End-to-End Distance for a Four-Dimensional Self-Avoiding Walk[1]

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[1]Based on a lecture given at *Mathematical Quantum Theory*, a summer school in mathematical physics held in August 1993 at the University of British Columbia, Vancouver.

[2]Supported in part by the National Science Foundation under grants DMS 91-20962 and DMS 91-96161.

In this lecture I will present a method for obtaining the end-to-end distance of a self-avoiding walk, assuming properties of its Green's function are known. The construction is joint work with D. Brydges, and we apply it to the case of hierarchical self-avoiding walk in four dimensions.

In four dimensions there are logarithmic corrections to Brownian behavior, and our method is sensitive enough to pick up the power of the logarithm.

The needed Green's function estimates have not yet been proven for the ordinary (non-hierarchical) self-avoiding walk (SAW). However, there is reason to hope that the estimates can be proven, as closely related models have been treated already in the literature [GK, FMRS, HT, IM].

As we shall see, the end-to-end distance will be recovered from Green's functions by a contour integral. Why not analyze the end-to-end distance directly? The answer lies in the fact that the precise RG calculations needed to compute logarithmic corrections are best done in the field theoretic representation of SAW (commonly known as the zero component limit of the N-vector model). In field theory the length of the walk is integrated over, as, for example, in the Green's funciton of random walk:

$$G(\beta, x) = \int_0^\infty dT e^{-\beta T} e^{T\Delta}(0,x) = (-\Delta + \beta)^{-1}(0,x).$$

Here Δ is the lattice Laplacian in $\mathbf{Z}^d$ and $\beta \geq 0$. The operator $e^{-T(-\Delta+\beta)}$ describes the propagation for time T of a walk that undergoes nearest neighbor transitions and is killed off at rate β. If one wishes to describe properties of the walk at fixed T, one must invert the Laplace transform:

$$\begin{aligned} P(T,x) &= \int_{a-i\infty}^{a+i\infty} \frac{d\beta}{2\pi i} e^{\beta T} G(\beta, x) \qquad (a > 0) \\ &= \int \frac{d\beta}{2\pi i} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{p^2+\beta} e^{-ipx} e^{\beta T}. \end{aligned}$$

Now evaluating the β integral by taking the residue of the pole at $\beta = p^2$, we obtain

$$P(T,x) = \int \frac{d^d p}{(2\pi)^d} e^{-p^2 T} e^{ipx} \simeq \text{ const } T^{-\frac{d}{2}} e^{-\frac{x^2}{4T}}.$$

(Here we need T large since $x \in \mathbf{Z}^d$ and $p \in [-\pi, \pi]^d$.) Clearly the width of the distribution is of order $T^{\frac{1}{2}}$, which, of course, is the end-to-end distance for a simple random walk.

1 Definition of the model

The hierarchical lattice $\mathcal{G}$ is the direct sum of infinitely many copies of $\mathbf{Z}_n$, where $n = L^4$ for some integer $L > 1$. Each $x \in \mathcal{G}$ is of the form

$$(\cdots 0, 0, x_{N-1}, x_{N-2} \cdots x_0)$$

with $x_i \in \mathbf{Z}_n = \{0, \ldots, n-1\}$. The length of x is defined as

$$|x| = \begin{cases} 0 & \text{if } x = (\cdots 0); \\ L^N & \text{if } x = (\cdots x_{N-1} \cdots x_0). \end{cases}$$

This lattice is best thought of as $\mathbf{Z}^d$ with an unusual notion of distance and a nonstandard Laplacian.

Define a Levy process on $\mathcal{G}$ such that

$$P(\omega(t+dt) = y | \omega(t) = x) = c|x-y|^{-6} dt, \text{ if } x \neq y.$$

Then the Green's function for this process is defined as

$$G(\beta, x) = \int_0^\infty dT e^{-\beta T} E(\mathbb{1}_{\omega(T)=x}),$$

where E denotes the expectation for the process starting at the origin. In [BEI] an explicit formula for $G\beta, x)$ is given, which at $\beta = 0$ reduces to

$$G(0, x) = |x|^{-2} \qquad (x \neq 0),$$

Thus $G(0, x)$ mimicks the behavior of $(-\Delta)^{-1}$ on the standard Euclidean lattice.

The function $G(\beta, x)$ is a meromorphic function of β with poles at $\beta < 0$. Thus $P(T, x)$ may be evaluated explicitly by inverse Laplace transform. This leads to the following proposition.

Proposition 1. (End-to-end distance for the unperturbed walk on the hierarchical lattice.) Let $0 < \alpha < 2$. Then

$$\lim_{m \to \infty} \frac{1}{\sqrt{L^{2m} T}} E(|w(L^{2m} T)|^\alpha)^{\frac{1}{\alpha}} = f(T)$$

exists for each $T > 0$ with $0 < f(T) < \infty, \ f(T) = f(L^2 T)$.

This gives a precise sense in which $|\omega(T)| \sim T^{\frac{1}{2}}$ for the simple random walk on the hierarchical lattice. Higher moments do not exist because of the $|x|^{-6}$ tail in the jump distribution.

We seek a similar statement for a weakly self-avoiding walk, which we proceed to define. First, let

$$\tau(x) = \tau^{(T)}(x) \equiv \int_0^T ds \mathbb{1}_{\{\omega(s)=x\}}$$

denote the local time at x, the time in $[0, T]$ that the walk spends at x. We write the interaction as

$$\tau^2(\mathcal{G}) = \sum_{x \in \mathcal{G}} dx \tau^2(x) = \int ds\, dt\, \mathbb{1}_{\{\omega(s)=\omega(t)\}},$$

where the last expression shows it is a measure of the self-intersection of the walk.

From this we define an unnormalized probability distribution for the walk at time T,

$$P_\lambda(T,x) = E\left(e^{-\lambda\tau^2(\mathcal{G})}\mathbb{1}_{\{\omega(T)=x\}}\right).$$

Finally, the normalized expectation is

$$E_\lambda^T(\cdot) = \frac{\sum_x \cdot P_\lambda(T,x)}{\sum_x P_\lambda(T,x)}.$$

Theorem. Let $0 < \alpha < 2$ be fixed. Let $L >> 1,\ \lambda << 1$. Then

$$E_\lambda^T(|\omega(T)|^\alpha)^{\frac{1}{\alpha}} = (1+\epsilon_1(T))E(|\omega(\ell^{\frac{1}{4}}T)|^\alpha)^{\frac{1}{\alpha}},$$

where

$$\ell = C(L,\lambda)\log T + \epsilon_2(T)$$

$$\begin{aligned} |\epsilon_1(T)| &= \mathcal{O}\left(\frac{1}{\log T}\right) \\ |\epsilon_2(T)| &= \mathcal{O}(\log\log T) \end{aligned}$$

$$C(L,\lambda) > 0.$$

This theorem shows that

$$\begin{aligned} \omega(T) &\sim \ell^{\frac{1}{8}}T^{\frac{1}{2}} \sim (\log T)^{\frac{1}{8}}T^{\frac{1}{2}} \\ &\sim \text{const } (\log T)^{\frac{1}{8}}T^{\frac{1}{2}} \end{aligned}$$

in the same sense that the noninteracting $\omega(T)$ goes as a constant time $T^{\frac{1}{2}}$.

As indicated earlier, the idea is to derive the behavior of $P_\lambda(T,x)$ from its Laplace transform, the interacting Green's function:

$$G_\lambda(a,x) = \int_0^\infty dt e^{-aT}P_\lambda(T,x).$$

It was shown in [BEI] that when $a = a_c(\lambda)$ the Green's function decays as $|x|^{-2}$. This is the critical value of the killing rate: for larger a the decay will be shown to be at the rate $|x|^{-6}$, which is the analog for the hierarchical model of exponential decay.

(The long tail in the jump distribution prevents true exponential decay.) We shift to coordinates in $\mathbf{C}$ centered at $a_c(d)$ by putting

$$\beta = a - a_c(\lambda),$$

and writing $G_\lambda(\beta, x)$ instead of $G_\lambda(\beta + a_c(\lambda), x)$.

The proof of the above theorem breaks into two parts: First, to determine the behavior of $G_\lambda(\beta, x)$, and second, to recover $P_\lambda(T, x)$ from $G_\lambda(\beta, x)$.

2 Behavior of $G_\lambda(\beta, x)$.

We need the behavior of $G_\lambda(\beta, x)$ for $|\arg \beta| < \frac{3\pi}{4} - 2b$ for some $b > 0$. The free Green's function $G(\beta, x)$ has poles on the negative β axis but is analytic in the above pie-shaped region. In this region it behaves as

$$G(\beta, x) \sim \frac{1}{(1 + |x|^2)(1 + |\beta|\,|x|^2)^2},$$

which shows the $|x|^{-2}$ behavior for $\beta = 0$ and the $|x|^{-6}$ behavior for $\beta \neq 0$. The crossover from one behavior to the other takes place when $|x|$ is of the order $\beta^{-\frac{1}{2}}$, so G has "range" $\beta^{-\frac{1}{2}}$ just like the Green's function of the nonhierarchical walk.

Proposition 2. (Field-theoretic representation of G_λ.) The interacting Green's function may be represented as

$$G_\lambda(\beta, x) = \int d\mu_{G(\beta)}(\Phi) e^{-\mathcal{A}(\Phi)} \bar{\phi}_0 \phi_x,$$

where $\Phi = (\phi, \bar{\phi}, \psi, \bar{\psi})$ is a vector consisting of a pair of conjugate complex fields and a pair of Grassmann fields, and where $d\mu_{G(\beta)}(\Phi)$ is the Gaussian measure with covariance $G(\beta)$ (both for $\phi, \bar{\phi}$ and for $\psi, \bar{\psi}$. The action is

$$\mathcal{A}(\Phi) = \sum_x [\lambda \Phi^4(x) + a_c(\lambda) \Phi^2(x)],$$

where $\Phi^2 = \phi\bar{\phi} + \psi\bar{\psi}$ and $\Phi^4 = (\Phi^2)^2$.

This representation was used extensively in [BEI] and corresponds to the $N = 0$ component quartic field theory. The right-hand side needs to be defined as a limit of volume cut-off quantities. Proving this proposition would take this lecture too far afield, but is is worth mentioning that it follows from the general fact

$$\int d\mu_{G(0)}(\Phi) F(\Phi^2) \bar{\phi}_0 \phi_x = \int dT E\left(F(\tau^{(T)}) \mathbb{1}_{w(T)=x}\right).$$

Taking $F(\Phi^2) = e^{-\mathcal{A}(\Phi)}$ we obtain the interaction $\tau^2(\mathcal{G})$ defined earlier, plus $a_c(\lambda)\tau(\mathcal{G}) = a_c(\lambda)T$. When the Laplace transform is taken on the RHS instead of $\int dT$, the additional term changes the covariance on the left to $G(\beta)$.

We analyze the quartic field theory using renormalization group methods. The first step is to write

$$G(\beta, x) = L^{-2}G\left(L^2\beta, \frac{x}{L}\right) + \Gamma\left(\beta, \frac{x}{L}\right),$$

and integrate out the Γ covariance. After rescaling and rewriting of the interaction, we find that the quadratic and quartic coefficients have changed via the following recursion:

$$\begin{aligned}
\lambda_{j+1} &= \lambda_j - 8B\lambda_j^2 \frac{1+2\beta_j}{(1+\beta_j)^2} + O\left(\frac{\lambda_j^3}{(1+\beta_j)^2}\right) \\
\beta_{j+1} &= L^2\beta_j\left(1 - 2B\frac{\lambda_j}{1+\beta_j} + O\left(\frac{\lambda_j^2}{1+\beta_j}\right)\right).
\end{aligned}$$

Note that the first recursion is effectively the familiar $\beta = 0$ recursion, at least until β has grown to be $\mathcal{O}(1)$. The coefficient β blows up by a factor L^2 per iteration with a slight retardation due to λ. This retardation is responsible for the logarithmic corrections to the noninteracting model.

The coupling constant recursion can be solved, leading to accurate upper and lower bounds on λ_j and β_j. We find that

$$|\lambda_j| \sim \frac{\lambda}{1 + 8B\lambda \min\{j, \log(1+|\beta|^{-1})\}},$$

which means that λ_j decreases as $(8Bj)^{-1}$ until β_j has grown to be $\mathcal{O}(1)$. Now define

$$\hat{\beta}_j = L^{-2j}\beta_j,$$

to cancel out the trivial expansion of β_j with each step. Then

$$\begin{aligned}
\frac{\hat{\beta}_j}{\beta} \sim \prod_{i=1}^{j}(1 - 2B\lambda_j) &\sim \exp\left(-\sum_{i=1}^{j}\frac{1}{4j}\right) \\
&\sim \exp\left(-\frac{1}{4}\log j\right) = j^{-\frac{1}{4}}.
\end{aligned}$$

We may put $\hat{\beta}_\infty = \lim_{j\to\infty}\hat{\beta}_j$, since after $j \sim \log(1+|\beta|^{-1})$ both recursions settle down to fixed points. From the above we find

$$|\hat{\beta}_\infty| \sim \log\left(1 + \frac{1}{|\beta|}\right)^{-\frac{1}{4}} |\beta|,$$

and the exponent of the log traces to the ratio of the two coefficients $2B, 8B$ in the recursion relation.

If we define

$$\ell^{-\frac{1}{4}}(\beta) = \frac{\hat{\beta}_\infty}{\beta},$$

then

$$\ell(\beta) = C(L, \lambda) \log |\beta|^{-1} + \epsilon(\beta),$$

with $\epsilon(\beta) = \mathcal{O}(\log\log|\beta|^{-1})$ as per our main theorem.

The renormalization group analysis may be extended to the Green's function, with the result that $G_\lambda(\beta, x)$ is well approximated by $G_0(\hat{\beta}_{N(x)}, x)$ where $\hat{\beta}_{N(x)}$ is the effective killing rate after $N(x)$ RG steps. Here $N(x) = \log_L |x|$ is the number of steps needed to bring $0, x$ to the same point.

Proposition 3. Let $N(x) = \log_L |x|$. Then

$$|G_\lambda(\beta, x) - (1 + \delta(\lambda))G_0(\hat{\beta}_{N(x)}, x)| \leq \mathcal{O}(\lambda_{N(x)})|G_0(\hat{\beta}_{N(x)}, x)|.$$

Remark. The reader will recall that $\arg\beta$ may be larger than $\frac{\pi}{2}$, and so in the course of proving these estimates on G_λ and on the recursion there will be stability problems. These are sidestepped by rotating $\Phi \to \Phi e^{-i\arg\beta/6}$. Since $|\arg\beta| < \frac{3\pi}{4}$ we preserve positivity of the measure $(\Phi, G_0^{-1}\Phi)$ and of $\lambda\Phi^4$, which pick up phases less than $\frac{\pi}{4}, \frac{\pi}{2}$, respectively. The mass term, $\beta\Phi^2$, however, is rotated back to the right half-plane and we have a single-minimum ϕ^4 potential to work with. One can consider the broken phase of this model by putting β on the negative axis and working carefully with the infinite volume limit. This leads to a new set of phenomena relating to the collapse of the polymer, see [GI].

3 Recovering $P_\lambda(T, x)$ from $G_\lambda(\beta, x)$

We wish to see how in writing the inverse Laplace transform

$$P_\lambda(T, x) = \int \frac{d\beta}{2\pi i} e^{\beta T} G_\lambda(\beta, x),$$

the logarithmic behavior in β becomes the desired logarithm in T. By Proposition 3, we can replace $G_\lambda(\beta, x)$ with $G_0(\hat{\beta}_{N(x)}, x)$, up to a small error. Now we wish to make a further replacement of $G_0(\hat{\beta}_{N(x)}, x)$ with $G_0(\beta\ell^{-\frac{1}{4}}(T^{-1}), x)$. Allowing this for the moment, we find that

$$P_\lambda(T, x) = \frac{1 + \delta(\lambda)}{\ell^{-\frac{1}{4}}} P_0\left(\frac{T}{\ell^{-\frac{1}{4}}}, x\right),$$

since rescaling β is equivalent to rescaling T in $\int \frac{d\beta}{2\pi i} e^{\beta T} G_0(\beta \ell^{-\frac{1}{4}}, x)$. (We write $\ell = \ell(T^{-1})$.) After normalization, the width of $P_\lambda(T,x)$ is seen to be $(T\ell^{\frac{1}{4}})^{\frac{1}{2}} = T^{\frac{1}{2}}\ell^{\frac{1}{8}}$, and this leads to the behavior of $E^T_\lambda(\omega(T)^\alpha)^{\frac{1}{\alpha}}$ claimed in the theorem.

Going back to the replacement of $\hat{\beta}_{N(x)}$ with $\beta\ell^{-\frac{1}{4}}$, we analyze the error term on the keyhole-shaped contour

$$\{\beta : |\beta| = T^{-1} \text{ and } |\arg\beta| \le \frac{3\pi}{4} - 2b\}$$
$$\cup\ \{\beta : |\beta| > T^{-1} \text{ and } |\arg\beta| = \frac{3\pi}{4} - 2b\}.$$

For large values of β, the exponential decay of $e^{\beta T}$ suppresses the contribution. For moderate values of β, the replacement is valid except for the N-dependence of $\hat{\beta}_N$. But the N-dependent part contains an extra $\frac{1}{\log T}$ from differentiating $\ell^{-\frac{1}{4}}$. Similarly, the error term in Proposition 3 contains a factor λ_N which leads to an extra $\frac{1}{\log T}$.

References

[BEI] Brydges, D., Evans, S., and Imbrie, J.: Self-Avoiding Walk on a Hierarchical Lattice in Four Dimensions. *Ann. Prob.* **20**, 82-124 (1992).

[FMRS] Feldman, J. Magnen, J., Rivasseau, V., and Sénéor, R.: Construction and Borel Summability of Intrared Φ^4_4 by a Phase Space Expansion. *Commun. Math. Phys.* **109**, 437 (1987).

[GK] Gawedzki, K. and Kupiainen, A.: Massless Lattice ϕ^4_4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model. *Commum. Math. Phys.* **99**, 197(1985).

[GI] Golowich, S., and Imbrie, J.: The Broken Symmetry Phase of a Self-Avoiding Random Walk. Preprint.

[HT] Hara, T., and Tasaki, H.: A Rigorous Control of Logarithmic Corrections in Four-Dimensional ϕ^4 Spin Systems. *Jour. Stat. Phys.* **47**, 57-121 (1987).

[IM] Iagolniter, D., and Magnen, J.: Polymers in a Weak Random Potential in Dimension Four: Rigorous Renormalization Group Analysis. Preprint.