Space-Dependent Dirac Operators and Effective Quantum Field Theory for Fermions[*]

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Abstract. For the operator $i\partial\!\!\!/ + m(x)$, where $m(x)$ can change sign, we develop a cluster expansion for computing the determinant and Green's functions. We use a local chiral transformation to relate the space-dependent case to the ordinary Dirac operator.

1. Introduction

The study of multi-phase field theories with generalized Yukawa interactions provides a natural structure for studying Dirac operators with space-dependent mass. Different phases of such a model will have different effective fermion masses. If one attempts to analyze such a model via a cluster expansion, different cluster will be in different phases and have different masses for the fermions.

Our specific motivation for studying operators like $i\partial\!\!\!/ + m(x)$ comes from trying to understand the phase structure of two-dimensional Wess–Zumino models. While the single phase case has been studied [15] and much is known for the system in finite volume [9–12], the infinite volume multiphase problem remains unexplored.

A first step to understanding the behavior of the Wess–Zumino model is to study a simpler toy model with almost no bosonic field. By "almost no" field we mean that the only remnant of the boson is a restriction that each block of space-time is in a particular phase. This results in the study of a Dirac operator $i\partial\!\!\!/ + m(x)$ in two dimensions where $m(x)$ takes on a small number of values.

Dirac Operators with Space Dependent Masses. To obtain a view of the technical

problems involved, consider first the Laplacian with a space dependent potential

$$- \Delta + m^2(x),$$

where

$$m_0 \geq m(x) > \varepsilon > 0. \tag{1.1}$$

Exponential decay for the Green's function $(- \Delta + m^2(x))^{-1}$ can be established using the Neumann series

$$(- \Delta + m^2(x))^{-1} = [1 + (- \Delta + m_0^2)^{-1}(m^2(x) - m_0^2)]^{-1}(- \Delta + m_0^2)^{-1}$$

$$= \sum_{k=0}^{\infty} [(- \Delta + m_0^2)^{-1}(m_0^2 - m^2(x))]^k (- \Delta + m_0^2)^{-1} \tag{1.2}$$

where convergence is guaranteed since $|m_0^2 - m^2(x)| < m_0^2$.

Now consider the Dirac operator $i\partial\!\!\!/ + m(x)$ on $\mathbb{R}^2$. This is the operator defined by

$$i\partial\!\!\!/ = i\gamma_0^E \partial_{x_0} + i\gamma_1^E \partial_{x_1}, \tag{1.3}$$

where γ_0^E, γ_1^E are 2×2 matrices

$$\gamma_0^E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1^E = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.4}$$

The Neumann series technique described above again works so long as $m(x)$ satisfies a bound of the form (1.1). However in this case of the Dirac operator the restriction to positive mass is no longer trivial; in fact the effective quantum field theories of [11, 12] give rise to Dirac operators with $m(x)$ taking different signs in different space time regions.

In order to handle this problem we begin by remarking that the operators $i\partial\!\!\!/ + m$ and $i\partial\!\!\!/ - m$ are related via a (chiral) unitary transformation by $\gamma_5 = \gamma_0 \gamma_1$, where $\gamma_0 = i\gamma_0^E$ and $\gamma_1 = \gamma_1^E$:

$$\gamma_5 \gamma_0 (i\partial\!\!\!/ + m)\gamma_5 = \gamma_0 (i\partial\!\!\!/ - m). \tag{1.5}$$

Similarly we have, with $U = e^{i\alpha(x)\gamma_5}, \alpha \in C_0^\infty(\mathbb{R}^2)$,

$$U^{-1}\gamma_0(i\partial\!\!\!/ + m)U = \gamma_0(i\partial\!\!\!/ + e^{2i\alpha(x)\gamma_5}m - \partial\!\!\!/\alpha(x)\gamma_5). \tag{1.6}$$

If we choose $\alpha(x) = 0$ in one region (region A), and $\alpha(x) = \pi/2$ in region B (with a boundary layer between), then

$$U^{-1}\gamma_0(i\partial\!\!\!/ + m)U = \begin{cases} \gamma_0(i\partial\!\!\!/ + m), & \text{in region } A \\ \gamma_0(i\partial\!\!\!/ - m), & \text{in region } B. \end{cases} \tag{1.7}$$

Thus the Dirac operator with masses of differing signs is related to the Dirac operator with constant mass via a unitary transformation.

This is our main technical device for handling the Dirac operator with differing signs for the masses. For simplicity we will consider the case where the function $m(x) = \pm 1$. We discuss the more general case in the appendix, where we combine our methods with a Neumann series of the type discussed above (1.2). We could also easily consider the case where the mass was chiral, i.e. proportional to

$\cos\theta + i\gamma_5 \sin\theta$, $\theta \in \mathbb{R}$. Although we consider only the two dimensional case, we believe an extension of our methods should work in any even dimension.

Before stating our main theorem, however, we must make the following remark. Let $\gamma_0(i\partial + m(x))$ be the Dirac operator on $\mathscr{H}_{1/2} \oplus \mathscr{H}_{1/2}$, where $\mathscr{H}_\alpha$ is the Sobolev space $\mathscr{H}_\alpha = L^2(\mathbb{R}^2, (p^2 + 1)^\alpha d^2 p)$. Let $S = (\gamma_0(i\partial + 1))^{-1}$ be the operator on $\mathscr{H} \equiv \mathscr{H}_{-1/2} \oplus \mathscr{H}_{-1/2}$ determined by the kernel

$$S(x, y) = \int \frac{-\not{p} + 1}{p^2 + 1} e^{-ip(x-y)} \frac{d^2 p}{(2\pi)^2} \gamma_0, \tag{1.8}$$

i.e. the kernel (2.5) without cutoff. We wish to prove estimates on the putative Green's function

$$[\gamma_0(i\partial + m(x))]^{-1}. \tag{1.9}$$

In fact it is not difficult to write down functions $m(x)$ for which this inverse will not exist. To circumvent this problem we compute the "reduced" Green's function, formally given by

$$[\gamma_0(i\partial + m(x))]^{-1} \det_3(S[\gamma_0(i\partial + m(x))]) = (1 + K)^{-1} S \det_3(1 + K), \tag{1.10}$$

where $K = S\gamma_0(m(x) - 1)$ is an operator on $\mathscr{H}$. The object (1.10) is well defined and given by a Fredholm series [14], whether or not $\gamma_0(i\partial + m(x))$ is invertible. Convergence of the Fredholm series is guaranteed by Lemma 2.1. Note that where $\gamma_0(i\partial + m(x))$ is invertible, the reduced Green's function differs from the inverse by an irrelevant constant.

We may now state our main theorem.

Theorem 1. *Let* $g, h \in C_0^\infty(\mathbb{R}^2)$ *be supported in unit squares* Δ_g, Δ_h. *Then there exists* $c > 0$ *such that*

$$|\langle g, (1 + K)^{-1} S \det_3(1 + K)h \rangle|$$
$$\leq \|g\|_{\mathscr{H}} \|h\|_{\mathscr{H}} \exp[-cd(\Delta_g, \Delta_h)] \exp[2|\Lambda_{-1}|/\pi + O(|\partial \Lambda_{-1}|)],$$

where

$$\Lambda_{-1} = \operatorname{supp}(1 - m(x)).$$

A similar result holds for higher Fredholm minors. We also have the following estimate for the regularized determinant (Proposition 2.7):

$$|\det_3(1 + K)| \leq \exp[2|\Lambda_{-1}|/\pi + O(|\partial \Lambda_{-1}|)], \tag{1.11}$$

where the coefficient of $|\Lambda_{-1}|$ has the expected perturbative value.

The main idea of the proof is to make use of the unitary transformation U to relate $\gamma_0(i\partial + m(x))$ to an operator equal to $\gamma_0(i\partial + 1) + \zeta$, where ζ is supported on $\partial \Lambda_{-1}$. The correction $2|\Lambda_{-1}|/\pi$ arises since, unlike a true determinant, $\det_3$ is not invariant under unitary conjugation.

This result gives the main contribution to the weights of the contour expansion for the multiphase $N = 2$ Wess–Zumino$_2$ models. We would hope that a slightly different version of this work, replacing determinants by Pfaffians, should give a similar result for the $N = 1$ models. The cluster expansion for these quantum field theories will be the subject of future papers [8]. For the reader interested in the quantum field theories this paper should serve as a simple introduction to the

necessary cluster expansion techniques. We hope however that this result may be of interest in itself.

2. Estimates for the Determinant

Before turning to estimates on Fredholm minors, let us consider the determinant alone. We want to be able to compute determinant ratios, such as that formally given by

$$\det_3 \gamma_0(i\partial + m(x))/\det_3 \gamma_0(i\partial + 1). \tag{2.1}$$

The existence of the determinant is guaranteed by the following, which is an immediate consequence of Lemma 5.4:

Lemma 2.1. *Let $\mathscr{L}_3(\mathscr{H})$ denote the 3-Schatten ideal, i.e. the ideal of operators $\mathscr{K}$ on $\mathscr{H}$ such that*

$$\mathrm{Tr}(\mathscr{K}^*\mathscr{K})^{3/2} < \infty.$$

Then

$$K \in \mathscr{L}_3(\mathscr{H}).$$

The fact that it is regularized determinants which appear complicates our analysis. While (formally)

$$\frac{\det \gamma_0(i\partial - 1)}{\det \gamma_0(i\partial + 1)} = \det(1 - 2S\gamma_0) = 1 \tag{2.2}$$

because of (1.5), we (unfortunately) have

$$\det_3(1 - 2S\gamma_0) = \exp \lim_{\kappa \to \infty} T_\kappa, \tag{2.3}$$

where

$$T_\kappa = \mathrm{Tr}[2S_\kappa\gamma_0 + 2S_\kappa\gamma_0 S\gamma_0]; \tag{2.4}$$

here S_κ is the operator on $\mathscr{H}_{-1/2} \oplus \mathscr{H}_{-1/2}$ given by the kernel

$$S_\kappa(x, y) = \int \frac{-\not{p} + 1}{p^2 + 1} e^{-ip(x-y)} \chi_\kappa(p) \frac{d^2p}{(2\pi)^2} \gamma_0, \tag{2.5}$$

where

$$\chi_\kappa(p) = e^{-p^2/\kappa^2}. \tag{2.6}$$

Let $\varXi_\kappa$ be the operator on $\mathscr{H}$ defined by

$$\varXi_\kappa f = \int \chi_\kappa(p) \hat{f}(p) \frac{d^2p}{(2\pi)^2}. \tag{2.7}$$

Then $S_\kappa = \varXi_\kappa S = S\varXi_\kappa$.

It is easily seen that in a finite volume Λ

$$\lim_{\kappa \to \infty} T_\kappa = 2|\Lambda|/\pi + O(|\partial\Lambda|). \tag{2.8}$$

Thus in the local case we expect (finite) $O(|\Lambda|)$ corrections from the difference between det and $\det_3$, as well as boundary corrections.

Let the function $m(x)$ take the values ± 1; let

$$\zeta = \gamma_0(m(x)e^{2i\alpha\gamma_5} - 1 - \partial\alpha\gamma_5),\tag{2.9}$$

where we choose α such that $m(x)e^{2i\alpha(x)} = 1$ except in a neighborhood of $\partial\Lambda_{-1}$. Let

$$R_\kappa = \operatorname{Tr} \Xi_\kappa[S\gamma_0(1-m(x)) + (S\gamma_0(1-m(x))^2/2] + \operatorname{Tr}\Xi_\kappa[S\zeta - (S\zeta)^2/2].\tag{2.10}$$

Then we have the following:

Proposition 2.2

(a) *Then limit* $\lim\limits_{\kappa\to\infty} R_\kappa$ *exists, and*

$$R \equiv \lim_{\kappa\to\infty} R_\kappa = 2|\Lambda_{-1}|/\pi + O(|\partial\Lambda_{-1}|).\tag{2.11}$$

(b) *The determinant* $\det_3[1 + S\gamma_0(m(x)-1)]$ *satisfies*

$$\det_3[1 + S\gamma_0(m(x)-1)] = \det_3[1 + S\zeta]\exp(R+r),\tag{2.12}$$

where

$$r = \frac{1}{4\pi}\|\partial\alpha\|_{L^2}^2.\tag{2.13}$$

Remark. The factor $\exp R$ is the ratio of the determinants expected from a naive product formula. The additional term $\exp r$ is a correction resulting from the need to place cutoffs on operators before such a formula can apply.

Proof. Part (a) follows by a straightforward calculation. We devote the remainder of this section to proving (b). We begin with the following lemma:

Lemma 2.3. *Let* $A, B \in \mathscr{L}_3(\mathscr{H})$. *Then*

$$C = A + B + AB = (1+A)(1+B) - 1 \in \mathscr{L}_3(\mathscr{H}),\tag{2.14}$$

and

$$\det_3(1+A)\det_3(1+B) = \det_3(1+C)\exp -\operatorname{Tr} T,\tag{2.15}$$

where

$$T = (AB)^2/2 + AB^2 + BA^2 \in \mathscr{L}_1(\mathscr{H}).\tag{2.16}$$

Proof. If $A, B \in \mathscr{L}_1(\mathscr{H})$, then

$$\operatorname{Tr} T = \operatorname{Tr}[A + B - C - A^2/2 - B^2/2 + C^2/2],\tag{2.17}$$

and the lemma follows by the product formula for Fredholm determinants and the definition of $\det_3$. Since operators in $\mathscr{L}_3(\mathscr{H})$ may be approximated by finite rank operators, the lemma follows.

We now apply the above lemma to our case. Let $U_t = \exp[i\alpha\gamma_5 t]$, and let

$$\zeta_t = \gamma_0(m(x)e^{2i\alpha\gamma_5 t} - 1 - \partial\alpha\gamma_5 t).\tag{2.18}$$

Then

$$1 + S\zeta_t = SU_t^{-1}\gamma_0(i\partial + m(x))U_t$$
$$= SU_t^{-1}S^{-1}U_tU_t^{-1}S\gamma_0(i\partial + m(x))U_t$$
$$= (1 + S\zeta_t^0)(1 + U_t^{-1}KU_t), \tag{2.19}$$

where

$$\zeta_t^0 = \gamma_0(e^{2i\alpha\gamma_5 t} - 1 - \partial\alpha\gamma_5 t). \tag{2.20}$$

As a consequence of Lemma 2.3 we have the following:

Corollary 2.4. *Let*

$$T_t = (S\zeta_t^0 U_t^{-1}KU_t)^2/2 + (S\zeta_t^0)^2 U_t^{-1}KU_t + (U_t^{-1}KU_t)^2 S\zeta_t^0. \tag{2.21}$$

Then

$$\det_3[1 + S\zeta_t] = \det_3[1 + S\zeta_t^0]\det_3[1 + U_t^{-1}KU_t]\exp\operatorname{Tr}T_t$$
$$= \det_3[1 + S\zeta_t^0]\det_3[1 + K]\exp\operatorname{Tr}T_t. \tag{2.22}$$

On the other hand, some algebraic manipulation shows that

$$\operatorname{Tr}T \equiv \operatorname{Tr}T_t|_{t=1}$$
$$= \lim_{\kappa\to\infty}\operatorname{Tr}\varXi_\kappa T$$
$$= \lim_{\kappa\to\infty}\operatorname{Tr}(R_\kappa + E_\kappa + F_\kappa), \tag{2.23}$$

where

$$E_\kappa = E_{\kappa,t}|_{t=1}, \quad F_\kappa = F_{\kappa,t}|_{t=1} \tag{2.24}$$

with

$$E_{\kappa,t} = \operatorname{Tr}\varXi_\kappa(S\zeta_t^0 - (S\zeta_t^0)^2/2), \tag{2.25}$$

and

$$F_{\kappa,t} = \tfrac{1}{2}\operatorname{Tr}\varXi_\kappa[S\zeta_t^0, U_t^{-1}KU_t]. \tag{2.26}$$

The term R_κ was estimated in Proposition 2.2, part (a). For $E_{\kappa,t}$ and $F_{\kappa,t}$ we have the following results:

Lemma 2.5.
(a) $E(t) = \lim_{\kappa\to\infty} E_{\kappa,t}$ *exists, and*

$$\det_3[1 + S\zeta_t^0]\exp E(t) = \exp\frac{-t^2}{4\pi}\|\partial\alpha\|_2^2; \tag{2.27}$$

(b) $\lim_{\kappa\to\infty} F_{\kappa,t} = 0.$

To prove this we make use of the following lemma:

Lemma 2.6 *The operator* $U_t^{-1}\varXi_\kappa U_t$ *may be written as a sum*

$$U_t^{-1}\varXi_\kappa U_t = \varXi_\kappa + \mathcal{O}_\kappa^5\gamma_5 + \mathcal{O}_\kappa^I, \tag{2.28}$$

where for any $\varepsilon > 0$,

$$\|\mathcal{O}_\kappa^I\|_{1+\varepsilon} \to 0 \quad \text{as} \quad \kappa \to \infty, \tag{2.29}$$

$$\|\mathcal{O}_\kappa^5\|_{2+\varepsilon} \to 0 \quad \text{as} \quad \kappa \to \infty. \tag{2.30}$$

Proof. We first demonstrate (2.30). The operator $\mathcal{O}_\kappa^5$ is given by the kernel

$$\mathcal{O}_\kappa^5(x, y) = \frac{i\gamma_5 \kappa^2}{4\pi} \sin(\alpha(x) - \alpha(y)) e^{-\kappa^2(x-y)^2/4}. \tag{2.31}$$

We write $\mathcal{O}_\kappa^5 = L_1^5 + L_2^5$, where L_1^5, L_2^5 are given by the kernels

$$L_1^5(x, y) = \frac{i\gamma_5 \kappa^2}{4\pi} \partial\alpha(x) \cdot (x - y) e^{-\kappa^2(x-y)^2/4}, \tag{2.32}$$

$$L_2^5(x, y) = \mathcal{O}_\kappa^5(x, y) - L_1^5(x, y). \tag{2.33}$$

We have

$$\|\mathcal{O}_\kappa^5\|_{2+\varepsilon} \leq \|L_1^5\|_{2+\varepsilon} + \|L_2^5\|_2. \tag{2.34}$$

Using the notation of Lemma 2.1 of [13], L_1^5 is of the form

$$\text{const} \times f(p)g(x), \tag{2.35}$$

where

$$f(p) = p\kappa^{-2} e^{-p^2/\kappa^2}, \tag{2.36}$$

$$g(x) = \partial\alpha(x). \tag{2.37}$$

Thus

$$\|L_1^5\|_{2+\varepsilon} \leq \text{const} \times \|f\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \|g\|_{L^{2+\varepsilon}(\mathbb{R}^2)}$$
$$\leq O(|\partial\Lambda_{-1}|)\kappa^{-\varepsilon/(2+\varepsilon)}. \tag{2.38}$$

The Hilbert–Schmidt norm of L_2^5 is easily estimated using pointwise bounds, yielding (2.30).

To deal with (2.29) we must get around the fact that Lemma 2.1 of [13] is not directly applicable to $(1+\varepsilon)$-norms. The kernel of $\mathcal{O}_\kappa^I$ is given by

$$\mathcal{O}_\kappa^I(x, y) = (\cos(\alpha(x) - \alpha(y)) - 1) \frac{\kappa^2}{4\pi} e^{-\kappa^2(x-y)^2/4}. \tag{2.39}$$

Let χ be the characteristic function of Λ_{-1}. Then

$$\mathcal{O}_\kappa^I = \chi\mathcal{O}_\kappa^I + \mathcal{O}_\kappa^I\chi - \chi\mathcal{O}_\kappa^I\chi, \tag{2.40}$$

and

$$\|\mathcal{O}_\kappa^I\|_{1+\varepsilon} \leq 3 \|\chi\mathcal{O}_\kappa^I\|_{1+\varepsilon}. \tag{2.41}$$

Now we write

$$\|\chi\mathcal{O}_\kappa^I\|_{1+\varepsilon} \leq \|\chi S\gamma_0(i\partial + 1)\mathcal{O}_\kappa^I\|_{1+\varepsilon}$$
$$\leq \|\chi S\|_{2+\delta} \|(i\partial + 1)\mathcal{O}_\kappa^I\|_{2+\delta}$$
$$\leq \text{const} \times \|(i\partial + 1)\mathcal{O}_\kappa^I\|_{2+\delta}. \tag{2.42}$$

Now

$$(i\partial + 1)\mathcal{O}_\kappa^I(x, y) = K_1^I(x, y) + K_2^I(x, y), \tag{2.43}$$

where the leading term is

$$K_1^I(x, y) = \frac{-i\kappa^2}{4\pi} \partial\alpha(x) \cdot (x - y)\partial\alpha(x)e^{-\kappa^2(x-y)^2}, \tag{2.44}$$

the kernel of an operator K_1^I whose $(2 + \delta)$-norm may be estimated similarly to (2.38). The remainder $K_2^I(x, y)$ is the kernel of an operator whose Hilbert–Schmidt norm vanishes as $\kappa \to \infty$, yielding the lemma. ∎

Proof of Lemma 2.5. To show (b), we note that

$$F_{\kappa,t} = \tfrac{1}{2}\operatorname{Tr} S(\Xi_\kappa \zeta_t^0 - \zeta_t^0 \Xi_\kappa)(U_t^{-1} K U_t). \tag{2.45}$$

This can be expressed as a sum of terms, some of which are of the form $\operatorname{Tr}(\Xi_\kappa \zeta_t^0 - \zeta_t^0 \Xi_\kappa)\mathcal{O}$, with $\mathcal{O} \in \mathcal{L}_1$, and hence vanish as $\kappa \to \infty$. The remaining terms may be estimated by

$$\int \frac{1}{(p^2 + 1)^{1/2}} \frac{1}{(q^2 + 1)^{1/2}} |\hat{f}(p - q)|(1 - m(y))|\chi_\kappa(p) - \chi_\kappa(q)| d^2p\, d^2q\, d^2y, \tag{2.46}$$

where $\hat{f}$ is the Fourier transform of ζ_t^0. Since α is smooth, $\hat{f}$ decays rapidly. We also have

$$|\chi_\kappa(p) - \chi_\kappa(q)| \leq 2\frac{|p - q|}{\kappa^2}((|p| + \kappa)\chi_\kappa(p) + (|q| + \kappa)\chi_\kappa(q)), \tag{2.47}$$

and thus $F_{\kappa,t}$ is $O(1/\kappa)$.

We are left with (a). First we note that

$$\det_3[1 + S\zeta_t^0] \exp E(t)|_{t=0} = 1. \tag{2.48}$$

To prove (a) we will show that

$$\frac{\partial}{\partial t}\log(\det_3[1 + S\zeta_t^0]\exp E(t)) = \frac{-t}{2\pi}\|\partial\alpha\|_2^2 \tag{2.49}$$

so that

$$\det_3[1 + S\zeta_t^0]\exp E(t)|_{t=1} = \frac{-1}{4\pi}\|\partial\alpha\|_2^2 \tag{2.50}$$

as needed.

Now

$$\lim_{\kappa \to \infty} \frac{\partial}{\partial t}(\det_3[1 + S\zeta_t^0]\exp \operatorname{Tr} \Xi_\kappa(S\zeta_t^0 - (S\zeta_t^0)^2/2))$$

$$= \lim_{\kappa \to \infty}\left\{\left(\operatorname{Tr}\frac{1}{1 + S\zeta_t^0}(S\zeta_t^0)^2 S(\zeta_t^0)' + \operatorname{Tr} S_\kappa(\zeta_t^0)' - \operatorname{Tr} S_\kappa \zeta_t^0 S(\zeta_t^0)'\right)\right.$$

$$\left. \cdot \det_3[1 + S\zeta_t^0]\exp E_{\kappa,t}\right\}$$

$$= \lim_{\kappa \to \infty}\left\{\operatorname{Tr}\left(\frac{1}{1 + S\zeta_t^0}S_\kappa(\zeta_t^0)' + S\zeta_t^0 S_\kappa(\zeta_t^0)' - S_\kappa \zeta_t^0 S(\zeta_t^0)'\right)\right.$$

$$\cdot \det_3 [1 + S\zeta_t^0] \exp E_{\kappa,t} \Bigg\}$$

$$= \lim_{\kappa \to \infty} \Bigg\{ \mathrm{Tr}\Bigg(\frac{1}{1 + S\zeta_t^0} S_\kappa(\zeta_t^0)'\Bigg) \det_3 [1 + S\zeta_t^0] \exp E_{\kappa,t} \Bigg\} \qquad (2.51)$$

since the difference $\mathrm{Tr}[S\zeta_t^0 S_\kappa(\zeta_t^0)' - S_\kappa \zeta_t^0 S(\zeta_t^0)']$ can be estimated in the same fashion as $F_{\kappa,t}$ was in Lemma 2.5(b), and thus vanishes as $\kappa \to 0$. Explicitly writing out the last line of (2.51) above we see that the derivative (2.51) is equal to

$$\lim_{\kappa \to \infty} \Bigg\{ \det_3 [1 + S\zeta_t^0] \exp E_{\kappa,t} \cdot \mathrm{Tr}\, U_t^{-1} S U_t S^{-1} \Xi_\kappa S \frac{\partial}{\partial t}(U_t^{-1} S^{-1} U_t)\Bigg\}. \qquad (2.52)$$

Now

$$\frac{\partial}{\partial t}(U_t^{-1} S^{-1} U_t) = -i\alpha\gamma_5 U_t^{-1} S^{-1} U_t + i U_t^{-1} S^{-1} U_t \alpha\gamma_5, \qquad (2.53)$$

so that (using trace cyclicity)

$$\mathrm{Tr}\, U_t^{-1} S U_t S^{-1} \Xi_\kappa S \frac{\partial}{\partial t}(U_t^{-1} S^{-1} U_t) = \mathrm{Tr}\,[\Xi_\kappa(-i\alpha\gamma_5) + S U_t \Xi_\kappa U_t^{-1} S^{-1}(i\alpha\gamma_5)]$$

$$= i\,\mathrm{Tr}\, S U_t \Xi_\kappa U_t^{-1} S^{-1} \alpha\gamma_5, \qquad (2.54)$$

since γ_5 is traceless.

Writing

$$S^{-1}\alpha\gamma_5 = \gamma_5 \gamma_0 (i\partial\!\!\!/ - 1)\alpha$$

$$= \gamma_5(\alpha S^{-1} - 2\gamma_0\alpha + i\gamma_0\partial\!\!\!/\alpha) \qquad (2.55)$$

we see that

$$\mathrm{Tr}\, S U_t \Xi_\kappa U_t^{-1} S^{-1}\alpha\gamma_5 = \mathrm{Tr}(\Xi_\kappa\gamma_5\alpha + 2S\gamma_0 U_t^{-1}\Xi_\kappa U_t\gamma_5\alpha - iS\gamma_0 U_t^{-1}\Xi_\kappa U_t\gamma_5(\partial\!\!\!/\alpha)). \qquad (2.56)$$

The first term is clearly zero, and the second term goes to zero as $\kappa \to \infty$ since it can be bounded, e.g., by $\|\mathcal{O}_\kappa^5\|_3$. Similar computations with the third term lead to

$$\frac{\partial}{\partial t}(\det_3 [1 + S\zeta_t^0] \exp E(t)) = \det_3 [1 + S\zeta_t^0] \exp E(t) \lim_{\kappa \to \infty} \mathrm{Tr}\, S\gamma_0 \mathcal{O}_\kappa^5(\partial\!\!\!/\alpha). \qquad (2.57)$$

We can write

$$\mathcal{O}_\kappa^5(y, x) = \frac{i\kappa^2}{4\pi} e^{-\kappa^2(x-y)^2/4} \sin t(\alpha(x) - \alpha(y))$$

$$= \frac{-it\kappa^2}{4\pi} e^{-\kappa^2(x-y)^2/4}[(y-x)\cdot\partial\alpha + O((x-y)^2)], \qquad (2.58)$$

and the singularity of the fermionic covariance is

$$S\gamma_0(x, y) \sim \frac{-i}{2\pi|x-y|^2}(x_\mu - y_\mu)\gamma_\mu^E, \qquad (2.59)$$

so that the trace in (2.57) can be evaluated:

$$\lim_{\kappa \to \infty} \text{Tr } S\gamma_0 \mathcal{O}_\kappa^5(\partial\alpha) = \lim_{\kappa \to \infty} \frac{-t\kappa^2}{4\pi 2} \int d^2x \, d^2y \, e^{-\kappa^2(x-y)^2/4} \frac{1}{|x-y|^2} [(x-y) \cdot \partial\alpha(x)]^2$$

$$+ \text{ higher order terms which vanish}$$

$$= \frac{-t}{2\pi} \int |\partial\alpha|^2 d^2x, \tag{2.60}$$

completing the proof.

Finally, we obtain the following simple estimate on $\det_3 [1 + S\gamma_0(m(x) - 1)]$:

Proposition 2.7.

$$\det_3 [1 + S\gamma_0(m(x) - 1)] \leq \exp [2|\Lambda_{-1}|/\pi + O(|\partial\Lambda_{-1}|)].$$

Proof. Let $\tilde{K} = S\zeta$. In view of Proposition 2.2, we need only estimate

$$|\det_3 (1 + \tilde{K})| = |\det_4 (1 + \tilde{K}) \exp [- \text{Tr } \tilde{K}^4/4]|$$

$$\leq \exp \text{const} \| \tilde{K} \|_4^4$$

$$\leq \exp O(|\partial\Lambda_{-1}|) \tag{2.61}$$

by Lemma 5.4.

3. The Cluster Expansion

Let us consider the Fredholm minor

$$\det_{jk}(g_j, [\gamma_0(i\partial + m(x))]^{-1} h_k)_{L^2} \det_3 [1 + S\gamma_0(m(x) - 1)], \tag{3.1}$$

where $g_j, h_k \in C_0^\infty(\mathbb{R}^2)$. The reduced Green's function of Theorem 1 corresponds to the special case where there are only two test functions, denoted g, h.

Now the expression (3.1) is clearly equal to

$$\det_{jk} (g_j, U^{-1}[1 + S\zeta]^{-1} S U h_k)_{L^2} \det_3 [1 + S\zeta] \frac{\det_3 [1 + S\gamma_0(m(x) - 1)]}{\det_3 [1 + S\zeta]}. \tag{3.2}$$

The ratio of regularized determinants was calculated in Lemma 2.2, and is equal to

$$\exp [2|\Lambda_{-1}|/\pi + O(|\partial\Lambda_{-1}|)]; \tag{3.3}$$

hence we will only need to perform the cluster expansion on the Fredholm minor

$$\det_{jk}(g_j, U^{-1}[1 + S\zeta]^{-1} S U h_k)_{L^2} \det_3 [1 + S\zeta]. \tag{3.4}$$

It will simplify the analysis considerably to work in $\mathcal{H}' = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ instead of $\mathcal{H} = \mathcal{H}_{-1/2} \oplus \mathcal{H}_{-1/2}$. This results in replacing the operator $\tilde{K} = S\zeta$ by $D^{1/2} S\zeta D^{-1/2}$, where $D = (-\Delta + 1)^{1/2}$. Thus we have

$$\det_{jk}(g_j, U^{-1}[1 + S\zeta]^{-1} S U h_k)_{L^2} \det_3 [1 + S\zeta]$$

$$= \det_{jk}(\tilde{g}_j, [1 + S\zeta]^{-1} S\tilde{h}_k)_{L^2} \det_3 [1 + S\zeta]$$

$$= \det_{jk}(D^{-1/2}\tilde{g}_j, [1 + D^{1/2} S\zeta D^{-1/2}]^{-1} D^{1/2} S\tilde{h}_k)_{L^2} \det_3 [1 + D^{1/2} S\zeta D^{-1/2}], \tag{3.5}$$

with

$$\tilde{g}_j = U^{-1}g_j, \quad \tilde{h}_\mu = Uh_k. \tag{3.6}$$

We wish to express the Fredholm minor (3.4) as a trace of an antisymmetric product of $[1 + S\zeta]^{-1}$ multiplied by an operator on $\overset{k}{\wedge} \mathcal{H}'$. This structure will be maintained throughout, and is necessary in order to obtain the proper estimates on the terms produced by differentiation. We write the expression (3.5) as

$$\mathcal{J}! \operatorname{Tr}_{\wedge \mathcal{H}'}\left(\overset{\mathcal{J}}{\underset{j=1}{\wedge}} [1 + D^{1/2}S\zeta D^{-1/2}]^{-1}P_j\right)\det_3[1 + D^{1/2}S\zeta D^{-1/2}], \tag{3.7}$$

where P_j is the operator on $\mathcal{H}'$ given by

$$P_j \equiv (D^{-1/2}\tilde{g}_j, \cdot)D^{1/2}S\tilde{h}_j. \tag{3.8}$$

We now introduce decoupling parameters on the bonds of the lattice $l\mathbb{Z}^2$; for all $s_b = 1$ we have the same equation as before, while we have complete decoupling across bonds with $s_b = 0$. We only decouple bonds that are far away $(O(l)$ – the specific distance will be chosen later) from regions where the mass is changing, i.e. far from the region where $\alpha(x)$ is changing. We denote the subset of bonds on which we decouple by $\mathcal{B}(\alpha)$.

We use the decoupling method of Balaban and Gawędzki [1], with Δ on the l-lattice. Let Δ, Δ' be l-lattice squares, and take $s \in [0, 1]^{\mathcal{B}(\alpha)}$. Then define

$$H(s, \Delta, \Delta') = \sum_{\text{finite } \gamma \subset \mathcal{B}(\alpha)} \prod_{b \in \gamma} s_b \prod_{b \notin \gamma}(1 - s_b)\frac{\bar{C}^{\mathcal{B}(\alpha)\backslash\gamma}(\Delta, \Delta')}{\bar{C}(\Delta, \Delta')}, \tag{3.9}$$

where

$$\bar{C}^\Gamma(\Delta, \Delta') = \int_\Delta dx \int_{\Delta'} dy\, C^\Gamma(x, y), \quad C^\Gamma = (-\Delta_\Gamma^D + m_c^2)^{-1}; \tag{3.10}$$

Δ_Γ^D is the Laplacian with Dirichlet boundary conditions on Γ and $m_c > 0$ is a sufficiently small constant to be chosen later.

Definition 3.1. *Let A be an operator on $\mathcal{H}'$. Then we define*

$$A_s \equiv \sum_\Delta \chi_\Delta A \chi_\Delta + \sum_{\Delta \neq \Delta'} H(s, \Delta, \Delta')\chi_\Delta A \chi_{\Delta'}.$$

The decoupled version of Eq. (3.7) is

$$\mathcal{J}! \operatorname{Tr}\left(\overset{\mathcal{J}}{\underset{j=1}{\wedge}} [1 + K(s)]^{-1}P_j\right)\det_3[1 + K(s)] \equiv Z(s), \tag{3.11}$$

where

$$K(s) \equiv (D^{1/2}SD^{1/2})_s(D^{-1/2})_s\zeta(D^{-1/2})_s, \tag{3.12}$$

and where

$$P_j(s) \equiv ((D^{-1/2})_s\tilde{g}_j, \cdot)(D^{1/2}S)_s\tilde{h}_j. \tag{3.13}$$

Note that $Z(s)$ factors on connected regions whose boundary consists of bonds with $s_b = 0$.

The Glimm–Jaffe–Spencer [5, 6] cluster expansion is essentially an application of the fundamental theorem of calculus. For a partially decoupled function $F(s)$

we have

$$F(1) = \sum_{\Gamma \in \mathscr{B}(\alpha)} \int_0^1 ds_\Gamma \partial_s^\Gamma F(s), \tag{3.14}$$

where

$$\partial_s^\Gamma = \prod_{b \in \Gamma} \partial_{s_b}, \quad ds_\Gamma = \prod_{b \in \Gamma} ds_b \tag{3.15}$$

and we set $s_b = 0$ for all $b \notin \Gamma$. We can put this all together with the following:

Proposition 3.2 (*Cluster Expansion*). *The correlation function* (3.11) *satisfies the following convergent expansion*:

$$Z(1) = \sum_{\Gamma \in \mathscr{B}(\alpha)} \int ds_\Gamma \partial_s^\Gamma Z(s), \tag{3.16}$$

and for some constants c and δ' obeys the bound

$$|Z(1)| \leq c^{\mathscr{I}} e^{-\delta' d(\{g_J\}, \{h_k\})} e^{O(|\partial \Lambda - 1|)}. \tag{3.17}$$

The remainder of this paper will be dedicated to proving the above proposition; in addition to the factoring across bonds with $s_b = 0$ the main fact we will need is a bound on cluster activities that is exponential in the size of the cluster.

A typical term in our expansion will have the form

$$k! \operatorname{Tr} \left(\bigwedge^k [1 + K(s)]^{-1} \cdot G \right) \det_3 [1 + K(s)] \equiv \tau_k(G), \tag{3.18}$$

where G is an operator on $\bigwedge^k \mathscr{H}'$ given by antisymmetric products of operators of the type A, E and P defined below. The result of performing one differentiation on $\tau_k(G)$ is then [3]

$$\frac{\partial}{\partial s} \tau_k(G) = \tau_k \left(\frac{\partial G}{\partial s} \right) + \tau_{k+1}(G \wedge A_s) - \tau_k(G \cdot d \wedge^k E_s), \tag{3.19}$$

where

$$A_s = K^2(s) \frac{\partial K(s)}{\partial s}, \quad E_s = (1 - K(s)) \frac{\partial K(s)}{\partial s}, \tag{3.20}$$

and the operator $d \wedge^k$ identifies E_s with an element of $\bigwedge^k \mathscr{H}'$:

$$d \wedge^k E = kE \wedge \mathbf{I}^{k-1}, \quad \mathbf{I}^{k-1} = 1 \wedge \underbrace{\cdots}_{k-1 \text{ times}} \wedge 1. \tag{3.21}$$

We can now write down the generic term in the cluster expansion,

$$\partial_s^\Gamma Z(s) = \partial_s^\Gamma \tau_{\mathscr{I}} \left(\bigwedge_{j=1}^{\mathscr{I}} P_j(s) \right)$$

$$= \sum_{\pi \in \mathscr{P}(\Gamma)} \sum_{\text{decomp } \pi} \partial^{\pi_0} \tau_r(P(\pi_P) \wedge A(\pi_A) \cdot d \wedge^r E(\pi_E)). \tag{3.22}$$

Here the set of derivatives that have been applied to the Fredholm minor is decomposed according to whether they result in a factor of type A, E or P, or produce a higher derivative of such a term, as follows:

$$\pi = \pi_0 \cup \pi_P \cup \pi_A \cup \pi_E, \tag{3.23}$$

and $\mathscr{P}(\Gamma)$ denotes the set of partitions of Γ. The other terms appearing in (3.22) are:

$$P(\pi_P) = \left(\prod_{\gamma \in \pi_P} \partial_s^\gamma \right) P_1(s) \wedge \cdots \wedge P_{\mathscr{I}}(s) \tag{3.24}$$

(where each ∂_s^γ acts on a different $P_i(s)$),

$$A(\pi_A) = \bigwedge_{\gamma \in \pi_A} K^2(s) \partial_s^\gamma K(s), \tag{3.25}$$

$$d \wedge {}^r E(\pi_E) = \prod_{\gamma \in \pi_E} d' \wedge {}^r (1 - K(s)) \partial_s^\gamma K(s), \tag{3.26}$$

where $r = \mathscr{I} + |\pi_A|$ and the $d' \wedge$ means that terms where E derivatives precede A derivatives (according to an arbitrary ordering of the bonds) are omitted, and finally

$$\partial^{\pi_0} = \prod_{\gamma \in \pi_0} \partial_s^\gamma; \tag{3.27}$$

each ∂_s^γ acts on an undifferentiated $K(s)$ in an A or E factor.

We will be estimating (3.22) by fixing all the localization squares in the decoupling (Definition 3.1). To indicate a term so localized we write A_l, E_l, etc. We re-write (3.22) to make this explicit:

Lemma 3.3. *The derivative $\partial_s^\Gamma Z(s)$ may be written as the following sum over localized terms*:

$$\partial_s^\Gamma Z(s) = \sum_{\pi \in \mathscr{P}(\Gamma)} \sum_{\substack{\text{decomp } \pi \\ }} \sum_{\substack{\text{localizations} \\ \{\Delta_i\}}} \partial^{\pi_0} \tau_r(P_l(\pi_P) \wedge A_l(\pi_A) \cdot d \wedge {}^r E_l(\pi_E)).$$

4. Combinatoric Estimates

We will estimate the sums in Lemma 3.3, primarily using the following lemma:

Lemma 4.1 (*Method of Combinatoric Factors*). *Given two sequences $\{a_n\}$ and $\{c_n\}$, if $\sum_n |c_n|^{-1} < 1$, then*

$$\sum_n |a_n| \leq \sup_n |c_n| |a_n|.$$

First we see that

$$|\partial_s^\Gamma Z(s)| \leq \sum_\pi 4^{|\pi|} \sup_{\text{decomp}} \sum_{\text{localizations}} |\partial^{\pi_0} \tau_r(P_l \wedge A_l \cdot d' \wedge {}^r E_l)|. \tag{4.1}$$

We can localize the A factors (this means choosing a particular square from each of the sums over characteristic functions implicit in the definition of the decoupling) with a combinatoric factor

$$O(1) |\gamma|^{O(1)} \prod_k \exp \varepsilon d(\Delta_k, \gamma) \tag{4.2}$$

since

$$\frac{1}{|\gamma|} \sum_\Delta \exp[-\varepsilon d(\Delta, \gamma)] < O(1) \quad \text{for} \quad \varepsilon l > 1. \tag{4.3}$$

Similarly we can localize the P factors with combinatoric factors

$$O(1) \exp \varepsilon [d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)], \tag{4.4}$$

where $\bar{\Delta}_j$ and $\bar{\Delta}'_j$ are the supports of the test functions h_j and g_j.

For the E factors, in addition to fixing a localization we wish to fix a choice of terms in the exterior derivative. For the localization we proceed just as we did for the A factors, yielding a combinatoric factor like (4.2).

Combining the above we get the following:

Proposition 4.2.

$$|\partial_s^\Gamma Z(s)| \leq \sum_\pi O(1)^{|\Gamma|} \sup_{\substack{\text{decomps,} \\ \text{localizations}}} |\partial^{\pi_0} \tau_r(P_l \wedge A_l \cdot d' \wedge {}^r E_l)|$$

$$\cdot \prod_{\gamma \in \pi_A \cup \pi_E} \left(O(1)|\gamma|^{O(1)} \prod_k \exp \varepsilon d(\Delta_k, \gamma) \right)$$

$$\cdot \prod_{j=1}^{\mathscr{J}} O(1) \exp \varepsilon [d(\Delta_j, \bar{\Delta}_j) + d(\Delta'_j, \bar{\Delta}'_j)].$$

We now want to use additional combinatoric factors to restrict the internal structure of the A, P and E terms. These estimates involve exponential pinning; see for example [1] p. 301. To fix a term in the exterior derivative, let $e_L(\Delta)$ be the number of E terms with left-most localization square Δ. Then the number of terms is bounded by

$$2^{\mathscr{J}+|\Gamma|} \prod_\Delta e_L(\Delta)! \leq 2^{\mathscr{J}} O(1)^{|\Gamma|} \prod_{\gamma \in \pi_E} \exp \varepsilon d(\gamma, \Delta_L), \tag{4.5}$$

where Δ_L is the left-most localization square corresponding to the E factor being differentiated (by $\gamma \ni b$). Now we consider restricting the sum over the P derivatives. There are $\mathscr{J}$ factors and $|\pi_P|$ derivatives, so there are less than $(2\mathscr{J})^{|\pi_P|}$ possibilities (the 2 comes from the choice of the bra or ket). We have the estimate

$$(2\mathscr{J})^{|\pi_P|} \leq O(1)^{|\pi_P| + \mathscr{J}} \prod_{\gamma \in \pi_P} \exp [\varepsilon \min \{d(\gamma, \Delta_l), d(\gamma, \bar{\Delta}_l)\}], \tag{4.6}$$

where Δ_l is the localization square corresponding to γ chosen via (4.4) and the choice of the bra or ket, and $\bar{\Delta}_l$ is the corresponding test function square. We also restrict our choice for the distribution of derivatives of π_0. There are less than $[O(1)(|\pi_A| + |\pi_E|)]^{|\pi_0|}$ factors, so we get an estimate similar to (4.6),

$$[O(1)(|\pi_A| + |\pi_E|)]^{|\pi_0|} \leq O(1)^{|\pi|} \prod_{\gamma \in \pi_0} \exp [\varepsilon \min \{d(\gamma, \Delta_l), d(\gamma, \Delta'_l)\}], \tag{4.7}$$

where Δ_l and Δ'_l are the localization squares chosen via (4.2) or its equivalent that surround the factor being differentiated by γ.

Finally, because each $K(s)$ has three decouplings each term can actually be differentiated a number of ways, so that the partitions π_A, π_E and π_0 must be divided into sub-partitions. However, this simply gives a factor of $O(1)^{|\Gamma|}$, and we will leave our notation unchanged for the time being.

The above discussion can be summarized by the following proposition:

Proposition 4.3.

$$|\partial_s^\Gamma Z(s)| \leq \sum_\pi O(1)^{|\Gamma| + \mathscr{I}} \sup_{\substack{\text{decomps,} \\ \text{localizations}}} \sup_{\substack{\pi_P\text{-derivatives} \\ \pi_0\text{-derivatives} \\ \text{exter. derivs}}} |\partial^{\pi_0} \tau_r (P'_l \wedge A'_l \cdot d' \wedge {}'E'_l)|$$

$$\cdot \prod_{\gamma \in \pi_A \cup \pi_E} \left(O(1) |\gamma|^{O(1)} \prod_k \exp \varepsilon d(\Delta_k, \gamma) \right)$$

$$\cdot \prod_{k=1}^{\mathscr{I}} O(1) \exp \varepsilon [d(\Delta_k, \bar{\Delta}_k) + d(\Delta'_k, \bar{\Delta}'_k)]$$

$$\cdot \prod_{\gamma \in \pi_E} \exp \varepsilon d(\gamma, \Delta_L) \prod_{\gamma \in \pi_P} \exp [\varepsilon \min \{d(\gamma, \Delta_l), d(\gamma, \bar{\Delta}_l)\}]$$

$$\cdot \prod_{\gamma \in \pi_0} \exp [\varepsilon \min \{d(\gamma, \Delta_l), d(\gamma, \Delta'_l)\}],$$

where the primes on π_0, P_l, A_l and $d' \wedge {}'E_l$ indicate that we take only one term from each summation.

5. Estimates on Kernels

In order to prove convergence of the cluster expansion we must now prove analytical estimates on the expressions $|\partial^{\pi_0} \tau_r (P'_l \wedge A'_l \cdot d' \wedge {}'E'_l)|$ appearing in Proposition 4.3 above. Each of these terms is of the form

$$r! \operatorname{Tr} (\wedge^r [1 + K(s)]^{-1}, Q_1^{\gamma_1} \wedge Q_2^{\gamma_2} \wedge \cdots \wedge Q_r^{\gamma_r}) \det_3 [1 + K(s)], \tag{5.1}$$

where each $Q_i^{\gamma_i}$ takes the form

$$Q_i^{\gamma_i} = A^{\gamma_i^1} E^{\gamma_i^2} E^{\gamma_i^3} \cdots E^{\gamma_i^{k_i}} \tag{5.2}$$

or

$$Q_i^{\gamma_i} = P^{\gamma_i^1} E^{\gamma_i^2} E^{\gamma_i^3} \cdots E^{\gamma_i^{k_i}} \tag{5.3}$$

and each A, P or E is fully localized as indicated above.

Let

$$G_1(\gamma, \delta) \equiv \sum_{\sigma \in S_{|\gamma|}} \exp - \delta |l_\sigma(\gamma)|, \tag{5.4}$$

where S_n denotes the permutation group on n elements, and where the "size" $|l_\sigma(\gamma)|$ of the linear ordering of γ determined by the permutation $\sigma \in S_{|\gamma|}$ is defined in [3], page 12, and let

$$G_P(Q_i^{\gamma_i}) = \begin{cases} 1, & \text{if } Q_i^{\gamma_i} \text{ is of the form (5.2)} \\ \exp \left(-\delta_6 \sum_i d(\Delta_{ij}^1, \bar{\Delta}_i) + d(\Delta_{ij}^2, \bar{\Delta}_i) \right), & \text{if } Q_i^{\gamma_i} \text{ is of the form (5.3)} \end{cases} \tag{5.5}$$

where $\Delta_{ij}^1, \Delta_{ij}^2$ are the localizations associated with the P factors.

The required estimate is as follows:

Proposition 5.1. *Choose m_c, the decoupling mass in (3.10), sufficiently small. Then*

there exist $\delta_1, \ldots, \delta_6 > 0$ *such that (uniformly in s)*

$$r! |\mathrm{Tr}(\wedge^r [1 + K(s)]^{-1}, Q_1^{\gamma_1} \wedge Q_2^{\gamma_2} \wedge \cdots \wedge Q_r^{\gamma_r}) \det_3 [1 + K(s)]|$$

$$\leqq O(1) \exp(O(|\partial \Lambda_{-1}|)) \prod_{\{\gamma_i^j\}} \left(G_1(\gamma_i^j, \delta_1) \exp(-\delta_2 l |\gamma_i^j|) \exp(-\delta_3 d(\gamma_i^j, \partial \Lambda_{-1})) G_P(Q_i^{\gamma_i}) \right.$$

$$\left. \cdot \exp\left(-\delta_4 \sum_k d(\Delta_{ij}^k, \gamma_i^j)\right) \exp(\delta_5 |\gamma_i^j|) \exp(O(l)) \right),$$

where $\{\Delta_{ij}^k\}$ *are the localization squares associated with the differentiation of* γ_i^j, *and*
$$\gamma_i = \bigcup_j \gamma_i^j.$$

Note that the γ_i^j correspond to the different ways the derivatives in γ_i can be applied within each factor, as explained preceding Proposition 4.3. Also, since the P factors do not contribute to decay away from $\partial \Lambda_{-1}$, we indicate by a prime on the product that the δ_3 term is absent for differentiations arising from P factors.

Proof. We have, using Lemma A.2 of [3],

$$r! |\mathrm{Tr}(\wedge^r [1 + K(s)]^{-1}, Q_1^{\gamma_1} \wedge Q_2^{\gamma_2} \wedge \cdots \wedge Q_r^{\gamma_r}) \det_3 [1 + K(s)]|$$

$$\leqq \| \wedge^r [1 + K(s)]^{-1} \cdot \det_3 [1 + K(s)] \| \prod_i \| Q_i^{\gamma_i} \|_1. \tag{5.6}$$

We complete the proof with the following two lemmas. The first contains the nonperturbative bound on the determinant; the second is a perturbative bound on the kernels Q_i.

Lemma 5.2. *There exist* $c_1, c_2 > 0$, *independent of s, so that*

$$\| \wedge^r [1 + K(s)]^{-1} \cdot \det_3 [1 + K(s)] \| \leqq c_1^r \exp c_2 |\partial \Lambda_{-1}|.$$

Lemma 5.3. *For* m_c *sufficiently small the kernels* $Q_i^{\gamma_1}$ *satisfy*

$$\| Q_i^{\gamma_1} \|_1 \leqq \prod_j G_1(\gamma_i^j, \delta_1) \exp(-\delta_2 l |\gamma_i^j|) \exp(-\delta_3 d(\gamma_i^j, \partial \Lambda_{-1})) G_P(Q_i^{\gamma_1})$$

$$\cdot \exp\left(-\delta_4 \sum_k d(\Delta_{ij}^k, \gamma_i^j)\right) \exp(\delta_5 |\gamma_i^j|) \exp(O(|\partial \Lambda_{-1}|)) \exp(O(l)),$$

where $\delta_1 \cdots \delta_6 > 0$ ($\delta_3 = 0$ *if* γ_i *comes from a P factor) are independent of s.*

The fundamental estimates required to complete the proofs of the above lemmas are the estimates on kernels given in [1]. We quote them here for convenience.
Let

$$K(\Delta, \Delta', \Delta'', \Delta'''; f) = \chi_\Delta(-\not P + 1) D^{-1} \chi_{\Delta'} D^{-1/2} f \chi_{\Delta''} D^{-1/2} \chi_{\Delta'''}. \tag{5.7}$$

Lemma 5.4 (*Proposition A.I.1 of* [1]). *Let* $p > 2$. *There exists* $\varepsilon > 0$ *such that*

$$\| K(\Delta, \Delta', \Delta'', \Delta'''; f) \|_p \leqq \| f \chi_{\Delta''} \|_{L_\infty} \exp - \varepsilon [d(\Delta, \Delta') + d(\Delta', \Delta'') + d(\Delta'', \Delta''')].$$

Proof. [1], page 388.

Lemma 5.5. *Let* $H(s, \Delta, \Delta')$ *be given by* (3.9). *Then*

(a) $0 \leq H(s, \Delta, \Delta') \leq 1$,
(b) $|\partial_s^\gamma H(s, \Delta, \Delta')| \leq \exp(O(l)) \exp(2m_c d(\Delta, \Delta') - \delta m_c d(\gamma, \Delta, \Delta')) G_1(\gamma, \delta_1) \exp(c - \delta_2 l)|\gamma|$.

Proof. [1], page 314.

Proof of Lemma 5.2. By standard determinant estimates, we have

$$
\begin{aligned}
\| \wedge^r [1 + K(s)]^{-1} \cdot \det_3 [1 + K(s)] \| & \\
= \| \wedge^r [1 + K(s)]^{-1} \cdot \det_4 [1 + K(s)] \| \exp[-\operatorname{Tr} K^4(s)/4] & \\
\leq c_1^r \exp c_2 \| K(s) \|_4^4. &
\end{aligned}
\tag{5.8}
$$

Now

$$
\begin{aligned}
\| K(s) \|_4^4 &= \operatorname{Tr} K^*(s) K(s) K^*(s) K(s) \\
&= \sum_{\Delta_{ij} \subset \mathbb{R}^2} \operatorname{Tr}[K^*(s; \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}; \delta) K(s; \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{24}; \delta) \\
&\qquad \cdot K^*(s; \Delta_{31}, \Delta_{32}, \Delta_{33}, \Delta_{34}; \delta) K(s; \Delta_{41}, \Delta_{42}, \Delta_{43}, \Delta_{44}; \delta)] \\
&\qquad \cdot \prod_k H(s, \Delta_{k1}, \Delta_{k2}) H(s, \Delta_{k2}, \Delta_{k3}) H(s, \Delta_{k3}, \Delta_{k4})
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\leq \sum_{\substack{\Delta_{ij}: \Delta_{14} = \Delta_{21}, \\ \Delta_{24} = \Delta_{31}, \Delta_{34} = \Delta_{41}}} \prod_j \| K(s; \Delta_{j1}, \Delta_{j2}, \Delta_{j3}, \Delta_{j4}; \delta) \|_4 \\
\cdot \prod_k H(s, \Delta_{k1}, \Delta_{k2}) H(s, \Delta_{k2}, \Delta_{k3}) H(s, \Delta_{k3}, \Delta_{k4}).
\end{aligned}
\tag{5.10}
$$

Now by Lemma 5.5(a) and Lemma 5.4, this is less than

$$
\sum_{\substack{\Delta_{ij}: \Delta_{14} = \Delta_{21}, \\ \Delta_{24} = \Delta_{31}, \Delta_{34} = \Delta_{41}}} \prod_j \| \zeta \chi_{\Delta_{j2}} \|_{L^\infty} \exp - \varepsilon[d(\Delta_{j1}, \Delta_{j2}) + d(\Delta_{j2}, \Delta_{j3}) + d(\Delta_{j3}, \Delta_{j4})]
\tag{5.11}
$$

$$
\leq O(|\partial \Lambda_{-1}|),
$$

since $\operatorname{supp} \zeta \subset \partial \Lambda_{-1}$.

Proof of Lemma 5.3. We estimate

$$
\| A_l' E_l' E_l' \cdots E_l' \|_1 \leq \| A_l' \|_1 \| E_l' \| \cdots \| E_l' \|,
\tag{5.12}
$$

$$
\| P_l' E_l' E_l' \cdots E_l' \|_1 \leq \| P_l' \|_1 \| E_l' \| \cdots \| E_l' \|.
\tag{5.13}
$$

Recall that

$$
A = \partial^{\gamma_1} K \partial^{\gamma_2} K \partial^{\gamma_3} K,
$$

where γ_1, γ_2 or both may be empty;

$$
E = (1 - K) \partial^\gamma K \quad \text{or} \quad E = -\partial^{\gamma_1} K \partial^{\gamma_2} K,
$$

and $P = P_j(s)$ or derivatives thereof. Thus

$$\|A_l'\|_1 = \left\| \sum K(s; \Delta_1, \Delta_2, \Delta_3, \Delta_4; \delta) K(s; \Delta_4, \Delta_5, \Delta_6, \Delta_7; \delta) K(s; \Delta_7, \Delta_8, \Delta_9, \Delta_{10}; \delta) \right\|_1$$

$$\cdot \prod_{\substack{\{i: \gamma_1' \cup \gamma_2' \cup \gamma_3' = \gamma_1 \\ \gamma_4' \cup \gamma_5' \cup \gamma_6' = \gamma_2 \\ \gamma_7' \cup \gamma_8' \cup \gamma_9' = \gamma_3\}}} \partial^{\gamma_i'} H(s, \Delta_i, \Delta_{i+1}) \tag{5.14}$$

$$\leq \sum \|K(s; \Delta_1, \Delta_2, \Delta_3, \Delta_4; \delta)\|_3 \|K(s; \Delta_4, \Delta_5, \Delta_6, \Delta_7; \delta)\|_3$$

$$\cdot \prod \|K(s; \Delta_7, \Delta_8, \Delta_9, \Delta_{10}; \delta)\|_3 |\partial^{\gamma_i'} H(s, \Delta_i, \Delta_{i+1})|. \tag{5.15}$$

Using Lemma 5.4 and Lemma 5.5(b) this is bounded by

$$\exp - \varepsilon[d(\Delta_1, \Delta_2) + d(\Delta_2, \Delta_3) + \cdots + d(\Delta_9, \Delta_{10})]$$
$$\cdot \prod \exp[2m_c d(\Delta_i, \Delta_{i+1}) - \delta m_c d(\gamma_i', \Delta_i, \Delta_{i+1})]$$
$$\cdot \exp((c - \delta_2 l)|\gamma_i'|) G_1(\gamma_i', \delta_1). \tag{5.16}$$

The E terms are estimated by

$$\|(1 - K)\partial^\gamma K\| \leq \|\partial^\gamma K\| + \|K\partial^\gamma K\|$$
$$\leq \|\partial^\gamma K\|_3 + \|K\partial^\gamma K\|_3. \tag{5.17}$$

Using the same techniques as above, this is bounded by

$$\exp - \varepsilon[d(\Delta_1, \Delta_2) + d(\Delta_2, \Delta_3) + \cdots + d(\Delta_7, \Delta_8)]$$

$$\cdot \prod_{\{i: \gamma_1' \cup \gamma_2' \cup \gamma_3' = \gamma\}} \exp[2m_c d(\Delta_i, \Delta_{i+1}) - \delta m_c d(\gamma_i', \Delta_i, \Delta_{i+1})]$$

$$\cdot \exp((c - \delta_2 l)|\gamma_i'|) G_1(\gamma_i', \delta_1). \tag{5.18}$$

Finally, to estimate $\partial^\gamma P_l'$ we note that (recall that the $\tilde{h}$'s and $\tilde{g}$'s are localized in blocks $\bar{\Delta}'$, and $\gamma = \gamma_1 \cup \gamma_2$)

$$\partial^\gamma P_l'(\cdot) = (\chi_{\Delta_2} D^{-1/2} \tilde{g}_j, \cdot) \chi_{\Delta_1} D^{1/2} S \tilde{h}_j \partial^{\gamma_1} H(s, \Delta_1, \bar{\Delta}) \partial^{\gamma_2} H(s, \Delta_2, \bar{\Delta}')$$
$$= (\tilde{G}_j, \cdot) \tilde{H}_j \partial^{\gamma_1} H(s, \Delta_1, \bar{\Delta}) \partial^{\gamma_2} H(s, \Delta_2, \bar{\Delta}'), \tag{5.19}$$

where

$$\tilde{H}_j \equiv \chi_{\Delta_1} D^{1/2} S \tilde{h}_j, \tag{5.20}$$

$$\tilde{G}_j \equiv \chi_{\Delta_2} D^{-1/2} \tilde{g}_j. \tag{5.21}$$

Now by straightforward arguments (e.g. Theorem 2.2 of [13])

$$\|\tilde{H}_j\|_{\mathscr{H}'} \leq \exp - \varepsilon d(\Delta_1, \bar{\Delta}) \|\tilde{h}_j\|_{\mathscr{H}}, \tag{5.22}$$

$$\|\tilde{G}_j\|_{\mathscr{H}'} \leq \exp - \varepsilon d(\Delta_2, \bar{\Delta}') \|\tilde{g}_j\|_{\mathscr{H}}; \tag{5.23}$$

while

$$\|\partial^\gamma P\|_1 \leq \|\tilde{H}_j\|_{\mathscr{H}'} \|\tilde{G}_j\|_{\mathscr{H}'} |\partial^{\gamma_1} H(s, \Delta_1, \bar{\Delta})| |\partial^{\gamma_2} H(s, \Delta_2, \bar{\Delta}')|. \tag{5.24}$$

Combining (5.22)–(5.24), we get by Lemma 5.5

$$\|\partial^\gamma P\|_1 \leq \exp(2m_c - \varepsilon)[d(\Delta_1, \bar{\Delta}) + d(\Delta_2, \bar{\Delta}')]$$
$$\cdot \exp[-\delta m_c(d(\gamma_i, \Delta_1, \bar{\Delta}) + d(\gamma_i, \Delta_2, \bar{\Delta}'))]$$
$$\cdot \|h_j\|_{\mathscr{H}} \|g_j\|_{\mathscr{H}} \prod_{i=1,2} \exp((c - \delta_2 l)|\gamma_i|) G_1(\gamma_i, \delta_1) \qquad (5.25)$$

as needed.

6. Convergence of the Expansion

We now want to complete our proof of the convergence of the cluster expansion. First we take m_c small so that, for example, $(2m_c - \varepsilon) < 0$ in (5.16), (5.25). Then we take l sufficiently large so that $(c - \delta_2 l) < 0$ in (5.16) and (5.25) with enough left over to beat the $O(1)^{|\Gamma|}$ in Proposition 4.3. Then we can combine Proposition 4.3 and 5.1 to give

$$|\partial_s^\Gamma Z(s)| \leq \exp((-\delta_2' d(\Delta_g, \Delta_h))) \exp(O(|\partial\Lambda_{-1}|) - \delta_2'|\Gamma|)$$
$$\cdot \sum_\pi \sup \prod_{\{\gamma_i^j\}}' [G_1(\gamma_i^j, \delta_1) \exp(-\delta_3 d(\gamma_i^j, \partial\Lambda_{-1})) \exp(O(l))]. \qquad (6.1)$$

Here we consider for simplicity the case of the reduced Green's function, as in Theorem 1, with only two test functions g, h supported in Δ_g, Δ_h. The decay between Δ_g, Δ_h can be extracted from the factors in Proposition 5.1, assuming we allow for an increase in $O(|\partial\Lambda_{-1}|)$ to compensate for a possible lack of connections going all the way from Δ_g to Δ_h.

Let us insert this bound in the starting expansion (3.16), so that we may complete the proof of Proposition 3.2. We obtain

$$|Z(\mathbf{1})| \leq \exp(-\delta' d(\Delta_g, \Delta_h)) \exp(O(|\partial\Lambda_{-1}|)) \sum_\Gamma \exp(-\delta'|\Gamma|) \sum_\pi \sup \prod_{\{\gamma_i^j\}}'$$
$$\cdot \left[\sum_\sigma \exp(-\delta'|l_\sigma(\gamma_i^j)|) \exp(-\delta' d(\gamma_i^j, \partial\Lambda_{-1})) \exp(O(l)) \right], \qquad (6.2)$$

where we have written G_1 out using (5.4). Now rather than sum over π we can just as well sum over all partitions of Γ into $\{\gamma_i^j\}$ – the original partition π can be reconstructed from the structure of differentiations in $\{\gamma_i^j\}$. Furthermore, we sum freely over the γ_i^j's, reconstructing Γ as their union. To sum over a single γ_i^j, we use the estimate

$$\sum_{\gamma_i^j} \sum_\sigma \exp(-\delta'|l_\sigma(\gamma_i^j)|) \exp(-\delta' d(\gamma_i^j, \partial\Lambda_{-1})) \exp(O(l)) \leq O(|\partial\Lambda_{-1}|). \qquad (6.3)$$

The first bond yields the factor $O(|\partial\Lambda_{-1}|)$, since it is localized to $\partial\Lambda_{-1}$ only. Subsequent bonds are localized by $|l_\sigma(\gamma_i^j)|$. The factor $\exp(O(l))$ is beaten by the decay in $d(\gamma_i^j, \partial\Lambda_{-1})$ – for this we choose $\mathscr{B}(\alpha)$ so that $d(\mathscr{B}(\alpha), \partial\Lambda_{-1}) > O(l)$. This may not be possible at Δ_g, Δ_h, however, so there will be two $\exp(O(l))$ factors left over.

Now in summing freely over N γ_i^j's, we count each partition of Γ $N!$ times. Hence we have an estimate

$$|Z(\mathbf{1})| \leq \exp(O(l)) \exp(-\delta' d(\Delta_g, \Delta_h)) \exp(O(|\partial \Lambda_{-1}|)) \sum_N \frac{1}{N!} O(|\partial \Lambda_{-1}|)^N. \quad (6.4)$$

This completes the proof of Proposition 3.2. It is worth noting that all differentiations were ultimately linked to $\partial \Lambda_{-1}$, there being no interaction away from $\partial \Lambda_{-1}$ after the chiral transformation. Hence it was not necessary to exploit factorization of $Z(s)$ or to compute a ratio of expansions with and without observables.

To complete a proof of Theorem 1, we have only to multiply back the ratio of regularized determinants that we split off in (3.2). This yields the additional factor $\exp(2|\Lambda_{-1}|/\pi)$.

Appendix: General Masses

When $|m(x)|$ takes on values other than 1, we can prove a result analogous to Theorem 1. We suppose $m(x)$ takes on finitely many values; for definiteness suppose $m(x)$ is real and constant on cubes of some size. The chiral case $m(x) = \mu(x) \exp(i\gamma_5 \theta(x))$ can be treated similarly. By rescaling we can assume the largest $|m(x)|$ is 1. Corresponding to each value of $m(x)$ there is a region in $\mathbb{R}^2$, where $m(x)$ takes that value. Each of these regions now has a different vacuum energy (but explicitly calculable, as in [7], p. 333), and the bound in Theorem 1 has to be adjusted by the appropriate factors exponential in the volumes. For simplicity we assume $m(x) = 1$ at infinity, but the more general case can be handled as well by taking a limit on volumes, normalizing so as to cancel the vacuum energy of the phase at infinity. Constants and decay rates in Theorem 1 will of course not be uniform as the minimum $|m(x)|$ tends towards zero.

The chiral transformation can be performed as in the constant $|m(x)|$ case. We obtain a Fredholm minor as in (3.4), only now the function ζ, as defined by (2.9), no longer vanishes away from $\text{suppt}\, \partial\alpha$. To remedy this, we perform a Neumann series to bring the mass into the propagator S. Putting $\delta(x) = \gamma_0(1 - |m(x)|)$, we can define

$$S' = (S^{-1} - \delta)^{-1} = S + S\delta S + S\delta S\delta S + \cdots. \quad (A.1)$$

This is convergent since $\|S\| \leq 1$ and $|1 - |m(x)|| \leq \delta_{\max} < 1$. Then with $\zeta' = \zeta + \delta$ we put $A = S'\zeta'$, $B = -S\delta$, $C = A + B + AB = S\zeta$ and by Lemma 2.3,

$$\det_3[1 + S\zeta] = \det_3[1 + S'\zeta'] \det_3[1 - S\delta] \exp(\text{Tr}\, T), \quad (A.2)$$

where

$$T = (AB)^2/2 + AB^2 + BA^2. \quad (A.3)$$

As in [7], Sect. 5.5, we can analyze the series for $\log \det_3[1 - S\delta] + \text{Tr}\, T$ and break it into a sum over the values of $m(x)$ of vacuum energies × volumes, with corrections associated with the length of boundary.

It remains to analyze $\det_3[1 + S'\zeta']$ and the corresponding minors with a cluster expansion. The new perturbation ζ' vanishes except near boundary regions. The

only essential new feature is the interpolation of S'. By analogy with (3.12) we put

$$K(s) = (D^{1/2}SD^{1/2})_s(D^{-1/2})_s\left(\sum_{n=0}^{\infty}(\delta S)^n\right)_s \zeta'(D^{-1/2})_s. \qquad (A.4)$$

Note that in contrast with [7], we decouple after the mass shift. While this is inconvenient for some purposes it does enable us to get the necessary propagator estimates. Specifically, the new estimates required are on $\|\chi_\Delta(\delta S)^n\chi_{\Delta'}\|$. Using the Combes–Thomas method [2], we conjugate δS by $\exp(\eta x)$ ($x = x_0$ or x_1). For small enough η the resulting operator still has norm less than 1. (Near the diagonal the conjugation is multiplication by $1 + O(\eta)$, while far from the diagonal the decay of S takes over.) Hence the series converges and a decay $\exp(-O(\eta)d(\Delta, \Delta'))$ can be extracted.

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