_{s \geq \tau} \|G\|_{X_{(s, s+1)}}, \|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} \right\} \leq \varepsilon.$$

We prove that (4.5) holds for such τ . Suppose not, that is, suppose that

$$T := \inf\{t > \tau : \sup_{x \in \mathbb{R}^N} u(x, t) = 3\varepsilon\} < \infty.$$

Since $3\varepsilon < \varepsilon_\gamma^*$, by (3.9) one has $c(x, t) \leq -\gamma$ for any $(x, t) \in \mathbb{R}^N \times [\tau, T]$. An application of Corollary 3.6 b) with $Q = \mathbb{R}^N \times (\tau, T)$ yields

$$\sup_{\mathbb{R}^N} u(\cdot, T) \leq e^{-\gamma(T-\tau)} \|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} + \frac{C}{1 - e^{-\gamma}} \sup_{s \geq \tau} \|G\|_{X_{(s, s+1)}} \leq 2\varepsilon < 3\varepsilon,$$

a contradiction. Thus, (4.3) holds true, or equivalently there are constants $s, T > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} > s \quad (t \in (T, \infty)).$$

By (2.5), we can replace \mathbb{R}^N in the previous inequality by $B_R \cup B$ for a sufficiently large R independent of T . Then, an application of Corollary 3.11 with $(d, \lambda, D, \tau, \vartheta) = (1, -\infty, B_R \cup B, t, 1)$ yields

$$u(x, t) \geq \kappa s - \kappa_1 \|G\|_{X_{(t-4, t)}} \quad ((x, t) \in (\bar{B}_R \cup B) \times (T + 4, \infty)),$$

where κ, κ_1 depend on R, N, α_0, β_0 . Since the second term in the previous inequality converges to 0 as $t \rightarrow \infty$, we obtain for sufficiently large \tilde{T}

$$u(x, t) \geq k(B_R \cup B) := \frac{\kappa s}{2} \quad ((x, t) \in (\bar{B}_R \cup B) \times (\tilde{T}, \infty)). \quad \square$$

Recall, that for any $x = (x_1, x') \in \mathbb{R}_\lambda^N$ we already defined $x^\lambda = (2\lambda - x_1, x')$. Now, for any function $g : \mathbb{R}^N \rightarrow \mathbb{R}$, let

$$V_\lambda g(x) := g(x^\lambda) - g(x) \quad (x \in \mathbb{R}_\lambda^N, \lambda \in \mathbb{R}),$$

and for the solution u of (1.1) let

$$w^\lambda(x, t) := V_\lambda u(x, t) := u(x^\lambda, t) - u(x, t) \quad ((x, t) \in \mathbb{R}_\lambda^N \times (0, \infty), \lambda \in \mathbb{R}).$$

As shown in Example 3.4, the function w^λ satisfies (3.3) such that (3.5), (3.7), (3.11), and (3.12) hold. Hence, the results of Subsection 3.2 are applicable to w^λ . We use this observation below, often without notice.

In the process of the moving hyperplanes we examine the following statement

$$\liminf_{t \rightarrow \infty} \inf_{x \in D} w^\lambda(x, t) \geq 0 \quad \text{for all bounded } D \subset \mathbb{R}_\lambda^N, \quad (4.6)$$

which by the compactness of $\{u(\cdot, t) : t \geq 0\}$ in $C_0^1(\mathbb{R}^N)$ is equivalent to

$$V_\lambda z(x) \geq 0 \quad (x \in \mathbb{R}_\lambda^N, z \in \omega(u)). \quad (4.6^*)$$

The next lemma states an criterion for (4.6) to hold.

Lemma 4.2. *Consider g and $\delta = \delta(N, \alpha_0, \beta_0, \gamma) > 0$ such that Lemma 3.7 is satisfied with $(\Theta, \varepsilon) = (\frac{\beta_0}{\alpha_0} + 1, \frac{\gamma}{2\alpha_0})$. For fixed $\lambda > 0$ consider a domain $D_0 \subset \subset \mathbb{R}_\lambda^N$ such that*

$$B_{\rho_\gamma^* + 2|\lambda|} \cap \{x \in \mathbb{R}_\lambda^N : x_1 > \lambda + \delta\} \subset D_0, \quad (4.7)$$

where ρ_γ^* was defined in (2.7). Then (4.6) holds, provided there exist $\eta > 0$ and $t_0 > 0$ with

$$(w^\lambda)^+(x, t) \geq \eta \quad ((x, t) \in D_0 \times (t_0, \infty)). \quad (4.8)$$

Remark 4.3. Notice that (4.8) is equivalent to assumptions in [27, Lemma 3.2]:

$$\liminf_{t \rightarrow \infty} \|w^\lambda(\cdot, t)\|_{L^\infty(D_0)} > 0, \quad (4.9)$$

$$w^\lambda(x, t) > 0 \quad ((x, t) \in D_0 \times (t_0, \infty)). \quad (4.10)$$

Proof of Lemma 4.2. Fix a bounded domain $D^* \subset \subset \mathbb{R}_\lambda^N$ with $D_0 \subset D^*$ and denote $d := \text{dist}(D^*, H_\lambda)$.

If we transform (3.12) as described in Remark 3.8, then

$$\tilde{w}^\lambda(x, t) := \frac{w^\lambda(x, t)}{g(x_1 - \lambda)} \quad ((x, t) \in \mathbb{R}_\lambda^N \times (0, \infty))$$

satisfies (3.19) with $\hat{L}^\lambda \in E(\alpha_0, 5\beta_0, \mathbb{R}_\lambda^N \times (0, \infty))$, $\|\hat{c}^\lambda\|_{L^\infty(\mathbb{R}_\lambda^N \times (0, \infty))} \leq 5\beta_0$, and \hat{f}^λ satisfies (3.7). Moreover, by (3.11) one has $c(x, t) < -\gamma$ for each $(x, t) \in (\mathbb{R}_\lambda^N \setminus B_{\rho_\gamma^* + 2|\lambda|}) \times (0, \infty)$, and consequently (3.20) yields

$$\hat{c}(x, t) \leq -\frac{\gamma}{2} \quad ((x, t) \in (\mathbb{R}_\lambda^N \setminus D_0) \times (0, \infty)).$$

By (4.8), any connected component Q of the set $\{(x, t) : \tilde{w}^\lambda(x, t) < 0, t \geq t_0\}$ is contained in $(\mathbb{R}_\lambda^N \setminus D_0) \times (t_0, \infty)$, and in particular $\hat{c}(x, t) \leq -\gamma/2$ for any $(x, t) \in Q$. Then Corollary 3.6 b) implies

$$\|(\tilde{w}^\lambda)^-\|_{L^\infty(Q_t)} \leq e^{-\frac{\gamma}{2}(t-t^*)} \|(\tilde{w}^\lambda)^-\|_{L^\infty(Q_{t^*})} + \frac{C}{1 - e^{-\frac{\gamma}{2}}} \sup_{s \geq t^*} \|f^\lambda\|_{X_{(s, s+1)}^\lambda} \quad (t_0 < t^* < t), \quad (4.11)$$

where C depends on N , α_0 , and β_0 . This, (4.8), and an application of Corollary 3.11 with $\vartheta = \frac{1}{4}$ imply

$$\begin{aligned}
\tilde{w}^\lambda(x, t+1) &\geq \kappa \|(\tilde{w}^\lambda)^+\|_{L^\infty(D^* \times (t+\frac{1}{4}, t+\frac{1}{2}))} - e^{\beta_0} \sup_{\partial_P(\mathbb{R}_\lambda^N \times (t, t+1))} (\tilde{w}^\lambda)^- \\
&\quad - \kappa_1 \|f^\lambda\|_{X_{(t, t+1)}^\lambda} \\
&\geq \kappa \eta - e^{\beta_0} e^{-\frac{7}{2}(t-t^*)} \|(\tilde{w}^\lambda)^-(\cdot, t^*)\|_{L^\infty(\mathbb{R}_\lambda^N)} \\
&\quad - \left(\kappa_1 + \frac{e^{\beta_0} C}{1 - e^{\frac{7}{2}}} \right) \sup_{s \geq t^*} \|f^\lambda\|_{X_{(t, t+1)}^\lambda} \quad ((x, t) \in D^* \times (t^*, \infty)),
\end{aligned} \tag{4.12}$$

where $\kappa, \kappa_1 > 0$ depends on N , α_0 , β_0 , d and $\text{diam } D^*$. Then, by (2.4) and (3.7), one can choose large enough t^* such that $\tilde{w}^\lambda(x, t+1) \geq \kappa \frac{\eta}{2}$ for any $(x, t) \in D^* \times (2t^*, \infty)$. Since $D^* \subset \mathbb{R}_\lambda^N$ was arbitrary, (4.6) follows. \square

The following lemma shows that the method of moving hyperplanes can get started, that is, (4.6) is true for large λ . The proof is similar to [27, Lemma 3.3] and we omit it here.

Lemma 4.4. *There exists λ_1 such that (4.6) holds for all $\lambda > \lambda_1$.*

Now, we move the hyperplane H_λ to the left (decrease λ) as far as (4.6) is satisfied and we investigate properties of the limiting position:

$$\lambda_\infty := \inf\{\mu : (4.6) \text{ holds for all } \lambda \geq \mu\}. \tag{4.13}$$

Lemma 4.5. *Let λ_1 be as in Lemma 4.4. Then:*

- (i) $-\infty < \lambda_\infty \leq \lambda_1$.
- (ii) $V_{\lambda_\infty} z \geq 0$ for all $z \in \omega(u)$.
- (iii) There exists $\hat{z} \in \omega(u)$ such that $V_{\lambda_\infty} \hat{z} \equiv 0$.
- (iv) For each $z \in \omega(u)$ one has $\partial_{x_1} z < 0$ in $\mathbb{R}_{\lambda_\infty}^N$.

Proof. The proofs of (i) and (ii) are analogous to [27, Lemma 3.4 (i), (ii)].

To prove (iii), we proceed by contradiction, that is, we assume $V_{\lambda_\infty} z \not\equiv 0$ for each $z \in \omega(u)$. By (ii), one has $V_{\lambda_\infty} z \geq 0$ for each $z \in \omega(u)$. By the compactness of $\omega(u)$ we can assume the existence of a bounded open set $D_0 \subset \subset \mathbb{R}_{\lambda_\infty}^N$ and $b > 0$ such that

$$\|(V_{\lambda_\infty} z)^+\|_{L^\infty(D_0)} > 2b \quad (z \in \omega(u)). \tag{4.14}$$

This remains valid if we enlarge $D_0 \subset \subset \mathbb{R}_{\lambda_\infty}^N$. We make D_0 so large that it satisfies assumptions of Lemma 4.2 for any $\lambda < \lambda_\infty$ sufficiently close to λ_∞ . By (2.9) and (4.14), there is $t^* > 0$ such that

$$\|(w^{\lambda_\infty})^+(\cdot, t)\|_{L^\infty(D_0)} > b \quad (t \geq t^*).$$

Consequently, Corollary 3.11 with $\vartheta = \frac{1}{4}$ yields

$$w^{\lambda_\infty}(x, t) \geq \kappa b - e^{\beta_0} \sup_{\mathbb{R}_{\lambda_\infty}^N} (w^{\lambda_\infty})^-(\cdot, t-1) - \kappa_1 \|f^{\lambda_\infty}\|_{X_{(t-1, t)}^\lambda} \quad (x \in D_0, t \geq t^*),$$

where κ and κ_1 depend on N , α_0 , β_0 , $\text{dist}(D_0, H_\lambda)$ and $\text{diam}(D_0)$. Since $V_{\lambda_\infty} z \geq 0$ for each $z \in \omega(u)$ and (3.7) holds true, the last two terms decay to 0 as $t \rightarrow \infty$. Hence, for any sufficiently large t

$$w^{\lambda_\infty}(x, t) \geq \frac{1}{2} \kappa b \quad (x \in D_0).$$

Since ∇u is bounded, the previous inequality holds with λ_∞ replaced by λ for any $\lambda < \lambda_\infty$ sufficiently close to λ_∞ . Then, Lemma 4.2 implies (4.6) for any λ sufficiently close to λ_∞ , a contradiction.

The statement (iv) is proved by analogous arguments as in [27, Proposition 3.5]. We only modify the application of the Harnack inequality in the same way as we did in the proof of (iii). \square

This lemma finishes the proof of the first part of Theorem 2.2.

Before we proceed we state a lemma analogous to Lemma 4.5. Define $v : \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}$ as $v(x, t) := u(-x_1, x', t)$ for all $(x_1, x', t) = (x, t) \in \mathbb{R}^N \times (0, \infty)$, and observe that v satisfies (1.1), (2.4), and (2.5) with G changed to $\tilde{G}(x, t) := G(-x_1, x', t)$. Then \tilde{G} satisfies (G1), and Lemma 4.5 applied to v yields to following result.

Lemma 4.6. *There exists λ_∞^- such that*

- (i) $-\infty < \lambda_\infty^- \leq \lambda_\infty$,
- (ii) $V_{\lambda_\infty^-} z \leq 0$ for all $z \in \omega(u)$,
- (iii) There exists $\tilde{z} \in \omega(u)$ such that $V_{\lambda_\infty^-} \tilde{z} \equiv 0$,

(iv) For each $z \in \omega(u)$ one has $\partial_{x_1} z > 0$ in $(\mathbb{R}_{\lambda_\infty}^N)^- := \{x = (x_1, x') \in \mathbb{R}^N : x_1 < \lambda_\infty^-\}$.

To prove the second part of Theorem 2.2, it suffices to show $\lambda_\infty = \lambda_\infty^-$. Indeed, then Lemma 4.5 (ii), (iv) and Lemma 4.6 (ii), (iv) imply that all functions $z \in \omega(u)$ are symmetric with respect to H_{λ_∞} and decreasing in x_1 for $x_1 > \lambda_\infty$.

Lemma 4.7. *If (G2) holds, then $\lambda_\infty = \lambda_\infty^-$.*

The basic idea of the proof, already introduced in [27], is to move a hyperplane H_λ beyond the natural limit H_{λ_∞} , that is, to consider $\lambda < \lambda_\infty$, and investigate the behavior of sign-changing functions w^λ . One of the crucial steps is to estimate $(w^\lambda)^+$ from below. This is done by the comparison of w^λ with a subsolution, similar to one constructed in [27, Lemma 3.8]. Its properties are listed in the following lemma.

Lemma 4.8. *Given any domain $D_0 \subset\subset \mathbb{R}_{\lambda_\infty}^N$ and any $\theta > 0$, there exist $\lambda_2 < \lambda_\infty$, $t_0 > 0$, a domain D , and a function $\phi : \bar{D} \times [t_0, \infty) \rightarrow \mathbb{R}$ with the following properties:*

- (i) $D_0 \subset\subset D \subset\subset \mathbb{R}_{\lambda_\infty}^N$,
- (ii) $\phi \in C^{2,1}(\bar{D} \times [t_0, \infty))$,
- (iii) $e^{\theta t} \phi(x, t) \geq C_2 > 0$ for any $(x, t) \in D_0 \times (t_0, \infty)$ and some C_2 independent of t_0 and t ,
- (iv) $\phi < 0$ in $\partial D \times (t_0, \infty)$,
- (v) one has

$$\frac{\|\phi^+(\cdot, t)\|_{L^\infty(D)}}{\|\phi^+(\cdot, s)\|_{L^\infty(D)}} \geq C e^{-\theta(t-s)} \quad (t \geq s \geq t_0), \quad (4.15)$$

for some constant $C > 0$ independent of t and s ,

- (vi) for each $\lambda \in [\lambda_2, \lambda_\infty]$, ϕ satisfies

$$\phi_t < a_{ij}(x, t)\phi_{x_i x_j} + b_i^\lambda(x, t)\phi_{x_i} + c^\lambda(x, t)\phi + C' e^{-\theta t} |f^\ell(x, t)|, \\ (x, t) \in D \times (t_0, \infty),$$

where $\ell > \lambda_\infty$, is a fixed number close to λ_∞ , and C' depends on the L^∞ bound of u and ℓ .

Sketch of the proof of Lemma 4.8. Since the proof closely follows the proof of [27, Lemma 3.8], we only outline differences. We define

$$\phi(x, t) = e^{-\theta t} v^\alpha(x, t) + s(-e^{-\theta t} (x_1 - \ell)^\beta) = w_1 + s w_2, \quad (4.16)$$

where $v := w^\ell$, $\ell > \lambda_\infty$ is sufficiently close to λ_∞ , and $\alpha > 1 > \beta$ with α, β sufficiently close to 1. We remark that [27, Lemma 3.8] uses μ instead of ℓ . Then by calculations similar to those in [27, Lemma 3.8] one obtains for any $\lambda < \lambda_\infty$ sufficiently close to λ_∞ :

$$\begin{aligned} e^{\theta t} (\partial_t w_1 - a_{ij}(x, t)(w_1)_{x_i x_j} + b_i^\lambda(x, t)(w_1)_{x_i} + c^\lambda(x, t)w_1) &\leq -\frac{\theta}{8} v^\alpha + v^{\alpha-1} f^\ell \\ &\leq -\frac{\theta}{8} v^\alpha + C f^\ell. \end{aligned}$$

The rest of the proof remains unchanged. Notice that (iii) immediately follows from [27, (3.31)]. \square

Proof of Lemma 4.7. We proceed by contradiction, that is, we assume $\lambda_\infty > \lambda_\infty^-$. Since (G2) holds, f^λ satisfies (3.8) (and in particular (3.7)) for each $\lambda \in \mathbb{R}$. Then, by Lemma 4.5 and Lemma 4.6, there exist \hat{z} and $\tilde{z} \in \omega(u)$ monotone in $\mathbb{R}_{\lambda_\infty}^N$ and $\mathbb{R}_{\lambda_\infty^-}^N$ respectively, with $V_{\lambda_\infty} \hat{z} \equiv V_{\lambda_\infty^-} \tilde{z} \equiv 0$. Hence,

$$\begin{aligned} V_\lambda \hat{z}(x) &< 0 & (x \in \mathbb{R}_\lambda^N, \lambda \in (\lambda_\infty^-, \lambda_\infty)), \\ V_\lambda \tilde{z}(x) &> 0 & (x \in \mathbb{R}_\lambda^N, \lambda \in (\lambda_\infty^-, \lambda_\infty)). \end{aligned} \quad (4.17)$$

Fix sequences $(\hat{t}_n)_{n \in \mathbb{N}}$ and $(\tilde{t}_n)_{n \in \mathbb{N}}$ such that $\tilde{t}_n < \hat{t}_n < \tilde{t}_{n+1}$ for all $n \in \mathbb{N}$ and

$$u(\hat{t}_n, \cdot) \rightarrow \hat{z}, \quad u(\tilde{t}_n, \cdot) \rightarrow \tilde{z} \quad \text{with the convergence in } C^1(\mathbb{R}^N).$$

Let $\delta = \delta(N, \alpha_0, \beta_0, \gamma) > 0$ be such that Lemma 3.7 is satisfied with $(\Theta, \varepsilon) = (\frac{\beta_0}{\alpha_0} + 1, \frac{\gamma}{2\alpha_0})$. Fix a domain $D_0 \subset \subset \mathbb{R}_{\lambda_\infty}^N$ with $B_{\rho_\gamma^* + 2|\lambda_\infty|}^{\lambda_\infty + \frac{\delta}{2}} \subset D_0$. Consequently,

$$B_{\rho_\gamma^* + 2|\lambda|}^{\lambda + \delta} \subset D_0 \quad \left(\lambda \in \left[\lambda_\infty - \frac{\delta}{2}, \lambda_\infty \right] \right). \quad (4.18)$$

Let $\lambda_2 < \lambda_\infty$ and D be such that Lemma 4.8 holds with D_0 and

$$\theta := \frac{1}{4} \min \left\{ \frac{\gamma}{2}, \alpha_0, \mu \right\}, \quad (4.19)$$

where μ and γ are defined in (G2) and (N4) respectively. Fix any λ with

$$\max\{\lambda_2, \lambda_\infty - \frac{\delta}{2}\} < \lambda < \lambda_\infty.$$

Then by (4.17), there is $q > 0$ such that

$$\begin{aligned} V_\lambda \hat{z}(x) &< -q & (x \in \bar{D}), \\ V_\lambda \tilde{z}(x) &> q & (x \in \bar{D}), \end{aligned} \tag{4.20}$$

and therefore for large $n \in \mathbb{N}$ we have

$$\begin{aligned} w^\lambda(x, \hat{t}_n) &< -q & (x \in \bar{D}), \\ w^\lambda(x, \tilde{t}_n) &> q & (x \in \bar{D}). \end{aligned} \tag{4.21}$$

Then, there exists $T_n \in (\tilde{t}_n, \hat{t}_n)$ with

$$\begin{aligned} w^\lambda(x, t) &> 0 & ((x, t) \in \bar{D} \times (\tilde{t}_n, T_n)), \\ w^\lambda(x_0, T_n) &= 0 & \text{for some } x_0 \in \bar{D}. \end{aligned} \tag{4.22}$$

We claim that the following three statements are true.

(C1) $\lim_{n \rightarrow \infty} T_n - \tilde{t}_n = \infty$.

(C2)

$$\lim_{n \rightarrow \infty} \sup_{t \in [\tilde{t}_n, T_n]} e^{2\theta(t - \tilde{t}_n)} \|(w^\lambda)^-(\cdot, t)\|_{L^\infty(\mathbb{R}_\lambda^N)} = 0.$$

(C3) For any sufficiently large n and any $\tilde{t}_n \leq t_1 < t_2 \leq T_n$ one has

$$\begin{aligned} \sup_{x \in \bar{D}} w^\lambda(x, t) &\geq C_0 e^{-\theta(t - t_1)} \inf_{x \in \bar{D}} w^\lambda(x, t_1) \\ &\quad - C_1 e^{\beta_0(t - t_1)} e^{-\mu t_1} \quad (t \in [t_1, t_2]), \end{aligned}$$

where C_0 is independent of t_1, t_2 and n , C_1 depends on $t_2 - t_1$, but it is independent of t_1 and n .

Let us first prove (C1). Fix $M > 0$ and for each $n \in \mathbb{N}$ we define

$$w_n^\lambda(x, t) := w^\lambda(x, t) - w^\lambda(x, \tilde{t}_n) \quad ((x, t) \in \mathbb{R}_\lambda^N \times (\tilde{t}_n, \infty)).$$

Then w_n^λ is a classical bounded solution of

$$\begin{aligned} (w_n^\lambda)_t &= L^\lambda(x, t)w_n^\lambda + c^\lambda(x, t)w_n^\lambda + h_n^\lambda(x, t), & (x, t) \in \mathbb{R}_\lambda^N \times (\tilde{t}_n, \infty), \\ w_n^\lambda(x, t) &= 0, & (x, t) \in \partial_P(\mathbb{R}_\lambda^N \times (\tilde{t}_n, \infty)), \end{aligned}$$

where $L^\lambda \in E(\alpha_0, \beta_0, \mathbb{R}^N \times (\tilde{t}_n, \infty))$, $\|c^\lambda\|_{L^\infty(\mathbb{R}^N \times (\tilde{t}_n, \infty))} \leq \beta_0$ and

$$h_n^\lambda(x, t) := f^\lambda(x, t) + L^\lambda(x, t)w^\lambda(x, \tilde{t}_n) + c^\lambda(x, t)w^\lambda(x, \tilde{t}_n) \\ ((x, t) \in \mathbb{R}_\lambda^N \times (\tilde{t}_n, \infty)).$$

Consequently, by Corollary 3.6 a) and the boundedness of coefficients of L^λ , one has

$$\begin{aligned} \sup_{\mathbb{R}_\lambda^N \times (\tilde{t}_n, \tilde{t}_n + \vartheta)} (w_n^\lambda)^- &\leq C \|h_n^\lambda\|_{X_{(\tilde{t}_n, \tilde{t}_n + \vartheta)}^\lambda} \\ &\leq C (\|f^\lambda\|_{X_{(\tilde{t}_n, \tilde{t}_n + \vartheta)}^\lambda} + \vartheta^{\frac{1}{N+1}} \beta_0 \|w^\lambda(\cdot, \tilde{t}_n)\|_{C^2(\mathbb{R}_\lambda^N)}) \quad (\vartheta \in [0, 1]), \end{aligned} \quad (4.23)$$

where C depends on N , α_0 and β_0 . Now, choosing ϑ sufficiently small (independent of n) and n sufficiently large, we can by (2.4) and (3.8) achieve $(w_n^\lambda)^- \leq \frac{q}{2}$ in $\mathbb{R}_\lambda^N \times [\tilde{t}_n, \tilde{t}_n + 4\vartheta]$. Then, by the definition of w_n^λ and (4.21) one has

$$w^\lambda(x, t) \geq \frac{q}{2} \quad ((x, t) \in \bar{D} \times [\tilde{t}_n, \tilde{t}_n + 4\vartheta]). \quad (4.24)$$

Next, an application of Corollary 3.10 with constants $(D, \tau, \theta, \tau_1, \tau_2, \tau_3, \tau_4) = (D, t_n, \vartheta, t_n + 2\vartheta, t_n + 3\vartheta, t_n + 4\vartheta, t_n + M)$ yields

$$\begin{aligned} w^\lambda(x, t) &\geq \kappa \|(w^\lambda)^+\|_{L^\infty(D \times (t_n + 2\vartheta, t_n + 3\vartheta))} - e^{2\beta_0} \sup_{\mathbb{R}_\lambda^N} (w^\lambda)^-(\cdot, t_n) \\ &\quad - \kappa_1 \|f^\lambda\|_{X_{(t_n, t_n + M)}^\lambda} \quad ((x, t) \in \bar{D} \times (\tilde{t}_n + 4\vartheta, \tilde{t}_n + M)), \end{aligned}$$

where κ and κ_1 depend on N , α_0 , β_0 , $\text{diam } D$, ϑ and M . By (4.17) and (3.8), the last two terms in the previous inequality converge to 0 as $n \rightarrow \infty$, whereas the first one is bounded from below by $\kappa q/2$. Therefore $w^\lambda(x, t) \geq \kappa \frac{q}{8}$ for all $(x, t) \in \bar{D} \times [\tilde{t}_n + 4\vartheta, \tilde{t}_n + M]$ and sufficiently large n . This and (4.24) yields the desired result, since M was arbitrary.

To prove (C2) it is enough to show that for any $\varepsilon' > 0$, there is n_0 such that

$$\sup_{(x, t) \in \mathbb{R}_\lambda^N \times [\tilde{t}_n, T_n]} v_n(x, t) \leq \varepsilon' \quad (n \geq n_0), \quad (4.25)$$

where

$$v_n(x, t) := e^{2\theta(t-\tilde{t}_n)} \frac{(w^\lambda)^-(x, t)}{g(x_1 - \lambda)} \quad ((x, t) \in \mathbb{R}_\lambda^N \times [\tilde{t}_n, T_n]),$$

and g is as in Lemma 3.7 with $(\Theta, \varepsilon) = (\frac{2\beta_0}{\gamma} + 1, \frac{\gamma}{2})$. Since $w^\lambda > 0$ in $D \times [\tilde{t}_n, T_n)$,

$$U^n := \{(x, t) : v_n(x, t) > 0, t \in [\tilde{t}_n, T_n)\} \subset (\mathbb{R}_\lambda^N \setminus D) \times [\tilde{t}_n, T_n). \quad (4.26)$$

Observe that $(w^\lambda)^- = -w^\lambda$ on \bar{U}^n for each $n \in \mathbb{N}$. Thus Remark 3.8 yields

$$\begin{aligned} (v_n)_t - \hat{L}^\lambda(x, t)v_n &= \tilde{c}^\lambda(x, t)v_n + \tilde{f}^\lambda(x, t), & (x, t) \in U^n, \\ v_n(x, t) &= 0, & (x, t) \in \partial_P U^n \setminus (U^n)_{\tilde{t}_n}, \\ v_n(x, \tilde{t}_n) &= \frac{(w^\lambda)^-(x, \tilde{t}_n)}{g(x_1 - \lambda)}, & x \in (U^n)_{\tilde{t}_n}, \\ \limsup_{|x| \rightarrow \infty} \sup_{t \in (0, \infty)} |v(x, t)| &= 0, \end{aligned} \quad (4.27)$$

where $\hat{L}^\lambda \in E(\alpha_0, 5\beta_0, U^n)$,

$$\tilde{c}^\lambda := \hat{c}^\lambda + 2\theta, \quad \text{and} \quad \tilde{f}^\lambda(x, t) := -e^{2\theta(t-\tilde{t}_n)} \frac{f^\lambda(x, t)}{g(x_1 - \lambda)}.$$

By Remark 3.8 we have $\|\hat{c}^\lambda\|_{L^\infty(U^n)} \leq 5\beta_0$, and therefore

$$\|\tilde{c}^\lambda\|_{L^\infty(U^n)} \leq 7\beta_0.$$

Moreover, by (3.20) and (4.19)

$$\tilde{c}^\lambda(x, t) \leq \hat{c}^\lambda(x, t) + 2\theta \leq -\frac{\gamma}{2} + 2\theta \leq -\theta \quad (x_1 \in [\lambda, \lambda + \delta], t > 0), \quad (4.28)$$

and by (3.11) one has

$$\tilde{c}^\lambda(x, t) \leq -\gamma + 2\theta \leq -\theta \quad (|x| \geq \rho_\lambda^* + 2\lambda, t > 0).$$

Since $B_{\rho_\gamma^* + 2|\lambda|} \subset D$ and (4.26) holds true, $c^\lambda < -\gamma$ for any $(x, t) \in U^n$, $n > 0$. Also, $\mu > 2\theta$ and (3.8) implies, that there exists $t_{\varepsilon'}$ such that

$$\|\tilde{f}^\lambda\|_{X_{(t_{\varepsilon'}, \infty)}^\lambda} < \varepsilon' \frac{1 - e^{-\vartheta}}{2C},$$

where $C = C(N, \alpha_0, 7\beta_0)$ is the constant from Corollary 3.6 b). Then Corollary 3.6 b) yields

$$\begin{aligned} \sup_{U_t^n} v_n &\leq e^{-\theta(t-\tilde{t}_n)} \sup_{U_{\tilde{t}_n}^n} v_n + C \frac{1}{1-e^{-\theta}} \|\tilde{f}^\lambda\|_{X_{(t_\varepsilon, \infty)}^\lambda} \\ &\leq \sup_{U_{\tilde{t}_n}^n} v_n + \frac{\varepsilon'}{2} \quad (t \in [\tilde{t}_n, T_n], \tilde{t}_n > t_\varepsilon). \end{aligned}$$

Since $\|v_n(\cdot, \tilde{t}_n)\|_{L^\infty(U_{\tilde{t}_n}^n)} \rightarrow 0$ as $n \rightarrow \infty$, we obtain that (4.25) holds for sufficiently large n_0 .

Let us prove (C3). Recall that D was fixed such that Lemma 4.8 holds with D_0 and θ . Let ϕ be the corresponding subsolution. Denote

$$\eta := \frac{\inf_{x \in \bar{D}} w^\lambda(x, t_1)}{\|\phi^+(\cdot, t_1)\|_{L^\infty(D)}} > 0 \quad \text{and} \quad v := w^\lambda - \eta\phi.$$

Lemma 4.8 (iii) and (2.4) imply that $e^{-\theta t}\eta$ is bounded by a constant independent of t_1 .

Then

$$\begin{aligned} v_t &\geq L^\lambda(x, t)v + c^\lambda(x, t)v + (f^\lambda - C'e^{-\theta t}\eta|f^\ell|), & (x, t) \in D \times (t_1, t_2), \\ 0 &< v(x, t), & (x, t) \in \partial D \times (t_1, t_2), \\ 0 &\leq v(x, t_1), & x \in \bar{D}. \end{aligned}$$

Consequently, (3.8), Corollary 3.6 a), and positivity of w^λ in $D \times [t_1, t_2]$ yield

$$\begin{aligned} C_1 e^{\beta_0(t-t_1)} e^{-\mu t_1} &\geq C e^{\beta_0(t-t_1)} \|f^\lambda - C'e^{-\theta t}\eta|f^\ell|\|_{X_{(t_1, t)}^\lambda} \geq \sup_{x \in D} (v(x, t))^- \\ &\geq -\sup_{x \in D} w^\lambda(x, t) + \eta \sup_{x \in D} \phi(x, t) \quad (t \in [t_1, t_2]), \end{aligned}$$

where C_1 depends on $t_2 - t_1$, but is independent of t_1 . Since by Lemma 4.8 (v) and the definition of η one has

$$\eta \sup_{x \in D} \phi(x, t) \geq \eta C e^{-\theta(t-t_1)} \|\phi(\cdot, t_1)\|_{L^\infty D} \geq C e^{-\theta(t-t_1)} \inf_{x \in \bar{D}} w^\lambda(x, t_1),$$

(C3) follows.

We will complete the proof of the lemma by showing that (C1)–(C3) lead to a contradiction.

By (C1) we have $T_n - \tilde{t}_n \rightarrow \infty$ as $n \rightarrow \infty$. Let C_0 be as in (C3), let κ, κ_1 be as in Corollary 3.11 for already fixed D and $\vartheta = \frac{1}{4}$. Denote

$$\hat{C} := \frac{\kappa C_0 e^{\frac{\theta}{2}}}{2}.$$

Fix $K > 2$ such that

$$e^{-\theta K} \leq \hat{C} \quad (4.29)$$

and let $C_1 := C_1(t_2 - t_1)$ be as in (C3) with $t_2 - t_1 = K$. By (C2) there is $n_0 > 0$ such that

$$e^{2\theta} e^{\beta_0} e^{-2\theta(t - \tilde{t}_n)} \|(w^\lambda)^-(\cdot, t)\|_{L^\infty(\mathbb{R}_\lambda^N)} \leq \frac{q}{2} \quad (t \in [\tilde{t}_n, T_n], n \geq n_0). \quad (4.30)$$

Enlarge n_0 if necessary, such that $T_n - \tilde{t}_n > K$ and

$$(\kappa_1 + C_1(K) e^{\beta_0 K}) C_\mu e^{\theta K} e^{-\mu \tilde{t}_n} \leq \frac{1}{2} q \hat{C} \quad (n \geq n_0). \quad (4.31)$$

Now, fix $n \geq n_0$. We prove by the mathematical induction that for any $i \in \mathbb{N} \cup \{0\}$ with $i \leq \frac{T_n - \tilde{t}_n}{K}$, one has

$$w^\lambda(x, \tau_i) \geq q e^{-\theta i K} \hat{C}^i \quad (x \in \bar{D}), \quad (4.32)$$

where $\tau_i := iK + \tilde{t}_n$.

For $i = 0$ the statement follows from (4.21). Next assume that (4.32) is true for some $i \in \mathbb{N} \cup \{0\}$ such that $(i+1)K \leq T_n - \tilde{t}_n$. We show that (4.32) holds with i replaced by $i+1$. Indeed, Corollary 3.11 with $(\tau, \vartheta) = (\tau_{i+1} - 1, \frac{1}{4})$, (C3), (4.30), and (3.8) yield

$$\begin{aligned} w^\lambda(x, \tau_{i+1}) &\geq \kappa \|(w^\lambda)^+\|_{L^\infty(D \times (\tau_{i+1} - \frac{3}{4}, \tau_{i+1} - \frac{1}{2}))} e^{\beta_0} \|(w^\lambda)^-\|_{L^\infty(\mathbb{R}_\lambda^N \times (\tau_{i+1} - 1, \tau_{i+1}))} \\ &\quad - \kappa_1 \|f^\lambda\|_{X_{(\tau_{i+1} - 1, \tau_{i+1})}^\lambda} \\ &\geq \kappa e^{-\theta K} C_0 e^{\frac{\theta}{2}} \inf_{x \in \bar{D}} w^\lambda(x, \tau_i) - \frac{q}{2} e^{-2\theta(\tau_{i+1} - \tilde{t}_n)} - (\kappa_1 + C_1 e^{\beta_0 K}) C_\mu e^{-\mu \tau_i} \quad (x \in \bar{D}). \end{aligned}$$

Consequently, (4.32), (4.29), and (4.31) imply

$$\begin{aligned} w^\lambda(x, \tau_{i+1}) &\geq 2\hat{C} e^{-\theta K} q e^{-\theta i K} \hat{C}^i - \frac{q}{2} e^{-\theta(i+1)K} (e^{-\theta K})^{i+1} \\ &\quad - (\kappa_1 + C_1 e^{\beta_0 K}) C_\mu e^{-\theta i K} (e^{-\theta K})^i e^{-\mu \tilde{t}_n} \\ &\geq q e^{-\theta(i+1)K} \hat{C}^{i+1} \left(2 - \frac{1}{2} - \frac{(\kappa_1 + C_1 e^{\beta_0 K}) C_\mu e^{\theta K}}{q \hat{C}} e^{-\mu \tilde{t}_n} \right) \\ &\geq q e^{-\theta(i+1)K} \hat{C}^{i+1} \quad (x \in \bar{D}). \end{aligned}$$

Thus if $i_0 \in \mathbb{N}$ is such that $i_0 K \leq T_n - \tilde{t}_n < (i_0 + 1)K$, then (4.32) holds with $i = i_0$. If we replace τ_{i+1} by T_n and τ_i by τ_{i_0} in the previous calculation, we obtain by the same reasoning

$$w^\lambda(x, T_n) \geq qe^{-\theta(i+1)K} \hat{C}^{i+1} > 0 \quad (x \in \bar{D}),$$

a contradiction to the definition of T_n . This finishes the proof of the lemma. \square

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