O(3) SYMMETRIC MERONS IN AN SU(3) YANG–MILLS THEORY*

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ABSTRACT. We investigate irreducible, $O(3)$ symmetric multiple-meron solutions to the classical $SU(3)$ Yang–Mills equations in four-dimensional Euclidean space. The solutions have topological charge density equal to a sum of delta-functions with integer coefficients, and correspond to solutions of a system of two coupled singular elliptic equations. We prove the existence of two-meron solutions of the coupled system.

Infinite action solutions to the classical Yang–Mills equations in Euclidean space with topological charge density concentrated at points have been used by Glimm and Jaffe [1], and Callen *et al.* [2] in models for quark confinement. The first such meron solution was found by deAlfaro *et al.* [3] and was generalized to multiple merons on a line by Glimm and Jaffe [4]. Merons have only been studied in an $SU(2)$ gauge theory, which can, of course, be embedded in the $SU(3)$ theory believed to describe strong interactions. In this paper, we use the $O(3)$ symmetric ansatz of Bais and Weldon [5] to investigate $SU(3)$ merons which do not arise from an embedding of $SU(2)$ in $SU(3)$. We call these irreducible.

We find some new phenomena in $SU(3)$. Finding irreducible meron solutions is reduced to solving a pair of coupled elliptic equations, analogous to the equation $r^2 \Delta\psi = \psi^3 - \psi$ in the $SU(2)$ case [4]. We prove that the equations have solutions corresponding to two-meron configurations. The charge density is a sum of unit delta-functions instead of the half delta-functions found in the $SU(2)$ case. Although the word meron originated from fractional charges, we use it for integer charges as well because the solutions are similar to fractionally charged merons in other respects. They have regular, singular points and nonintegrable action.

Following Bais and Weldon [5], we define the following Hermitian, traceless matrices (a, b, l, m take values 1, 2, 3):

$$\begin{aligned}(L_a)_{lm} &= i\epsilon_{lam}, \\ (Q_{ab})_{lm} &= \delta_{al}\delta_{bm} + \delta_{am}\delta_{bl} - \tfrac{2}{3}\,\delta_{ab}\delta_{lm}.\end{aligned} \tag{1}$$

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$$M^+ = \tfrac{1}{2} L_a \hat{r}_a + \tfrac{1}{4} Q_{ab}\hat{r}_a\hat{r}_b,$$
$$M^- = \tfrac{1}{2} L_a \hat{r}_a - \tfrac{1}{4} Q_{ab}\hat{r}_a\hat{r}_b,$$
$$K_a^+ = (\delta_{ab} - \hat{r}_a\hat{r}_b)(\tfrac{1}{2} L_b + \tfrac{1}{2} Q_{bc}\hat{r}_c),$$
$$K_a^- = (\delta_{ab} - \hat{r}_a\hat{r}_b)(\tfrac{1}{2} L_b - \tfrac{1}{2} Q_{bc}\hat{r}_c). \tag{2}$$

Whenever ± appears in an equation, the equation is to be interpreted as two equations: one with ± replaced by +, and one with ± replaced by −. We denote by $\vec{K}^{\pm}$ the vector whose components are $K_a^{\pm}$. The following relations among the $M^{\pm}$ and $\vec{K}^{\pm}$ will be needed [5] :

$$[\vec{K}^{\pm}, M^{\pm}] = i\hat{r} \times \vec{K}^{\pm},$$
$$[\vec{K}^-, M^+] = [\vec{K}^+, M^-] = [M^+, M^-] = \vec{K}^+ \cdot \vec{K}^- - \vec{K}^- \cdot \vec{K}^+ = 0,$$
$$\vec{K}^+ \times \vec{K}^+ = i\hat{r}(2M^+ - M^-), \qquad \vec{K}^- \times \vec{K}^- = i\hat{r}(2M^- - M^+),$$
$$\vec{K}^+ \times \vec{K}^- = 0,$$
$$\vec{\nabla} M^{\pm} = \frac{1}{r}\vec{K}^{\pm},$$
$$\vec{\nabla} \times \vec{K}^{\pm} = \frac{\hat{r}}{r} \times \vec{K}^{\pm},$$
$$\vec{\nabla} \cdot \vec{K}^+ = \frac{1}{r}(-4M^+ + 2M^-), \qquad \vec{\Delta} \cdot \vec{K}^- = \frac{1}{r}(-4M^- + 2M^+),$$
$$\mathrm{Tr}\, M^{\pm}M^{\pm} = 2/3, \qquad \mathrm{Tr}\, M^+M^- = 1/3,$$
$$\mathrm{Tr}\, K_a^{\pm}K_b^{\pm} = \delta_{ab} - \hat{r}_a\hat{r}_b,$$
$$\mathrm{Tr}\, K_a^+K_b^- = 0. \tag{3}$$

The $O(3)$ symmetric ansatz for the gauge field [5] is

$$A_0 = iM^+a_0^+ + iM^-a_0^-,$$
$$\vec{A} = iM^+\hat{r}a_r^+ - i\hat{r} \times \vec{K}^+ [(1 + \phi_1^+)/r] + i\vec{K}^+(\phi_0^+/r) + iM^-\hat{r}a_r^- - i\hat{r} \times \vec{K}^-[(1 + \phi_1^-)/r)] + i\vec{K}^-(\phi_0^-/r), \tag{4}$$

where $a_0^{\pm}$, $a_r^{\pm}$, $\phi_0^{\pm}$, and $\phi_1^{\pm}$ depend only on $r = |\vec{r}|$ and t. The ansatz is consistent with gauge transformations $\exp(i\alpha^+M^+)$, $\exp(i\alpha^-M^-)$ where $\alpha^{\pm}$ depends on r and t only. This is a natural generalization of Witten's ansatz in $SU(2)$ [6]. We can regard $\vec{\phi}^{\pm} = \begin{pmatrix}\phi_0^{\pm}\\ \phi_1^{\pm}\end{pmatrix}$ as a two component Higgs field interacting with a $U(1)$ gauge field $a^{\pm} = a_0^{\pm}\mathrm{d}t + a_r^{\pm}\mathrm{d}r$ in the half plane $\mathbb{R}^2_+ = \{(r, t)|r \geqslant 0\}$ with metric $r^{-2}(\mathrm{d}r^2 + \mathrm{d}t^2)$. The covariant derivatives are defined as $\mathrm{D}^{\pm}\vec{\phi}^{\pm} = \mathrm{d}\vec{\phi}^{\pm} + a^{\pm}\epsilon\vec{\phi}^{\pm}$ where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The $SU(3)$ field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ break into an electric part $E_i = F_{0i}$ and a magnetic part $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ [5]:

$$\begin{aligned}
\vec{E} &= iM^+\hat{r}f^+_{0r} - i\frac{\hat{r}}{r}\times\vec{K}^+\mathrm{D}_0^+\phi_1^+ + i\frac{\vec{K}^+}{r}\mathrm{D}_0^+\phi_0^+ + \\
&\quad + iM^-\hat{r}f^-_{0r} - i\frac{\hat{r}}{r}\times\vec{K}^-\mathrm{D}_0^-\phi_1^- + i\frac{\vec{K}^-}{r}\mathrm{D}_0^-\phi_0^-, \\
\vec{B} &= iM^+\frac{\hat{r}}{r^2}(2|\vec{\phi}^+|^2 - |\vec{\phi}^-|^2 - 1) + i\frac{\hat{r}}{r}\times\vec{K}^+\mathrm{D}_r^+\phi_0^+ + \frac{i\vec{K}^+}{r}\mathrm{D}_r^+\phi_1^+ - \\
&\quad - iM^-\frac{\hat{r}}{r^2}(2|\vec{\phi}^-|^2 - |\vec{\phi}^+|^2 - 1) + i\frac{\hat{r}}{r}\times\vec{K}^-\mathrm{D}_r^-\phi_0^- + \frac{i\vec{K}^-}{r}\mathrm{D}_r^-\phi_1^-.
\end{aligned} \tag{5}$$

Here $f^{\pm} = \mathrm{d}a^{\pm}$ are the field strengths defined in $\mathbb{R}^2_+$, i.e. $f^{\pm}_{0r} = \partial_0 a_r^{\pm} - \partial_r a_0^{\pm}$. The $SU(3)$ topological charge density is

$$\Omega(\vec{r}, t) = \frac{-1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma} = \frac{1}{4\pi^2 r^2}(f^+_{0r} + k^+_{0r} + f^-_{0r} + k^-_{0r}). \tag{6}$$

Here, $k^{\pm} = \mathrm{d}I^{\pm}$ in $\mathbb{R}^2_+$, with $I^{\pm} = \phi_1^{\pm}\mathrm{D}^{\pm}\phi_0^{\pm} - \phi_0^{\pm}\mathrm{D}^{\pm}\phi_1^{\pm}$, i.e. $k^{\pm}_{0r} = \partial_0 I_r^{\pm} - \partial_r I_0^{\pm}$. The charge density is the 4-divergence of the current

$$\begin{aligned}
J_\mu &= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma), \\
\vec{J} &= -\frac{1}{4\pi^2 r^2}(a_0^+ + I_0^+ + a_0^- + I_0^-)\hat{r}, \\
J_0 &= \frac{1}{4\pi^2 r^2}(a_r^+ + I_r^+ + a_r^- + I_r^-).
\end{aligned} \tag{7}$$

After integrating over angular variables, the charge density in $\mathbb{R}^2_+$ is

$$q(r, t) = \frac{1}{\pi}(k^+_{0r} + f^+_{0r} + k^-_{0r} + f^-_{0r}). \tag{8}$$

The Yang–Mills equations $\partial_\mu \check{F}_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0$ become the following in our ansatz:

$$\begin{aligned}
*\mathrm{d}{*f^+} &= 4I^+ - 2I^-, \\
*\mathrm{d}{*f^-} &= 4I^- - 2I^+.
\end{aligned} \tag{9}$$

$$*D^+ * D^+ \phi_i^+ - \phi_i^+(2|\vec{\phi}^+|^2 - |\vec{\phi}^+|^2 - 1) = 0,$$
$$*D^- * D^- \phi_i^- - \phi_i^-(2|\vec{\phi}^-|^2 - |\vec{\phi}^+|^2 - 1) = 0, \tag{10}$$

where $(*d*f^{\pm})_0 = r^2 \partial_r f^{\pm}_{r0} + 2rf^{\pm}_{r0}$, $(*d*f^{\pm})_r = r^2 \partial_0 f^{\pm}_{0r}$, and $*D^{\pm} *D^{\pm} = r^2 D^{\pm}_{\alpha} D^{\pm}_{\alpha}$, $(\alpha = 0, r)$.

We look for solutions to (9) and (10) with

$$Q(\vec{r}, t) = \sum_{i=1}^{n} \alpha_i \delta(\vec{r}) \delta(t - t_i),$$
$$q(r, t) = \sum_{i=1}^{n} 2\alpha_i \delta(r) \delta(t - t_i). \tag{11}$$

t_i are the positions of the merons on the t-axis, $(-\infty = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty)$ and α_i are their charges (allowing $\alpha_i = 0$). Then (8) and $k^{\pm} = dl^{\pm}$ imply that (9) and (11) are equivalent to the following equations:

$$l^+ = \tfrac{1}{6} *d*(2f^+ + f^-), \qquad l^- = \tfrac{1}{6} *d*(2f^- + f^+). \tag{12}$$

$$\left[\left(\frac{d*d*}{2\pi} + \frac{1}{\pi} \right) (f^+ + f^-) \right]_{0r} = q(r, t) = \sum_{i=1}^{n} 2\alpha_i \delta(r) \delta(t - t_i). \tag{13}$$

One solution of (12) and (13) is

$$f^{\pm}_{0r} = \sum_{i=1}^{n} 2\pi \beta_i^{\pm} \delta(r) \delta(t - t_i), \qquad l^{\pm} = 0, \qquad \beta_i^+ + \beta_i^- = \alpha_i, \tag{14}$$

because for this $f^{\pm}$, $*d*f^{\pm} = 0$ due to the r^2 in the metric. In what follows, we assume $f^{\pm}_{0r}$ is given by (14), and reduce the remaining Yang–Mills equations (10) to two coupled elliptic equations for two real-valued functions of r and t. We will find that in gauges consistent with (4), $\beta_i^{\pm}$ must be integers in order for A_μ to be regular away from the merons.

Following Glimm and Jaffe [4], we find an $a^{\pm}$ such that $f^{\pm} = da^{\pm}$ where $f^{\pm}$ is as in (14):

$$a^{\pm} = d\theta^{\pm}, \qquad \theta^{\pm}(r, t) = \sum_{i=1}^{n} \beta_i^{\pm} \arg(-t + t_i + ir). \tag{15}$$

$\vec{\phi}^{\pm}$ can be rotated into a single direction by defining

$$\vec{\psi}^{\pm} = \Theta^{\pm} \vec{\phi}^{\pm}, \qquad \Theta^{\pm} = \begin{pmatrix} \cos\theta^{\pm} & \sin\theta^{\pm} \\ -\sin\theta^{\pm} & \cos\theta^{\pm} \end{pmatrix}. \tag{16}$$

$\Theta^{\pm}$ rotates the covariant derivative into the ordinary derivative because $d\Theta^{\pm} = a^{\pm}\epsilon\Theta^{\pm}$ and $\Theta^{\pm}\epsilon(\Theta^{\pm})^{-1} = \epsilon$, so that $\Theta^{\pm}D(\Theta^{\pm})^{-1} = \Theta^{\pm}(d + a^{\pm}\epsilon)(\Theta^{\pm})^{-1} = d$. Moreover, $\Theta^{\pm} * (\Theta^{\pm})^{-1} = *$, and $|\vec{\phi}^{\pm}|^2 = |\vec{\psi}^{\pm}|^2$ so that (10) becomes

$$
\begin{aligned}
&*d*d\psi_i^+ - \psi_i^+(2|\vec{\psi}^+|^2 - |\vec{\psi}^-|^2 - 1) = 0,\\
&*d*d\psi_i^- - \psi_i^-(2|\vec{\psi}^-|^2 - |\vec{\psi}^+|^2 - 1) = 0,
\end{aligned}
\tag{17}
$$

where $*d*d = r^2\Delta$. The circulation density of $\vec{\psi}^{\pm}$ is

$$
\epsilon_{ij}\psi_i^{\pm}d\psi_j^{\pm} = \epsilon_{ij}\phi_i^{\pm}D^{\pm}\phi_j^{\pm} = I^{\pm} = 0. \tag{18}
$$

This implies that $\vec{\psi}^{\pm}$ maintains a constant direction in connected sets where $\vec{\psi}^{\pm} \neq 0$. If $\vec{\psi}^{\pm}$ solves (17), it can, at worst, change direction by π across curves where $\vec{\psi}^{\pm} = 0$. This means that (17) is equivalent to the following system:

$$
\begin{aligned}
&r^2\Delta\psi^+ - \psi^+(2(\psi^+)^2 - (\psi^-)^2 - 1) = 0,\\
&r^2\Delta\psi^- - \psi^-(2(\psi^-)^2 - (\psi^+)^2 - 1) = 0,
\end{aligned}
\tag{19}
$$

where $\psi^{\pm}$ is the component of $\vec{\psi}^{\pm}$ along a fixed direction which is one of the two possible directions of $\vec{\psi}^{\pm}$.

The action density for the $SU(3)$ field is given in [5]:

$$
\begin{aligned}
L(\vec{r}, t) &= -\tfrac{1}{4}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu}\\
&= r^{-2}(D_\alpha^+\phi_i^+)^2 + r^{-2}(D_\alpha^-\phi_i^-)^2 + \tfrac{1}{3}(f_{0r}^+)^2 + \tfrac{1}{3}(f_{0r}^-)^2 + \tfrac{1}{3}f_{0r}^+f_{0r}^- +\\
&\quad + r^{-4}(1 - |\vec{\phi}^+|^2 - |\vec{\phi}^-|^2 - |\vec{\phi}^+|^2|\vec{\phi}^-|^2 + |\vec{\phi}^+|^4 + |\vec{\phi}^-|^4).
\end{aligned}
\tag{20}
$$

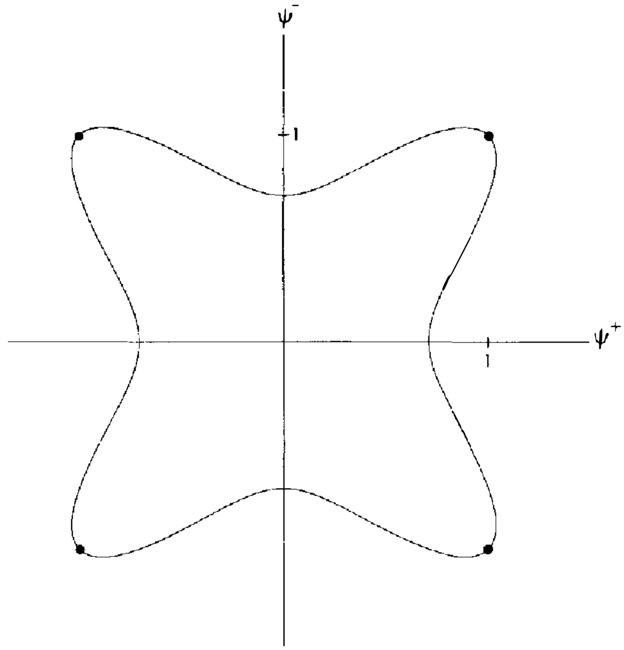
For the case $f_{0r}^{\pm} = 0$ at $r > 0$, we integrate $L(\vec{r}, t)$ over angular variables and express the result in terms of ψ^+ and ψ^-:

$$
\begin{aligned}
&l(r, t) = 8\pi(\tfrac{1}{2}|\vec{\Delta}\psi^+|^2 + \tfrac{1}{2}|\vec{\Delta}\psi^-|^2 + (r^{-2})V(\psi^+, \psi^-),\\
&V(\psi^+, \psi^-) = \tfrac{1}{2}(1 - (\psi^+)^2 - (\psi^-)^2 - (\psi^+)^2(\psi^-)^2 + (\psi^+)^4 + (\psi^-)^4) \geqslant 0.
\end{aligned}
\tag{21}
$$

The potential achieves its minimum $V = 0$ at the four points $|\psi^+| = |\psi^-| = 1$. Along the line $\psi^+ = \chi\cos\omega$, $\psi^- = \chi\sin\omega$, V is the double-well potential

$$
V(\omega, \chi) = \frac{1}{2} - \frac{1}{a^2} + \frac{a^2}{16}\left(\frac{4}{a^2} - \chi^2\right)^2, \tag{22}
$$

where $a^2 = 5 + 3\cos 4\omega$ (Figure 1).



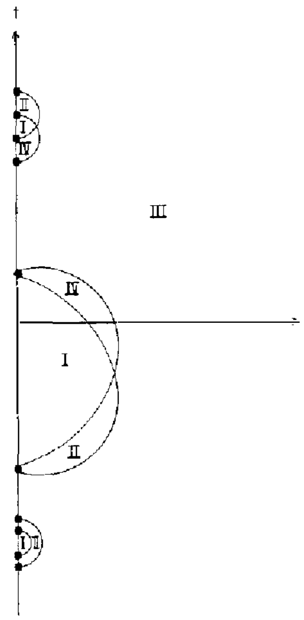


Fig. 1. The potential $V(\psi^+, \psi^-)$. Dots indicate absolute minima. Curve indicates minima along rays through the origin.
Fig. 2. A possible set of strings connecting merons. The central two merons have ψ^+ and ψ^- changing sign simultaneously. Roman numerals indicate the quadrant of $\arg(\psi^+ + i\psi^-)$.

In order that the $SU(3)$ field strengths be regular at $r = 0$ between the merons, we must require that $|\psi^+| = |\psi^-| = 1$ at $r = 0$ so that the term in V in the action density (21) is not singular. We can assume that the set $\{t_i | 1 \leqslant i \leqslant n\}$ includes every point where ψ^+ or ψ^- changes sign. The condition that A_μ in (4) be regular at $r = 0$ is

$$\lim_{r\to 0} 1 + \phi_1^\pm = \lim_{r\to 0} \phi_0^\pm = 0. \tag{23}$$

For each segment (t_i, t_{i+1}), it will be possible to find a gauge such that (23) is satisfied, as long as $|\psi^+| = |\psi^-| = 1$ at $r = 0$. However, the charges $\beta_i^\pm$ can be changed by any amount by gauge transformations unless we restrict our attention to gauges (4) which are regular on both (t_{i-1}, t_i) and (t_i, t_{i+1}). By (23), $\vec{\psi}^\pm$ does not change direction across t_i in such gauges. According to (16), $\vec{\psi}^\pm$ changes direction by $\beta_i^\pm\pi$ for any $\theta^\pm$ consistent with (14). Therefore, when $\psi^\pm$ is constant across t_i, $\beta_i^\pm$ must be an even integer, and when $\psi^\pm$ changes sign across t_i, $\beta_i^\pm$ must be an odd integer. Although $\beta_i^\pm$ is still not gauge invariant, $\beta_i^\pm$ mod 2 is invariant.

In summary, finding meron solutions with $SU(3)$ topological charge density $Q(\vec{r}, t) = \Sigma_{i=1}^n (\beta_i^+ + \beta_i^-)\delta(\vec{r})\delta(t - t_i)$ is reduced to solving (19) with boundary conditions

$$\psi^\pm = (-1)^{1+\Sigma_{i=1}^k \beta_i^\pm} \text{ at } r = 0, \qquad t_k < t < t_{k+1}. \tag{24}$$

The charge is quantized in integers, instead of half-integers as in $SU(2)$, and is defined only modulo 2, instead of modulo 1 as in $SU(2)$. The integer charge is not an artifact of a normalization convention. $SU(2)$ merons embedded in an $SU(3)$ theory have half-integer charges. For a solution to (19) and (24), we introduce strings as the lines for which ψ^+ or ψ^- is zero. We expect that the strings will end at the merons, and divide the plane into disjoint regions where $\omega = \arg(\psi^+ + i\psi^-)$ is within one of the four quadrants of the unit circle (Figure 2).

If $\beta_i^+ \equiv \beta_i^- \equiv 0 \bmod 2$ for some i, the boundary conditions (24) are as if there were no meron at t_i. The action density (21) would also be affected (if solutions to (19) and (24) exist). Such t_i are artifacts of a singular gauge transformation, not merons. The three other possibilities for $\beta_i^\pm \bmod 2$ affect the boundary conditions in such a way that the action density is changed on sets of positive measure in $\mathbb{R}^4$ (if solutions to (19) and (24) exist). For these cases, t_i is a genuine meron, not a gauge artifact. This includes the case $\beta_i^+ = -\beta_i^- \equiv 1 \bmod 2$ which has total charge $\beta_i^+ + \beta_i^- = 0$ at t_i.

We now treat the existence of solutions to (19) and (24). Note that (19) are the variational equations of (21). For the case $\beta_i^+ \equiv \beta_i^- \bmod 2$ for all i, $\psi^+ = \psi^-$ is a solution, provided

$$r^2 \Delta\psi^+ = (\psi^+)^3 - \psi^+. \tag{25}$$

Note (25) is the equation for merons in $SU(2)$ [4]. A proof of existence of solutions to (25) and (24) was given by Jonsson *et al.* [7]. Their proof depended on a knowledge of the behavior of ψ^+ in a neighborhood of each meron. This knowledge was provided by a closed-form two-meron solution (one meron at $t = 0$, one at $t = \infty$). No solution to (19) and (24) with $\beta_i^+ \neq \beta_i^- \bmod 2$ is known in closed form. However, in what follows, we prove that such a solution exists in the two-meron case. This leads us to expect that an existence proof for (19) and (24) in the general case could be given along the lines of the proof of Jonsson *et al.* for (25). An alternative proof, using super- and sub-solution methods, has been given by Baker and Zirilli [8] and such methods may also be applicable here.

If ψ^+ and ψ^- depend only on $\theta = \arg(-t + ir)$, then (19) becomes

$$\begin{aligned} &\sin^2\theta(\psi^+)'' - \psi^+(2(\psi^+)^2 - (\psi^-)^2 - 1) = 0, \\ &\sin^2\theta(\psi^-)'' - \psi^-(2(\psi^-)^2 - (\psi^+)^2 - 1) = 0, \end{aligned} \tag{26}$$

where $'$ denotes differentiation with respect to θ. The two-meron boundary conditions that we will study are

$$\lim_{\theta\to 0} \psi^- = \lim_{\theta\to 0} \psi^+ = \lim_{\theta\to\pi} \psi^- = \lim_{\theta\to\pi} (-\psi^+) = 1, \tag{27}$$

corresponding to charges $\beta_1^+ \equiv 1$, $\beta_1^- \equiv 0 \bmod 2$. We note that $\psi^+ = \psi^- = \cos\theta$ is a solution of (26) with boundary conditions corresponding to $\beta_1^+ \equiv \beta_1^- \equiv 1 \bmod 2$.

THEOREM. *There exist two bounded, continuous functions ψ^+ and ψ^- defined in $(0, \pi)$ satisfying* (26) *and* (27). *$\psi^\pm$ possesses two bounded, continuous derivatives and* $\lim_{\theta\to 0} (\psi^\pm)' = \lim_{\theta\to\pi} (\psi^\pm)' = 0$.

The proof adapts a standard method for solving singular boundary value problems to the case of a system of equations. See Bailey *et al.* [9] for the method in the case of a single equation.

Proof. It is sufficient to work in the interval $(0, \pi/2]$ with boundary conditions $\psi^+ = (\psi^-)' = 0$ at $\pi/2$ because setting $\psi^+(\pi/2 + \theta) = -\psi^+(\pi/2 - \theta)$ and $\psi^-(\pi/2 + \theta) = \psi^-(\pi/2 - \theta)$ will extend a solution and its first two derivatives to continuous functions on $(0, \pi)$.

We define four functions, $u_1 < u_2$ and $v_1 < v_2$ which satisfy the boundary conditions at 0 and at $\pi/2$:

$$u_1(\theta) = \cos\theta, \qquad u_2(\theta) = 1 - (2\theta/\pi)^3,$$
$$v_1(\theta) = 1 - \tfrac{1}{2}\sin^2\theta, \qquad v_2(\theta) = 1. \tag{28}$$

It will turn out that the solution satisfies $u_1 < \psi^+ < u_2$ and $v_1 < \psi^- < v_2$. The following differential inequalities hold for $\theta \in (0, \pi/2)$:

$$\sin^2\theta u_1'' - u_1(2u_1^2 - v_1^2 - 1) > 0$$
$$\sin^2\theta v_1'' - v_1(2v_1^2 - u_1^2 - 1) > 0$$
$$\sin^2\theta u_2'' - u_2(2u_2^2 - v_2^2 - 1) < 0 \tag{29}$$
$$\sin^2\theta v_2'' - v_2(2v_2^2 - u_2^2 - 1) < 0.$$

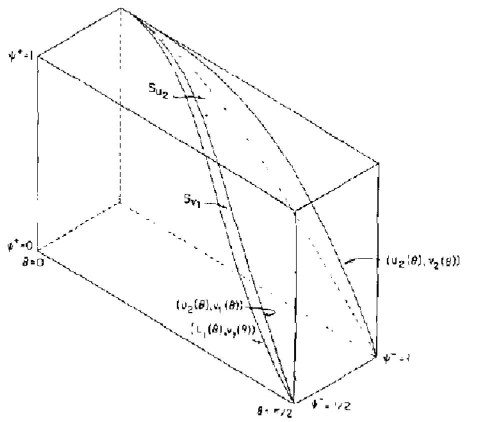
The third inequality is proved by dividing by θ^3 and replacing $(\sin^2\theta)/\theta^2$ by the smaller function $1 - (4\theta/\pi)(1 - 2/\pi)$. The result is a sixth degree polynomial in θ which is everywhere less than $-(1/4)$. The other three inequalities are simple to prove.

We use u_i and v_i to define some regions in $[0, \pi/2] \times \mathbb{R}^2$ (Figure 3): $D = \{(\theta, \psi^+, \psi^-) | 0 < \theta < \pi/2, u_1(\theta) < \psi^+ < u_2(\theta), v_1(\theta) < \psi^- < v_2(\theta)\}$, $S = \partial D - [\{\pi/2\} \times \mathbb{R}^2]$. Let D_0 be a neighborhood of $\bar{D}$ and $D_\delta = D_0 \cap [(\delta, \pi/2] \times \mathbb{R}^2]$ for $\delta > 0$. The equations (26) satisfy a Lipshitz condition in D_δ if $\delta > 0$. Therefore, the initial value problem (26) with initial conditions $\psi^+ = (\psi^-)' = 0$, $(\psi^+)' = b$, $\psi^- = c$ at $\pi/2$ has a unique twice continuously differentiable solution as far as the boundary of D_δ. This solution is a curve γ_{bc} in D_0, where $\gamma_{bc}(\theta) = (\theta, \psi^+(\theta), \psi^-(\theta))$. γ_{bc} depends continuously on (b, c).

For $(b, c) \in \Gamma = (-(6/\pi)' - 1) \times (1/2, 1)$, we have $\gamma_{bc}(\pi/2) \in \partial D$ and $\gamma_{bc}(\pi^-/2) \in D$. For each $(b, c) \in \Gamma$ there is associated a unique point $T(b, c)$ where γ_{bc} first intersects S. In the case $\gamma_{bc}(\theta) \in D$ for all $\theta \in (0, \pi/2)$, we define $T(b, c) = (0, 1, 1) = \gamma_{bc}(0)$. As long as $\gamma_{bc}(\theta) \in D$, we can divide (26) by $\sin^2\theta$ to obtain a uniform bound on $(\psi^\pm)''$ of order unity. Since $u_i'(0) = v_i'(0) = 0$, such a bound implies that $\lim_{\theta \to 0} (\psi^\pm)' = 0$. Therefore γ_{bc} is a solution to our problem if $T(b, c) = (0, 1, 1)$.

We prove that the map $T: \Gamma \to S$ is continuous. Since γ_{bc} depends continuously on (b, c), the only way T could fail to be continuous is if γ_{bc} were tangent to S at $T(b, c)$. For example, suppose $\psi^+ = u_1$ and $(\psi^+)' = u_1'$ at $\theta_0 > 0$. Since $\psi^- \geqslant v_1$ and $u_1 \geqslant 0$, (29) implies that $u_1'' \geqslant (\sin^2\theta_0)^{-1} \psi^+(2(\psi^+)^2 - (\psi^-)^2 - 1) = (\psi^+)''$ (strict inequality if $\theta_0 < \pi/2$). Therefore, $\psi^+ < u_1$

for $\theta \neq \theta_0$ in some neighborhood of θ_0. This contradicts the definition of $T(b, c)$ as the first point where γ_{bc} intersects S. The case where γ_{bc} is tangent to one of the other three faces of S are handled in the same fashion. Since $T(b, c)$ cannot be a point of tangency, T must be continuous.



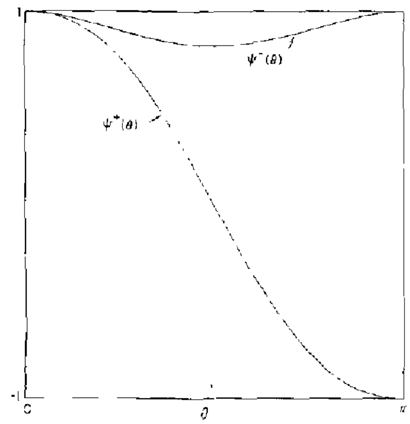


Fig. 3. The surface S, its faces S_{u_i} and S_{v_i} and the curves that define them.
Fig. 4. A numerical solution of the two-meron equation.

In order to conclude that there is a $(b, c) \in \Gamma$ such that $T(b, c) = (0, 1, 1)$, we must show that there is a curve $C \subsetneq \Gamma$ such that $T(C)$ surrounds the point $(0, 1, 1)$ in S. We outline a proof here. The face of S consisting of points (θ, ψ^+, ψ^-) with $\psi^+ = u_i(\theta)$ will be denoted by S_{u_i}. The face consisting of points (θ, ψ^+, ψ^-) with $\psi^- = v_i(\theta)$ will be denoted by S_{v_i} (Figure 3). If $(b_0, c_0) \in \partial\Gamma$, then $\gamma_{b_0c_0}$ is tangent to ∂D at $\pi/2$. The remarks in the above paragraph on solutions tangent to ∂D apply, and we conclude that $\gamma_{b_0c_0}$ is outside D for θ near $\pi/2$. Since the differential equations satisfy a Lipshitz condition, the curve γ_{bc} and its derivative converge uniformly to $\gamma_{b_0c_0}$ and its derivative as (b, c) approaches (b_0, c_0). Therefore, there is a δ such that for $(b, c) \in B_\delta((b_0, c_0)) \cap \Gamma$ the point $T(b, c)$ has θ-coordinate close to $\pi/2$. If $b_0 = -6/\pi$ there is a $\delta' \leqslant \delta$ such that for $(b, c) \in B_{\delta'}((b_0, c_0)) \cap \Gamma$ the curve γ_{bc} does not reach S until it exits by S_{u_2} or at a point no more than one-third of the way from S_{u_2} to S_{u_1} on S_{v_1} or S_{v_2}. Similarly, if $b_0 = -1$, γ_{bc} does not reach S until it exits by S_{u_1} or at a point no more than one-third of the way from S_{u_1} to S_{u_2}. Here, we have used uniform convergence of the derivative. If $c_0 = 1/2$, γ_{bc} exits by S_{v_1} or close to it, and if $c_0 = 1$, γ_{bc} exits by S_{v_2} or close to it (for (b, c) in an appropriate neighborhood of (b_0, c_0)). These neighborhoods of the points of $\partial\Gamma$ form an open cover which admits a finite subcover by the compactness of $\partial\Gamma$. Thus there is a δ'' such that every point of Γ within δ'' of $\partial\Gamma$ is covered. Let C be a rectangle in Γ closer to $\partial\Gamma$ than δ''. By construction, the four sides of C are mapped by T into four regions which surround the point $(0, 1, 1)$ in S. Since T is continuous, $T(C)$ must surround the point $(0, 1, 1)$. If $T(\Gamma)$ did not include $(0, 1, 1)$, then $T(C)$ would have to jump as C is contracted to a point. Since T is continuous, we conclude that there is a (b, c) such that $T(b, c) = (0, 1, 1)$, and the theorem is proved.

In the above proof, we found that the solution to (26) and (27) lies in a domain which does not include the function $\psi^- = 1$. Thus, both ψ^+ and ψ^- exhibit singular behavior at the origin of $\mathbb{R}^2_+$, even though ψ^- obeys constant boundary conditions. This can also be seen in Figure 4, which displays a numerical solution of (26) and (27).

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