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The Mass Gap for Higgs Models on a Unit Lattice

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An isomorphism is established between certain compact and noncompact formulations of Abelian gauge theory on a lattice. For weak coupling, the mass gap predicted by the Higgs mechanism is then established.

1. INTRODUCTION

The Higgs mechanism in the context of Abelian lattice gauge theories has already been extensively investigated [1, 2]. We return to the subject with two objectives: (1) to comment on relationships between various different actions, (2) to adapt to this simple context the Glimm–Jaffe–Spencer cluster expansion [3] which is better for investigating continuum limits (weak coupling regime) than the cluster expansion in [1, 2]. Indeed this paper is to be regarded as a prelude to an analysis of the continuum limit (in preparation). The basic program to study the continuum limit is to perform renormalization transformations which ultimately map an ε-lattice model onto a unit lattice model, with properties similar to the model studied here.

The two parts of this paper enable us to analyze a different region of the coupling constant space and to analyze a Higgs model which was not included in the previous papers [1, 2]. We show that at weak coupling, i.e., in the neighborhood of a lattice Gaussian theory, all gauge invariant correlations decay exponentially.

Our first result, Theorem 3.1, states an equivalence between noncompact Abelian gauge theory on a lattice (finite difference electromagnetism) and a compact Abelian Higgs theory with an action similar to the Villain model. It contains vortices or

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vortex lines as new fields. There is a remark in [4] about such a relationship but it is not correct as stated.

Our second result, Theorem 4.1, exhibits the exponential decay of correlations. The proof can be understood to show that cluster expansions cancel the partition function (and thereby vacuum energy volume divergences) out of expectations. Thus finite volume decay properties can be carried over to infinite volume.

There are two standard approximations in which such a cancellation is trivial: (a) independent sites (bonds), (b) Gaussian expectations. High temperature expansions as in Osterwalder and Seiler [1] or in the Ising model relay on being in a neighborhood in coupling constant space of (a). Field theory cluster expansions [3] work in a neighborhood of (b) and therefore have a larger domain of convergence.

2. Basic Definitions and Notation

We consider a complex scalar field $\phi(x)$ and a real-valued vector field A_b defined on sites and bonds, respectively, of a finite lattice, Λ,

$$\Lambda \subset \mathbb{Z}^d; \qquad d \geqslant 2. \tag{2.1}$$

A bond $b \subset \Lambda$ is an ordered pair of nearest neighbor sites, $b = \langle x, y \rangle$, $x, y \in \Lambda$, and we require that $A_b = -A_{-b}$ where $-b$ is the pair b in the opposite order, $-b = \langle y, x \rangle$. The set of bonds in Λ will be denoted Λ^*. The set of all plaquettes (oriented squares whose corners are four neighboring points in Λ) is denoted Λ^{**}; 3-cubes are denoted Λ^{***}; etc.

The choices for Yang–Mills action fall into two classes, compact and noncompact, as summarized below.

Compact Formalism

Here $\exp(ieA_b)$ takes values in $U(1)$, and the Lie algebra variable A_b takes values in $[-\pi/e, \pi/e]$. The gauge field action S_{YM} is periodic in A_b for each b, and often it is local—namely it has the form

$$S_{YM}(A) = \sum_{p \subset \Lambda} f(dA(p)), \tag{2.2}$$

for some periodic function f with period $2\pi/e$. Here d denotes the unit lattice exterior derivative or coboundary operator

$$(dA)(p) = \sum_{b \subset \partial p} A_b,$$

taking functions on bonds p to functions on plaquettes p. Here ∂p is the oriented boundary of p. Furthermore, the restriction of the sum (2.2) to plaquettes p lying inside Λ imposes a Neumann-type boundary condition on $\partial\Lambda$.

The Wilson action arises from the choice $f(x) = \cos x$. The Villain action corresponds to the choice of $f(x)$ as given by the relation

$$e^{-(1/g^2)f(x)} = \sum_{n \in (2\pi/e)\mathbb{Z}} e^{-(1/2)(x-n)^2}. \tag{2.3}$$

But we also require a nonlocal, periodic or compact action $S^{(C)}$ which is not a sum of the form (2.2). This action is defined by the relation

$$\exp\left[-\frac{1}{g^2} S^{(C)}_{YM}(A)\right] = \sum_{v:dv=0} \exp\left[-\frac{1}{2}\sum_{p \subset \Lambda} (dA(p) - v(p))^2\right]. \tag{2.4}$$

Here v is a map from plaquettes inside Λ into the lattice $(2\pi/e)\mathbb{Z}$, such that

$$(dv)(c) = \sum_{p \subset \partial c} v(p) \tag{2.5}$$

vanishes for all cubes $c \subset \Lambda$.

The interpretation of (2.5) is a conservation law requiring that no flux be created in any cube $c \subset \Lambda$. This means that v can be considered a configuration of vortex lines with conserved vorticity. In dimension 2, we set $dv = 0$ by convention, and in that case $S^{(C)}_{YM}$ agrees with the Villain action.

The partition function is defined by

$$Z^{(C)}_{YM} = \int_C \exp(-S^{(C)}_{YM}(A))\, \mathscr{D}A,$$

where

$$\mathscr{D}A = \mathscr{D}A_\Lambda = \prod_{b \in \Lambda} dA_b$$

denotes the product Lebesgue measure on the interval $[-\pi/e, \pi/e]$. To be specific, we also use $\int_C$ to denote integration over this interval.

Noncompact Formalism

In the noncompact case, A_b takes all real values (it is Lie algebra valued) and the action function we choose is quadratic:

$$S^{(NC)}_{YM}(A) = \tfrac{1}{2} \sum_{p \subset \Lambda} ((dA)(p))^2. \tag{2.6}$$

Again we use Neumann-like boundary conditions. In the noncompact case, $\mathscr{D}A$ denotes the product Lebesgue measure on the real line $\mathbb{R}$, and $\int_{NC}$ specifies this integration. Since (2.6) is gauge invariant, the function $\exp(-S^{(NC)}_{YM})$ is not integrable over the noncompact space of A's. To avoid this difficulty, we introduce a gauge-fixing function $G(A)$. Such a function does depend on the gauge function $\theta_x: x \in \Lambda \to \mathbb{R}$. Let

$$\mathscr{D}\theta_\Lambda = \prod_{x \in \Lambda} d\theta_x,$$

where $d\theta_x$ here denotes Lebesgue measure on $\mathbb{R}$. The gauge fix $G(A)$ must satisfy

$$\int e^{-G(A+d\theta)} \mathscr{D}\theta_{\Lambda \setminus x_0} = 1. \tag{2.7}$$

In (2.7), $d\theta$ denotes the lattice derivative of θ and $x_0 \in \Lambda$ is a point we choose later.

After choosing a gauge fix $G(A)$, the partition function is defined by

$$Z^{(NC)} = \int_{NC} \exp[-S_{YM}^{(NC)}(A) - G(A)] \, \mathscr{D}A_\Lambda.$$

Note that expectations of functions $F(A)$ which are gauge invariant (i.e., functions of dA) have expectations

$$\langle F \rangle^{(NC)} = (Z^{(NC)})^{-1} \int_{NC} F(A) \exp[-S_{YM}^{(NC)}(A) - G(A)] \, \mathscr{D}A_\Lambda,$$

which are independent of the gauge fix $G(A)$.

An example of an appropriate gauge function is $G(A) = G_0(A) + c$, with

$$G_0(A) = \sum_{x \in \Lambda} \frac{\alpha}{2} (d^*A)^2 (x).$$

Here $A_b \equiv 0$ for $b \notin \Lambda^*$, and d^* denotes the lattice divergence operator (the adjoint of d in the standard l_2 inner product on lattice forms). The positive constant α can give a one-parameter family of such G's. The constant c is chosen so (2.7) holds, namely

$$c = \ln \int \exp[-G_0(A + d\theta)] \, \mathscr{D}\theta_{\Lambda \setminus x_0}.$$

With our choices, c is easily seen to be finite and independent of A.

The Higgs Field

The action for the Higgs field minimally coupled to the gauge field A is defined by

$$S_\phi(\phi, A) \equiv \tfrac{1}{2} \sum_{\langle xy \rangle \in \Lambda} |\exp(ieA_{\langle xy \rangle}) \, \phi(y) - \phi(x)|^2 + \sum_{x \in \Lambda} (\lambda \, |\phi(x)|^4 - \tfrac{1}{4}\mu^2 \, |\phi(x)|^2 - E). \tag{2.8}$$

Here $\langle xy \rangle$ denotes the bond from x to y. Also $\lambda > 0$ and $\mu^2 > 0$. The constant E is chosen so that the minimum of $S_\phi(\phi, A)$ is zero—i.e., there is no classical vacuum energy.

Abelian Higgs Theory on a Lattice

We define the total action of the Abelian Higgs model by

$$S(\phi, A) = S_{YM}(A) + S_\phi(\phi, A) + G(A), \tag{2.9}$$

where S_{YM} is any of the actions discussed above. In the compact case, we choose $G(A) = 0$. The total partition function is

$$Z = \int \exp[-S(\phi, A)] \, \mathscr{D}A \, \mathscr{D}\phi$$

and expectations of gauge invariant functions $F = F(\phi, A)$ are

$$\langle F \rangle = Z^{-1} \int F(\phi, A) \exp[-S(\phi, A)] \, \mathscr{D}A \, \mathscr{D}\phi. \tag{2.10}$$

Here $\mathscr{D}\phi$ denotes the product integral over the complex plane $\mathbb{C}$ for each field component $\phi(x)$.

3. Equivalence of the Periodized Compact and Noncompact Formalisms

We consider *integer charge observables*, which we define as functions $F(\phi, A)$ of ϕ and A which are both gauge invariant and periodic. Gauge invariance means

$$F(e^{-ie\theta}\phi, A + d\theta) = F(\phi, A). \tag{3.1a}$$

The condition of periodicity means that F is periodic in each component A_b with period $2\pi/e$. Thus with δ_b denoting the characteristic function of the bond b, we require that for every b,

$$F(\phi, A + \delta_b 2\pi/e) = F(\phi, A). \tag{3.1b}$$

In general, we restrict attention to polynomial functions of $\exp(ieA_b)$, $b \in \Lambda^*$ and $\phi(x)$ and $\overline{\phi(x)}$, $x \in \Lambda$. Such functions include Wilson loops,

$$W(C) = \prod_{b \in C} \exp(ieA_b),$$

for C a closed curve composed of lattice bonds, as well as string variables,

$$\overline{\phi(x)} \prod_{b \in \Gamma_{xy}} \exp(ieA_b) \, \phi(y),$$

where Γ_{xy} is a contour along lattice bonds from x to y. We now consider a finite lattice Λ and expectations with Neumann boundary conditions in the compact state $\langle \cdot \rangle^{(C)}$ and the noncompact state $\langle \cdot \rangle^{(NC)}$.

THEOREM 3.1. *Let F be an integral charge observable. Then*

$$\langle F \rangle^{(C)} = \langle F \rangle^{(NC)}. \tag{3.2}$$

In addition

$$Z^{(C)} = (2\pi/e)^{|\Lambda| - 1} Z^{(NC)}.$$

Proof. We consider the expectations as a ratio of numerator to denominator. The numerator of $\langle F \rangle^{(NC)}$ is defined as

$$Z^{(NC)} \langle F \rangle^{(NC)} = \int_{NC} F e^{-S} \mathscr{D}A \, \mathscr{D}\phi \tag{3.3}$$

where $\mathscr{D}A$ denotes the product measure over A_b as b ranges over lattice bonds and $\mathscr{D}\phi$ denotes the product integral over $\phi(x)$ as x ranges over lattice sites. Write $S = S - G + G$, where G is the gauge fix. Both $S - G$ and F are gauge invariant, and the integration measure $\mathscr{D}A \mathscr{D}\phi$ is also gauge invariant. Thus the value of (3.3) remains unchanged if we replace $\exp(-G(A))$ in the integrand by its gauge transform $\exp(-G(A + d\theta))$, or by its average over θ.

Let $\mathscr{D}A_\Lambda$ denote the product Lebesgue integral for each $\phi(x)$, $x \in \Lambda$. We define the compact gauge average of $\exp(-G)$ by

$$\langle\langle \exp(-G(A)) \rangle\rangle = (2\pi/e)^{-|\Lambda|} \int_C \exp(-G(A + d\theta)) \, \mathscr{D}\theta_\Lambda$$

where C denotes integration of each $\theta(x)$ over the compact interval $[-\pi/e, \pi/e]$. Then

$$Z\langle F \rangle = \int_{NC} F \exp[-S_{YM}^{(NC)}(A) - S_\phi(\phi, A)] \langle\langle \exp(-G(A)) \rangle\rangle \, \mathscr{D}A \mathscr{D}\phi.$$

Since $S_\phi(\phi, A)$ is an integral charge observable (and hence periodic in each A_b with period $2\pi/e$) we obtain

$$\begin{aligned} Z\langle F \rangle = \sum_a \int_C \mathscr{D}A \int d\phi \, F \exp\left[-\tfrac{1}{2} \sum_p |dA + da|^2 - S_\phi(\phi, A) \right] \\ \times \langle\langle \exp(-G(A + a)) \rangle\rangle. \end{aligned} \tag{3.4}$$

Here a denotes a function from the bonds $b \in \Lambda$ to $(2\pi/e)\,\mathbb{Z}$.

To evaluate the sum over a, break it into two parts:

$$\sum_a = \sum_v \sum_{a : da = v} .$$

Since our lattice has trivial cohomology, with v fixed, any two choices for a differ by a coboundary ds, and s is uniquely determined by the restriction $s(x_0) = 0$. Here x_0 is the site specified in the gauge fixing conditions (2.7. Thus if a_v is one solution to $da = v$, any other solution

$$a = a_v + ds$$

yields s by integration of $a - a_v$ from x_0 to x, and the integral of ds around any closed loop vanishes. Then

$$Z\langle F\rangle = \sum_v \int_{\mathrm{C}} \mathscr{D}A \int \mathscr{D}\phi\, F \exp\left[-\tfrac{1}{2}\sum_p |dA + v|^2 - S_\phi(\phi, A)\right]$$
$$\times \sum_{da=v} \langle\!\langle \exp(-G(A+a))\rangle\!\rangle.$$

The gauge fixing condition (2.7) ensures that for fixed v,

$$\begin{aligned}\sum_{da=v} \langle\!\langle \exp(-G(A+a))\rangle\!\rangle &= \sum_s \langle\!\langle \exp(-G(A+a_v+ds))\rangle\!\rangle \\ &= \sum_s \int_{\mathrm{C}} \exp(-G(A+a_v+d\theta))\, \mathscr{D}\theta_\Lambda (2\pi/e)^{-|\Lambda|} \\ &= (2\pi/e)^{-|\Lambda|+1} \int_{\mathrm{NC}} \exp(-G(A+a_v+d\theta))\, \mathscr{D}\theta_{\Lambda\setminus x_0} \\ &= (2\pi/e)^{-|\Lambda|+1}.\end{aligned}$$

Thus

$$Z^{(\mathrm{NC})}\langle F\rangle^{(\mathrm{NC})} = (2\pi/e)^{-|\Lambda|+1} \sum_{v:dv=0} \int_{\mathrm{C}} \mathscr{D}A \int \mathscr{D}\phi\, F$$
$$\times \exp\left[\tfrac{1}{2}\sum_p |dA + v|^2 - S_\phi(\phi, A)\right].$$

Since our lattice has trivial cohomology, we have replaced the condition that v be exact ($v = da$) with the equivalent condition that v be closed ($dv = 0$). Hence taking $F = 1$, we can evaluate $Z^{(\mathrm{NC})} = (2\pi/e)^{-|\Lambda|+1} Z^{(\mathrm{C})}$, and therefore $\langle F\rangle^{(\mathrm{NC})} = \langle F\rangle^{(\mathrm{C})}$.

4. The Higgs Mechanism

The Higgs mechanism boils down to the assertion that for certain ranges of the coupling constants the models we have just discussed, despite the massless appearance of their actions, will only have massive spectrum. We address this assertion by proving that the correlations decay exponentially. The method we use applies to any of the models just discussed, but we only work through the noncompact case. This has not yet been treated in the literature, and indeed it would be difficult without using the equivalence proved in Section 3. The corresponding compact action in the equivalence of Theorem 3.1 is more complicated than the other compact actions, because it is non local.

The Model and Results

The coupling constants appearing in our model, which will be defined below, are the electric charge, e, of the boson field and the coupling, λ, for the quartic boson self-interaction. Define

$$m_A = \mu e/\sqrt{8\lambda}.$$

This is the classical prediction for the gauge field mass. The classical boson mass is μ.

For us, the meaning of *weak coupling* is fix μ and m_A strictly positive and take λ (and hence e) small depending on μ, m_A. In this paper we make no attempt to analyze carefully the dependencies on μ and m_A. This is one way of describing how our analysis must be improved to discuss the continuum limit.

We will see that the weak coupling limit is a lattice Gaussian model describing a vector field with mass m_A and a real scalar boson field of mass μ. Our expansion is in principle able to produce a very detailed analysis of a neighborhood of this limit (perturbation theory with symmetry breaking, etc.) but we have settled for a very small part of this.

The model is

$$\begin{aligned}\langle F\rangle_\Lambda = Z_\Lambda^{-1} \int_{NC} \mathscr{D}A \int \mathscr{D}\phi \exp\left\{-\tfrac{1}{2} \sum_{p\subset\Lambda} (dA(p))^2 - G(A)\right\} \\ \cdot \exp\left\{-\tfrac{1}{2} \sum_{\langle xy\rangle\subset\Lambda} |e^{ieA\langle xy\rangle}\phi(y) - \phi(x)|^2\right\} \\ \cdot \exp\left\{-\sum_{x\in\Lambda} [\lambda\,|\phi(x)|^4 - \tfrac{1}{4}\mu^2\,|\phi(x)|^2 + E]\right\} \\ \cdot F(\phi, A).\end{aligned}$$

The theorem is

THEOREM 4.1. *Let F be any integral charge observable depending only on fields ϕ, A at sites and bonds in a bounded subset of $\mathbb{Z}^d$, $d \geqslant 2$. Given μ and $m_A = \mu e/\sqrt{8\lambda}$, let λ be sufficiently small. Then the infinite volume expectation*

$$\langle F\rangle \equiv \lim_{\Lambda\to\mathbb{Z}^d} \langle F\rangle_\Lambda$$

exists. Furthermore, let F and G be two such observables, and let G_x denote the translate of G by $x \in \mathbb{Z}^d$. Then with the same conditions on μ, m_A, and λ, it follows that

$$|\langle FG_x\rangle - \langle F\rangle\langle G_x\rangle| \leqslant \exp(-m\,|x|),$$

where $m > 0$ is independent of F, G.

Remark. Gauge invariant observables that are not integral charge observables, such as dA, could be treated by our methods with some additional work. They would acquire dependence on the vortices in the same manner as the terms in the action.

5. General Outline of Proof, Notation

The strategy is in large part borrowed from [5]. We start in Section 6 with the standard change of variables into the unitary gauge. This shows, at least formally, that the model is a small perturbation of a massive Gaussian model. It is easiest for the subsequent analysis to work with the compact model, by taking advantage of Theorem 3.1.

In Section 7 we use a partition of unity on the space of field configurations to isolate the regions of Λ where the fields are so large that the Gaussian approximation breaks down. This idea appeared in [6] (using Peierls contours) in the analysis of scalar fields. However, it is important that it is implemented differently in this context, because the deviation from the Gaussian measure in gauge field theories is more drastic than for scalar theories. It is related to having to use several coordinate patches to describe the space in which fields take their values (circles in this simple context). In particular, as in [5], we postpone the integration over the fields which are large. The large field regions set boundary conditions (*conditioning*) for the complementary region, Λ, where we carry out a Glimm–Jaffe–Spencer cluster expansion. This is done in Section 9.

The cluster expansion has some not-so-standard aspects because the large field regions alter normalization factors and impose nonzero Dirichlet boundary conditions. This will be discussed further in Section 8. Eventually we wind up with a formula for the log of the partition function which displays its analytic properties. We evaluate derivatives with respect to "sources" using Cauchy's formula and thereby obtain formulas for truncated correlations which exhibit their exponential decay and prove our main theorem.

We conclude with a summary of notation; it is generally consistent with [5].

Given a bond $b' = \langle x'y'\rangle$ with x', y' in the coarse lattice $(L\mathbb{Z})^d$ we let

$$B(b') = \{\langle uv\rangle \subset \mathbb{Z}^{d^*} : u \in B(x'), v \in B(y')\}.$$

Thus $B(b')$ can be thought of as a *face* between two L-blocks centered on the ends of b'. We shall sometimes identify b' with its associated face.

Given a subset $\Gamma \subset (L\mathbb{Z})^{d^*}$, we denote by $X(\Gamma)$ the union of L-blocks *linked* by Γ,

$$X(\Gamma) \equiv \bigcup_{x \in \text{some}\, b' \in \Gamma} B(x).$$

We say Γ is connected if $X(\Gamma)$ is connected.

6. Classical Aspects

Using Theorem 3.1, rewrite the expectation in the form

$$\begin{aligned}\langle F \rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{v: dv = 0} \int_{-\pi/e}^{\pi/e} \mathscr{D}A \int \mathscr{D}\phi \\ \cdot \exp\left\{ -\tfrac{1}{2} \sum_{p \subset \Lambda} (dA(p) + v(p))^2 \right\} \\ \cdot \exp\left\{ -\tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda} |e^{ieA\langle xy \rangle} \phi(y) - \phi(x)|^2 \right\} \\ \cdot \exp\left\{ - \sum_{x \in \Lambda} [\lambda |\phi(x)|^4 - \tfrac{1}{4}\mu^2 |\phi(x)|^2 + E] \right\} \\ \cdot F(\phi, A).\end{aligned}$$

We exhibit the Higgs mechanism on the classical level by using the standard change of variables

$$\phi = \rho e^{i\theta}, \qquad A \to A - \frac{1}{e} d\theta.$$

Then

$$\begin{aligned}\langle F \rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{v: dv = 0} \int_{C} \mathscr{D}A \int_0^\infty \mathscr{D}\rho \\ \cdot \exp\left\{ -\tfrac{1}{2} \sum_{p \subset \Lambda} (dA(p) + v(p))^2 \right\} \\ \cdot \exp\left\{ -\tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda} |e^{ieA\langle xy \rangle} \rho(y) - \rho(x)|^2 \right\} \\ \cdot \exp\left\{ - \sum_{x \in \Lambda} [\lambda \rho^4(x) - \tfrac{1}{4}\mu^2 \rho^2(x) + E - \log 2\pi\, \rho(x)] \right\} F.\end{aligned}$$

The log ρ comes from the Jacobian of the change of variables. The minimum of the exponent (excluding log ρ) occurs at

$$\rho_0 = \mu/\sqrt{8\lambda}, \qquad A = 0, v = 0.$$

After the translation $\rho \to \rho_0 + \rho$, the action has the quadratic piece

$$S_0(\rho, A, v) \equiv \tfrac{1}{2} \sum_{p \subset \Lambda} (dA + v)^2 (p) + \tfrac{1}{2} m_A^2 \sum_{b \subset \Lambda} A_b^2 + \tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda} (\rho(y) - \rho(x))^2 + \tfrac{1}{2}\mu^2 \sum_{x \in \Lambda} \rho^2(x) \tag{6.1}$$

and an interaction piece

$$\begin{aligned} V_1(\rho, A) \equiv \rho_0^2 & \sum_{\langle xy \rangle \subset \Lambda} (1 - \cos eA_{\langle xy \rangle} - \tfrac{1}{2} e^2 A_{\langle xy \rangle}^2) \\ & + \rho_0 \sum_{\langle xy \rangle \subset \Lambda} (\rho(y) + \rho(x))(1 - \cos eA_{\langle xy \rangle}) \\ & + \sum_{\langle xy \rangle \subset \Lambda} \rho(y) \rho(x)(1 - \cos eA_{\langle xy \rangle}) \\ & + \sum_{x \in \Lambda} (\lambda \rho^4(x) + \sqrt{2\lambda}\, \mu \rho^3(x) - \log\left(1 + \frac{\rho(x)}{\rho_0}\right). \end{aligned} \tag{6.2}$$

We have altered the normalization by subtracting $\log(2\pi \rho_0)|\Lambda|$. The expectation is unaffected and is

$$\langle F \rangle_\Lambda = \frac{1}{Z} \sum_{v: dv = 0} \int_C \mathscr{D}A \int_{-\rho_0}^{\infty} \mathscr{D}\rho e^{-S_0 - V_1} F.$$

Here $Z = Z_\Lambda$ normalizes the expectation to unity.

Formally V_1 tends to zero in the weak coupling limit, so the expectation is expected to become the massive Gaussian with the action (6.1) in the weak coupling limit.

Rather than obtaining expansions for $\langle F \rangle$, we add to the interaction an extra "source" term, V_2, where

$$V_2 \equiv \sum_f \alpha_f e^{ie \sum_b f_b A_b} + \sum_x \alpha_x \rho(x). \tag{6.3}$$

Here f runs over some finite set of functions on bonds which are nonzero for finitely many bonds, and the parameters α_f, α_x are complex numbers subject to the bounds

$$|\alpha_f| \leqslant 1; \qquad |\alpha_x| \leqslant \lambda^\varepsilon, \qquad \varepsilon > 0. \tag{6.4}$$

We can use f's that lead to non-gauge-invariant terms in V_2 because the gauge is fixed at this point. We let $V = V_1 + V_2$ and define

$$S \equiv S_0 + V_1 + V_2 \equiv S_0 + V$$

$$Z \equiv \int_C \mathscr{D}A \int_{-\rho_0}^{\infty} \mathscr{D}\rho \, e^{-S}.$$

We will find an expansion for log Z and obtain $\langle F \rangle$ by applying a suitable differential operator in the α's which are then evaluated at $\alpha = 0$. The derivatives will be expressed by Cauchy's formula.

The bounds (6.4) ensure the V_2 is small in the same sense that V_1 is small, and the combination V is treated as a single entity. This way of getting at expectations does not give bounds on $\langle F \rangle$ which are asymptotically correct as $\lambda \to 0$, but it is sufficient to prove our main theorem.

Our expansion is defined in terms of the coarse lattice Λ'. We assume that L, the block size, is chosen so that factors $\exp(ie \sum f_b A_b)$ in V_2 do not couple disjoint blocks. This can be achieved for arbitrarily large L.

7. Large and Small Fields, Conditioning

An important part of this expansion is to use separate methods on large and small fields respectively. The number $p(\lambda)$ given by

$$p(\lambda) \equiv |\log \lambda|^{d+1}$$

will be the borderline between large and small; i.e., A_b is *large* if $|A_b| \gtrsim p(\lambda)$, $\rho(x)$ is *large* if $|\rho(x)| \gtrsim p(\lambda)$. The discrete vortex field $v(p)$ is *large* iff $v(p) \neq 0$.

Partition of Unity of Field Configurations

We choose a function χ of one variable as an approximate characteristic function, obeying

(1) $\chi \in C^\infty$, $\chi(t) = \chi(-t)$,

(2) $|\chi| \leqslant 1, \chi(t) = 1$ if $|t| \leqslant 1/2$,
$\quad = 0$ if $|t| \leqslant 1$

(3) For each $n = 1,2,\ldots,$

$$\left| \frac{d^n}{dt^n} \chi \right| \leqslant c^n (n!)^p \tag{7.1}$$

for some c and p independent of t. Let

$$\zeta = (1 - \chi)$$

so that

$$1=\prod_{x\in\Lambda}(\chi(\rho(x)/p(\lambda))+\zeta(\rho(x)/p(\lambda)))$$
$$=\sum_{X\subset\Lambda}\prod_{x\in X}\chi(\rho(x)/p(\lambda))\prod_{x\notin X}\zeta(\rho(x)/p(\lambda))$$

and similarly

$$1=\sum_{Y\subset\Lambda^*}\prod_{b\in Y}\chi(A_b/p(\lambda))\prod_{b\notin Y}\zeta(A_b/p(\lambda))$$
$$1=\sum_{Z\subset\Lambda^{**}}\prod_{p\in Z}1_{\{v(p)=0\}}(v)\prod_{p\notin Z}1_{\{v(p)\neq 0\}}(v).$$

Here 1_E denotes the characteristic function of the event E. We combine these three partitions of unity by multiplying them. For a given term in the resulting partition of unity, let Λ_0 be the largest union of L-blocks $\subset\Lambda$ with the property that every site, bond, and plaquette in Λ_0 is selected by χ's to be small field. Set

$$\Lambda_0^c\equiv\Lambda\backslash\Lambda_0$$

and resum the partition of unity holding Λ_0 fixed. We write the result in the form

$$1=\sum_{\Lambda_0\subset\Lambda}\chi_{\Lambda_0}\zeta_{\Lambda_0^c}$$

where χ_{Λ_0} is the characteristic function for the event that ϕ, A are small at sites and bonds in Λ_0 and $v=0$ on plaquettes in Λ_0. $\chi_{\Lambda_0^c}$ is the characteristic function for the event: For each L-block $B(x)\subset\Lambda_0^c$, at least one field associated with a site touching $B(x')$, or a bond, or plaquette, intersecting $B(x)$, is large.

Finally, it is convenient to include in $\zeta_{\Lambda_0^c}$ the characteristic function of the range of A- and ρ-integration. We will suppose this is done and not change notation. (The χ-functions already force ρ and A to be inside the integration range if λ is sufficiently small.)

Conditioning

We defer the integration over the fields in large field regions as selected by the partition of unity, and by multiplying and dividing by a suitable factor, normalize the remaining integrals so that the Gaussian part is a probability measure.

Define, for any subset $X\subset\Lambda$, the Gaussian action $S_0^*(X)$ by throwing away, in the action S_0 defined above, all terms labelled by sites, bonds, or plaquettes which do not lie in or intersect X; i.e.,

$$S_0^*(X) \equiv \tfrac{1}{2} \sum_{p \subset \Lambda, p \cap X \neq \varnothing} (\mathscr{D}A(p) + v(p))^2$$

$$+ \tfrac{1}{2} m_A^2 \sum_{b \subset \Lambda, b \cap X \neq \varnothing} A_b^2 + \tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda, \langle xy \rangle \cap X \neq \varnothing}$$

$$\cdot (\rho(y) - \rho(x))^2 + \tfrac{1}{2}\mu^2 \sum_{x \in X} \rho^2(x). \tag{7.2}$$

Define $S^*(X)$ and $V^*(X)$ by applying the same operations to S and V. Then

$$S = S^*(V) + S(X^c), \tag{7.3}$$

so that

$$Z = \sum_{\Lambda_0 \subset \Lambda} \sum_{v: dv = 0} \int \mathscr{D}A_{\Lambda_0^{*c}} \int \mathscr{D}\rho_{\Lambda_0^c} e^{-S(\Lambda_0^c)} \zeta_{\Lambda_0^c} \int \mathscr{D}A_{\Lambda_0} \int \mathscr{D}\rho_{\Lambda_0} e^{-S^*(\Lambda_0)} \chi_{\Lambda_0}. \tag{7.4}$$

Define a normalized Gaussian measure with nonzero mean by

$$d\mu_X = \frac{1}{Z_0^*(X)} \mathscr{D}A_X \mathscr{D}\rho_X e^{-S_0^*(X, v=0)}. \tag{7.5}$$

Then

$$Z = \sum_{\Lambda_0 \subset \Lambda} \sum_{v: dv = 0} \int \mathscr{D}A_{\Lambda_0^{*c}} \int \mathscr{D}\rho_{\Lambda_0^c} e^{-S(\Lambda_0^c)} \zeta_{\Lambda_0^c} Z_0^*(\Lambda_0)$$

$$\cdot \int d\mu_{\Lambda_0} e^{-V^*(\Lambda_0)} \chi_{\Lambda_0} \tag{7.6}$$

$$= \sum_{\Lambda_0 \subset \Lambda} \sum_{v: dv = 0} \int \mathscr{D}A_{\Lambda_0^{*c}} \int \mathscr{D}\rho_{\Lambda_0^c} e^{-S(\Lambda_0^c)} \zeta_{\Lambda_0^c} Z_0^*(\Lambda_0)\, \Xi(\Lambda_0), \tag{7.7}$$

where

$$\Xi(X) \equiv \int d\mu_X e^{-V^*(X)} \chi_X. \tag{7.8}$$

8. Cluster Expansions

These are analogous to high temperature expansions of statistical mechanics and are applied to the factors $Z_0^*(\Lambda_0)$, $\Xi(\Lambda_0)$ produced by the conditioning. Our presentation is quite complete but neverthcless some review of [3] is probably a useful prelude to this part of the analysis.

There are three operations: (1) an expansion for $\log Z_0^*$; (2) an expansion for Ξ;

(3) the union of (1), (2), and conditioning into one expansion. The expansions of (1) and (2) are only rapidly convergent far away from the break-down of the Gaussian approximation. Accordingly we define, for any union of L-blocks X the set

$$X_1 \equiv \bigcup_{x' : \|x' - X'\| \leqslant r(\lambda)} B(x')$$

where

$$r(\lambda) \equiv |\log \lambda|^2.$$

This operation is applied to the large field region $\Lambda_\bullet^c$ to get $(\Lambda_\bullet^c)_1 \equiv \Lambda_1^c$ and with an incompatible notation which should not cause confusion

$$\Lambda_1 \equiv (\Lambda_1^c)^c.$$

The expansions are localized in Λ_1.

The choice of $r(\lambda)$ is dictated by two requirements: (a) the large action caused by even one large field produces a small factor $\exp(-O(p^2(\lambda)))$ which dominates a factor $O(r(\lambda)^d)$ arising from crude estimates of the vacuum energy in an $r(\lambda)$ neighborhood. (b) the effects of large fields are exponentially damped away from Λ_0^c so that an expansion in Λ_1 is perturbed by the conditioning at the boundary of $\Lambda_\bullet^c$ by effects of size of the order of $p^2(\lambda)\exp(-r(\lambda)\min(\mu, m_A))$ which tends to zero rapidly with λ.

Expansion for Z_0^*

We introduce an interpolated Gaussian action $S_0^*(\mathbf{s})$ and a corresponding Gaussian partition function $Z_0^*(\mathbf{s})$ where $\mathbf{s}$ is a collection of parameters, each in $[0, 1]$, one for each bond $b' \in \Lambda_1^{\prime c^* c}$, labelling a face between L-blocks. We define this action by multiplying all terms in $S_0^*(v=0)$ which couple blocks $B(x')$, $B(y')$ by $s_{\langle x'y'\rangle}$ so that $s_{\langle x'y'\rangle}$ controls the strength of coupling between $B(x')$ and $B(y')$. Thus

$$S_0^*(\mathbf{s}) \equiv \tfrac{1}{2}\sum_p \sigma_p(dA(p))^2 + \tfrac{1}{2}m_A^2 \sum_b A_b^2 + \tfrac{1}{2}\sum_{\langle xy\rangle} \sigma_{\langle xy\rangle}(\rho(y) - \rho(x))^2 + \tfrac{1}{2}\mu^2 \sum_x \rho^2(x),$$

where the sums are over p, b, $\langle xy\rangle$ which touch Λ_0 and x is summed over sites in Λ_0. Furthermore

$$\begin{aligned} \sigma_p &= s_{\langle x'y'\rangle} && \text{if } p \text{ couples blocks } B(x'), B(y') \\ &= 1 && \text{if } p \text{ does not couple } L\text{-blocks or if } p \\ & && \text{intersects four } L\text{-blocks} \\ \sigma_{\langle xy\rangle} &= s_{\langle x'y'\rangle} && \text{if } \langle xy\rangle \in B'(\langle x'y'\rangle) \\ &= 1 && \text{if } \langle xy\rangle \text{ does not couple } L\text{-blocks;} \end{aligned} \tag{8.1}$$

thus we have left bonds and plaquettes which do not couple L-blocks undisturbed and in addition no bonds near $\partial\Lambda_0$ are altered. In particular, when $s=0$ all L-blocks in Λ_1 are decoupled from each other and from Λ_1^c. Each bond coupling two blocks is decoupled from everything else (except for bonds in a plaquette intersecting four blocks—they are coupled only within that plaquette).

Following [3] we interpolate between all $\mathbf{s}=1$ and $\mathbf{s}=0$ using the fundamental theorem of calculus

$$\begin{aligned}\log Z_0^*(\Lambda_0) &= \log Z_0^*(\mathbf{s}=\mathbf{1})\\ &= \sum_{\Gamma} \int d\mathbf{s}_\Gamma \partial^\Gamma \log Z_0^*(\mathbf{s}_\Gamma)\end{aligned} \tag{8.2}$$

where Γ is summed over all subsets of bonds in Δ_1', including the null set (which means that ∂^Γ and the s-integration are omitted),

$$\partial^\Gamma \equiv \prod_{b'\in\Gamma} \frac{\partial}{\partial s_{b'}}$$

$$\mathbf{s}_\Gamma \equiv (s_{b'}) \qquad \text{with } s_{b'}=0 \text{ if } b'\notin\Gamma.$$

We define

$$W_1(\Gamma) \equiv \int d\mathbf{s}_\Gamma \partial^\Gamma \log Z_0^*(\mathbf{s}_\Gamma)$$

so that (8.2) becomes

$$Z_0^*(\Lambda_0) = Z_0^*(\Lambda_0, \mathbf{s}=0) \exp\left(\sum_{\Gamma\neq\varnothing} w_1(\Gamma)\right).$$

We divide by the same expansion (i.e., decoupling on faces in $\Lambda_1'^{c*c}$) for $Z_0(\Lambda)$, so that

$$Z_0^*(\Lambda_0) = Z_0(\Lambda) \frac{Z_0{}^*(\Lambda_0, \mathbf{s}=0)}{Z_0(\Lambda, \mathbf{s}=0)} e^{\Sigma_{\Gamma\neq\varnothing} W_2(\Gamma)}$$

where

$$W_2(\Gamma) = W_1(\Gamma, \Lambda_0) - W_1(\Gamma, \Lambda).$$

Note that $W_2(\Gamma)=0$ unless Γ touches Λ_1^c. Also, if Γ is not connected, then $Z_0^*(\mathbf{s}_\Gamma)$ factors and again $W_2(\Gamma)=0$.

We substitute this expansion for Z_0^* into $Z(\Lambda)$ as expressed in (7.6) and obtain

$$\begin{aligned}Z(\Lambda) = Z_0(\Lambda) \sum_{\Lambda_0} \int &\mathscr{D}A_{\Lambda_0^{*c}} \mathscr{D}\rho_{\Lambda_0^c} \rho_1(\Lambda_0^c)\\ &\cdot \exp\left(\sum_\Gamma W_2(\Gamma)\right) \Xi(\Lambda_0)\end{aligned} \tag{8.3}$$

where $\rho_1(\Lambda_0^c)$ is the function of fields ρ, A associated with Λ_0^c defined by

$$\rho_1(X) \equiv \sum_{v:dv=0} e^{-S_0(X)-V(X)}\zeta_X Z_0^*(X^c, \mathbf{s}=0)/Z(\Lambda, \mathbf{s}=0).$$

Expansion for Ξ

Now we will obtain a polymer expansion for the factor

$$\Xi(\Lambda_0) \equiv \int d\mu\, e^{-V^*}\chi.$$

As in the last section we introduce the Gaussian measure $d\mu_{\mathbf{s}}$ by interpolating the terms in S_0^* that couple across faces in $\Lambda_1'^{c*c}$. In an analogous way we introduce s-dependence in V^*, defining $V^*(\mathbf{s})$ by making the replacement

$$\sum_{\langle xy\rangle\subset\Lambda_1^{c*c}} (\cdots) \to \sum_{\langle xy\rangle\subset\Lambda_1^{c*c}} \sigma_{\langle xy\rangle}(\cdots)$$

in the definition of $V^*(\Lambda_0)$. The σ-factors were defined in (8.1). Note that V_2 is not affected because the coarse lattice is chosen to avoid couplings by V_2. By interpolating between $\mathbf{s}=1$ and $\mathbf{s}=0$ using the fundamental theorem of calculus we obtain, as in [3],

$$\Xi(\Lambda_0) = \sum_{\Gamma} \int ds_\Gamma\, \partial^\Gamma \Xi(\Lambda_0, s_\Gamma)$$

where, if $\Gamma=\varnothing$, the s_Γ-integration and ∂^Γ are omitted, and

$$\Xi(\Lambda_0, \mathbf{s}) \equiv \int d\mu_{\mathbf{s}} e^{-V^*(\mathbf{s})}\chi.$$

Let us define $\Xi_0(\Lambda_0)$ to be $\Xi(\Lambda_0, 0)$ but with $s_b=0$ for all $b\subset\Lambda_0'$ and with conditioning equal to zero. We then put

$$K_1(\Gamma) \equiv \int ds_\Gamma\, \partial^\Gamma \Xi(\Lambda_0, \mathbf{s}_\Gamma)/\Xi_0(\Lambda_0).$$

We call Γ Λ_1-connected if either

(a) Γ is connected,

(b) the connected components of Γ may be connected by adjoining connected components of $\Lambda_1^{c'}$.

Then $K_1(\Gamma)$ factors across connected components:

$$K_1(\Gamma) = \prod_{\tilde\Gamma\in\Lambda_1\text{-connected components of }\Gamma} K_1(\tilde\Gamma)$$

and

$$\Xi(\Lambda_0) = \Xi_0(\Lambda_0) \sum_{\Gamma} K_1(\Gamma)$$

where the sum over Γ includes the null set.

The definition of K_1 is a ratio of large volume (Λ_0) quantities, but since $s_{b'} = 0$ for all b' not in Γ, the numerator and denominator factor into contributions from blocks filling $\Lambda_1 \backslash \{\text{blocks connected by } \Gamma\}$. These blocks cancel out, including integrals over gauge fields associated with bonds in between blocks, so that $K_1(\Gamma)$ is really a "small" volume (blocks connected by Γ and associated "in between" bonds and plaquettes) quantity.

When this expansion for Ξ is substituted into our expression (8.3) for $Z(\Lambda)$ we obtain

$$Z(\Lambda) = Z_0(\Lambda)\, \Xi_0(\Lambda) \sum_{\Lambda_0} \int \mathscr{D}A_{\Lambda_0^{*c}} \mathscr{D}\rho_{\Lambda_0^c} \rho_2(\Lambda_0^c) \cdot \exp\left(\sum_{\Gamma} W_2(\Gamma)\right) \sum_{\Delta} K_1(\Delta) \tag{8.4}$$

where we have changed Γ to Δ in the K_1 sum to avoid confusion with the W_2 sum, and ρ_2 is ρ_1 with Ξ-factors included:

$$\rho_2(X) \equiv \sum_{v:dv=0} e^{-S_0(X) - V(X)} \zeta_X \cdot (Z_0^*(X^c, \mathbf{s}=0)/Z(\Lambda, \mathbf{s}=0))\, \Xi_0(X^c)/\Xi_0(\Lambda).$$

Uniting the Expansions

The large field regions represented by the ρ-factors transmit information via the conditioning at the boundary of Λ_0^c to any quantities $W_2(\Gamma)$, $K_1(\Gamma)$ occurring in the expansions for Z_0^* and Ξ such that Γ touches Λ_1^c. In this section we collect the components of Λ_1^c along with any Γ's touching Λ_1^c components into clusters C so that different clusters do not touch directly or indirectly through W_2 or K_1 factors. Then there is complete factorization of $\mathscr{D}\rho$, $\mathscr{D}A$ integrals attached to different C's.

First we expand the exponential in W_2 in (8.4) according to

$$\exp\left(\sum W_2(\Gamma)\right) = \sum_{\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_n)} \frac{1}{n!} \prod_{\Gamma \in \mathbf{\Gamma}} W_2(\Gamma)$$

where the sum runs over all *ordered collections* ($\equiv$ ordered sets which are allowed repeated elements) of connected subsets of $\Lambda_1'^*$. In this way Z is expressed as a sum over *configurations* Λ_0^c, $\mathbf{\Gamma}$, Δ.

A configuration (Λ_0^c, $\mathbf{\Gamma}, \Delta$) is said to form a *cluster on* C, which is a subset of Λ, defined by

$$C \equiv \Lambda_1^c \cup X(\mathbf{\Gamma}) \cup X(\Delta)$$

iff C is connected.

Not every configuration occurring in the expansion for Z is a cluster on some C but we can uniquely decompose any configuration into clusters on sets $C \in \{C_1, C_2, \ldots, C_n$: n arbitrary$\} \equiv \mathbf{C}$. We define $K_3(C)$ to be the sum over clusters on C:

$$K_3(C) = \sum_{R,\,\boldsymbol{\Gamma},\,\Delta \in \text{clusters on } C} \frac{1}{|\boldsymbol{\Gamma}|!} \int \mathscr{D}A \mathscr{D}\rho\, \rho_2(R) \prod_{\Gamma \in \boldsymbol{\Gamma}} W_2^R(\Gamma) K_1^R(\Delta)$$
$$(= 0 \text{ if sum is empty}). \tag{8.5}$$

Then

$$Z(\Lambda) = Z_0(\Lambda)\, \Xi_0(\Lambda) \sum_{\mathbf{C}} \prod_{C \in \mathbf{C}} K_3(C). \tag{8.6}$$

The superscripts R on W_2 and K_1 mean that Λ_0^c has been changed to R, Λ_1^c to R_1 in their definition (8.5). We can do this because $W_2(\Gamma)$ does not depend on connected components of Λ_1^c not touched by Γ, and K_1 factors across connected components. $\mathbf{C}$ is summed over all sets whose elements are connected unions of L-blocks and no two elements touch.

9. Taking the Logarithm of Z

Z has been expanded in the form

$$Z = Z_0 \Xi_0 \sum_{\mathbf{C}} \prod_{C \in \mathbf{C}} K_3(C)$$

where the sum over $\mathbf{C}$ ranges over sets whose elements are unions of L-blocks such that no two elements touch. Furthermore K_3 vanishes unless its argument is connected.

We take the logarithm by a now standard procedure [7] of regarding this expansion without the two multiplicative factors Z_0, Ξ_0 as a grand canonical ensemble of a hard core gas of connected subsets and applying the Mayer expansion.

The procedure and our main estimate are given in more detail in the Appendix. The results are

$$\log Z = \log(Z_0 \Xi_0) + \sum_{X \subset \Lambda} K_4(X),$$

where

$$K_4(\bar{X}) \equiv \sum_{\mathbf{X}: \cup \mathbf{X} = \bar{X}} \frac{1}{|\mathbf{X}|!} \prod_{X \in \mathbf{X}} K_3(X)$$
$$\sum_{G \in \{\text{connected graphs on } \mathbf{X}\}} \prod_{XY \in G} (U(X, Y) - 1).$$

Here U is defined by

$$U(X, Y) = 0 \quad \text{if } X, Y \text{ touch}$$
$$= 1 \quad \text{otherwise}$$

and $\mathbf{X}$ is summed over ordered collections of connected unions of L-cubes.

To discuss convergence we define the c-norm by

$$\|K\|_c \equiv \sup_{x' \in (L\mathbb{Z})^d} \sum_{X,\, X' \text{ touches } x'} |K(X)| \exp(c\,|X'|).$$

The result is, for any $c \geqslant 0$,

$$\|K_4\|_c \leqslant \sum_{n=1}^{\infty} \|K_3\|^n_{c+1+\varepsilon} \left(1 + \frac{1}{\varepsilon}\right)^n.$$

Furthermore when the right-hand side is convergent, the analyticity properties of K_3 as a function of parameters in its action extend to K_4—because it is a uniform limit of analytic functions.

Proof of Theorem 4.1. We can now give the proof, assuming an estimate on the c-norm of K_3 which will be proved in the rest of this paper.

We obtain a formula for the expectation of F by applying a suitable set of derivatives with respect to the α-parameters occurring in V_2 to $\log Z$. (See (6.3).) For example,

$$\begin{aligned}
&|\langle |\phi(x)|\,|\phi(y)|\rangle - \langle |\phi(x)|\rangle\langle |\phi(y)|\rangle| \\
&= \left| \frac{\partial}{\partial \alpha_x} \frac{\partial}{\partial \alpha_y} \log Z|_{\alpha_x = \alpha_y = 0} \right| \\
&= \left| \sum_X \frac{\partial}{\partial \alpha_x} \frac{\partial}{\partial \alpha_y} K_4(X)|_{\alpha_x = \alpha_y = 0} \right| \\
&= \left| \sum_{X \supset \{x,y\}} \left(\frac{1}{2\pi i}\right)^2 \oint \frac{d\alpha_x}{\alpha_x^2} \oint \frac{d\alpha_y}{\alpha_y^2} K_4(X) \right| \qquad (9.1) \\
&\leqslant \sum_{X \supset \{x,y\}} \frac{1}{|\alpha_x|} \frac{1}{|\alpha_y|} |K_4(X)|\, e^{c|X'|} e^{-c|X'|} \\
&\leqslant \frac{1}{|\alpha_x|} \frac{1}{|\alpha_y|} e^{-c|x-y|/L} \|K_4\|_c \\
&\leqslant \lambda^{-2\varepsilon} e^{-c|x-y|/L} \sum_{n=1}^{\infty} \|K_3\|^n_{c+1+\varepsilon} \left(1 + \frac{1}{\varepsilon}\right)^n. \qquad (9.2)
\end{aligned}$$

Here we take $|\alpha_x|^{-1}$, $|\alpha_y|^{-1} = \lambda^{-\varepsilon}$, as allowed in (6.4). The restrictions $X \supset \{x, y\}$ may be imposed because $K_4(X)$ will vanish when the α-derivatives are imposed if

$X \not\supset \{x, y\}$. We assume x, y are in different blocks so that the derivatives vanish when applied to $\log \Xi_0$. In the next section we will prove that, for any c if L is large enough,

$$\|K_3\|_c \to 0 \qquad \text{as } \lambda \to 0.$$

Thus exponential decay for this correlation is proved for λ small, uniformly in Λ. Other correlations may be handled by variations of the same argument.

To see that the infinite volume limit exists we note that in (9.1) there is a non explicit constraint $X \subset \Lambda$ on the sum and this is the only dependence on Λ. It is clear that a change in Λ affects the sum by adding or subtracting terms of the form $K_4(X)$, with

$$X \supset \{x, y\} \qquad \text{and} \qquad X \cap \Lambda^c \neq \varnothing.$$

These additional terms are exponentially small in the distance from $\{x, y\}$ to $\partial\Lambda$. We use

$$\sum_{X, X \supset \{x,\, y\}, X \cap \Lambda^c \neq \varnothing} |K_4(X)| \leqslant e^{-c \operatorname{dist}(\{x,y\}, \Lambda^c)/L} \|K_4\|_c$$

to see that the infinite volume limit exists, for λ small.

The same type of argument demonstrates exponential decay for correlations of Wilson loops. At first one obtains exponential decay only for Wilson loops translated by multiples of L because the coarse lattice must be chosen to contain each Wilson loop within single blocks. However, the argument can be made for a finite number of different L's and thereby extended to arbitrary translates of Wilson loops.

10. Combinatoric Aspects of Convergence

In this section we will give the combinatoric aspects of the proof that

$$\|K_3\|_c \to 0 \qquad \text{with } \lambda,$$

deferring until a later section some estimates which require a detailed grasp of Gaussian integrals.

Given a function f on subsets of Λ'^* we define its c-norm by

$$\|f\|_c \equiv \sup_{x'} \sum_{\Gamma : \Gamma \text{ touches } x'} |f(\Gamma)| \, e^{c|X(\Gamma)'|}.$$

If f is a function on unions of L-blocks then we define its c-norm by

$$\|f\|_c \equiv \sup_{x'} \sum_{X : X \text{ touches } x'} |f(X)| \, e^{c|X'|}.$$

The following two lemmas contain the basic combinatorial ideas, then we state the three propositions which contain the more technical analysis and combine them to prove that $\|K_3\|_c \to 0$ with λ.

PROPOSITION 10.1 [3, p. 218]. *There are constants c_1, c_2 depending only on dimension such that*

(a) $\sum_X e^{-c_1|X'|} < \infty$

(b) $\sum_\Gamma e^{-c_2|\Gamma|} < \infty$

where X is summed over all connected unions of L-blocks touching the origin, and Γ is summed over all connected sets of bonds $b' \subset (L\mathbb{Z})^d$ touching some fixed site x' in the coarse lattice. In part (b), *$|\Gamma|$ can be replaced by $|X'(\Gamma)|$ with a different c_2.*

LEMMA 10.2. *Let $f(X)$ be a function on unions of L-blocks and let*

$$g(\mathbf{X}) \equiv \prod_{X \in \mathbf{X}} f(X); \qquad (=1 \text{ if } \mathbf{X} = \varnothing),$$

where $\mathbf{X}$ is a collection of sets X. Then

(a) $\sum_{X:X \text{ touches } Y} |f(X)| \leqslant \|f\|_0 \, |Y'|$

(b) $\sum_{\mathbf{X}:\text{each } X \text{ touches } Y} (1/|\mathbf{X}|!) \, |g(\mathbf{X})| \leqslant \exp(\|f\|_0 \, |Y'|)$

(c) $\sum_{\mathbf{X} \neq \varnothing:\text{each } X \text{ touches } Y} (1/|\mathbf{X}|!) \, |g(\mathbf{X})| \leqslant A^{-1} \exp(A \, \|f\|_0 \, |Y'|)$

where $A \geqslant 1$. Analogous statements with X, $\mathbf{X}$ replaced by Γ, $\mathbf{\Gamma}$ hold if $f = f(\Gamma)$.

Proof. (a) is trivial and implies (b) because

$$\sum_{\mathbf{X}} \frac{1}{|\mathbf{X}|!} \, |g(\mathbf{X})| = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n} \prod_1^n |f(X_i)|$$
$$= \exp\left(\sum_{X:X \text{ touches } Y} |f(X)| \right).$$

For (c), we apply (b) with f replaced by Af.

PROPOSITION 10.3. *For all L and all c, ρ_3 which is defined by*

$$\rho_3(X) \equiv \sum_{R \in \{R : R_1 = X, R_1 \text{ connected}\}} \int_C \mathscr{D}A_{R^{c} \cdot c} \mathscr{D}\rho_R \, \rho_2(R)$$

tends to zero in c-norm as $\lambda \to 0$.

PROPOSITION 10.4. *Given c_1, for L sufficiently large and λ sufficiently small, it follows that*

$$\|W_2^R\|_{c_1} \leqslant c_2 L^{d-1}.$$

For the next proposition we define K_2 by

$$K_2^R(\Gamma) = K_1^R(\Gamma) \qquad \text{if } \Gamma \text{ is connected}$$
$$= 0 \qquad \text{otherwise}$$

PROPOSITION 10.5. *For all c, if L is sufficiently large, there are constants $c_1(\lambda)$, c_2 where $c_1(\lambda)$ tends to zero with λ and such that for λ small*

$$\sum_{\Gamma \neq \varnothing} |K_1^R(\Gamma)| \, e^{c|X(\Gamma)|} \leqslant c_1(\lambda) \, e^{c_2|R_1'|}$$

where Γ is summed so that all its connected components touch R_1'. If $R = \varnothing$ then this reduces to

$$\|K_2^{\varnothing}\|_c \leqslant c_1(\lambda).$$

If $\Gamma = \varnothing$ then for Γ small, depending on L,

$$|K_1^R(\Gamma)| \leqslant e^{c_2|R_1'|}.$$

With the aid of these lemmas we now prove that for any c, if L is large enough,

$$\|K_3\|_c \to 0 \qquad \text{as } \lambda \to 0.$$

Define $\tilde{K}_1^R$, $\tilde{W}_2^R$ by

$$\tilde{K}_1^R(\Gamma) \equiv |K_2^R(\Gamma)| \, e^{c|X'(\Gamma)|}$$

together with an analogous formula for $\tilde{W}_2^R$. Also define $\tilde{K}_3$ and $\tilde{\rho}$ by

$$\tilde{K}_3(C) \equiv |K_3(C)| \, e^{c|C'|}$$
$$\tilde{\rho}(R) \equiv \rho_2(R) \, e^{c|R_1'|}.$$

The definition of K_3, and in particular the connectedness of C, ensures that

$$\tilde{K}_3^R(C) \leqslant \sum_{R, \boldsymbol{\Gamma}, \Delta} \frac{1}{|\boldsymbol{\Gamma}|!} \int \mathscr{D}A \mathscr{D}\rho \; \tilde{\rho}(R) \prod_{\Gamma \in \boldsymbol{\Gamma}} \tilde{W}_2^R(\Gamma) \, \tilde{K}_1^R(\Delta).$$

We estimate the $\boldsymbol{\Gamma}$ sum using Lemma 10.2. It is less than $\exp(\|W_2^R\|_c |R_1'|)$, because every Γ must touch R_1' or else $W_2^R(\Gamma) = 0$. By Proposition 10.4 this in turn is less than $\exp(c_2 L^{d-1} |R_1'|)$ provided L is large and λ is small (depending on c).

We hold R and the components of Δ which touch R_1' fixed and estimate the sum over Δ subject to these constraints. The sum over Δ is, equivalently, the sum over the collection of connected components of Δ. By Lemma 10.2 it is less than

$$\exp(\|K_2\|_c |C'|).$$

(We have replaced R by the null set in K_2 because if Γ does not touch R_1' then $K_2^R(\Gamma) = K_2^{\varnothing}(\Gamma)$.) By Proposition 10.5 this is less than $\exp(c_1(\lambda)|C'|)$.

Next we use Proposition 10.5 to bound the sum over the connected components of Δ that do touch R_1'. There is the possibility that this part of Δ is null. Thus we obtain the bound $[1 + c_3(\lambda)]\, e^{c_4|R_1'|}$. If λ is sufficiently small this is less then $2e^{c_4|R_1'|}$, so that

$$\tilde{K}_3(C) \leqslant 2 \left(\sum_{R \subset C} \int \mathscr{D}A \mathscr{D}\rho \; \rho_2(R)\, e^{\gamma|R_1'|} \right) e^{c_2(\lambda)|C'|},$$

provided L is large and λ is small, depending on c. Here $\gamma = \gamma(L, c)$ is given by $\gamma = c_1 L^{d-1} + c_4 + c$.

This bound is improved by considering, in the foregoing analysis, the possibilities $R = \varnothing$, $R \neq \varnothing$ separately. If $R = \varnothing$ then the only way to form a cluster on C is to let $\Gamma = \varnothing$ and $X(\Delta) = C$. By separating out these terms we obtain

$$\begin{aligned} \tilde{K}_3(C) &\leqslant \sum_{\Delta, X(\Delta) = C} \tilde{K}_1^{\varnothing}(\Delta) + 2 \left(\sum_{R \neq \varnothing} \int \mathscr{D}A \mathscr{D}\rho\, \rho_2(R)\, e^{\gamma|R_1'|} \right) e^{c_2(\lambda)|C'|} \\ &= \sum_{\Delta, X(\Delta) = C} \tilde{K}_1^{\varnothing}(\Delta) \\ &\quad + 2 \sum_{X \subset C, X \neq \varnothing} \prod_{Y \in \{\text{conn.cpts.of} X\}} (\rho_3(Y)\, e^{\gamma|Y'|})\, e^{c_2(\lambda)|C'|}, \end{aligned}$$

where ρ_3 is the quantity that appears in Proposition 10.3.

By Lemma 10.2 applied to the sum over X, rewritten as a sum over the collection of connected components of X,

$$\tilde{K}_3(C) \leqslant \sum_{\Delta, X(\Delta) = C} \tilde{K}_1^{\varnothing}(\Delta) + 2A^{-1} \exp(A\, \|\rho_3\|_\gamma |C'| + c_2(\lambda)|C'|).$$

If λ is sufficiently small we can achieve

$$A = \|\rho_3\|^{-1}, \qquad c_2(\lambda) < 1,$$

so that

$$\tilde{K}_3(C) \leqslant \sum_{\Delta, X(\Delta) = C} \tilde{K}_1^{\varnothing}(\Delta) + 2\, \|\rho_3\|_\gamma e^{2|C'|};$$

i.e.,

$$|K_3(C)| \leqslant \sum_{\Delta, X(\Delta) = C} |K_2^{\varnothing}(\Delta)| + 2\, \|\rho_3\|_\gamma e^{(2-c)|C'|},$$

and therefore for any c, if L is large,

$$\|K_3\|_c \to 0 \qquad \text{as } \lambda \to 0$$

by Lemma 10.1 and Propositions 10.3 and 10.5.

11. Large Field Regions; Proof of Proposition 10.3

In this section we quantify the idea that large fields are suppressed by action density by proving Proposition 10.3.

LEMMA 11.1 (Stability). *There exists a constant $\gamma(\mu) > 0$ independent of m_A, λ, e such that on the range of integration,*

$$S_0 + V_1 \geqslant \gamma S_0.$$

Proof. It is easiest to work with the untranslated ρ and A. Insert into $S_0 + V_1$ the elementary bounds

(a) $\lambda\rho^4 - \frac{1}{4}\mu^2\rho^2 + E \geqslant \frac{1}{8}\mu^2(\rho - \rho_0)^2$,

(b) $\frac{1}{2}\,|e^{ieA_{\langle xy\rangle}}\rho(y) - \rho(x)|^2 \geqslant \frac{1}{2}(\rho(y) - \rho(x))^2 + \frac{1}{2}\gamma_1\rho(y)\,\rho(x)(eA_{\langle xy\rangle})^2$,

where γ_1 is independent of μ, m_A, λ, e.

Case 1. If $\rho(y)\,\rho(x) \geqslant \frac{1}{2}\rho_0^2$, then we continue (b) with

$$\begin{aligned} &\geqslant \tfrac{1}{2}(\rho(y) - \rho(x))^2 + \tfrac{1}{2}\gamma_1\,\tfrac{1}{2}\rho_0^2 e^2 A^2_{\langle xy\rangle} \\ &= \tfrac{1}{2}(\rho(y) - \rho(x))^2 + \tfrac{1}{4}\gamma_1\, m_A^2 A^2_{\langle xy\rangle}. \end{aligned}$$

Case 2. If on the other hand, $\rho(y)\,\rho(x) < \frac{1}{2}\rho_0^2$, then $\rho = \rho(y)$ or $\rho(x)$ is less than $\rho_0/\sqrt{2}$ and we continue (a) with the bounds

$$\begin{aligned} &\geqslant \frac{1}{16}\mu^2(\rho - \rho_0)^2 + \frac{1}{16}\mu^2\left(1 - \frac{1}{\sqrt{2}}\right)^2 \rho_0^2 \\ &\geqslant \frac{1}{16}\mu^2(\rho - \rho_0)^2 + \frac{1}{16}\mu^2\left(1 - \frac{1}{\sqrt{2}}\right)^2 \rho_0^2(eA_{\langle xy\rangle}/\pi)^2 \\ &= \frac{1}{16}\mu^2(\rho - \rho_0)^2 + \frac{1}{16}\mu^2\left(1 - \frac{1}{\sqrt{2}}\right)^2 (m_A^2/\pi^2)\,A^2_{\langle xy\rangle}, \end{aligned}$$

valid on the range of A, and in either case the requisite lower bound follows.

LEMMA 11.2. *Let X be a union of L-blocks in Λ. Then there exists c_1, c_2 so that*

$$\int \mathscr{D}\rho_X \mathscr{D}A_{X^{c*c}}\,\rho_2(X) \leqslant c_1^{|X_1|}\, e^{-c_2 p^2(\lambda)|X'|}.$$

Proof. By the Stability Lemma 11.1,

$$\operatorname{Re} S(X) \geqslant \gamma(\mu)\, S_0(X) + \operatorname{Re} V_2(X) \geqslant c_1 S_0(X) - c_2\,|X|$$

for some c_1, $c_2 > 0$ and λ small. For any c

$$\zeta_X e^{-(1/2)cS_0(X)} \leqslant e^{-c'(c)p^2(\lambda)|X'|}$$

since ζ forces each L-block $B(x')$, $x' \in X'$, together with its nearest neighbor sites to contain at least one large field. (When $v(p) \neq 0$ then either $(dA+v)^2\,(p) > cp^2(\lambda)$ or else $A(b)^2 > cp^2(\lambda)$ for some $b \in p$.) We substitute this bound into the definition of ρ_2 along with, for λ small depending on L,

$$|\Xi(\Lambda_0 = X^c, \mathbf{s}=0, \text{cond.}=0)/\Xi(\Lambda, \mathbf{s}=0)| \leqslant \exp(c\,|X_1'|).$$

To obtain this estimate we note that regions outside X_1 cancel exactly. Inside X_1 we use

$$\chi\,|\exp(-V)| \leqslant \exp(c\,|X_1'|)$$

in the numerator and bound the denominator using

$$\left|\int d\mu\,\chi e^{-V}\right| = \prod_{\text{components } C} \left|\int d\mu_C \chi_C e^{-V(C)}\right| \geqslant \prod_C \left(\int d\mu_C \chi_C \operatorname{Re} e^{-V(C)}\right)$$
$$\geqslant \prod_C \left(\int d\mu_C \chi_C \cdot \tfrac{1}{2}\right) \geqslant e^{-c|X_1'|}.$$

The last type of inequality is discussed more fully at the end of Section 14, below.

Now we are reduced to proving that for each $c_1 \geqslant 0$ there exists c_2 such that

$$\int \mathscr{D}\rho_X \int_C \mathscr{D}A_{X^{c*c}} \sum_{v:dv=0} e^{-c_1 S_0(X)} Z_0^*(X^c, \mathbf{s}=0)/Z_0(\Lambda, \mathbf{s}=0) \leqslant c_2^{|X_1|}. \qquad (11.1)$$

We drop the constraint $dv=0$. As before each $v(p) \neq 0$ produces a factor $O(M_A^2/e^2)$ or $O(v(p)^2/e^2)$ in the exponent. After summing over $v(p) \neq 0$, the exponent is still bounded below by a constant times the $v=0$, A-field form. Thus

$$\sum_{v:dv=0} \exp\left(-\tfrac{1}{2}c_1 \sum_{p \subset X} (dA(p)+v(p))^2 - \tfrac{1}{2}c_1 m_A^2 \sum_{b \subset X} A_b^2\right)$$
$$\leqslant 2^{|X^{**}|} \exp\left(-\tfrac{1}{2}c(m_A) \sum_{p \subset X} (dA(p))^2 - \tfrac{1}{2}c(m_A)\, m_A^2 \sum_{b \subset X} A_b^2\right)$$

which holds at sufficiently weak coupling in the range of A-integration. Consequently the left-hand side of (11.1) is less than

$$2^{|X|} \int_{NC} \mathscr{D}\rho_X \mathscr{D}A_{X^{c*c}} e^{-c(m_A)S_0(X, v=0)} Z_0^*(X^c, \mathbf{s}=0)/Z_0(\Lambda, \mathbf{s}=0).$$

When we substitute in the fact that Z_0^*, Z are integrals over $\exp(-S_0^*)$, $\exp(-S_0)$ and make use of the decoupling caused by $\mathbf{s}=0$ to cancel out contributions from X_1^c we obtain a bound

$$2^{|X|}\int_{NC}\mathscr{D}\rho_{X_1}\mathscr{D}A_{X_1}e^{-c(m_A)S_0(X_1,v=0)}\Big/\int_{NC}\mathscr{D}\rho_{X_1}\mathscr{D}A_{X_1}e^{-S_0(X_1,v=0)}$$

which is less than $c_2^{|X_1|}$ for some c_2 by scaling ρ and A in the numerator.

Proof of Proposition 10.3. By Lemma 11.2,

$$\begin{aligned}\rho_3(X)&\leqslant\sum_{R\in\{R:R_1=X\}}c_1^{|R_1|}e^{-c_2p^2(\lambda)|R'|}\\&\leqslant e^{-c|X|}\sum_{R\in\{R:R_1=X\}}c_1^{|R_1|}e^{c|R_1|}e^{-c_2p^2(\lambda)|R'|}.\end{aligned}$$

By the definition of R_1,

$$\begin{aligned}|R_1'|&\leqslant(c_3r(\lambda))^d\,|R'|\\&\leqslant\varepsilon p^2(\lambda)|R'|,\end{aligned}$$

for any $\varepsilon>0$ if λ is small enough. Thus for a decreased constant c_2

$$\rho_3(X)\leqslant e^{-c|X'|}\sum_{R\subset X}e^{-c_2p^2(\lambda)|R'|}.$$

We estimate the sum over R by rewriting it as a sum over the set of connected components of R and applying Lemma 10.2 with

$$\begin{aligned}f(R)&=e^{-c_2p^2(\lambda)|R'|}, &&\text{if } R \text{ is connected},\\&=0, &&\text{otherwise},\end{aligned}$$

so that

$$\rho_3(X)\leqslant e^{-c|X'|}A^{-1}e^{A\|f\|_0|X'|}.$$

Since, by Lemma 10.1, $\|f\|_0\to 0$ with λ, we can choose A so that $A\,\|f\|_0=1$. We find

$$\rho_3(X)\leqslant\|f\|_0\,e^{-(c-1)|X'|}$$

which, by Lemma 10.1, proves the proposition.

12. Properies of Covariances; Formulas for Differentiating Gaussian Expectations with Respect to s-Factors

This section is very close in spirit to parts of [3]. We want to establish analogues to the formulas of [3] for a conditional Gaussian measure. The covariance for the gauge field also needs discussion. We establish here the basic estimates used to prove Propositions 10.4 and 10.5.

We will use $\langle \cdot \rangle$ to denote Gaussian expectations with respect to the Gaussian measures (7.5) or one of its s-dependent analogs.

Estimates on Covariances.

We shall set

$$C_\rho(x, y) \equiv \langle \rho(x)\, \rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle,$$
$$C_A(b, c) \equiv \langle A(b)\, A(c) \rangle - \langle A(b) \rangle \langle A(c) \rangle.$$

Since $\langle\ \rangle$ is Gaussian, we can evaluate

$$C_\rho(x, y) = [\mu^2 - \Delta(\mathbf{s})]^{-1}\,(x, y)$$
$$C_A(b, c) = [m_A^2 + \delta d(\mathbf{s})]^{-1}\,(b, c),$$

where $\Delta(\mathbf{s})$ is the linear operator on $l^2(\Lambda_0)$ defined by

$$(\theta, -\Delta(s)\,\theta) = \tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda} \sigma_{\langle xy \rangle}(\theta(y) - \theta(x))^2,$$

with $\theta = 0$ on Λ_0^c. The σ's depend on $\mathbf{s}$; see (8.1). Likewise $\delta d(\mathbf{s})$ is the linear operator on $l^2(\Lambda_0^*)$ defined by

$$(\psi, \delta d(\mathbf{s})\,\psi) = \tfrac{1}{2} \sum_{p \subset \Lambda} \sigma_p (d\psi)^2\,(p)$$

with $\psi(b) = 0$ if $b \not\subset \Lambda_0$.

Lemma 12.1. *There exists $m = m(\mu, m_A)$, nonzero if $m_A, \mu > 0$, and $c = c(\mu, m_A)$ such that*

$$C_\rho(x, y) \leqslant c e^{-m \|x - y\|}$$
$$C_A(b, c) \leqslant c e^{-m\, \mathrm{dist}(b, c)}$$

uniformly in $\mathbf{s}$, Λ_0.

Proof. The easiest method to prove this seems to be an idea of Combes and

Thomas. (See [8] and also [9].) We will work through the C_A case. C_ρ is the same with obvious variations. Define a multiplication operator $V = V(a)$ on $l^2(\Lambda_0^*)$ by

$$f(b) \xrightarrow{V(a)} e^{a \cdot x_b} f(b)$$

where x_b is the midpoint of bond b. Then, letting δ_b denote the element of $l^2(\Lambda_0^*)$ which equals one on bond b, and equals zero everywhere else,

$$\begin{aligned} |C_A(b, c)| &= |(\delta_b, [m_A^2 + \delta d(\mathbf{s})]^{-1} \delta_c)| \\ &= |(V\delta_b, V^{-1}(m_A^2 + \delta d(\mathbf{s}))^{-1} VV^{-1}\delta_c)| \\ &\leqslant e^{a(x_b - x_c)} \|(m^2 + V^{-1}\delta d(\mathbf{s})\, V)^{-1}\|. \end{aligned}$$

Next we use the fact that if A is an operator on $l^2(\Lambda_0^*)$, such that for all $\psi \in l^2(\Lambda_0^*)$,

$$(\psi, A\psi) \geqslant c\, \|\psi\|^2,$$

then $\|A^{-1}\| \leqslant c^{-1}$. Thus

$$\begin{aligned} &(\psi, [m_A^2 - V^{-1}\delta d(\mathbf{s})\, V]\, \psi) \\ &\quad = m_A^2 \sum_b |\psi(b)|^2 + \sum_p \sigma_p \left(\sum_{b \in \partial p} e^{-ax_b} \psi(b) \right) \left(\sum_{b \in \partial p} e^{ax_b} \psi(b) \right), \end{aligned}$$

which is a strictly positive operator if $\|a\|$ is not too large. This proves the lemma.

Derivatives of Covariances

We evaluate derivatives of C_ρ, C_A using

$$\frac{\partial}{\partial s_b} [\mu^2 - \Delta(\mathbf{s})]^{-1} = [\mu^2 - \Delta(\mathbf{s})]^{-1} \left(\frac{\partial}{\partial s_b} \Delta(\mathbf{s}) \right) [\mu^2 - \Delta(\mathbf{s})]^{-1}$$

repeatedly. The result can be wriiten in the form

$$(\partial^\Gamma C_\rho)(x, y) = \sum_{\omega : x \to y} \prod_{\text{steps} \in \omega} C_\rho(\text{step})$$

where ω is summed over all *walks* from x to y which consist of a sequence of *steps* of the form

$$(x, b_1),\ (b_1, b_2),\ (b_2, b_3), \ldots,\ (b_n, y)$$

where each b_i, $i = 1, \ldots, n$, belongs to a face in Γ, so that each face is visited once and only once. Thus $n = |\Gamma|$. To define C_ρ (step), let b, c be the bonds $b = \langle tu \rangle$, $c = \langle vw \rangle$ and set

$$\begin{aligned} C_\rho(x, c) &\equiv C_\rho(x, v) - C_\rho(x, w), \\ C_\rho(b, c) &\equiv C_\rho(t, c) - C_\rho(u, c), \\ C_\rho(b, x) &\equiv C_\rho(t, x) - C_\rho(u, x). \end{aligned}$$

This suffices to define C_ρ (step) since a step is either a pair of bonds or a bond and a site.

An analogous formula holds for C_A except that sites get replaced by bonds and bonds by plaquettes which link L-blocks linked by Γ. These formulas enable us, in conjunction with Lemma 12.1, to read off the following estimate:

LEMMA 12.2. *There exist $c_i = c_i(\mu, m_A)$, $i = 1, 2$, $m = m(\mu, m_A)$ such that*

$$|(\partial^\Gamma C_\rho)(x, y)| \leqslant c_1 c_2^{|\Gamma|} e^{-md(x,y,\Gamma)},$$
$$|(\partial^\Gamma C_A)(b, c)| \leqslant c_1 c_2^{|\Gamma|} e^{-md(b,c,\Gamma)},$$

where $d(x, y, \Gamma)$ is the length of the shortest path which joins x to y and visits a bond in each face in Γ. Define $d(b, c, \Gamma)$ analogously.

By increasing L, the exponential decay dominates $c^{|\Gamma|}$. Finally note that

$$\langle \rho(x) \rangle = \sum_{\langle yz \rangle \in \partial \Lambda_1, z \notin \Lambda_1} C_\rho(x, y)\, \rho(z).$$

Together with an analogous formula for $\langle A(b) \rangle$, we can estimate derivatives

$$\begin{aligned} |\partial^\Gamma \langle \rho(x) \rangle| &\leqslant c_1 c_2^{|\Gamma|} p(\lambda)\, e^{-md(x,\partial\Lambda_1,\Gamma)}, \\ |\partial^\Gamma \langle A(b) \rangle| &\leqslant c_1 c_2^{|\Gamma|} p(\lambda)\, e^{-md(b,\partial\Lambda_1,\Gamma)}, \end{aligned} \tag{12.1}$$

where $c_i = c_i(m, \mu)$, $i = 1, 2$ and $d(x, \partial\Lambda_1, \Gamma)$ is the distance to the boundary $\partial\Lambda_1$ from x, via Γ. Here $d(b, \partial\Delta_1, \Gamma)$ is defined analogously.

The formula for differentiating an s-parameter appearing in the Gaussian measure is

$$\frac{\partial}{\partial s_{b'}} \langle P \rangle = \langle (K_\rho^{b'} + K_A^{b'})\, P \rangle$$

where the K's are differential operators

$$\begin{aligned} K_\rho^{b'} \equiv{}& \frac{1}{2} \sum_{x,y} \frac{\partial}{\partial s_{b'}} C_\rho(x, y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} \\ &+ \sum_x \frac{\partial}{\partial s_{b'}} \langle \rho(x) \rangle \frac{\partial}{\partial \rho(x)}, \\ K_A^{b'} \equiv{}& \frac{1}{2} \sum_{b,c} \frac{\partial}{\partial s_{b'}} C_A(b, c) \frac{\partial}{\partial A(b)} \frac{\partial}{\partial A(c)} \\ &+ \sum_b \frac{\partial}{\partial s_{b'}} \langle A(b) \rangle \frac{\partial}{\partial A(b)}. \end{aligned}$$

These formulas are the same as the standard one in [3], except for the first order terms which arise because the measure does not have mean zero. They may be proved by first translating so that the measure does have mean zero, using the standard formula, and translating back again.

Following [3] we can write a formula for a multiple s-derivative:

$$\partial^{\Gamma} \langle P \rangle = \sum_{\pi \in P(\Gamma)} \left\langle \prod_{\gamma \in \pi} (K^{\gamma}) P \right\rangle \tag{12.2}$$

with

$$K_{\rho}^{\gamma} \equiv \frac{1}{2} \sum_{x,y} (\partial^{\gamma} C(x,y)) \frac{\partial}{\partial \rho(x)} + \sum_{x} (\partial^{\gamma} \langle \rho(x) \rangle) \frac{\partial}{\partial \rho(x)},$$

$$K_{A}^{\gamma} \equiv \frac{1}{2} \sum_{b,c} \partial^{\gamma} C_{A}(b,c) \frac{\partial}{\partial A(b)} \frac{\partial}{\partial A(c)} + \sum_{b} \partial^{\gamma} \langle A(b) \rangle \frac{\partial}{\partial A(b)},$$

$$K^{\gamma} \equiv K_{\rho}^{\gamma} + K_{A}^{\gamma}, \tag{12.3}$$

and $P(\Gamma)$ is the set of partitions of Γ.

13. Proof of Proposition 10.4

We begin by estimating $I \equiv \partial^{\Gamma} \log Z_0^*(\mathbf{s}_{\Gamma})$.
If we pick one bond b' in Γ and differentiate with respect to $s_{b'}$, we obtain

$$I = -\frac{1}{2} \partial^{\Gamma \setminus \{b'\}} \left\langle \frac{\partial S_0^*}{\partial s_{b'}} \right\rangle.$$

Now we perform the remaining derivatives, $\partial^{\tilde{\Gamma}}$, using the formula (12.2) for differentiating the expectation:

$$I = -\frac{1}{2} \sum_{\pi} \left\langle \prod_{\gamma \in \pi} K^{\gamma} \frac{\partial S_0^*}{\partial s_{b'}} \right\rangle.$$

Since S_0^* is quadratic, the only partitions π which can contribute are those with one or two elements. There are less than $2^{|\Gamma|-1}$ such partitions, so

$$|I| \leqslant \frac{1}{4} 2^{|\Gamma|} \sup_{\pi} \left| \left\langle \prod_{\gamma \in \pi} K^{\gamma} \frac{\partial S_0^*}{\partial s_{b'}} \right\rangle \right|.$$

Lemma 12.2 and (12.1) tell us that there exists $m > 0$ such that

$$|I| \leqslant c_1 c_2^{|\Gamma|} L^{d-1} [1 + \rho(\lambda) e^{-mr(\lambda)}]^2 e^{-md(\Gamma)},$$

where $d(\Gamma)$ is the shortest path connecting faces in Γ. Since $p\exp(-mr)$ tends to zero with λ, for λ small and a new c_1,

$$|I| \leqslant c_1 c_2^{|\Gamma|} L^{d-1} e^{-md(\Gamma)}.$$

By considering faces in Γ perpendicular to each coordinate axis it is not difficult to prove that

$$d(\Gamma) \geqslant \left(\frac{1}{2d}|\Gamma| - 1\right) L.$$

Therefore, by Lemma 10.1, if L is large enough, depending on c, then $\|\exp(-md(\Gamma))\| < \infty$, and Proposition 10.4 is proved.

14. Proof of Proposition 10.5

If Γ is the null set, then $K_1(\Gamma)$ has the form

$$\Xi(R^c, \mathbf{s} = 0 \text{ in } R_1^c)/\Xi_0(R^c).$$

Since $\mathbf{s} = 0$, the couplings across boundaries of (connected components of $R_1)\backslash R$ and L-blocks in R_1^c are zero, and numerator and denominator factor. They would cancel exactly, except for the conditioning in the numerator and the extra $s = 0$ bonds. Consequently contributions from (connected components of $R_1)\backslash R$ do not cancel. We bound the uncancelled factors in the numerator above and the factors in the denominator below, just as in the proof of Lemma 11.2, and obtain, for λ small depending on L,

$$|K_1^R(\Gamma = \varnothing)| \leqslant e^{c|R_1^c|}.$$

The definition of K_1 implies

$$\sum_{\Gamma \neq \varnothing} |K_1(\Gamma)| \, e^{c|\Gamma|} \leqslant \sum_{\Gamma \neq \varnothing} \sup_{s_\Gamma} |\partial^\Gamma \Xi(R^c, \mathbf{s}_\Gamma)/\Xi_0(R^c)| \, e^{c|\Gamma|},$$

where the sum over Γ is constrained so that each connected component of Γ touches R. Since, by the definition of $\mathbf{s}_\Gamma$, all coupling across faces not in Γ or in R_1 is absent, if we set

$$X \equiv (R_1 \backslash R) \cup X(\Gamma),$$

then both Ξ's in $K_1(\Gamma)$ factor across ∂K and the factors corresponding to X^c (and ∂X) cancel in the ratio so that

$$\partial^\Gamma \Xi(R^c, \mathbf{s}_\Gamma)/\Xi_0(\Lambda_0) = \partial^\Gamma \langle e^{-V^*(X, \mathbf{s}_\Gamma)} \chi_X \rangle / \Xi_0(X).$$

Furthermore $\Xi_0(X)$ factors across all boundaries of L-blocks in $X\backslash R_1$ and across connected components of X. As in the proof of Lemma 11.2, we find

$$|\Xi_0(X)| \geqslant (\tfrac{1}{2})^{|X'|}$$

for λ sufficiently small, depending on L. Set

$$I_1 \equiv I_1(\Gamma) \equiv |\partial^\Gamma \langle e^{-V^*(X, s_\Gamma)} \chi_X \rangle|$$

so that

$$\sum_\Gamma{}' |K_1(\Gamma)| \, e^{c|\Gamma|} \leqslant \sum_\Gamma e^{c|X'|} 2^{|X'|} \sup_{s_\Gamma} I_1. \tag{14.1}$$

Here $X\backslash R_1$ is a disjoint union of sets of the form $\tilde{X} \equiv X(\tilde{\Gamma})$ where $\tilde{\Gamma}$ is a connected component of Γ. Accordingly $|\tilde{X}'| \leqslant 2|\tilde{\Gamma}|$ and $|X'| \leqslant |R_1'| + 2|\Gamma|$. This estimate is used to eliminate $|X'|$ in (14.1). We find that in order to prove the proposition, it is sufficient to prove that given c_2, if L is sufficiently large and λ is sufficiently small, then

$$I_1(\Gamma) \leqslant c_1(\lambda) \, e^{-c_2|\Gamma|} e^{c_3|R_1|}. \tag{14.2}$$

Here $c_1(\lambda) \to 0$ with λ. By Lemmas 10.1 and 10.2, if c_1 is large enough, then

$$\sum_\Gamma e^{-c_1|\Gamma|} \leqslant \exp(c_2 |R_1'|)$$

for some c_2.

Besides the s-dependence of $\langle \ \rangle$, the V is also a function of s; each derivative in ∂^Γ can act on either dependence, so we write

$$I_1 = |(\partial_{\langle\rangle} + \partial_V)^\Gamma \langle e^{-V} \chi \rangle|,$$

where the subscripts $\langle\rangle$, V specify where ∂ acts. The expectation is analytic as a function of the s-dependence in V because χ limits the size of the fields. We estimate the ∂_V derivatives by Cauchy's formula: For any $A > 0$ (and we choose A large for λ small)

$$I_1 \leqslant 2^{|\Gamma|} \sup_{\Gamma_1 \subset \Gamma} A^{-|\Gamma \backslash \Gamma_1|} \sup_{s_V \in \partial D_A} |\partial^{\Gamma_1}_{\langle\rangle} \langle e^{-V} \chi \rangle|, \tag{14.3}$$

where $2^{|\Gamma|}$ is the number of terms in the expansion of $(\partial_V + \partial_{\langle\rangle})^\Gamma$, and D_A is the contour corresponding to the polydisc of radius A centered at the origin in complex s-space. Now we concentrate on estimating

$$I_2 \equiv |\partial^\Gamma_{\langle\rangle} \langle e^{-V} \chi \rangle|$$

for arbitrary Γ. By the formula for differentiating expectations (12.2) we find

$$I_2 \leqslant \sum_{\pi \in P(\Gamma)} \left| \left\langle \prod_{\gamma \in \pi} K^{\gamma} e^{-V} \chi \right\rangle \right| \equiv \sum_{\pi \in P(\Gamma)} I_3.$$

Each K in I_3 is split into its first and second order parts and into its $\partial/\partial A, \partial/\partial \rho$ parts. Let I_4 be the largest term in the resulting sum

$$I_3 \leqslant 4^{|\Gamma|} I_4.$$

Next we write each partial differential operator as a sum over the positions (x, b) of its ρ or A derivatives. We take absolute values inside these sums and bound s-differentiated covariances and $\langle \rho \rangle$, $\langle A \rangle$ factors using Lemma 12.2, and (12.1):

$$I_4 \leqslant c^{|\Gamma|} \left(\prod_{\gamma \in \pi} \sum_{r(\gamma)} \right) \prod_{\gamma \in \pi} e^{-md(r(\gamma), \gamma)} |\langle D(e^{-V^*} \chi) \rangle|.$$

$r(\gamma)$ is, depending on γ, either a pair of sites, a pair of bonds, a single site or a single bond. The $r(\gamma)$ label positions of field derivatives which are collectively denoted by D. Here $d(r(\gamma), \gamma)$ is the distance along the shortest path that links the element(s) of $r(\gamma)$ and the faces in γ. We have used $|\langle \rho(x) \rangle|$, $|\langle A(b) \rangle| \leqslant 1$ for λ sufficiently small. This follows from (12.1) and $p(\lambda) \exp(-r(\lambda)) \to 0$, as $\lambda \to 0$.

By Leibnitz' rule,

$$|\langle D(e^{-V^*} \chi) \rangle| \leqslant 4^{|\Gamma|} |\langle D_1 (e^{-V^*}) D_2 \chi \rangle|,$$

where the derivatives in D are assigned to either D_1 or D_2 so as to maximize.

We estimate derivatives of $\exp(-V^*)$ with, for small fields, and λ sufficiently small depending on L,

$$|D_1 e^{-V^*}| \leqslant \prod_{x'} (n(x')!)^p \, (q(\lambda))^{N_1} c^{|X'|},$$

where $n(x')$ is the number of derivatives localized in $B(x')$; N_1 is the total number of derivatives in D_1; p, c are constants and

$$q(\lambda) = O(\lambda^{\beta})$$

with $\beta > 0$ a constant. This estimate is easy and the proof is omitted.

The derivatives of χ are bounded with

$$\langle |D_2 \chi| \rangle \leqslant \prod_{x} (n(x)!)^p \prod_{b} (n(b)!)^p \, c^{N_1} q(\lambda)^{N_2 - M} \tag{14.4}$$

where p, c are constants; $q(\lambda) = O(\lambda^{\beta})$; N_2 is the order of D_2; $n(x)$, $n(b)$ are, respectively, the number of derivatives at site x, bond b; and M is the number of derivatives within distance $r(\lambda)/2$ of R_1. The proof of this estimate is deferred to the end of our argument.

We combine the last two estimates with, for $\varepsilon > 0$,

$$\prod_{x'} (n(x')!)^{p_1} \prod_{x} (n(x)!)^{p_2} \prod_{b} (n(b)!)^{p_3} \leqslant c_\varepsilon^N \prod_{\gamma} e^{-(m-\varepsilon)d(r(\gamma),\gamma)}.$$

This estimate can be found on p. 240 of [3]. It is not hard. The idea is that if one of the factorials is large then most of the corresponding derivatives are attached by covariances to distant locations. By the last five estimates we obtain (with different $m > 0$, and different c)

$$I_4 \leqslant c^{N+|\Gamma|+|X'|} \left(\prod_{\gamma \in \pi} \sum_{r(\gamma)} \right) \prod_{\gamma \in \pi} e^{-md(r(\gamma),\gamma)} (q(\lambda))^{N-M}$$

where N is the total number of derivatives. By decreasing M some more and increasing c, we may drop the $(q(\lambda))^{-M}$ factor because $\exp(-\varepsilon r(\lambda))\, p(\lambda) \to 0$ with λ faster than any power of λ for all $\varepsilon > 0$.

Let $d(\gamma)$ be the length of the shortest path connecting the faces in γ. We estimate the sums over $r(\gamma)$, γ fixed, using some of the exponential decay: with different m, c

$$I_3 \leqslant c^{|X'|+|\Gamma|} (q(\lambda))^{|\pi|} \prod_{\gamma \in \pi} |\gamma|^2\, e^{-md(\gamma)}.$$

We have eliminated N using $|\Gamma| \geqslant N \geqslant |\pi|$.

We now obtain an estimate on I_2 by doing the sum over $\pi \in P(\Gamma)$. Let

$$Q(\gamma) \equiv q(\lambda) |\gamma|^2\, e^{-md(\gamma)}\, e^{\xi|\gamma|} \eta^{-1}$$

where $\eta \leqslant 1$, $\xi \geqslant 0$. Then from our estimate on I_3,

$$I_2 \leqslant \eta c^{|X'|+|\Gamma|} e^{-\xi|\Gamma|} \left\{ \sum_{\pi \in P(\Gamma)} \prod_{\gamma \in \pi} Q(\gamma) \right\}.$$

We enlarge the sum over π to all collections (sets with repeated elements) of subsets of Γ so that it corresponds to an exponential:

$$\begin{aligned} I_2 &\leqslant \eta c^{|X'|+|\Gamma|} e^{-\xi|\Gamma|} \exp\left(\sum_{\gamma \subset \Gamma} Q(\gamma) \right) \\ &\leqslant \eta c^{|X'|+|\Gamma|} e^{-\xi|\Gamma|} \exp(q(\lambda)\, \eta^{-1}\, \| |\cdot|^2\, e^{-md} \|_\xi\, |\Gamma|). \end{aligned}$$

For L large enough depending on ξ the ξ-norm is bounded by a constant, c_1 say. Consequently if we choose $\eta = q(\lambda)$,

$$\begin{aligned} I_2 &\leqslant q(\lambda)\, c^{|X'|+|\Gamma|} e^{(-\xi+c_1)|\Gamma|} \\ &\leqslant q(\lambda)\, c^{|R_1'|+3|\Gamma|} e^{(-\xi+c_1)|\Gamma|} \end{aligned}$$

which yields (14.2) because, by taking L large, we can achieve any value for ξ.

The proof is complete except for the bound (14.4) on derivatives of χ. Since the expectation and the right-hand side of (14.4) factor into an A-field and a ρ-field part, we suppose without loss of generality that D_2 only contains ρ-derivatives. The M ρ-derivatives that take place at sites x closer than $r(\lambda)/2$ to R_1 are bounded by (7.1). Let $\tilde{D}$ be the remaining $N_2 - M$ derivatives. Let the set of sites where these take place be denoted by S. Since χ is constant except on the borderline between small and large fields we have

$$|\tilde{D}\chi| \leqslant \prod_{x \in S} ((\cosh(p(\lambda)/2))^{-1} \cosh \rho(x)) |\tilde{D}\chi|$$

so that by (7.1),

$$\langle |\tilde{D}\chi| \rangle \leqslant e^{-p(\lambda)|S|/2} \prod_{x \in S} c^{n(x)} (n(x)!)^p \left\langle \prod_{x \in S} \cosh \rho(x) \right\rangle.$$

Let $\langle\ \rangle_0$ denote the expectation with zero conditioning on R, then by translation

$$\left\langle \prod_{x \in S} \cosh \rho(x) \right\rangle = \left\langle \prod \cosh(\rho(x) + \langle \rho(x) \rangle) \right\rangle_0$$
$$\leqslant 2^{|S|} \prod_{x \in S} \cosh(\langle \rho(x) \rangle) \left\langle \prod_{x \in S} \cosh \rho(x) \right\rangle_0.$$

As $\lambda \to 0$, $\langle \rho(x) \rangle \to 0$ uniformly in x by (12.1), together with $p(\lambda) \exp(-r(\lambda)/2) \to 0$. Furthermore, by splitting each cosh into exponentials and explicitly evaluating the expectation, we find

$$\left\langle \prod_{x \in S} \cosh \rho(x) \right\rangle_0 \leqslant e^{c|S|}$$

for some constant c. All these estimates collect to give

$$\langle |\tilde{D}\chi| \rangle \leqslant \prod_{x \in S} c^{n(x)} (n(x)!)^p \, e^{-p(\lambda)|S|/2}$$

for λ small. This implies (14.4).

APPENDIX: ON THE MAYER EXPANSION

Let $\mathscr{L}$ be an abstract set with a reflexive binary relation "x touches y", $x, y \in \mathscr{L}$. We will say a subset $X \subset \mathscr{L}$ touches $x \in \mathscr{L}$ if some element in X touches x. Likewise two subsets X, $Y \subset \Lambda$ touch if they contain elements x, y which touch.

Given $f(X)$ a function on subsets of Λ set

$$\|f\|_c \equiv \sup_{x \in \mathscr{L}} \sum_{X, X \text{ touches } x} |f(X)| \, e^{c|X|}.$$

DEFINITION. $\mathbf{X}$ is used to denote an ordered collection of subsets of $\mathscr{L}$. A *collection* is a set which is allowed repeated elements. A *graph* G on $\mathbf{X}$ is a set of unordered pairs XY, called *lines*, with $X, Y \in \mathbf{X}$. Define $\bar{f}(\bar{X})$, where $\bar{X} \subset \mathscr{L}$ by

$$\bar{f}(\bar{X}) \equiv \sum_{\mathbf{X},\, \cup\{X : X \in \mathbf{X}\} = \bar{X}} \frac{1}{|\mathbf{X}|!} \prod_{X \in \mathbf{X}} f(X) \times \sum_{G \in \text{connected graphs on } \mathbf{X}} \prod_{XY \in G} (U(XY) - 1)$$

where $U(XY) = 0$ if X touches Y and is 1 otherwise.

The relationship between f and $\bar{f}$ is that if $\mathbf{X}$ is a set of subsets of $\Lambda \subset \mathscr{L}$, and if

$$Z \equiv \sum_{\mathbf{X}} \frac{1}{|\mathbf{X}|!} \prod_{X \in \mathbf{X}} f(X) \prod_{X, Y \in \mathbf{X}} U(XY)$$

is the grand canonical partition function of a hard core gas, then formally

$$Z = \exp\left[\sum_{X \subset \Lambda} \bar{f}(X)\right], \tag{A.1}$$

and furthermore, for all $c \geqslant 0$, $\varepsilon > 0$,

$$\|\bar{f}\|_c \leqslant \sum_{n=1}^{\infty} \|f\|_{c+1+\varepsilon}^n \left(1 + \frac{1}{\varepsilon}\right)^n. \tag{A.2}$$

A proof of (A.1) is given on p. 33 of [10]. Our proof of (A.2) is mostly extracted from [11].

Proof of the estimate. We will appeal to the following fact:

$$\left| \sum_{G \in \text{connected graphs on } \mathbf{X}} \prod_{XY \in G} (U(XY) - 1) \right| \leqslant \sum_{G \in \text{trees on } \mathbf{X}} \prod_{XY \in G} |U(XY) - 1|. \tag{A.3}$$

Here a *tree* is a connected graph which becomes disconnected if any line is removed. This estimate is proved in [12]. The proof of a more powerful theorem allowing many-body graphs is implicit in [13]. Let $X \in \mathscr{L}$ be arbitrary, then

$$\sum_{X \subset \mathscr{L}, \bar{X} \ni x} |\bar{f}(\bar{X})|\, e^{c|\bar{X}|} \leqslant \sum_{\mathbf{X},\, x \in \text{some } X \in \mathbf{X}} \frac{1}{|\mathbf{X}|!} \prod_{X \in \mathbf{X}} |f(X)|\, e^{c|X|} \cdot \sum_{C \in \text{trees on } \mathbf{X}} \prod_{XY \in G} |U(XY) - 1|$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_{X_1,\ldots,X_n \subset \mathscr{S},\ \text{some}\ X_1 \ni x} \prod_{i=1}^{n} |f(X_i)|\, e^{c|X_i|}$$

$$\cdot \sum_{G \in \text{trees on } \{1,\ldots,n\}} \prod_{ij \in G} |U(X_i X_j) - 1|. \tag{A.4}$$

Let d_i denote the *coordination number* of graph G at X_i. This is the number of lines that contain X_i. Cayley's theorem says that the number of trees on n vertices with prescribed coordination numbers $d_1,\ldots,d_n$ (adding up to $2(n-1)$) is equal to

$$(n-2)! \Big/ \left[\prod_{i=1}^{n} (d_i - 1)! \right] \qquad (n \geqslant 2). \tag{A.5}$$

In (A.4) we resum over $X_1,\ldots,X_n$ and G while holding n and the coordination numbers of G fixed. We estimate the sum subject to these constraints by the number of graphs times the supremum over graphs, so that (A.4) is less than

$$\sum_{n=1}^{\infty} \frac{(n-2)!}{(n-1)!} \sum_{d_1,\ldots,d_n} \sup_{G \in \text{trees on } \{1,2,\ldots,n\} \text{ with } d_i \text{ fixed}}$$

$$\sum_{X_1,\ldots,X_n \subset \mathscr{S},\ X_1 \ni x} \prod_{i=1}^{n} \left(\frac{1}{(d_i - 1)!} |f(X_i)|\, e^{c|X_i|} \right) \prod_{ij \in G} |U(X_i X_j) - 1|. \tag{A.6}$$

We perform the sums over the X_i starting with the "outer branches," i.e., those X_i for which $d_i = 1$, and working inwards using

$$\sum_{X \subset \mathscr{S}} |f(X)|\, e^{c|X|}\, |X|^p\, |U(XY) - 1| \leqslant |Y| \| f(\cdot)| \cdot |^p \|_c$$

and thereby obtain the bound

$$\sum_{n=1}^{\infty} \frac{(n-2)!}{(n-1)!} \sum_{d_1,\ldots,d_n} \prod_{i=1}^{n} \frac{1}{d(i)!} \| f(\cdot)| \cdot |^{d(i)} \|_c \tag{A.7}$$

where $d(i) = d_i - 1$ for all but one $i = j$, say, for which $d(i) = d_j$, j is arbitrary. Next we use

$$|X|^d \leqslant d! \left(\frac{1}{1+\varepsilon} \right)^d e^{(1+\varepsilon)|X|}$$

and sum over the d_i's to continue with

$$\leqslant \sum_{n=1}^{\infty} \frac{(n-2)!}{(n-1)!} \left(\frac{1+\varepsilon}{\varepsilon} \right)^n \| f \|_{c+1+\varepsilon}^n.$$

When $n = 1$, $(n-2)!$ is to be interpreted as 1 so this implies our claimed result (A.2).

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