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# A New Approach to the Long-Time Behavior of Self-Avoiding Random Walks*

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We study a $T$ step weakly self-avoiding random walk in $d>4$ dimensions by treating it as a one-dimensional statistical mechanical system. We use the polymer expansion along with the lace expansion to arrive at an expression for the Fourier-transformed propagator of the self-avoiding walk, valid in some neighborhood of the origin, that up to small edge effects is equal to that of some simple random walk. This yields simple proofs that, as $T \to \infty$, the variance of the endpoint is asymptotically linear in $T$ and the scaling limit of the endpoint is gaussian. 

## 1. INTRODUCTION AND RESULTS

We study a model for weakly self-avoiding random walks on a $d$-dimensional hypercubic lattice. The model is defined by the two-point function

$$C(x, T) \equiv \left(\frac{1}{2d}\right)^T \sum_{\substack{\omega: 0 \to x \\ |\omega| = T}} \prod_{0 \leqslant s < t \leqslant T} (1 - \lambda \delta_{\omega_s \omega_t}) \tag{1.1}$$

which is proportional to the probability of travelling from 0 to $x$ in $T$ steps. Here $d$ is the dimension of the lattice, $\omega$: $[0, T] \to \mathbf{Z}^d$ any path on the lattice beginning at the origin with nearest neighbor steps, and $\lambda$ the strength of the self-interaction. We also define the Fourier-transformed propagator by

$$C(k, T) \equiv \sum_x e^{ik \cdot x} C(x, T) \tag{1.2}$$

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with $k \in [-\pi, \pi]^d$ and, here and throughout, the arguments will serve to distinguish functions from their Fourier transforms.

The above model, when $\lambda = 0$, reduces to the simple random walk with nearest neighbor steps; i.e., the probability distribution $p(x, y)$ governing a single step is proportional to the nearest neighbor coupling matrix $J_{xy} \equiv \mathbf{1}_{|x-y|=1}$. In general, the distribution of the endpoint of a simple random walk after $T$ steps (which is a sum of displacements of single steps) is equal to the convolution $p^{*T}(0, x)$, which in Fourier space is just $p(k)^T$. So in the nearest neighbor case, we have $C_{\lambda=0}(k, T) = D(k)^T$, where

$$D(k) \equiv \frac{1}{d} \sum_{\mu=1}^{d} \cos(k_\mu). \tag{1.3}$$

We will exhibit an exact expression for $C(k, T)$ in some open neighborhood of the origin. Specifically, for any $T'$, we will define a function $\Gamma_{T'}(k)$ such that for $T \leqslant T'$,

$$C(k, T) = (D(k) e^{\Gamma_{T'}(k)})^T \cdot (\text{edge effects}). \tag{1.4}$$

We will show that for $k$ in some neighborhood of the origin (which depends on $T'$) $\Gamma_{T'}(k)$ and its first two derivatives are $O(\lambda)$ when $d \geqslant 5$ and that the edge effects are negligible (these statements will be made more precise below). This is a particularly revealing form since, for $k$ small, it mimics the form of the propagator for some simple random walk (up to edge effects) and hence makes explicit the fact that self-avoiding random walks behave like simple random walks above four dimensions. It also allows us to easily compute quantities of interest; for example, we find the mean square end-to-end distance:

$$\langle \omega(T)^2 \rangle = \sum_{\mu=1}^{d} \left. \frac{\partial_{k_\mu}^2 C(k, T)}{C(k, T)} \right|_{k=0} \tag{1.5}$$

$$\simeq T(1 + O(\lambda)); \tag{1.6}$$

i.e., the effect of self-avoidance is to add a small correction to the diffusion constant of the free walk.

This model has previously been studied by Brydges and Spencer in [1], who introduced the lace expansion and used it to prove diffusive behavior of the endpoint of a $T$-step walk and the Gaussian nature of the scaling limit. They studied the Laplace-transformed two-point function $C(k, z)$ (the generating function of $C(k, T)$), and used Cauchy's theorem to extract fixed-$T$ information. Their results were extended to, and other results proven about, the case of the standard self-avoiding walk above some high dimension $d_0$ in [9–11, 7], and in five or more dimensions in [6]. These works also used the lace expansion and generating functions, but in different ways. Related works [3–5, 8] used similar methods to prove results on percolation and lattice trees and animals. In this paper we also use the lace expansion, but with a view towards other models (e.g., branched polymers) we dispense with generating functions and work directly with the propagator in $(k, T)$-space using techniques drawn from statistical mechanics.

The principal technique we use is the polymer expansion. To see how such an expansion can arise, we rewrite the two-point function in a more suggestive form. We expand the product $\prod (1-\lambda\delta_{\omega_s\omega_t})$ into a sum of graphs on the interval $[0, T]$. These graphs split into connected components, where a graph that spans an interval $I$ is considered connected if every point in the interior of $I$ is spanned by some bond in the graph. For an interval $I$ we define the activity

$$\Pi(k, I)=\left(\frac{1}{2d}\right)^{|I|} \sum_x e^{ik\cdot x} \sum_{\substack{\omega: 0\to x\\ |\omega|=|I|}} \sum_{G^c} \prod_{(s,t)\in G^c} (-\lambda\delta_{\omega_s\omega_t}), \tag{1.7}$$

where $G^c$ are connected graphs consisting of bonds $(s, t)$ on $[0, T]$. Then it is easy to see that

$$\frac{C(k, T)}{D(k)^T}=\sum_{n=0} \sum_{\{I_i\}_{i=1}^n} \prod_{i=1}^n \frac{\Pi(k, I_i)}{D(k)^{|I_i|}}, \tag{1.8}$$

where the $\{I_i\}$ are nonoverlapping intervals in $[0, T]$. This is a form amenable to polymer expansion, but we cannot prove its convergence in this form. Instead, following [1] and using the idea of the renormalization group, we first consider walks that self-avoid only on short lengthscales and gradually work our way up to fully self-avoiding walks. Roughly, we define a sequence of times $T_l$ indexed by a number $l$, and write as $C_l(k, T)$ the propagator for walks that self-avoid only on times shorter than $T_l$. Then, defining $l(T)$ by $T_{l(T)}=T$, we write

$$\frac{C_{l(T)}}{C_0}=\frac{C_{l(T)}}{C_{l(T)-1}}\cdot\frac{C_{l(T)-1}}{C_{l(T)-2}}\cdots \tag{1.9}$$

and find an expansion for $C_l/C_{l-1}$ similar to (1.8). For an appropriate choice of $T_l$ we can prove convergence of such an expansion.

The structure of the rest of the paper is as follows. In Section 2 we will exhibit the expansion for $C_l(k, T)$, the convergence of which will be proven via an induction on timescales in Sections 3–6. In Section 7 we use the expansion to prove the main theorems, which we now state.

THEOREM 1.1. *For any $T'\geqslant 2$ there exist real-valued functions $\Gamma_{T'}(k)$, $\Gamma_{T'}^{\text{edge}}(k, T)$ and $T'$-independent constants $\lambda_0>0, f>0, \varepsilon'>0$, and $\mathscr{C}>0$ such that for any $T\leqslant T'$, with $0<\lambda<\lambda_0$ and $k$ such that $|D(k)|\geqslant T'^{-f/T'}$*

$$C(k, T)=(D(k)e^{\Gamma_{T'}(k)})^T\cdot e^{\Gamma_{T'}^{\text{edge}}(k, T)} \tag{1.10}$$

*and*

$$|\partial_k^u \Gamma_{T'}(k)| \leqslant \mathscr{C}\lambda \tag{1.11}$$

$$|\partial_k^u \Gamma_{T'}^{\text{edge}}(k, T)| \leqslant \mathscr{C}\lambda T'^{\max(|u|/2+2+\varepsilon'-d/2,0)}, \tag{1.12}$$

*where $u$ is a multi-index for $k$-derivatives, and $|u| \leqslant 2$.*

There is also convergence of $\Gamma_{T'}(k)$ in an appropriate sense near $k=0$ as $T' \to \infty$. This is used in the following theorem.

THEOREM 1.2.

$$\langle \omega(T)^2 \rangle = DT(1 + \lambda O(T^{-1/4})), \tag{1.13}$$

*where $D = 1 + \mathscr{C}\lambda$, with $\mathscr{C}$ independent of $T$. Also, the scaling limit of the endpoint is gaussian in the sense that*

$$\lim_{s \to \infty} \frac{C(k/\sqrt{s}, st)}{C(0, st)} = e^{-Dt(k^2/2d)} \tag{1.14}$$

*for any real $t$ and $k \in \mathbf{R}^d$, with the same $D$ as before.*

## 2. THE EXPANSION

We first define the propagator for the model that self-avoids only on scales smaller than $T_l$:

$$C_l(x, T) \equiv \left(\frac{1}{2d}\right)^T \sum_{\substack{\omega: 0 \to x \\ |\omega| = T}} \sum_{\substack{0 \leqslant s < t \leqslant T \\ |s-t| \leqslant T_l}} (1 - \lambda \delta_{\omega_s \omega_t}). \tag{2.1}$$

This gives rise to an activity $\Pi_l(k, I)$, just as in (1.7), except now the bonds $(s, t)$ in the connected graphs are restricted in length, $|s-t| \leqslant T_l$. We then define

$$\delta\Pi_l(k, T) \equiv \Pi_l(k, T) - \Pi_{l-1}(k, T) \tag{2.2}$$

and it is not hard to see that

$$\frac{C_l(k, T)}{C_{l-1}(k, T)} = \sum_{n=0} \sum_{\{I_i\}} \prod_{i=1}^{n} \left(\frac{\delta\Pi_l(k, I_i)}{C_{l-1}(k, I_i)}\right) \times \left(\frac{C_{l-1}(k, K_0) \prod_{j=1}^{n} [C_{l-1}(k, I_j)\, C_{l-1}(k, K_j)]}{C_{l-1}(k, T)}\right). \tag{2.3}$$

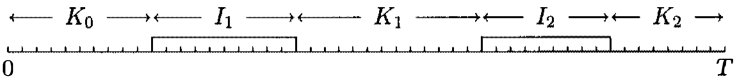


FIG. 1. Contribution to $C_l/C_{l-1}$.

Here $\{I_i\}$ are again nonoverlapping intervals on $[0, T]$, and $\{K_j\}$ are the complementary intervals; e.g., see Fig. 1. The last term here acts as an interaction between the $I_i$, so we must do a further expansion to obtain a form amenable to polymer expansion.

First, we make an inductive hypothesis about the result of the expansion:

$$\frac{C_{l-1}(k, T)}{C_{l-2}(k, T)} \equiv \exp(T\,\delta\Gamma_{l-1}(k) + 2\delta\Gamma^{\partial}_{l-1}(k) + \delta\Gamma^{\mathrm{IF}}_{l-1}(k, T)). \tag{2.4}$$

Note that this implies

$$C_{l-1}(k, T) = D(k)^T \exp(T\Gamma_{l-1}(k) + 2\Gamma^{\partial}_{l-1}(k) + \Gamma^{\mathrm{IF}}_{l-1}(k, T)), \tag{2.5}$$

where

$$\Gamma^{(\cdot)}_{l-1}(\cdot) \equiv \sum_{j=1}^{l-1} \delta\Gamma^{(\cdot)}_j(\cdot). \tag{2.6}$$

In the above, $\exp(\Gamma_l)$ is the multiplicative correction to $D(k)$, and $\Gamma^{\partial}_l$ and $\Gamma^{\mathrm{IF}}_l$ are edge effects arising from interactions with just one and both edges of $[0, T]$, respectively. This form holds trivially for $T_l < 2$, since this case reduces to simple random walk (the shortest subwalk that can intersect itself is two units long). In the remainder of this section, we will define $\delta\Gamma_l$, $\delta\Gamma^{\partial}_l$, and $\delta\Gamma^{\mathrm{IF}}_l$ so that $C_l/C_{l-1}$ is in the form of (2.4). We will then be able to prove estimates on these quantities in the following sections.

We begin by substituting (2.5) into (2.3), to find

$$\frac{C_l(k, T)}{C_{l-1}(k, T)} = e^{-\Gamma^{\mathrm{IF}}_{l-1}(k,T)} \sum_{n=0} \sum_{\{I_i\}} \prod_{i=1}^{n} (\delta\Pi^*_l(k, I_i)) \times \prod_{j=0}^{n} (1 + \delta\Pi^{**}_l(k, K_j)), \tag{2.7}$$

where we have defined

$$\delta\Pi^*_l(k, I) \equiv \frac{\delta\Pi_l(k, I)\, e^{2\Gamma^{\partial}_{l-1}(k)}}{D(k)^{|I|}\, e^{|I|\,\Gamma_{l-1}(k)}} \tag{2.8}$$

$$\delta\Pi^{**}_l(k, K) \equiv (e^{\Gamma^{\mathrm{IF}}_{l-1}(k,K)} - 1) \tag{2.9}$$

and $\{I_i\}$ are nonoverlapping intervals with integer endpoints in $[0, T]$, while $\{K_j\}$ are the intervals between the $I_i$'s, as in (2.3).

Now, we see that the interactions between the intervals $I_i$ are the terms $1+\delta\Pi_l^{**}$, so we must expand the product over $j$. The resulting sum of graphs splits into connected components of two types: those connected to the edges by a factor of $\delta\Pi_l^{**}$ and those that are not. We define activities for both types of connected components. The activity of an interval not connected to an edge is given by

$$\delta\tilde{\Pi}_l(k, J) \equiv \sum_{n=0} \sum_{\{I_i\}} \prod_{i=1}^{n} (\delta\Pi_l^{*}(k, I_i)\, \delta\Pi_l^{**}(k, K_i))\, \delta\Pi_l^{*}(k, I_n), \tag{2.10}$$

where the $\{I_i\}$ are nonoverlapping intervals on $J$ such that the left and right endpoints of $I_1$ and $I_n$, respectively, coincide with those of $J$. The activity of an interval connected to an edge is

$$\delta\tilde{\Pi}_l'(k, J) \equiv \sum_{n=1} \sum_{\{I_i\}} \prod_{i=1}^{n} (\delta\Pi_l^{**}(k, K_{i-1})\, \delta\Pi_l^{*}(k, I_i)), \tag{2.11}$$

where the $\{I_i\}$ are nonoverlapping intervals on $J$ such that the right endpoint of $I_n$ coincides with that of $J$. The expression for $C_l/C_{l-1}$ now splits into three parts:

$$\begin{aligned} \frac{C_l(k, T)}{C_{l-1}(k, T)} &= \{\text{no interaction with edges}\} \\ &\quad + \{\text{interaction with one edge only}\} \\ &\quad + \{\text{interaction with both edges}\}. \end{aligned} \tag{2.12}$$

In a moment, we will apply the polymer expansion to the piece that does not interact with the edges; for now, define

$$P(k, I) \equiv \sum_{n=0} \sum_{\{I_i\}} \prod_{i=1}^{n} (\delta\tilde{\Pi}_l(k, I_i)), \tag{2.13}$$

where the $\{I_i\}$ are nonoverlapping intervals on $I$, otherwise unrestricted. This allows us to write

$$\begin{aligned} \frac{C_l(k, T)}{C_{l-1}(k, T)} = e^{-\Gamma_{l-1}^{1F}(k, T)} \Bigg\{ & P(k, T) + 2 \sum_{K_L} \delta\tilde{\Pi}_l'(k, K_L)\, P(k, I) \\ & + \sum_{\substack{K_L, K_R \\ K_L \cap K_R = \varnothing}} \delta\tilde{\Pi}_l'(k, K_L)\, P(k, I)\, \delta\tilde{\Pi}_l'(k, K_R) \Bigg\}, \end{aligned} \tag{2.14}$$

where $K_L$, $K_R$ are intervals contained in $[0, T]$ with left and right endpoints at the left and right edges, respectively, of $[0, T]$, and $I$ is $[0, T] \backslash \{K_L \cup K_R\}$.

We now must recast this into the form of the inductive assumption. The first step is clearly to perform a polymer expansion on $P(k, I)$, as follows (see, e.g., [2]),

$$P(k, I) = \exp\left(\sum_n \frac{1}{n!} \sum_{\substack{(J_1,\dots,J_n)\\ \text{on } I}} \prod_{i=1}^{n} (\delta\tilde{\Pi}_l(k, J_i)) \sum \prod_{(s,t)\in G^c} A(J_s, J_t)\right), \tag{2.15}$$

where $(J_1, \dots)$ denotes an ordered collection of intervals on $I$ (otherwise unrestricted), $G^c$ are connected graphs on $\{1 \cdots n\}$, and $A$ is defined by

$$A(J_s, J_t) \equiv \begin{cases} -1 & \text{if } J_s \cap J_t \neq \varnothing \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Now, (2.15) becomes

$$P(k, I) = \exp(T\,\delta\Gamma_l(k) + 2\,\delta\Gamma_l^{\partial 1}(k) + \delta\Gamma^{\mathrm{IF}1}(k, I)) \tag{2.17}$$

if we make the appropriate definitions. Let

$$S \equiv \sum_n \frac{1}{n!} \sum_{\substack{(J_1,\dots,J_n)\\ \text{on } (-\infty,\infty)}}{}' \prod_{i=1}^{n} (\delta\tilde{\Pi}_l(k, J_i)) \sum_{G^c} \prod_{(s,t)\in G^c} A(J_s, J_t), \tag{2.18}$$

where the meaning of $\sum'$ depends on the quantity being defined:

$$\delta\Gamma_l(k) \equiv S, \tag{2.19}$$

where we require $\inf\{J_1 \cup \cdots \cup J_n\}$ to be the left endpoint of $I$,

$$2\delta\Gamma_l^{\partial 1}(k) \equiv -S, \tag{2.20}$$

where we require $\{J_1 \cup \cdots \cup J_n\}$ to intersect the right endpoint of $I$, and

$$\delta\Gamma_l^{\mathrm{IF}1}(k) \equiv S, \tag{2.21}$$

where we require $\{J_1 \cup \cdots \cup J_n\}$ to intersect both endpoints of $I$.

Note that we have begun to recover the inductive form, as $\delta\Gamma_l$ is one of the three functions we need. The other two will include contributions from the edge terms in (2.14) as well as $\delta\Gamma^{\partial 1}$ and $\delta\Gamma^{\mathrm{IF}1}$. To see what they are, we rewrite (2.14) as

$$\begin{aligned} \frac{C_l(k, T)}{C_{l-1}(k, T)} &= \exp(T\,\delta\Gamma_l(k) + 2\,\delta\Gamma_l^{\partial 1}(k) - \Gamma_{l-1}^{\mathrm{IF}}(k, T)) \\ &\quad \times [1 + 2A_2(k) + A_3(k) + B_1(k, T) + 2B_2(k, T) + B_3(k, T)], \end{aligned} \tag{2.22}$$

where

$$B_1(k, T) \equiv (e^{\delta\Gamma_l^{\mathrm{IF1}}(k,T)} - 1) \tag{2.23}$$

$$A_2(k) \equiv \sum_{|K|=1}^{\infty} \delta\tilde{\Pi}_l'(k, K) e^{-|K| \delta\Gamma_l(k)} \tag{2.24}$$

$$B_2(k, T) \equiv -\sum_{|K|=T+1}^{\infty} \delta\tilde{\Pi}_l'(k, K) e^{-|K| \delta\Gamma_l(k)} + \sum_{K_L} \delta\tilde{\Pi}_l'(k, K_L) e^{-|K_L| \delta\Gamma_l(k)} (e^{\delta\Gamma_l^{\mathrm{IF1}}(k,I)} - 1) \tag{2.25}$$

$$A_3(k) \equiv \left[ \sum_{|K|=1}^{\infty} \delta\tilde{\Pi}_l'(k, K) e^{-|K| \delta\Gamma_l(k)} \right]^2 \tag{2.26}$$

$$\begin{aligned} B_3(k, T) \equiv & -\sum_{K_L' \cap K_R' \neq \varnothing} [\delta\tilde{\Pi}_l'(k, K_L') e^{-|K_L'| \delta\Gamma_l(k)}][\delta\tilde{\Pi}_l'(k, K_R') e^{-|K_R'| \delta\Gamma_l(k)}] \\ & + \sum_{K_L \cap K_R = \varnothing} [\delta\tilde{\Pi}_l'(k, K_L) e^{-|K_L| \delta\Gamma_l(k)}] (e^{\delta\Gamma_l^{\mathrm{IF1}}(k,I)} - 1) \\ & \times [\delta\tilde{\Pi}_l'(k, K_R) e^{-|K_R| \delta\Gamma_l(k)}] \end{aligned} \tag{2.27}$$

with $K_L$, $I$, $K_R$ defined as in (2.14), and $K_L'$, $K_R'$ intervals on $(-\infty, \infty)$ that intersect the left and right endpoints of $[0, T]$, respectively, as well as each other. So we are led to define

$$\begin{aligned} 2\delta\Gamma_l^{\partial 2}(k) &\equiv \ln(1 + 2A_2 + A_3) \\ &= 2\ln(1 + A_2) \end{aligned} \tag{2.28}$$

$$\delta\Gamma_l^{\mathrm{IF2}}(k, T) \equiv \ln\left[ (1 + B_1 + 2B_2 + B_3) \times \left( 1 - \frac{(2A_2 + A_3)(B_1 + 2B_2 + B_3)}{(1 + 2A_2 + A_3)(1 + B_1 + 2B_2 + B_3)} \right) \right] \tag{2.29}$$

and we can finish recovering (2.4) by defining

$$\delta\Gamma_l^{\partial}(k) \equiv \delta\Gamma_l^{\partial 1}(k) + \delta\Gamma_l^{\partial 2}(k) \tag{2.30}$$

$$\delta\Gamma_l^{\mathrm{IF}}(k, T) \equiv \delta\Gamma_l^{\mathrm{IF2}}(k, T) - \Gamma_{l-1}^{\mathrm{IF}}(k, T) \tag{2.31}$$

which tells us that

$$\Gamma_l^{\mathrm{IF}}(k, T) = \delta\Gamma_l^{\mathrm{IF2}}(k, T). \tag{2.32}$$

## 3. Induction Started; Main Estimates

We now begin the proof of convergence of the expansion described in the previous section. We achieve this with an induction on $T_l$, the scale on which walks self-avoid, with the choice of

$$T_l \equiv 2^{l^{9/8}}. \tag{3.1}$$

Note that for $T_l < 2$ the model becomes the simple random walk, so for $l < 1$ we have $\Gamma_l^{(\cdot)}(\cdot) \equiv 0$. We make the following inductive assumptions:

$$\text{(i)} \qquad |\partial_k^u \, \delta\Gamma_l(k)| \leqslant C_{\text{I1}} \lambda T_l^{|u|/2} T_{l-1}^{1-d/2+\varepsilon} \tag{3.2}$$

$$\text{(ii)} \qquad |\partial_k^u \, \delta\Gamma_l^{\partial}(k)| \leqslant \mathscr{C}_{\text{I1}} \lambda T_l^{|u|/2} T_{l-1}^{2-d/2+\varepsilon} \tag{3.3}$$

$$\text{(iii)} \qquad |\partial_k^u \Gamma_l^{\text{IF}}(k, T)| \leqslant \begin{cases} (\mathscr{C}_{\text{I1}} \lambda)\, T_l^{|u|/2} T_{l-1}^{2-d/2+\varepsilon} & \text{for } T \leqslant T_{l-1} \\ (\mathscr{C}_{\text{I1}} \lambda)^{\lceil T/T_l \rceil}\, T_l^{|u|/2} T^{2-d/2+\varepsilon} & \text{for } T_{l-1} < T \leqslant N_3 T_l \\ (\mathscr{C}_{\text{I1}} \lambda)^{\lceil T/T_l \rceil}\, T_l^{|u|/2} T_l^{-a_3 T/T_l} & \text{for } T > N_3 T_l \end{cases} \tag{3.4}$$

$$\text{(iv)} \qquad \|\partial_k^u C_l(\cdot, T)\|_p^{k\text{-space}} \leqslant \mathscr{C}_{\text{I2}} T^{|u|/2 - d/2p} \exp(T\,\delta\Gamma_l(0) + 2\delta\Gamma_l^{\partial}(0) + \delta\Gamma_l^{\text{IF}}(0, T))$$
$$\text{for} \quad T \geqslant T_{l+1},\ 1 \leqslant p \leqslant 2, \tag{3.5}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to $\alpha$. Throughout this paper we will use the symbol $\mathscr{C}$ to denote positive real constants independent of induction step, $k$, and $T$ (they may depend on the dimension $d$). $\mathscr{C}'$ will denote a constant that is independent of $\mathscr{C}_{\text{I1}}$, while $\mathscr{C}$ may not be. (i)–(iii) are valid for $k \in U_l$, where

$$U_l \equiv \{k \colon |D(k)| \geqslant \delta_l\} \tag{3.6}$$

and

$$\delta_l \equiv \begin{cases} 0 & \text{for } l < 1 \\ T_l^{-f/T_l} & \text{for } l \geqslant 1; \end{cases} \tag{3.7}$$

$f$, $N_3$, and $a_3$ are constants to be determined in the course of the proof, and we choose $\varepsilon \equiv \frac{1}{8}$. $\lambda$ is fixed smaller than some $\lambda_0$ and $d \geqslant 5$; $u$ is a multi-index for derivatives with respect to $k_i$, and $|u| \equiv \sum u_i \leqslant 2$.

A comment on the origin of the restrictions on $k$ in (i)–(iii) is in order. We see that $D(k)^{|I|}$ appears in the denominator of (2.8), which must be cancelled by the decay of $\delta\Pi_l(k, I)$ with $I$. Clearly this decay becomes slower as $T_l$ increases, so we can expect the aforementioned cancellation to occur only for $|D(k)|$ close to one; i.e., for any given value of $k$ we can perform the expansion only out to some finite value of $T_l$ (which depends on $k$). Since the long-distance behavior of the walk is reflected in the small-$k$ region of $C(k, T)$, the fact that our expansion will only be valid in this region need not concern us. In proving the iteration of (iv) we will need to control the contributions to the norms from the large-$k$ region, but this is easily accomplished (Section 6).

At the beginning of the induction, $l<1$ and all of the $\delta\Gamma_0^{(\cdot)}$ are zero, so (i)–(iii) hold trivially. Also, it is easy to see that (iv) holds for $C_0(k,T)=D(k)^T$. So we must show that the hypotheses iterate.

PROPOSITION 3.1. *If* (i)–(iv) *hold for* $l-1$ *then they hold for* $l$, *any* $l\geqslant 1$.

We proceed in three stages. First we consider the contributions to (i)–(iii) arising from the polymer expansion (2.17), then the corrections due to edge effects (2.28) and (2.29). Once the proofs of (i)–(iii) are complete, we can prove the iteration of (iv).

Since the polymer expansion is in terms of the quantity $\delta\tilde{\Pi}_l$, we prove estimates on it first. The main result of this section is

LEMMA 3.2. *For any* $\delta'>0$ *there exists a choice of* $\mathscr{C}_{11}$ *so that*

$$|\partial_k^u\,\delta\tilde{\Pi}_l(k,T)|\leqslant\begin{cases}0 & \text{for } T\leqslant T_{l-1}/6\\ \delta'(\mathscr{C}_{11}\lambda)^{\lceil T/T_l\rceil}\,T_l^{|u|/2}T^{\varepsilon-d/2} & \text{for } T_{l-1}/6<T\leqslant N_2T_l\\ \delta'(\mathscr{C}_{11}\lambda)^{\lceil T/T_l\rceil}\,T_l^{|u|/2}T_l^{-a_2(T/T_l)} & \text{for } T>N_2T_l,\end{cases}\tag{3.8}$$

*where* $a_2$, $N_2$ *are constants to be determined and* $k\in U_l$.

Since $\tilde{\Pi}_l$ is defined in terms of $\delta\Pi^*$ and $\delta\Pi^{**}$ in (2.10), we prove estimates on these first. Now, $\delta\Pi^{**}$ is easy, since from its definition we see

$$\begin{aligned}|\delta\Pi_l^{**}(k,K)|&=|e^{\Gamma_{l-1}^{\mathrm{IF}}(k,K)}-1|\\&\leqslant\mathscr{C}'\,|\Gamma_{l-1}^{\mathrm{IF}}(k,K)|\end{aligned}\tag{3.9}$$

and we can use the inductive assumption (similarly for its derivatives).

$\delta\Pi_l^*$ will require more work. We prove

LEMMA 3.3.

$$|\partial_k^u\,\delta\Pi_l^*(k,I)|\leqslant\begin{cases}0 & \text{for } |I|\leqslant T_{l-1}/6\\ (\mathscr{C}'\lambda)^{\lceil|I|/T_l\rceil}\,T_l^{|u|/2}\,|I|^{\varepsilon-d/2} & \text{for } T_{l-1}/6<|I|\leqslant N_1T_l\\ (\mathscr{C}'\lambda)^{\lceil|I|/T_l\rceil}\,T_l^{|u|/2}T_l^{-a(|I|/T_l)} & \text{for } |I|>N_1T_l,\end{cases}\tag{3.10}$$

*where* $k\in U_l$, *fixing* $a_1\leqslant\frac{1}{2}(d/4-1)-(\mathscr{C}f+\mathscr{C}\lambda)-\delta$, *with any choice of* $\delta>0$, *and fixing* $N_1$ *such that* $a_1N_1>d/2-\varepsilon$.

*Proof.* First consider the case of $u\equiv 0$. We begin by obtaining bounds on $\delta\Pi_l$. Using the lace expansion of [1] (see the appendix for a proof of the first inequality below), we see that

$$
\begin{aligned}
|\delta \Pi_l(k, I)| &\leqslant \mathscr{C}' \sum_{m=1} \lambda^m \sum_{\{n_i\}_{i=1}^{2m-1}} \prod_{j=1}^{2m-1} \|C_{l(n_j)-1}(\cdot, n_j)\|_*^{k\text{-space}} \\
&\leqslant \sum_{m=1} (\mathscr{C}'\lambda)^m \sum_{\{n_i\}} \left( \prod_{j=1}^{2m-1} n_j^{-d/2*} \right) \\
&\quad \times \exp\left( \sum_{j=1}^{2m-1} (n_j \Gamma_{l(n_j)-1}(0) + 2\Gamma^{\partial}_{l(n_j)-1}(0) + \Gamma^{\mathrm{IF}}_{l(n_j)-1}(n_j, 0)) \right),
\end{aligned} \tag{3.11}
$$

where $l(n)$ is defined by $T_{l(n)} = n$, $\{n_i\}$ satisfies $\sum n_i = |I|$, at least one $n_i \geqslant T_{l-1}/6$, all $n_i \leqslant T_l$, and $* = 2$ or 1, with exactly one 1 in the product. In the second line we have substituted the inductive assumptions. We can now substitute this into (2.8), which yields

$$
\begin{aligned}
|\delta \Pi_l^*(k, I)| \leqslant \sum_{m=1} (\mathscr{C}'\lambda)^m \sum_{\{n_i\}} \Bigg\{ & \underbrace{\left[ \frac{\prod_{j=1}^{2m-1} n_j^{-d/2*}}{D(k)^{|I|}} \right]}_{\equiv t_1} \\
& \times \underbrace{\left[ \exp\left( -|I|\, \Gamma_{l-1}(k) + \sum_j n_j \Gamma_{l(n_j)-1}(0) \right) \right]}_{\equiv t_2} \\
& \times \underbrace{\left[ \exp\left( -2\delta\Gamma^{\partial}_{l-1}(k) + \sum_j \left( 2\Gamma^{\partial}_{l(n_j)-1}(0) + \Gamma^{\mathrm{IF}}_{l(n_j)-1}(n_j, 0) \right) \right) \right]}_{\equiv t_3} \Bigg\}.
\end{aligned} \tag{3.12}
$$

The important term here is $t_1$, while $t_2$ and $t_3$ are just small corrections. We prove this in

LEMMA 3.4.

$$
\sup_{\{n_i\}} t_2 t_3 \leqslant \exp(\mathscr{C}\lambda k^2\, |I| + m\mathscr{C}\lambda), \tag{3.13}
$$

*where $k \in U_l$.*

*Proof.* Fix a choice of $\{n_i\}$. Using (i), along with the fact that $\Gamma_l$ is even in $k$, we see that

$$
|\Gamma_{l-1}(k)| \leqslant \Gamma_{l-1}(0) + \mathscr{C}\lambda k^2. \tag{3.14}
$$

Inserting this into our expression for $t_2$, we find

$$
\begin{aligned}
|t_2| &\leqslant \exp\left( \mathscr{C}\lambda k^2\, |I| + \sum_j n_j \sum_{i=l(n_j)}^{l-1} \delta\Gamma_i(0) \right) \\
&\leqslant \exp\left( \mathscr{C}\lambda k^2\, |I| + \sum_i n_j \mathscr{C}\lambda T_{l(n_j)-1}^{1+\varepsilon-d/2} \right),
\end{aligned} \tag{3.15}
$$

where we have used (i) and performed the sum in the second line. Now, $T_{l(n_j)-1}^{1+\varepsilon-d/2} < \mathscr{C} T_{l(n_j)}^{-1}$, so $n_j T_{l(n_j)-1}^{1+\varepsilon-d/2} < \mathscr{C}$, and we have

$$|t_2| \leqslant e^{\mathscr{C}\lambda k^2 |I|} (e^{\mathscr{C}\lambda})^m. \tag{3.16}$$

The third term $t_3$ is even simpler as no cancellations need occur; we just directly substitute the inductive hypotheses to see that

$$\sup_{\{n_i\}} t_3 \leqslant (e^{\mathscr{C}\lambda})^m \tag{3.17}$$

and we are done. ∎

Next we must handle the main term, $t_1$. We first need

LEMMA 3.5.

$$\sum_{\substack{\{n_i\} \\ \text{on } I}} \prod_i n_i^{-d/2*} \leqslant \begin{cases} 0 & \textit{for} \quad |I| \leqslant T_{l-1}/6 \\ \mathscr{C}'^m \, |I|^{-d/2} & \textit{for} \quad T_{l-1}/6 < |I| \leqslant N_1 T_l \\ \mathscr{C}'^m \mathscr{C}'^{|I|/T_l} T_l^{(1/2)(1-d/4)(|I|/T_l)} & \textit{for} \quad |I| > N_1 T_l, \end{cases} \tag{3.18}$$

*where* $*$, $\{n_i\}$ *are defined as in* (3.11).

*Proof.* For $|I| \leqslant T_{l-1}/6$, we observe that at least one $n_i$ must be longer than $T_{l-1}/6$. For $(T_{l-1}/6) < |I| \leqslant N_1 T_l$, we note that at least one of the $n_i$ must be longer than $|I|/(2m-1)$. We use the condition $\sum n_i = |I|$ to fix the length of the longest interval given the other $2m-2$. Inserting a combinatorial factor of $(2m-1)$ for which interval is long and taking $* = 1$ on the long interval, we obtain

$$\begin{aligned} \sum_{\{n_i\}} \prod_i n_i^{-d/2*} &\leqslant (2m-1) \left( \frac{|I|}{2m-1} \right)^{-d/2} \left( \sum_{n=1} n^{-d/4} \right)^{2m-2} \\ &\leqslant \mathscr{C}'^m \, |I|^{-d/2}. \end{aligned} \tag{3.19}$$

For $|I| > N_1 T_l$, we set $* = 2$ on all $n_i$ as an upper bound. We note that at least $\lceil |I|/2T_l \rceil$ of the $n_i$ must be longer than $|I|/2(2m-1)$ in order to satisfy $\sum n_i = |I|$, subject to the constraint that $n_i \leqslant T_l$. Next insert a combinatorial factor of $\binom{2m-1}{\lceil |I|/2T_l \rceil}$ for which intervals are longest, and on these intervals use the bound

$$\begin{aligned} \sum_{n=\lceil |I|/2(2m-1) \rceil}^{T_l} n^{-d/4} &\leqslant \mathscr{C}' \left( \frac{|I|}{2(2m-1)} \right)^{1-d/4} \\ &\leqslant \mathscr{C}' \left( m \frac{T_l}{|I|} \right)^{d/4-1} T_l^{1-d/4}. \end{aligned} \tag{3.20}$$

The short intervals contribute a constant factor each. So

$$\sum_{\{n_i\}} \prod_i n_i^{-d/2*} \leqslant \mathscr{C}'^m \binom{2m-1}{\lceil |I|/2T_l \rceil} \left( \mathscr{C}' \left( m \frac{T_l}{|I|} \right)^{d/4-1} T_l^{1-d/4} \right)^{\lceil |I|/2T_l \rceil}. \tag{3.21}$$

Next use the general result $\binom{i}{j} \leqslant (\mathscr{C}'(i/j))^j$ to see that

$$\begin{aligned} \sum_{\{n_i\}} \prod_i n_i^{-d/2*} &\leqslant \mathscr{C}'^m \mathscr{C}'^{|I|/T_l} \left[ \left( \frac{mT_l}{|I|} \right)^{\lceil |I|/mT_l \rceil} \right]^{(m/2)(d/4)} T_l^{(1/2)(1-d/4)(|I|/T_l)} \\ &\leqslant \mathscr{C}'^m \mathscr{C}'^{|I|/T_l} T_l^{(1/2)(1-d/4)(|I|/T_l)} \end{aligned} \tag{3.22}$$

which completes the proof. ∎

Finally, we put the results on $t_1$, $t_2$, and $t_3$ together to bound $\delta\Pi_l^*$. We note that, for $k$ near 0,

$$D(k)^{-|I|} \sim e^{|I| (k^2/2d)} \tag{3.23}$$

and when $|D(k)| > \delta_l$ along with $k$ near 0, we have

$$k^2 < \mathscr{C} f \frac{\ln T_l}{T_l}. \tag{3.24}$$

We require $\lambda$ small and fix $f$ so that $N_1(\mathscr{C}f + \mathscr{C}\lambda) < \varepsilon$, and noting that the series in $\lambda$ starts at some $m \geqslant \lceil |I|/T_l \rceil$, we obtain the result for $u = 0$ (the $\delta$ in the definition of $a_1$ comes from the derivatives, which we consider next).

To handle the derivatives, we must consider their action on each of the four terms that constitute $\delta\Pi_l^*$ (and Leibnitz's rule gives us a factor in front that is dominated by the powers of $\lambda$). We easily see

$$\begin{aligned} |\partial_k^u e^{2\Gamma_{l-1}^{\partial}(k)}| &\leqslant T_l^{|u|/2} (\mathscr{C}\lambda) e^{2\Gamma_{l-1}^{\partial}(k)} \\ |\partial_k^u e^{-|I|\,\Gamma_{l-1}(k)}| &\leqslant \left( \mathscr{C}\lambda \left( \frac{|I|^2}{T_l} \ln T_l \right)^{|u|/2} + \mathscr{C}\lambda\, |I|^{|u|/2} \right) e^{-|I|\,\Gamma_{l-1}(k)} \\ |\partial_k^u \delta\Pi_l(k, I)| &\leqslant (\mathscr{C}\, |I|^{|u|/2}) \sum_{m=1} (\mathscr{C}\lambda)^m \sum_{\{n_i\}} \left( \prod_{j=1}^{2m-1} n_j^{-d/2*} \right) \\ &\quad \times \exp \left( \sum_{j=1}^{2m-1} (n_j \Gamma_{l(n_j)-1}(0) + 2\Gamma_{l(n_j)-1}^{\partial}(0) + \Gamma_{l(n_j)-1}^{\mathrm{IF}}(n_j, 0)) \right) \\ |\partial_k^u D(k)^{-|I|}| &\leqslant \left( \mathscr{C} \left( \frac{|I|^2}{T_l} \ln T_l \right)^{|u|/2} + \mathscr{C}\, |I|^{|u|/2} \right) D(k)^{-|I|}. \end{aligned} \tag{3.25}$$

Now, use

$$|I|^2 \ln T_l \leqslant \mathscr{C} T_l^{2 + \delta(|I|/T_l)}, \qquad \text{any } \delta > 0. \tag{3.26}$$

This allows us to absorb the extra factors of $|I|$ that the derivatives give into the exponential, which makes subsequent bounding of derivatives convenient: the action of a derivative on our estimates is just to bring down a factor of $T_l^{1/2}$. Note that the same is true for $\delta\Pi_l^{**}$. ▮

We can now combine the results on $\delta\Pi_l^*$ and $\delta\Pi_l^{**}$. We will need

LEMMA 3.6. *Define*

$$F(T) \equiv \sum_{|K|=0}^{T-1} \delta\Pi_l^*(k, I)\, \delta\Pi_l^{**}(k, K), \tag{3.27}$$

*where $k \in U_l$ and $I = [0, T] \backslash K$. Then for any $\delta' > 0$ there exists a choice of $\mathscr{C}_{11}$ so that*

$$|\partial_k^u F(T)| \leqslant \begin{cases} 0 & \text{for} \quad T < T_{l-1}/6 \\ \delta'(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil} T_l^{|u|/2} T_{l-1}^{3+\varepsilon-d/2} T^{\varepsilon-d/2} & \text{for} \quad T \geqslant T_{l-1}/6. \end{cases} \tag{3.28}$$

*Proof.* Fix $\delta' > 0$ and for now take $|u| = 0$. For $T < T_{l-1}/6$, we have $\delta\Pi_l^*(k, T) = 0$. We separate the $\lambda$ dependence from the $T$ dependence by noting that if we divide each term in the sum by $(\mathscr{C}'\lambda)^{\lceil |I|/T_l \rceil} (\mathscr{C}_{11}\lambda)^{\lceil |K|/T_{l-1} \rceil}$ we obtain, for $(T_{l-1}/6) \leqslant T \leqslant 2N_3 T_{l-1}$,

$$\begin{aligned} &\mathscr{C}' \left(\frac{T}{2}\right)^{\varepsilon-d/2} \sum_{K=1}^{N_3 T_{l-1}} K^{2+\varepsilon-d/2} + \mathscr{C}' \left(\frac{T}{2}\right)^{2+\varepsilon-d/2} \left( \sum_{I=T_{l-1}/6}^{N_1 T_l} I^{\varepsilon-d/2} + \sum_{I > N_1 T_l} T_l^{-a_1(I/T_l)} \right) \\ &\quad \leqslant \mathscr{C}' T_{l-1}^{3+\varepsilon-d/2} T^{\varepsilon-d/2}, \end{aligned} \tag{3.29}$$

where we have used (3.10) and (3.9), and for $T \geqslant 2N_3 T_{l-1}$,

$$[\mathscr{C}' T^{\varepsilon-d/2} T_{l-1}^{3+\varepsilon-d/2} + \mathscr{C}' T_{l-1}^{-a_3(T/2T_{l-1})+1+\varepsilon-d/2}] \leqslant \mathscr{C}' T_{l-1}^{3+\varepsilon-d/2} T^{\varepsilon-d/2}, \tag{3.30}$$

where we require $a_3$, $N_3$ to satisfy $a_3 N_3 > d/2 - 2 - \varepsilon$.

Now we bound the sum by multiplying this result by the leading order of $\mathscr{C}\lambda$, which we find by noting that the bonds in $\delta\Pi_l^*$ are $T_l$ units long while those in $\delta\Pi_l^{**}$ are only $T_{l-1}$, meaning that at most one power of $\lambda$ can come from the $\delta\Pi_l^{**}$ at leading order. Hence the leading $\lambda$ dependence is, for $T \leqslant T_l$,

$$(\mathscr{C}'\lambda)^2 + (\mathscr{C}'\lambda)(\mathscr{C}'\mathscr{C}_{11}\lambda) = (\mathscr{C}_{11}\lambda)^2 \left( \left(\frac{\mathscr{C}'}{\mathscr{C}_{11}}\right)^2 + \left(\frac{\mathscr{C}'}{\mathscr{C}_{11}}\right) \mathscr{C}' \right) \tag{3.31}$$

and for $T > T_l$ it becomes

$$(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil} \left( \left(\frac{\mathscr{C}'}{\mathscr{C}_{11}}\right) + \left(\frac{\mathscr{C}'}{\mathscr{C}_{11}}\right)^{\lceil T/T_l \rceil - 1} \mathscr{C}' \right). \tag{3.32}$$

We see that by taking $\mathscr{C}_{11}$ large enough we obtain the result.

For $|u|>0$, the derivatives bring down the desired factors of $T_l$, along with a harmless factor from the Leibnitz rule. ∎

We are now in a position to prove Lemma 3.2:

*Proof.* Fix $\delta'>0$. For the $n=1$ term, we have $\delta\tilde{\Pi}_l=\delta\Pi_l^*$, and so the result follows from Lemma 3.3 as long as we require $a_2\leqslant a_1$, $N_2\geqslant N_1$, with $\mathscr{C}_{\mathrm{I}1}$ large enough.

Now consider the $n>1$ terms. First let $|u|=0$. For $T<T_{l-1}/6$, all of the subintervals in the partition of $[0,T]$ are shorter than $T_{l-1}/6$ and hence the $\delta\Pi_l^*$ terms are zero.

For $T<N_2T_l$, we take $\{J_i\}$ a partition of $[0,T]$ and note that the longest interval has length at least $T/n$. So with $\mathscr{C}_{\mathrm{I}1}$ large enough we have

$$\begin{aligned}|\delta\tilde{\Pi}_l(k,T)|_{n>1\,\text{terms}}&\leqslant\sum_{n>1}\sum_{\{J_i\}}F(J_1)\cdots F(J_{n-1})\,\delta\Pi_l^*(J_n)\\&\leqslant\frac{\delta'}{2}\sum_{n>1}n(\mathscr{C}_{\mathrm{I}1}\lambda)^{\max(n,\lceil T/T_l\rceil)}\left(\frac{T}{n}\right)^{\varepsilon-d/2}(T_{l-1}^{4+2\varepsilon-d})^{n-1}\\&\leqslant\frac{\delta'}{2}(\mathscr{C}_{\mathrm{I}1}\lambda)^{\lceil T/T_l\rceil}T^{\varepsilon-d/2}.\end{aligned}\tag{3.33}$$

For $T>N_2T_\ell$, we look at

$$\begin{aligned}&|\delta\tilde{\Pi}_l(k,T)|_{n>1\,\text{terms}}\,T_l^{a_2(T/T_l)}\\&\leqslant\sum_{n>1}\sum_{\{I_i\}}\prod_{j=1}^{n-1}(|\delta\Pi_l^*(k,I_j)T_l^{a_2(|I_j|/T_l)}|\,|\delta\Pi_l^{**}(k,K_j)T_l^{a_2(|K_j|/T_l)}|)\\&\quad\times|\delta\Pi_l^*(k,I_n)T_l^{a_2(|I_n|/T_l)}|\\&\leqslant\frac{\delta'}{2}\sum_{n>1}(\mathscr{C}_{\mathrm{I}1}\lambda)^{\max(n,\lceil T/T_l\rceil)}(s_1)^n(s_2)^{n-1},\end{aligned}\tag{3.34}$$

where we have analyzed the $\lambda$ dependence as in the proof of Lemma 3.6, inserted our bounds for $\delta\Pi_l^*$ and $\delta\Pi_l^{**}$, and defined $s_1$, $s_2$ as

$$\begin{aligned}s_1&\equiv\sum_{|I|=1}^{N_1T_l}\mathscr{C}'\,|I|^{\varepsilon-d/2}\,T_l^{a_2(|I|/T_l)}+\sum_{|I|>N_1T_l}\mathscr{C}'T_l^{(a_2-a_1)(|I|/T_l)}\\&\leqslant\mathscr{C}'T_{l-1}^{2-d/2-2\varepsilon},\end{aligned}\tag{3.35}$$

where we fix $a_2$ such that $a_2N_1\leqslant 1-3\varepsilon$ and $(a_1-a_2)N_1>d/2-1+2\varepsilon$. Also,

$$\begin{aligned}s_2\equiv&\sum_{|K|=1}^{N_3T_{l-1}}\mathscr{C}'\,|K|^{2+\varepsilon-d/2}\,T_l^{a_2(|K|/T_l)}+\sum_{|K|>N_3T_{l-1}}\mathscr{C}'\lambda^{\mathscr{C}\lceil|K|/T_{l-1}\rceil}T_{l-1}^{-a_3(|K|/T_{l-1})}T_l^{a_2(|K|/T_l)}\\&\leqslant\mathscr{C}'T_{l-1}^{3+\varepsilon-d/2}.\end{aligned}\tag{3.36}$$

Note that we have used the fact that $\delta\Pi_l^{**}$ contains bonds of length at most $T_{l-1}$, so after we pull out a factor of $\lambda^{\lceil |K|/T_l \rceil}$ we are still left with a factor of $\lambda^{\mathscr{C}\lceil |K|/T_{l-1} \rceil}$ for $T_l \geqslant 2$. So the $n > 1$ terms give us

$$|\delta\tilde{\Pi}_l(k, T)|_{n>1\,\text{terms}}\, T_l^{a_2(T/T_l)} \leqslant \frac{\delta'}{2} \sum_{n>1} (\mathscr{C}_{11}\lambda)^{\max(n, \lceil T/T_l \rceil)} (T_{l-1}^{5-\varepsilon-d})^{n-1}\, T_{l-1}^{2-2\varepsilon-d/2}$$
$$\leqslant \frac{\delta'}{2} (\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil}. \tag{3.37}$$

For $|u| > 0$, again the derivatives bring down the expected factors of $T_l$, along with harmless factors from the Leibnitz rule. ∎

## 4. Convergence of Polymer Expansion

LEMMA 4.1. *For any $\delta' > 0$ there exists a choice of $\mathscr{C}_{11}$ such that*

$$\sum_{I \cap J} |\delta\tilde{\Pi}_l(k, T)|\, |I|^j \leqslant \delta' \mathscr{C}_{11} \mathscr{C}'^j \lambda j!\, T_l^j T_{l-1}^{1+\varepsilon-d/2} |J| \qquad \textit{for} \quad |J| \geqslant \frac{T_{l-1}}{6}, \tag{4.1}$$

*where $k \in U_l$ and the sum is over all intervals $I \subset (-\infty, \infty)$ that intersect some interval $J$.*

*Proof.* We write

$$\sum_{I \cap J} (\cdot) = \sum_{I \ni 0} (\cdot) + \sum_{j=1}^{|J|} \sum_{\substack{I \\ \inf\{I\} = j}} (\cdot) \tag{4.2}$$

$$= \sum_{|I|=1}^{\infty} |I|^{j+1} |\delta\tilde{\Pi}_l(k, I)| + |J| \sum_{|I|=1}^{\infty} |I|^j |\delta\tilde{\Pi}_l(k, I)|, \tag{4.3}$$

where we have defined the origin to be at the left endpoint of $J$. Apply Lemma 3.2, fixing $N_2$ so that $a_2 N_2 > d/2 - \varepsilon$. ∎

We are now in a position to prove convergence of the polymer expansion. We begin with $\delta\Gamma_l$, defined in (2.19). We overcount configurations of intervals by requiring only $J_1$ to satisfy the condition $\inf\{J_1\} = 0$ and inserting a combinatorial factor of $n$ for which interval has its endpoint fixed. We do not restrict the other intervals. We can now use standard techniques to bound the sum (we only sketch the argument here; see [2] for details): first note that, given a connected graph $G^c$ and a compatible configuration of intervals $\{J_i\}$, $\prod_{(s,t) \in G^c} A(J_s, J_t) = (-1)^{b(G^c)}$, where $b(G^c)$ is the number of bonds in $G^c$. Then we have

$$\left| \sum_{G^c} (-1)^{b(G^c)} \right| \leqslant \sum_{\substack{\text{Trees } T \\ \text{on } \{1, \dots, n\}}} 1; \tag{4.4}$$

i.e., for an upper bound we can discard bonds $(1+A)$ until the intervals are connected in a tree structure (see [2] for proof). Recalling that we are summing over ordered collections of intervals, we choose $J_1$ to be the root by thinking of it as attached to an additional vertex with coordination number fixed at one. We then form a tree of $n+1$ vertices with coordination numbers $(1, d_1+1, ..., d_\kappa+1)$, defining $\kappa$ to be the number of vertices with coordination number greater than 1. We choose $J_2, ..., J_{1+d_1}$ to be the intervals connected to $J_1$, then $J_{1+d_1+1}, ..., J_{1+d_1+d_2}$ to be those connected to $J_2$, etc., and multiply by a combinatorial factor of the number of trees with this structure $(n-1)!/(d_1! \cdots d_\kappa!)$ (given by Cayley's formula). Summing over all trees of this type, we arrive at

$$|\delta\Gamma_l(k)| \leqslant \sum_{n=1} \frac{1}{n}(n) \sum_{\kappa=1}^{n-1} \sum_{\substack{J_1 \\ \inf\{J_1\}=0}} \sum_{d_1=1} \sum_{(J)_{d_1}}$$

$$\times \sum_{d_2=1} \sum_{(J)_{d_2}} \cdots \sum_{d_\kappa=1} \sum_{(J)_{d_\kappa}}$$

$$\times \prod_{s_1=1}^{d_1} A(J_1, J_{1+s_1}) \prod_{s_2=1}^{d_2} A(J_2, J_{1+d_1+s_2}) \cdots \prod_{i=1}^{n} |\delta\tilde{\Pi}_l(J_i)|, \tag{4.5}$$

where we have defined

$$\sum_{(J)_{d_i}} \equiv \frac{1}{d_i!} \sum_{(J_{1+d_1+\cdots+d_{i-1}+1}, ..., J_{1+d_1+\cdots+d_{i-1}+d_i})} \tag{4.6}$$

and the $\{d_i\}$ satisfy

$$\sum_{i=1}^{\kappa} d_i = n-1. \tag{4.7}$$

Now fix $n$, $\kappa$, and the $\{d_i\}$ and perform the sums on the $\{J_i\}$. Apply Lemma 4.1 in the following manner: the terminal intervals (those furthest down the tree) give factors of $\mathscr{C}'\lambda T_{l-1}^{1+\varepsilon-d/2}|I|$, where $I$ is the node they are forced to intersect. So if $d_J$ terminal intervals overlap some $J$, and $J$ is forced to intersect some $J'$, the $\sum_{J\cap J'}$ gives

$$\sum_{J\cap J'} |\delta\tilde{\Pi}_l(k, J)| \, (\mathscr{C}'\lambda T_{l-1}^{1+\varepsilon-d/2}|J|)^{d_J} \leqslant (\mathscr{C}'\lambda)^{d_J+1}(d_J!)\, T_{l-1}^{1+\varepsilon-d/2}|J'|, \tag{4.8}$$

where we have used $T_{l-1}^{1+\varepsilon-d/2}T_l \leqslant \mathscr{C}'$. Note this gives the same power of $T_{l-1}$ multiplying the length of the interval $|J'|$ as did the terminal intervals. So we can proceed in this manner up the entire tree until we get to the root, which requires special treatment as it is a restricted sum: for $d_1=0$, 1 we can choose $\mathscr{C}_{11}$ so that for any $\delta'>0$

$$\sum_{|I_1|=1}^{\infty} |\delta\tilde{\Pi}_l(k, I_1)| \, (T_{l-1}^{1+\varepsilon-d/2}|I_1|)^{d_1} \leqslant \delta'\mathscr{C}_{11}\lambda T_{l-1}^{1+\varepsilon-d/2} \tag{4.9}$$

while for $d_1 \geqslant 2$ we find

$$\sum_{|I_1|=1}^{\infty} |\delta\tilde{\Pi}_l(k, I_1)| \, (T_{l-1}^{1+\varepsilon-d/2} |I_1|)^{d_1} \leqslant \delta' \mathscr{C}_{11} \lambda T_{l-1}^{d_1(1+\varepsilon-d/2)} T_l^{1+\varepsilon+d_1-d/2}$$

$$\leqslant \delta' \mathscr{C}_{11} \lambda T_{l-1}^{1+\varepsilon-d/2}. \tag{4.10}$$

So (4.5) becomes

$$|\delta\Gamma_l(k)| \leqslant \sum_{n=1} \delta' \mathscr{C}_{11} \mathscr{C}'^{n-1} \lambda^n T_{l-1}^{1+\varepsilon-d/2}$$

$$\times \sum_{k=1} \sum_{\{d_i\}} \left(\frac{1}{d_1! \cdots d_k!}\right) (d_1! \cdots d_k!). \tag{4.11}$$

Now, there are $\binom{n-2}{k-1}$ solutions to the equation $\sum_{i=1}^k d_i = n-1$, and $\sum_k \binom{n-2}{k-1} = 2^{n-2}$, so

$$|\delta\Gamma_l(k)| \leqslant \delta' \mathscr{C}_{11} \lambda T_{l-1}^{1+\varepsilon-d/2} \tag{4.12}$$

which is the desired result for $|u| = 0$. Derivatives act on the $\delta\tilde{\Pi}_l$ terms to bring down a factor of $T_l^{|u|/2}$ along with harmless factors from the Leibnitz rule, so finally for $\mathscr{C}_{11}$ large enough, any $\delta' > 0$,

$$|\partial_k^u \, \delta\Gamma_l(k)| \leqslant \delta' \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{1+\varepsilon-d/2}. \tag{4.13}$$

Clearly, by looking at (2.20) we see that $\delta\Gamma_l^{\partial 1}$ goes exactly like $\delta\Gamma_l$, except instead of requiring the root sum to have one endpoint fixed we only require it to intersect a given point. So the root sum gives a factor of $T_{l-1}^{2+\varepsilon-d/2}$, and we can obtain for any $\delta' > 0$

$$|\partial_k^u \, \delta\Gamma_l^{\partial 1}(k)| \leqslant \delta' \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{2+\varepsilon-d/2}. \tag{4.14}$$

We must treat $\delta\Gamma_l^{\mathrm{IF1}}$ a little more carefully because of its $T$ dependence. For $T \leqslant T_{l-1}$, we overcount contributing configurations by only requiring $\{J_1 \cup \cdots \cup J_n\}$ to intersect 0 instead of both 0 and $T$. This reduces it to the case of $\delta\Gamma_l^{\partial 1}$, so that result applies.

For $T_{l-1} < T \leqslant N_3 T_l$, we note that at least one of the intervals is longer than $T/n$, so we require the root to be at least this long and put in a combinatorial factor of $n$. Now, every configuration that contributes must have at least $\lceil T/T_l \rceil$ powers of $\lambda$, since they must all have connected sets of intervals that stretch from 0 to $T$. We overcount configurations by allowing some that do not stretch all the way, but we multiply these by enough powers of $\lambda$ so the total is $\lceil T/T_l \rceil$. We then proceed as before, and here the root sum yields, for $d_1 = 0$, any $\delta' > 0$ with $\mathscr{C}_{11}$ large enough,

$$\sum_{|I_1| \geqslant T/n}^{\infty} (\mathscr{C}_{11} \lambda)^{\max(\lceil T/T_l \rceil - \lceil |I_1|/T_l \rceil, 0)} |I_1| \, |\delta\tilde{\Pi}_l(k, I_1)| \leqslant \delta' (\mathscr{C}_{11} \lambda)^{\lceil T/T_l \rceil} \left(\frac{T}{n}\right)^{2+\varepsilon-d/2} \tag{4.15}$$

and for $d_1 \geqslant 1$

$$\sum_{|I_1| \geqslant T/n}^{\infty} (\mathscr{C}_{I1}\lambda)^{\max(\lceil T/T_l \rceil - \lceil |I_1|/T_l \rceil, 0)} |I_1| \, |\delta \tilde{\Pi}_l(k, I_1)| \, (T_{l-1}^{1+\varepsilon-d/2} |I_1|)^{d_1}$$
$$\leqslant \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil} T_{l-1}^{d_1(1+\varepsilon-d/2)} T_l^{2+\varepsilon+d_1-d/2}$$
$$\leqslant \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil} T^{2+\varepsilon-d/2} \tag{4.16}$$

which is the desired result.

For $T > N_3 T_l$, we show that for any $\delta' > 0$ with $\mathscr{C}_{I1}$ large enough,

$$|\delta \Gamma^{\mathrm{IF1}}(k, T)| \, T_l^{a_3(T/T_l)} \leqslant \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil} \tag{4.17}$$

which implies the desired result. To show this we will need

LEMMA 4.2. *For any $\delta' > 0$ there exists a choice of $\mathscr{C}_{I1}$ such that*

$$\sum_{I \cap J} |\delta \tilde{\Pi}_l(k, I)| \, |I|^j \, T_l^{a_3(|I|/T_l)}$$
$$\leqslant \delta' \mathscr{C}_{I1} \mathscr{C}'^j \lambda j! \, T_l^j T_{l-1}^{1+\varepsilon-d/2+1/4} |J| \qquad \textit{for} \quad |J| \geqslant \frac{T_{l-1}}{6}, \tag{4.18}$$

*where $k \in U_l$.*

*Proof.* We fix $a_3$ so that $a_3 N_2 \leqslant \frac{1}{4}$. Then everything proceeds as in Lemma 4.1. ∎

We now proceed as in the proofs of $\delta \Gamma_l$ and $\delta \Gamma_l^{\partial 1}$, finding

$$|\delta \Gamma^{\mathrm{IF1}}(k, T)| \, T_l^{a_3(T/T_l)} \leqslant \sum_n \delta'(\mathscr{C}_{I1}\lambda)^{\max(n, \lceil T/T_l \rceil)} 2^n T_{l-1}^{2+\varepsilon-d/2+1/4}$$
$$\leqslant \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil}. \tag{4.19}$$

Once again, derivatives yield a factor of $T_l^{|u|/2}$. So we have shown, for any $\delta' > 0$ and $\mathscr{C}_{I1}$ large enough,

$$|\partial_k^u \delta \Gamma^{\mathrm{IF1}}(k, T)| \leqslant \begin{cases} \delta' \mathscr{C}_{I1} \lambda T_l^{|u|/2} T_{l-1}^{2+\varepsilon-d/2} & \text{for} \quad T \leqslant T_{l-1} \\ \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil} T_l^{|u|/2} T^{2+\varepsilon-d/2} & \text{for} \quad T_{l-1} < T \leqslant N_3 T_l \\ \delta'(\mathscr{C}_{I1}\lambda)^{\lceil T/T_l \rceil} T_l^{|u|/2} T_l^{-a_3(T/T_l)} & \text{for} \quad T > N_3 T_l. \end{cases} \tag{4.20}$$

## 5. EDGE EFFECTS

In this section we obtain estimates on the edge effects (2.28) and (2.29), which allows us to complete the proof that (i)–(iii) iterate. It is clear from $|\ln(1+x)| \leqslant \mathscr{C}' |x|$ for small $x$ that we need only bound the $A_i$ and $B_j$ and the results for $\delta \Gamma_l^{\partial 2}$ and $\delta \Gamma_l^{\mathrm{IF2}}$ will follow.

LEMMA 5.1. *For any $\delta' > 0$ there exists a choice of $\mathscr{C}_{11}$ such that*

$$|\partial_k^u \delta \Gamma_l^{\partial 2}(k)| \leqslant \delta' \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{2+\varepsilon-d/2}, \tag{5.1}$$

*where $k \in U_l$.*

*Proof.* Fix $\delta' > 0$. Since $A_3 = A_2^2$, we need only prove the result for $A_2$. Recalling our definition of $F$ in (3.27), we see that

$$\delta \tilde{\Pi}_l'(k, K) = \sum_{n=1} \sum_{\{J_i\}} \prod_{i=1}^{n} F(J_i), \tag{5.2}$$

where $\{J_i\}$ is a partition of $K$. Using the same methods as in Lemma 3.2, we find for any $\delta'' > 0$ and $\mathscr{C}_{11}$ large enough,

$$|\partial_k^u \delta \tilde{\Pi}_l'(k, K)| \leqslant \begin{cases} 0 & \text{for } |K| \leqslant T_{l-1}/6 \\ \delta''(\mathscr{C}_{11}\lambda)^{\lceil |K|/T_l \rceil} T_l^{|u|/2} T_{l-1}^{3+\varepsilon-d/2} |K|^{\varepsilon-d/2} & \text{for } T_{l-1}/6 \leqslant |K| < N_2 T_l \\ \delta''(\mathscr{C}_{11}\lambda)^{\lceil |K|/T_l \rceil} T_l^{|u|/2} T_{l-1}^{3+\varepsilon-d/2} T_l^{-a_2(|K|/T_l)} & \text{for } |K| \geqslant N_2 T_l; \end{cases} \tag{5.3}$$

i.e., $\delta \tilde{\Pi}_l'$ obeys the same bounds as $\delta \tilde{\Pi}_l$ times a factor of $T_{l-1}^{3+\varepsilon-d/2}$. So immediately all the results we proved involving $\delta \tilde{\Pi}_l$ carry over, with this extra factor. Derivatives of $A_2$ can act on the exponential term as well as the $\delta \tilde{\Pi}_l'$ term. In this case, they yield an extra term of at most $\lambda(\mathscr{C}(|K|^2/T_l)^{|u|/2} + \mathscr{C}|K|^{|u|/2})$. Applying (5.3) to bound the sum, and using smallness of $\lambda$ to ensure $(a_2 - \mathscr{C}\lambda) N_2 > d/2 - \varepsilon$, we obtain

$$\begin{aligned} |\partial_k^u A_2| &\leqslant \frac{\delta'}{2} \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{3+\varepsilon-d/2} T_{l-1}^{1+\varepsilon-d/2} \\ &\leqslant \frac{\delta'}{2} \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{2+\varepsilon-d/2}. \end{aligned} \tag{5.4}$$

Inserting this into the definition of $\delta \Gamma_l^{\partial 2}$ immediately yields the result. ∎

LEMMA 5.2. *For any $\delta' > 0$ there exists a choice of $\mathscr{C}_{11}$ such that*

$$|\partial_k^u \delta \Gamma_l^{\mathrm{IF2}}(k, T)| \leqslant \begin{cases} \delta' \mathscr{C}_{11} \lambda T_l^{|u|/2} T_{l-1}^{2+\varepsilon-d/2} & \text{for } T \leqslant T_{l-1} \\ \delta'(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil} T_l^{|u|/2} T^{2+\varepsilon-d/2} & \text{for } T_{l-1} < T \leqslant N_3 T_l \\ \delta'(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil} T_l^{|u|/2} T_l^{-a_3(T/T_l)} & \text{for } T > N_3 T_l, \end{cases} \tag{5.5}$$

*where $k \in U_l$.*

*Proof.* We examine each of the $B_i$ in succession. The result is obvious for $B_1$ from (4.20).

Fix $\delta' > 0$. We write $B_2 \equiv v_1 + v_2$, where the $v_i$ are the two sums in (2.25). First look at $v_1$: for $T \leqslant T_{l-1}$, apply the result from $A_2$ an as upper bound. For $T_{l-1} < T \leqslant N_3 T_l$, an application of (5.3) yields the result. For $T > N_3 T_l$, we note two requirements on $N_3$ (which we will use below when we fix $N_3$): $N_3 > N_2$ and $(a_2 - a_3 - \mathscr{C}\lambda)N_3 > 4 + \varepsilon - d/2$. Applying (5.3) once again gives the result.

For $v_2$ and $T \leqslant T_{l-1}$, we note that $\delta\tilde{\Pi}_l'(K) = 0$ unless $|K| \geqslant T_{l-1}/6$, so we assume this. Then one of $K$ and $I$ is longer than $T_{l-1}/12$. We sum the shorter interval, and use the fact that the size of the longer interval is fixed by that of the shorter. When $I$ is the longer interval, we use

$$|\delta\Gamma_l^{\mathrm{IF1}}(k, I)| \leqslant \delta'\mathscr{C}_{11}\lambda T_{l-1}^{2+\varepsilon-d/2} \tag{5.6}$$

to bound the longer interval, along with

$$\sum_{|K|=1}^{T} |\delta\tilde{\Pi}_l'(k, K) e^{-|K|\,\delta\Gamma_l(k)}| \leqslant \delta'\mathscr{C}_{11}\lambda T_{l-1}^{4+2\varepsilon-d} \tag{5.7}$$

to bound the sum on the shorter interval. When $K$ is the longer interval, we use

$$\begin{aligned} |\delta\tilde{\Pi}_l'(k, K)| \exp(\mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}|K|) &\leqslant \delta'(\mathscr{C}_{11}\lambda)\, T_{l-1}^{3+\varepsilon-d/2}\,|K|^{\varepsilon-d/2} \\ &\leqslant \delta'(\mathscr{C}_{11}\lambda) T_{l-1}^{3+2\varepsilon-d} \end{aligned} \tag{5.8}$$

to bound the longer interval, and the sum on the shorter intervals is bounded by

$$\sum_{|I|=0}^{T} |\delta\Gamma_l^{\mathrm{IF1}}(k, I)| \leqslant \delta'\mathscr{C}_{11}\lambda T_{l-1}^{3+\varepsilon-d/2}.$$

Multiplying these bounds, we obtain the result. A similar argument gives the result for $T_{l-1} < T \leqslant N_3 T_l$.

For $T > N_3 T_l$, we examine

$$\begin{aligned} |v_2|\, T_l^{a_3(T/T_l)} \leqslant \mathscr{C}' \sum_{|K|=0}^{T-1} &[|\delta\tilde{\Pi}_l'(k, K)|\, T_l^{a_3(|K|/T_l)} \exp(\mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}|K|)] \\ &\cdot [|\delta\Gamma_l^{\mathrm{IF1}}(k, I)|\, T_l^{a_3(|I|/T_l)}] \end{aligned} \tag{5.9}$$

but we can bound the last term by $\delta'(\mathscr{C}_{11}\lambda)^{\lceil |I|/T_l \rceil}$ for $|I| > N_3 T_l$ (see (4.20)), and for any $|I| \leqslant N_3 T_l$ we have

$$\begin{aligned} |\delta\Gamma_l^{\mathrm{IF1}}(k, I)|\, T_l^{a_3(|I|/T_l)} &\leqslant \delta'(\mathscr{C}_{11}\lambda)^{\lceil |I|/T_l \rceil}\, |I|^{a_3 N_3 + 2 + \varepsilon - d/2} \\ &\leqslant \delta'(\mathscr{C}_{11}\lambda)^{\lceil |I|/2 \rceil}\, T_l^{a_3 N_3 + 2 + \varepsilon - d/2} \end{aligned} \tag{5.10}$$

so we see that

$$\begin{aligned} |v_2|\, T_l^{a_3(T/T_l)} &\leqslant \delta'^2(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil}\, T_l^{a_3 N_3 + 2 + \varepsilon - d/2} T_{l-1}^{3+\varepsilon-d/2} \\ &\quad \times \left[ \sum_{|K| = T_{l-1}/6}^{N_2 T_l} |K|^{\varepsilon - d/2}\, T_l^{a_3(|K|/T_l)} + \sum_{|K| > N_2 T_l} T_l^{-(a_2 - a_3 - \mathscr{C}\lambda)(|K|/T_l)} \right] \\ &\leqslant \delta'^2(\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil}\, [T_l^{a_3 N_3 + 2 + \varepsilon - d/2} T_{l-1}^{4+2\varepsilon-d+a_3 N_2} \\ &\quad + T_{l-1}^{3+\varepsilon-d/2} T_l^{3+\varepsilon-d/2+a_3 N_3 + a_3 N_2 - (a_2 - \mathscr{C}\lambda) N_2}]. \end{aligned} \tag{5.11}$$

So we fix $N_3$, subject to the above conditions along with $a_3 N_3 < d/2 - 3\varepsilon - \frac{5}{4}$, to obtain

$$|v_2|\, T_l^{a_3(T/T_l)} \leqslant \delta'^2 (\mathscr{C}_{11}\lambda)^{\lceil T/T_l \rceil}. \tag{5.12}$$

So we have completed $B_2$ for $|u| = 0$. The derivatives give the usual factor of $T_l^{|u|/2}$ by the same mechanisms as in Lemma 5.1.

$B_3$ can be handled in exactly the same manner as $B_2$. ▮

The proofs that (i)–(iii) iterate are now complete if we choose $\delta'$ small enough in the various lemmas and $\mathscr{C}_{11}$ large enough.

## 6. Induction Completed

We now close the induction by showing that (iv) holds for $l$. In this section we fix $T \geqslant T_{l+1}$. We estimate the $L_p$ norm by splitting $k$-space into $[l] + 1$ disjoint regions:

$$\begin{aligned} &U_l \equiv \{k\colon |D(k)| \geqslant \delta_l\} \\ &U_\gamma \equiv \{k\colon |D(k)| \in [\delta_\gamma, \delta_{\gamma+1})\} \qquad \text{for} \quad \gamma = l - [l], \ldots, l-1. \end{aligned} \tag{6.1}$$

We first handle $U_l$, looking at $k$ in the neighborhood of the origin (analogous results hold for the rest of $U_l$). For these values of $k$ we have the expression

$$C_l(k, T) = D(k)^T \exp(T\Gamma_l(k) + 2\Gamma_l^{\partial}(k) + \Gamma_l^{\mathrm{IF}}(k, T)) \tag{6.2}$$

which, since we have proven (i)–(iii) iterate, satisfies

$$|C_l(k, T)| \leqslant |D(k)|^T \exp(T\Gamma_l(0) + 2\Gamma_l^{\partial}(0) + \Gamma_l^{\mathrm{IF}}(0, T) + \mathscr{C}\lambda T k^2). \tag{6.3}$$

Also, from (i)–(iii) it is not hard to see that when $|u| = 1$,

$$\begin{aligned} &|\partial_k^u \exp(T\Gamma_l(k) + 2\Gamma_l^{\partial}(k) + \Gamma_l^{\mathrm{IF}}(k, T))| \\ &\qquad \leqslant \mathscr{C}\lambda T\, |k|\, \exp(T\Gamma_l(k) + 2\Gamma_l^{\partial}(k) + \Gamma_l^{\mathrm{IF}}(k, T)), \end{aligned} \tag{6.4}$$

and when $|u| = 2$ we have

$$\begin{aligned} &|\partial_k^u \exp(T\Gamma_l(k) + 2\Gamma_l^{\partial}(k) + \Gamma_l^{\mathrm{IF}}(k, T))| \\ &\qquad \leqslant \lambda(\mathscr{C}T^2 k^2 + \mathscr{C}T) \exp(T\Gamma_l(k) + 2\Gamma_l^{\partial}(k) + \Gamma_l^{\mathrm{IF}}(k, T)). \end{aligned} \tag{6.5}$$

Using these bounds, it is easy to see that

$$\left[\int_{k \in U_l} |\partial_k^u C_l(k, T)|^p \, d^d k\right]^{1/p} \leqslant \mathscr{C} T^{|u|/2 - d/2p} \exp(T\Gamma_l(0) + 2\Gamma_l^{\partial}(0) + \Gamma_l^{\mathrm{IF}}(0, T)). \tag{6.6}$$

Now, the constant here is some universal constant plus corrections of $O(\lambda)$, and only the corrections depend on $\mathscr{C}_{11}$ and $\mathscr{C}_{12}$. By taking lambda small and choosing $\mathscr{C}_{12}$ appropriately, we can arrange for the $\mathscr{C}$ in (6.6) to be $\frac{1}{2}\mathscr{C}_{12}$.

Next examine the integral over $U_\gamma$, $\gamma \leqslant l-1$. For now take $\gamma = l-1$ and $|u| = 0$, and work in the region of $U_\gamma$ closest to the origin (analogous arguments hold for the rest of $U_\gamma$). Recalling (2.3), we write

$$C_l(k, T)\exp(-T\Gamma_l(0) - 2\Gamma_l^\partial(0) - \Gamma_l^{\mathrm{IF}}(0, T))$$

$$= \exp(-2\Gamma_l^\partial(0) - \Gamma_l^{\mathrm{IF}}(0, T)) \sum_{n=0} \sum_{\{n_i\}} [C_{l-1}(k, K_0) e^{-|K_0|\Gamma_l(0)}]$$

$$\times \prod_{i=1}^{n} ([\delta\Pi_l(k, I_i) e^{-|I_i|\Gamma_l(0)}][C_{l-1}(k, K_i) e^{-|K_i|\Gamma_l(0)}]). \tag{6.7}$$

Now, in $U_{l-1}$ we can apply our expansion to $C_{l-1}$, so it is not hard to see that

$$|C_{l-1}(k, K)| e^{-|K|\Gamma_l(0)} \leqslant \mathscr{C}(D(k) e^{\mathscr{C}\lambda k^2 + \mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}})^{|K|}$$

$$\leqslant (e^{(-\mathscr{C}+\mathscr{C}\lambda)k^2 + \mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}})^{|K|}. \tag{6.8}$$

Also, we can use (3.11) to see that

$$|\delta\Pi_l(k, I)|\, e^{-|I|\Gamma_l(0)} \leqslant \begin{cases} 0 & \text{for } |I| \leqslant T_{l-1}/6 \\ (\mathscr{C}\lambda)^{\lceil |I|/T_l \rceil} |I|^{-d/2} & \text{for } T_{l-1}/6 < |I| \leqslant N_1 T_l \\ (\mathscr{C}\lambda)^{\lceil |I|/T_l \rceil} T_l^{-a_1(|I|/T_l)} & \text{for } |I| > N_1 T_l. \end{cases} \tag{6.9}$$

Next we use the fact that the longest interval of $\{I_i, K_j\}$ is longer than $T/(2n+1)$, and its length is fixed by those of the $2n$ shorter intervals. So as an upper bound we let the sums over the shorter intervals be unrestricted,

$$\sum_{|I|=1}^{\infty} |\delta\Pi_l(k, I)|\, e^{-|I|\Gamma_l(0)} \leqslant \mathscr{C}\lambda T_{l-1}^{1-d/2} \tag{6.10}$$

$$\sum_{|K|=0}^{\infty} |C_{l-1}(k, K)|\, e^{-|K|\Gamma_l(0)} \leqslant [\mathscr{C}k^2 - \mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}]^{-1}$$

$$\leqslant \left[\mathscr{C}\frac{\ln T_l}{T_l}\right]^{-1}, \tag{6.11}$$

where in the last line we have used $k^2 \geqslant \mathscr{C}(\ln T_l/T_l)$. Inserting these into (6.7), we find

$$|C_l(k, T)| \exp(-T\Gamma_l(0) - 2\Gamma_l^\partial(0) - \Gamma_l^{\mathrm{IF}}(0, T))$$

$$\leqslant (e^{-\mathscr{C}k^2 + \mathscr{C}\lambda T_{l-1}^{1+\varepsilon-d/2}})^T + \mathscr{C} \sum_{n=1}^{3(T/T_{l-1})} T_l^2 (e^{-\mathscr{C}(\ln T_l/T_l)})^{T/(2n+1)}$$

$$\times \left(\mathscr{C}\frac{T_l}{\ln T_l}\right)^n (\mathscr{C}\lambda T_{l-1}^{1-d/2})^n$$

$$\leqslant T_l^{-\mathscr{C}(T/T_l)} + \mathscr{C} \sum_{n=1}^{3(T/T_{l-1})} T_l^2 T_l^{-\mathscr{C}(T/nT_l)} (\mathscr{C}\lambda)^n, \tag{6.12}$$

where the upper limit on the sum arose, since $\delta\Pi_l(k, I) = 0$ if $I$ is short enough. We bound the sum by (number of terms) · (maximum value):

$$|C_l(k, T)| \exp(-T\Gamma_l(0) - 2\Gamma_l^{\partial}(0) - \Gamma_l^{\mathrm{IF}}(0, T))$$
$$\leqslant T_l^{-\mathscr{C}(T/T_l)} + \exp - \left[ -\mathscr{C} \ln(\mathscr{C}\lambda) \frac{T}{T_l} \ln T_l \right]^{1/2}. \qquad (6.13)$$

In the case $|u| > 0$, we can use (i)–(iii) to bound derivatives of all of the terms on the right-hand side of (6.7). It is evident that all factors due to these derivatives are dominated by the exponential in (6.13). Since $T \geqslant T_{l+1}$, with $\lambda$ small and an appropriate choice of $\mathscr{C}_{12}$, we can arrange for the contribution from $U_{l-1}$ to the norm to be less than any inverse power of $T$ times any inverse power of $l$, so in particular,

$$\left[ \int_{k \in U_{l-1}} |\partial_k^u C_l(k, T)|^p \, d^dk \right]^{1/p}$$
$$\leqslant \frac{1}{2} \mathscr{C}_{12} T^{|u|/2 - d/2p} \exp(T\Gamma_l(0) + 2\Gamma_l^{\partial}(0) + \Gamma_l^{\mathrm{IF}}(0, T)) \left( \frac{6}{l^2 \pi^2} \right). \qquad (6.14)$$

We can handle the other regions $U_\gamma$ in the same fashion: e.g., for $\gamma = l - 2$ we write

$$C_l(k, T) = \sum_{n=0} \sum_{\{I_i\}} C_{l-2}(k, K_0)$$
$$\times \prod_{i=1}^{n} ([\delta\Pi_l(k, I_i) + \delta\Pi_{l-1}(k, I_i)] \, C_{l-2}(k, K_i)) \qquad (6.15)$$

and proceed as before to find an expression like (6.14). We sum these terms, along with (6.6), to see that (iv) holds for $l$, and Proposition 3.1 is proven.

## 7. Results

The proofs of Theorems 1.1 and 1.2 are now almost trivial. With the completion of our induction we have an expression for the full propagator valid for $T \leqslant T'$,

$$C(k, T) = C_{l(T')}(k, T)$$
$$= (D(k) e^{\Gamma_{l(T')}(k)})^T \exp(2\Gamma_{l(T')}^{\partial}(k) + \Gamma_{l(T')}^{\mathrm{IF}}(k, T)) \qquad (7.1)$$

with $k \in U_{l(T')}$, along with bounds for the quantities in the last line. This proves Theorem 1.1 if we choose

$$\Gamma_{T'}^{\mathrm{edge}}(k, T) \equiv 2\Gamma_{l(T')}^{\partial}(k) + \Gamma_{l(T')}^{\mathrm{IF}}(k, T). \qquad (7.2)$$

We can also compute the mean square end-to-end distance (given by (1.5)). We find

$$\langle \omega(T)^2 \rangle = T - \sum_{\mu=1}^{d} \partial^2_{k_\mu}(T\Gamma_{l(T)}(k) + 2\Gamma^{\partial}_{l(T)}(k) + \Gamma^{\mathrm{IF}}_{l(T)}(k, T))|_{k=0}. \tag{7.3}$$

We have bounds on the edge terms, for any $|u| = 2$

$$|\partial^u_k(2\Gamma^{\partial}_{l(T)}(k) + \Gamma^{\mathrm{IF}}_{l(T)}(k, T))|_{k=0} \leqslant \mathscr{C}\lambda T^{\max(3+2\varepsilon-d/2,0)} \tag{7.4}$$

along with, for $T \leqslant T'$,

$$|\partial^u_k \Gamma_{l(T')}(k) - \partial^u_k \Gamma_{l(T)}(k)|_{k=0} \leqslant \mathscr{C}\lambda T^{2+2\varepsilon-d/2} \tag{7.5}$$

which tells us that $\partial^u_k \Gamma_{l(T)}(k)|_{k=0}$ converges to a limit $\partial^u_k \Gamma_\infty$ of order $\lambda$ as $T \to \infty$ (which is zero for the mixed partials), and hence that

$$\partial^2_{k_\mu} \Gamma_{l(T)}(k)|_{k=0} = \mathscr{C}\lambda + O(T^{2+2\varepsilon-d/2}). \tag{7.6}$$

Putting these results together yields the first result of Theorem 1.2. To prove the second, we note that for any fixed values of $k$, $t$ there exists an $s_0$ such that we can apply our expansion to the terms in the scaling limit of the endpoint for every $s \geqslant s_0$. So the limit becomes

$$\lim_{s \to \infty} \frac{(D(k/\sqrt{s})e^{\Gamma_{l(st)}(k/\sqrt{s})})^{st}\, e^{\Gamma^{\mathrm{edge}}_{st}(k/\sqrt{s}, st)}}{(e^{\Gamma_{l(st)}(0)})^{st}\, e^{\Gamma^{\mathrm{edge}}_{st}(0, st)}}. \tag{7.7}$$

Now, the edge terms cancel in the limit $s \to \infty$ (using Taylor's theorem along with Theorem 1.1), and $D(k/\sqrt{s})^{st} \Rightarrow \exp(k^2 t/2d)$. Also, by Taylor's theorem,

$$\Gamma_{l(st)}\left(\frac{k}{\sqrt{s}}\right) = \Gamma_{l(st)}(0) + \frac{1}{2} \sum_{|u|=2} \partial^u_k \Gamma_{l(st)}(\eta_s) \frac{k^u}{s}, \tag{7.8}$$

where $\eta_s$ is on the line segment stretching from 0 to $k/\sqrt{s}$. We can see that

$$\lim_{s \to \infty} \partial^u_k \Gamma_{l(st)}(\eta_s) = \partial^u_k \Gamma_\infty \tag{7.9}$$

by using the fact that $|\partial^u_k\, \delta\Gamma_l(k)|$ is continuous in $k$ for $k \in U_l$ and exponentially small in $l$, along with the comment following (7.5). This proves the second part of Theorem 1.2. ∎

## APPENDIX A

In this appendix we derive the bounds on $\partial_k^u \, \delta \Pi_l(k, T)$ used in (3.11). We closely follow a similar derivation (of the Laplace-transformed quantity) in [1]. We begin by bounding $\Pi_l(k, T)$, defined by

$$\Pi_l(k, T) = \left(\frac{1}{2d}\right)^T \sum_{|\omega| = T} e^{ik \cdot \omega(T)} \sum_{G^c} \prod_{(s,t) \in G^c} (-\lambda \delta_{\omega_s \omega_t}), \tag{A.1}$$

where $G^c$ are connected graphs consisting of bonds $(s, t)$ on $[0, T]$ satisfying $|s-t| \leqslant T_l$. Next we use the fact that every connected graph can be uniquely decomposed into a *lace* $\mathscr{L}$ and a set of bonds $\mathscr{K}(\mathscr{L})$ *compatible* with the lace. Here a lace is defined to be a minimally connected subgraph, i.e., a graph of the form shown in Fig. 2, and an $m$-lace is a lace consisting of $m$ bonds. Bonds $(i, j)$ compatible with a lace $\mathscr{L}$ are those such that the lace of $\mathscr{L} \cup (i, j)$ is equal to $\mathscr{L}$ (see [1] for a more precise definition). Hence, we can split the sum over connected graphs into a sum over laces followed by a sum over bonds compatible with the lace, which we then resum:

$$\begin{aligned}\sum_{G^c} \prod_{(s,t) \in G^c} (-\lambda \delta_{\omega_s \omega_t}) \\ = \sum_{m \geqslant 1} \sum_{\substack{m\text{-laces} \\ \mathscr{L}}} \prod_{(i,j) \in \mathscr{L}} (-\lambda \delta_{\omega_i \omega_j}) \prod_{(s,t) \in \mathscr{K}(\mathscr{L})} (1 - \lambda \delta_{\omega_s \omega_t}).\end{aligned} \tag{A.2}$$

Note that an $m$-lace is uniquely specified by a set of $2m-1$ integers $\{n_i\}$ subject to the restrictions

$$\begin{aligned} n_i &\geqslant 1 \qquad \text{for all } i \\ \sum_{i=1}^{2m-1} n_i &= T \\ n_1 + n_2 &\leqslant T_l \\ n_i + n_{i+1} + n_{i+2} &\leqslant T_l \qquad \text{for } i = 2, 4, ..., 2m-4 \\ n_{2m-2} + n_{2m-1} &\leqslant T_l. \end{aligned} \tag{A.3}$$


FIG. 2. A 3-lace.

As an upper bound, we discard those factors of $(1-\lambda\delta_{\omega_s\omega_t})$ that connect different intervals $n_i$. The effect of this is to break up our walk $\omega$ into $2m-1$ independent subwalks, each of which propagates according to $C_l$, and with restrictions on the various endpoints given by the lace:

$$|\Pi_l(k,T)| \leqslant \frac{1}{1-\lambda} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \sum_{\{x_i\}} C_l(x_1, n_1)\, C_l(x_2 - x_1, n_2) \cdots \\ \times C_l(x_{2m-1} - x_{2m-2}, n_{2m-1})\, \delta_{0x_2}\, \delta_{x_1x_4} \cdots \delta_{x_{2m-3}x_{2m-1}}. \tag{A.4}$$

The self-intersections of the walk, governed by the lace, occur in a simple pattern of which we can take advantage. Define the operators which act as multiplication and convolution by a propagator,

$$\mathbf{M}_n : f(x) \mapsto C_l(x, n)\, f(x) \tag{A.5}$$

$$\mathbf{C}_n : f(x) \mapsto \sum_y C_l(x-y, n)\, f(y) \tag{A.6}$$

which give us

$$|\Pi_l(k,T)| \leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} (\mathbf{C}_{n_1}\mathbf{M}_{n_2} \cdots \mathbf{C}_{n_{2m-3}}\mathbf{M}_{n_{2m-2}} C_l(\cdot, n_{2m-1}))(0) \\ \leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \prod_{j=1}^{2m-1} \|C_l(\cdot, n_j)\|^{x\text{-space}}_{* = 2 \text{ or } \infty,\ \text{one } \infty}. \tag{A.7}$$

(In the last line we have used Lemma 5.8 in [1], an application of Hölder's inequality). The norms here are all $L_2$ except for one which is $L_\infty$ (we have the freedom to choose which one). Next use $C_l(x,n) \leqslant C_{l'}(x,n)$ for all $x$, $n$, and $l \geqslant l'$. This, along with the fact that $n_i \leqslant T_l$ and the Hausdorff Young theorem, yields

$$|\Pi_l(k,T)| \leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \prod_{j=1}^{2m-1} \|C_{l(n_j)-1}(\cdot, n_j)\|^{x\text{-space}}_{* = 2 \text{ or } \infty,\ \text{one } \infty} \\ \leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \prod_{j=1}^{2m-1} \|C_{l(n_j)-1}(\cdot, n_j)\|^{k\text{-space}}_{* = 2 \text{ or } 1,\ \text{one } 1}. \tag{A.8}$$

Next consider $\delta\Pi_l$. The terms in $\Pi_l$ and $\Pi_{l-1}$ in which all bonds are of length $|s-t| \leqslant T_{l-1}/2$ exactly cancel each other. Hence each term in $\Pi_l$, $\Pi_{l-1}$ that contributes must have at least one long bond $|s-t| \geqslant T_{l-1}/2$. Denote by $\Pi'_l$ and $\Pi'_{l-1}$ the sum of such terms, and use the bound $|\delta\Pi_l| \leqslant |\Pi'_l| + |\Pi'_{l-1}|$. Next we note that the above condition implies that at least one $n_i \geqslant T_{l-1}/6$, so we require this to obtain an upper bound. We see that

$$|\delta\Pi_l(k,T)| \leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \prod_{j=1}^{2m-1} \|C_{l(n_j)-1}(\cdot, n_j)\|^{k\text{-space}}_{* = 2 \text{ or } 1,\ \text{one } 1}, \tag{A.9}$$

where, as an upper bound and for later convenience, we relax somewhat the conditions (A.3) on the $\{n_i\}$ and only require all $n_i \in [1, T_l]$, $\sum n_i = T$, and at least one $n_i \geqslant T_{l-1}/6$.

This bound can easily be modified to allow $k$-derivatives. We write the displacement of the full walk $\omega(T)$ in $\exp(ik \cdot \omega(T))$ as a sum of displacements of the subwalks, and act on this product according to the Leibnitz rule. We find

$$\begin{aligned} |\partial_k^u \delta\Pi_l(k, T)| &\leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \sum_{\{u_i\}} \prod_{j=1}^{2m-1} \|(\cdot)^{u_j} C_{l(n_j)-1}(\cdot, n_j)\|_{*=2 \text{ or } \infty, \text{ one } \infty}^{x\text{-space}} \\ &\leqslant \mathscr{C} \sum_{m \geqslant 1} \lambda^m \sum_{\{n_i\}} \sum_{\{u_i\}} \prod_{j=1}^{2m-1} \|\partial_k^{u_j} C_{l(n_j)-1}(\cdot, n_j)\|_{*=2 \text{ or } 1, \text{ one } 1}^{k\text{-space}}, \end{aligned} \tag{A.10}$$

where the sum over $\{u_i\}$, with $\sum |u_i| = |u|$, is over the possible ways of distributing the derivatives in applying the Leibnitz rule.

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