Breakup of degeneracies in disordered quantum systems

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Overview

For disordered quantum systems such as the Anderson model, degeneracies provide avenues for long-range tunneling, and hence are a barrier to localization. In order to control the likelihood of degeneracies or near-degeneracies, one needs to understand in detail the way eigenvalues and eigenvalue gaps depend on the disorder. Using multiscale analysis, one can build up smoothness of eigenvalue distributions even in the case of discrete disorder distributions.

Outline

Problems with degeneracies show up in the MBL Proof

Consider the following spin chain with random interactions and a weak transverse field on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^{K} h_i S_i^z + \sum_{i=-K}^{K} \gamma_i S_i^x + \sum_{i=-K-1}^{K} J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable.

Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

Need control of minimum level spacing in order to demonstrate failure of thermalization:

Assumption **LLA**(ν, C). *Consider the Hamiltonian H in boxes of size n. Its eigenvalues satisfy*

$$P\left(\min_{\alpha\neq\beta}|E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n,$$

for all $\delta > 0$ and all n.

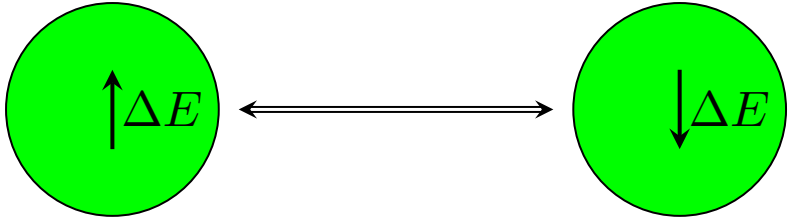
Theorem

Let ν, C be fixed. There exists a $\kappa > 0$ such that for γ sufficiently small, **LLA(ν,C)** *implies the following estimate:*

$$\mathbb{E}\,\mathrm{Av}_\alpha\,|\langle S_0^z\rangle_\alpha| = 1 - O(\gamma^\kappa), \tag{1}$$

where $\mathbb{E}$ denotes the disorder average, Av_α denotes an average over α, and $\langle\cdot\rangle_\alpha$ denotes the expectation in the eigenstate α.

Controlling degeneracies in subsystems



One needs to show that subsystems with near-degeneracies are rare. Perturbative methods break down in such regions.

To obtain that, one needs to know that energy differences within a block vary with the randomness.

This is a difficult open problem. But with Assumption **LLA**(ν, C), we have a minimum level spacing, with high probability. Then since all three terms of H are random, one can show that energy differences do vary with the randomness.

A challenge for minimum level spacing and localization: the Anderson model with discrete disorder

Consider the Anderson model in $\mathbb{Z}^d$:

$$H = -\gamma\Delta + v \text{ with } \gamma \ll 1.$$

The potentials $\{v_x\}_{x\in\mathbb{Z}^d}$ are iid random variables, each uniformly distributed on $\{0, \frac{1}{N-1}, \frac{2}{N-1}, \ldots, 1\}$, with $N \gg 1$.

For $N = 2$, this is the Anderson-Bernoulli model, the "Ising model" of random Schrödinger operators.

We are interested in showing exponential decay of eigenfunctions (localization). We would also like to control the probability of degeneracies or near-degeneracies in the spectrum of H.

Related work: Bourgain, Kenig, Invent. Math. 2005: Localization for the continuum version of Anderson-Bernoulli ($N = 2$, $\mathbb{Z}^d \to \mathbb{R}^d$) for energies near the bottom of the spectrum, $E = 0$.

The problem with discrete disorder

Consider a single site. For a continuous distribution for v, shrinking the energy interval shrinks the probability of spectrum ($= v$) in the interval.

Not so for a discrete distribution.

This forces one to use potentials at other sites to move spectrum away from E. Eigenvalues should exhibit at least a weak dependence on v_j, $j \neq i$.

On $\mathbb{R}^d$, unique continuation estimates $\implies$ an eigenfunction ψ cannot vanish in a small ball where the potential is moving. Consequently, the eigenvalue moves with the disorder.

Some recent results in dimensions 2 and 3 based on "unique continuation" in the lattice

Dimension 2:

Liouville theorem in $\mathbb{Z}^2$: A function harmonic and bounded on a $1-\epsilon$ fraction of sites must be constant. Buhovsky, Logunov, Malinnikova, Sodin, arXiv:1712.07902.

Localization near the edge for Anderson-Bernoulli in $\mathbb{Z}^2$: Ding, Smart, Invent. Math. 2019. (Unique continuation on $\mathbb{Z}^2$: an eigenfunction is supported on set of size $\sim L^{3/2}$ in any box of size L^2.)

Dimension 3:

Extension of Ding-Smart method to $\mathbb{Z}^3$: Li, Zhang, arXiv:1906.04350.

This talk:

We work in any dimension by making the most of a much thinner set of sites where an eigenfunction has a lower bound. We obtain also a minimum level-spacing result, but have not yet pushed the method to $N = 2$.

Large N result: Localization and minimum level spacing in any dimension

Let $I_\delta(E)$ denote the interval $[E-\delta, E+\delta]$, and let $\mathcal{N}(I)$ denote the number of eigenvalues of H in I.

Theorem [arXiv:1705.01916]. *Choose a sufficiently large p. Then for N sufficiently large (depending on p) and γ sufficiently small (depending on N),*

$$\mathbb{E}\,\mathcal{N}(I_\delta(E)) \leq |\Lambda|(\log_\gamma \delta)^{-p},$$

and

$$P\Big(\min_{\alpha\neq\beta} |E_\alpha - E_\beta| < \delta\Big) \leq |\Lambda|^2(\log_\gamma \delta)^{-p}.$$

for any rectangle Λ and any $\delta \in [\gamma^{\mathrm{diam}(\Lambda)/2}, 1]$.

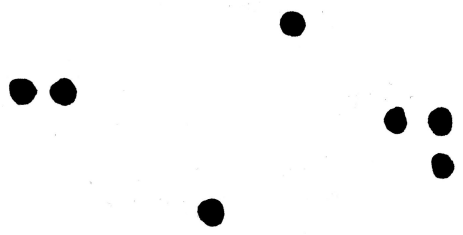
Also, $\sum_\alpha |\psi_\alpha(x)\psi_\alpha(y)|$ decays exponentially, and its disorder average is bounded by $C|x-y|^{-p}$.

Resonant set and block decomposition of H

Given an energy E, we expect eigenfunctions whose eigenvalues are close to E will be localized near sites where $v_x - E$ is small. Eigenfunctions should decay rapidly away from such sites.

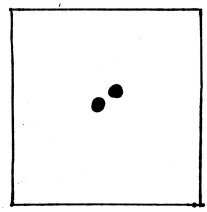
Resonant Set: Let $\varepsilon = \frac{1}{3N}$ and $\gamma \leq \varepsilon^{20}$. The resonant set for the first step is $R^{(1)} = \{i \in \Lambda : |v_i - E| \leq \varepsilon\}$.

Connected components of the resonant set: Decompose the resonant set into connected components or supersites using proximity conditions. The components are well separated:



Focus on a single component X

Work in a neighborhood of X. Call it $\bar{X}$. It should be a large neighborhood but still well-separated from other components. Let $H_{\bar{X}}$ be the Hamiltonian restricted to $\bar{X}$.



Block decomposition: $H_{\bar{X}} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$

where A, D are $H_{\bar{X}}$ restricted to X, $\bar{X} \setminus X$, respectively, and B, C contain the nearest-neighbor terms from the Laplacian that connect the two regions.

Schur complement: $F_\lambda \equiv A - B(D - \lambda)^{-1}C$.

Here we take λ near E, i.e. $|\lambda - E| \leq \varepsilon/2$.

Properties

Note that $(D-\lambda)^{-1}$ is the resolvent in $\bar{X} \setminus X$.

As D operates on the subspace associated with nonresonant sites, where $|v_i - E| \leq \varepsilon$, one has that $\|(D-\lambda)^{-1}\| \leq 3/\varepsilon$ for $|\lambda - E| \leq \varepsilon/2$.

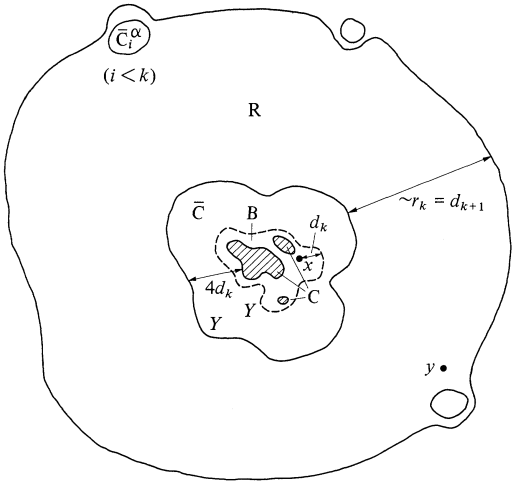
Combes-Thomas: $(D-\lambda)^{-1}(x,y)$ decays exponentially in $|x-y|$.

Schur complement properties: If ψ is an eigenvector of $H_{\bar{X}}$ with eigenvalue $\lambda \in I_{\varepsilon/2}(E)$ then $\psi = \begin{pmatrix} \varphi \\ -(D-\lambda)^{-1}\varphi \end{pmatrix}$ with φ an eigenvector of $F_\lambda = A - B(D-\lambda)^{-1}C$.

Decay of $(D-\lambda)^{-1}(x,y) \implies \psi(x)$ decays rapidly with $\operatorname{dist}(x,X)$.

Find eigenvalues of $H_{\bar{X}}$ in $I_{\varepsilon/2}(E)$ by solving $\lambda \in \operatorname{spec} F_\lambda$.

Fröhlich-Spencer-style analysis of resonant regions



From Fröhlich-Spencer 1983

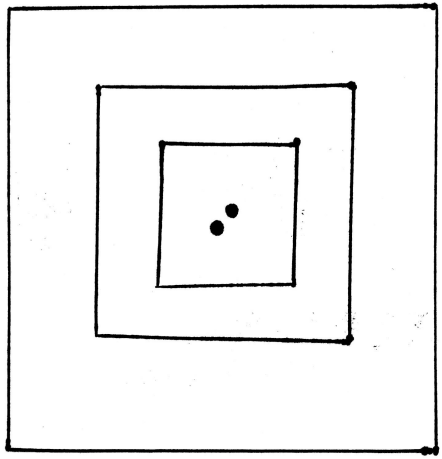
Initially, A, the upper-left block of H, is defined by restricting H to the sites with $|v_x + 2d\gamma - E| \leq \varepsilon_1 \equiv \frac{1}{3(N-1)}$.

In later steps, components of the resonant region are well separated, and H can be well approximated by a block diagonal matrix, each block based on a resonant component.

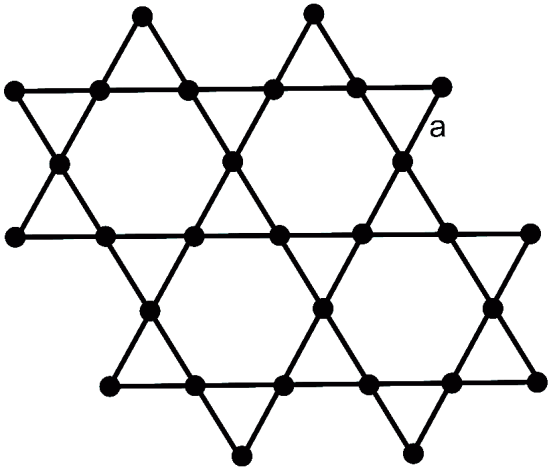
Obtain eigenvalues of H by sequential approximation

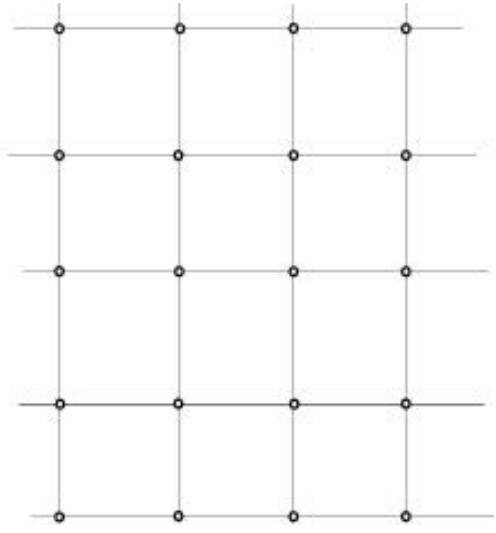
The error in the approximation drops exponentially with the size of the box.

Reason: the change in F_λ when going from $\bar{X}_{k-1}$ to $\bar{X}_k$ produces a connection from X to $\bar{X}_k \setminus \bar{X}_{k-1}$ (geometric resolvent identity.)



Eigenfunctions of $-\gamma\Delta + v$ cannot grow faster than exponentially on $\mathbb{Z}^d$





Kagome lattice: Compactly supported e-fns

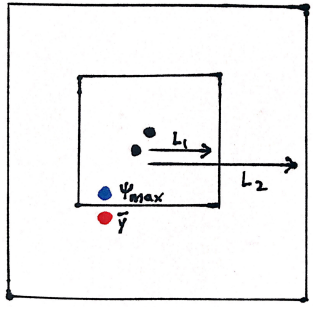
$\mathbb{Z}^d$: normalized e-fn $\geq \gamma^r$ somewhere on ∂B_r

Find the site in $\bar{X}_2 \setminus \bar{X}_1$ with the greatest influence on F_λ

As we go from one box to the next, we need to show that some site $\bar{y}$ is influential, i.e. $v_{\bar{y}}$ moves eigenvalues.

Fix the disorder in $\bar{X}_1$. We have an eigenfunction ψ of $H_{\bar{X}_1}$ with eigenvalue near E.

If we find a point where $|\psi|$ is maximal over the boundary of $\bar{X}_1$, then the neighbor of that site in $\bar{X}_2 \setminus \bar{X}_1$ will be the site of greatest influence. Call that site $\bar{y}$.



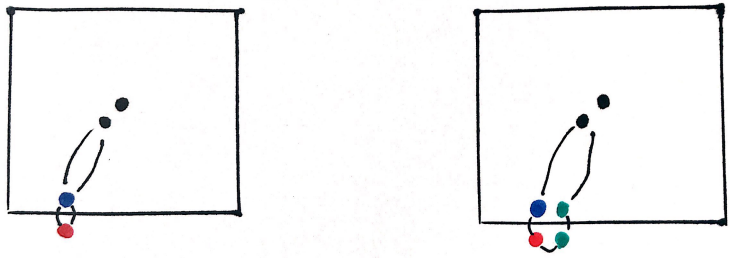
IMPORTANT: The choice of $\bar{y}$ is based only on H_{X_1}, so it does not depend on the potentials in $\bar{X}_2 \setminus \bar{X}_1$.

Upper bound on higher order graphs must fit inside lower bound on leading term

Consider the $v_{\bar{y}}$-dependent part of the change in F_λ as $\bar{X}_1$ is enlarged to $\bar{X}_2$.

Leading term (left) is $\psi_{\max}^2 \gamma^2$. Lower bound on $|\psi_{\max}|$ means $v_{\bar{y}}$ definitely moves eigenvalue.

Higher order term (right) is $\psi_{\max} \tilde{\psi} \gamma^3$, and since $|\tilde{\psi}| \leq |\psi_{\max}|$, it is smaller than the leading term.

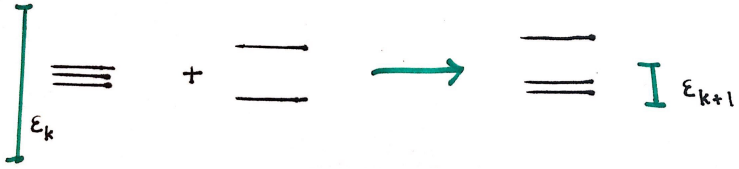


Breakup of degeneracies: the essence of the level-spacing estimate

In the general step k, we have $L_k = 2^k$ and the energy window is $\varepsilon_k = \gamma^{1.6L_k} = \varepsilon_{k-1}^2$.

We can use such a tiny window because corrections to F_λ when increasing $\bar{X}_{k-1} \to \bar{X}_k$ involve graphs going from X to $\bar{X}_k \setminus \bar{X}_{k-1}$ and back, at least $2L_{k-1}$ long.

Here we see graphically how a near-degeneracy in step k is broken down in response to the leading-order perturbation (from the expansion $\bar{X}_k \to \bar{X}_{k+1}$). This perturbation is rank 1. For at least $N-1$ choices of $v_{\bar{y}}$, the perturbation has size $\sim \gamma^2 |\psi_{\max}|^2 \gg \varepsilon_{k+1}$.



Weyl's inequality $\implies$ at least one eigenvalue is ejected from the new interval of size ε_{k+1} (with probability at least $1 - 1/N$).

IMPORTANT: Spectrum from outside the big interval cannot move into the small interval because the latter is centered at a point well inside the big interval.

Breakup of degeneracies: the essence of the level-spacing estimate, cont.

This is the essence of DOS estimates: randomness pushes spectrum out of an interval as it shrinks, so that $\mathbb{E}\mathcal{N}(I_\delta(E))$ decreases at $\delta \to 0$.

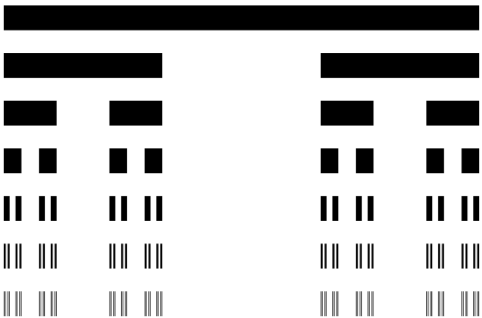
Also, we obtain level-spacing bounds because Borel-Cantelli $\implies$ the degeneracy keeps decreasing until the eigenvalue is isolated for all $k > k_0$.

Quantitatively we find that that the probability of finding spectrum in $I_{\varepsilon_k}(E)$ decreases as $N^{-k} = 2^{-k \log_2 N} = L_k^{-\log_2 N}$. Hence for N large enough, this will be summable. (This forms the basis for arguments as in Fröhlich-Spencer '83 establishing non-percolation of resonant sets.)

When looking at level-spacing estimates, we allow E to follow successive approximations to eigenvalues. Then a similar decay of probabilities with L_k occurs for the probability of finding a second eigenvalue in the interval.

Result: Cantor-like splitting of the eigenvalue distribution

But we can use only one potential per annulus of size $L_k = 2^k$, because we need upper bounds in step $k+1$ to fit inside the lower bounds from step k.



This leads to log-Hölder continuity of the density of states:

$$\mathbb{E}\,\mathcal{N}(I_\delta(E)) \leq |\Lambda|(\log_\gamma \delta)^{-p}.$$

Energy-following procedure: intelligent search

To control the minimum level-spacing, we need to check that each eigenvalue is suitably isolated. Blind search is impractical because probabilities are not small enough to sum over all possible intervals.

Intelligent search: Define a sequence of energies $E_1, E_2, \ldots$.

$E_1 = v_i$ for some $i \in \Lambda$.

The site i joins up with other nearby sites j such that $|v_j - E_1| \leq \varepsilon_1$.

The result is the block B_1 containing i.

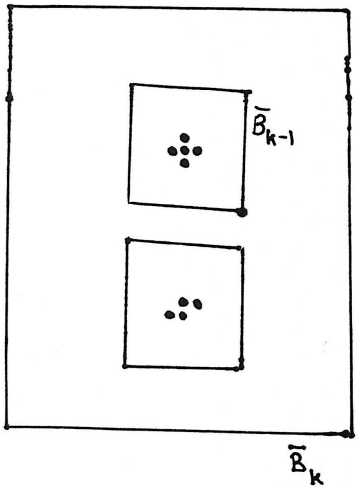
Perform the Schur complement and localize to the neighborhood $\bar{B}_1$.

Find the eigenvalues of $H_{\bar{B}_1}$ in the window $I_{\varepsilon_1/2}(E_1)$ by solving $\lambda \in \operatorname{spec} \tilde{F}^{(1)}_{\lambda}(B_1)$.

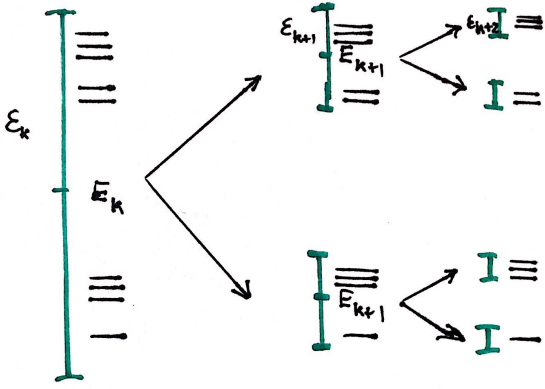
These are the choices for E_2 (group choices if within ε_2).

And so on ....

Energy-following procedure



Blocks may combine as proximity conditions are extended



Sum over choices for E_{k+1} so that ε_{k+1}-neighborhoods cover spectrum

Controlling the energy-following procedure

One can show that every eigenvalue of H can be obtained through a sequence of choices $E_1, E_2, \ldots$ arising in this procedure.

The tree of choices for $\{E_k\}$ and the possbilities for the expanding sequence of blocks $\{B_k\}$ are controlled inside the expectation $\mathbb{E}$ over the potentials.

Control comes from the small probability of remaining resonant to E_k when expanding the neighborhood $\bar{B}_k \to \bar{B}_{k+1}$.

The Anderson-Bernoulli model (work in progress with Svitlana Mayboroda)

$H = -\Delta + v$ with $v \in \{0, 1\}$ on $\mathbb{Z}^d$.

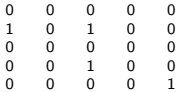
A plethora of degeneracies have the potential for spoiling localization.

To combat the problem, take $E > 0$ small (Lifshitz tail regime).

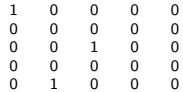
To get spectrum near E one needs a region of size $\ell_0 \sim E^{-1/2}$ with mostly 0's.

Large deviation arguments $\implies$ probability $\sim \exp(-E^{d/2})$

$\implies$ separation of resonant spots as in the large N case.



$\leftarrow \quad \exp(E^{d/2}) \quad \rightarrow$



Issues with $N = 2$

1. Recall that in the large N case, we used the potential at one site per annulus (because of the "Cantor" condition: we need the upper bound for next-generation effects $\gamma^{.85L_{k+1}} = \gamma^{1.7L_k}$ to fit inside the lower bound on the current effect γ^{L_k}.)

   We reach length scale L_k after k eigenvalue splittings (using the potential at one site per annulus), and then each atom has probability $\sim N^{-k} = 2^{-kp} = L_k^{-p}$, where $p = \log_2 N$.

   To prevent percolation of resonances we need $p > d$.
   But L_k^{-p} is no longer integrable when $N = 2$.

   So we need L_k to grow linearly, rather than exponentially with k.

   To retain the Cantor condition, use $\psi^2_{\max}$ as "gateway" so it works as a bound for next-generation effects.

2. Decay length is many lattice steps $\implies$ it is difficult to pin down the location of a site where the eigenfunction is guaranteed to be big (the location should not depend on the randomness in the "new" annulus).

   Pigeonhole principle: one choice will work with a positive probability.