DIMENSIONAL REDUCTION FOR ISOTROPIC AND DIRECTED BRANCHED POLYMERS

JOHN Z. IMBRIE

Department of Mathematics
University of Virginia
Charlottesville, VA 22904, USA
Email: ji2k@virginia.edu

I will describe an exact relation between self-avoiding branched polymers in $D+2$ continuum dimensions and the hard-core continuum gas at negative activity in D dimensions (joint work with David Brydges, [1, 2]). Our results explain why the critical behavior of branched polymers should be the same as that of the $i\varphi^3$ (or Yang-Lee edge) field theory in two fewer dimensions (as proposed by Parisi and Sourlas [3]). I will also discuss directed branched polymers in $D+1$ dimensions, and show that they, too, are related to the hard-core gas in D dimensions [4]. I will review conjectures and results on critical exponents for $D+2 = 2, 3, 4$ and show that they are corollaries of our results.

1. Introduction and Main Results

In this article we describe some beautiful identities which connect branched polymer models with repulsive gases in lower dimensions. For ordinary (isotropic) branched polymers (BP) the reduction in dimension is 2, and our results [1, 2] provide a rigorous version of [3] (in which the critical behavior of BP in $D+2$ dimensions is connected with that of the Yang-Lee edge in D dimensions). For directed branched polymers (DBP), the reduction in dimension is 1, and our results [4] parallel earlier work on directed lattice animal[a] (which have been related to dynamical models of hard-core lattice gases [5] and to the critical dynamics of the Yang-Lee edge [6, 7]; see also exact results in [8, 9, 10] and the review [11]). Forest-root formulas provide a unified picture for dimensional reduction for both BP and DBP. They are used to interpolate in the extra dimensions. They have other applications as well: we show how the one-dimensional forest-root formula generalizes some formulas of [12], used for interpolating in cluster expansions.

We now define generating functions for BP and DBP. Let T be a tree graph on $\{1, \ldots, N\}$. For BP, the k^{th} monomer is at position $y_k \in \mathbb{R}^{D+2}$ and we write $y_k = (w_k, x_k) \in \mathbb{C} \times \mathbb{R}^D$. For DBP, it is at $y_k = (t_k, x_k) \in \mathbb{R}_+ \times S$, where S is either $\mathbb{R}^D$ or $\mathbb{Z}^D$. While we must use continuous coordinates for BP, we allow discrete spatial coordinates for DBP, and for some models the time coordinate is discrete as well. For a unified treatment with DBP we consider here rooted BP (in contrast to [13, 1, 2], where BP are defined mod translations.

[a]Animals (for which loops are allowed) are generally believed to have the same critical behavior as BP (where loops are not permitted). Likewise for directed animals and DBP.

Fix the vertex 1 as the root, with $y_1 = 0$. For each pair $(i, j) \equiv ij$, we define

$$y_{ij} = (t_{ij}, x_{ij}) = \begin{cases} (w_i - w_j, x_i - x_j), & \text{for BP} \\ (|t_i - t_j|, x_i - x_j), & \text{for DBP} \end{cases}. \tag{1}$$

Each link of T connects a vertex j to a vertex i where i is one step closer than j to the root along T. For DBP, we require $t_j \geq t_i$.

The weight associated with each configuration depends on a linking weight $V(y)$ and a repulsive weight $U(y)$. The generating functions are written as

$$\begin{aligned} Z_{\mathrm{BP}}(z) &= \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \int_{(\mathbb{C}\times\mathbb{R}^D)^{N-1}} \prod_{ji\in T} [dy_{ji} V(|y_{ji}|^2)] \prod_{ji\notin T} U(|y_{ji}|^2), \\ Z_{\mathrm{DBP}}(z) &= \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \int_{(\mathbb{R}_+\times\mathbb{R}^D)^{N-1}} \prod_{ji\in T} [dy_{ji} V(y_{ji})] \prod_{ji\notin T} U(y_{ji}). \end{aligned} \tag{2}$$

Here each pair $\{i, j\}$ appears exactly once, either in $\prod_{ji\in T}$ or in $\prod_{ji\notin T}$. We assume that $U \to 1$ as $y_{ji} \to \infty$, so that the repulsion vanishes at infinity. For BP, U is evidently invariant under rotations in $\mathbb{R}^{D+2}$; for DBP we require $U(t, x) = U(t, -x)$. In order to get dimensional reduction, we require

$$\begin{aligned} V(t) &= 2U'(t), \quad \text{for BP}, \\ V(t, x) &= U'(t, x), \quad \text{for DBP}. \end{aligned} \tag{3}$$

where prime denotes the t-derivative. Note, however, that in the BP case t denotes a squared-radius variable, whereas for DBP t is a "time" variable. For positive weights, we require that U and V are positive. Furthermore, we assume that V is an integrable function of y so that (2) is well-defined.

We relate these generating functions to the density of a repulsive gas in D dimensions. Let $\Lambda \subset S$ and define the grand canonical partition function

$$Z_{\mathrm{HC}}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \prod_{i=1}^{N} dx_i \prod_{1\leq i<j\leq N} U_{ij}, \tag{4}$$

where $U_{ij} = U(|x_{ij}|^2)$ for BP and $U_{ij} = U(0, x_{ij})$ for DBP. Then define the pressure and density:

$$p(z) = \lim_{\Lambda\nearrow\mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{\mathrm{HC}}(z); \qquad \rho_{\mathrm{HC}}(z) = z\frac{d}{dz} p(z). \tag{5}$$

Theorem 1.1. *For all z such that the right-hand side converges absolutely,*

$$\rho_{\mathrm{HC}}(z) = \begin{cases} -2\pi Z_{\mathrm{BP}}\left(-\frac{z}{2\pi}\right), \\ -Z_{\mathrm{DBP}}(-z). \end{cases} \tag{6}$$

We can also prove a dimensional reduction formula for correlations. Let

$$\rho(\tilde{x}) = \sum_{i=1}^{N} \delta(\tilde{x} - x_i), \qquad \rho(\tilde{y}) = \sum_{i=1}^{N} \delta(\tilde{y} - y_i), \tag{7}$$

where $\tilde{x}, x_i \in \mathbb{R}^D$ and $\tilde{y}, y_i \in \mathbb{R}^{D+2}$ for BP or $\mathbb{R}_+ \times S$ for DBP. Then the density-density correlation functions of the three systems can be written as

$$G_{\mathrm{BP}}(0,\tilde{y};z) = \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \int_{(\mathbb{C}\times\mathbb{R}^D)^{N-1}} \rho(\tilde{y}) \prod_{ji\in T} [dy_{ji} V(|y_{ji}|^2)] \prod_{ji\notin T} U(|y_{ji}|^2),$$
$$G_{\mathrm{DBP}}(0,\tilde{y};z) = \sum_{N=1}^{\infty} \frac{z^N}{(N-1)!} \sum_T \int_{(\mathbb{R}_+\times S)^{N-1}} \rho(\tilde{y}) \prod_{ji\in T} [dy_{ji} V(y_{ji})] \prod_{ji\notin T} U(y_{ji}),$$
$$G_{\mathrm{HC}}(0,\tilde{x};z) = \lim_{\Lambda\nearrow S} \langle \rho(0)\rho(\tilde{x})\rangle_{\mathrm{HC},\Lambda}\,. \tag{8}$$

Here $\langle\cdot\rangle_{\mathrm{HC},\Lambda}$ is the expectation in the measure for which $Z_{\mathrm{HC}}(z)$ is the normalizing constant.

Theorem 1.2.

$$G_{\mathrm{HC}}(0,x;z) = \begin{cases} -2\pi \int d^2w\, G_{\mathrm{BP}}\left(0,y;-\frac{z}{2\pi}\right), & \textit{where } y=(w,x)\in\mathbb{C}\times\mathbb{R}^D \\ -\int_0^\infty dt\, G_{\mathrm{DBP}}(0,y;-z), & \textit{where } y=(t,x)\in\mathbb{R}_+\times S \end{cases}. \tag{9}$$

Here d^2w is defined as $du\,dv$, where $w = u+iv$.

2. Examples

A natural BP example is obtained by letting $U(t) = \vartheta(t-1)$, where ϑ is the usual step function. Then $V(|y_{ij}|^2) = \delta(|y_{ij}|-1)$, and we have a model of hard spheres linked together with kissing conditions determined by the tree graph T. See Fig. 1. One can also consider soft repulsion of the form $U(t) = e^{-v(t)}$. This is particularly interesting when $D=0$, because the sine-Gordon representation allows one to write

$$Z_{\mathrm{HC}}(z) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{v}\left(\tilde{z}e^{i\varphi} + \tfrac{1}{2}\varphi^2\right)\right] \frac{d\varphi}{\sqrt{2\pi v}}, \tag{10}$$

where $\tilde{z} = zve^{v/2}$. For small v, a steepest-descent analysis shows that (in two dimensions) the crossover to noninteracting (or mean-field) BP is given by an Airy function [14], see also [15].

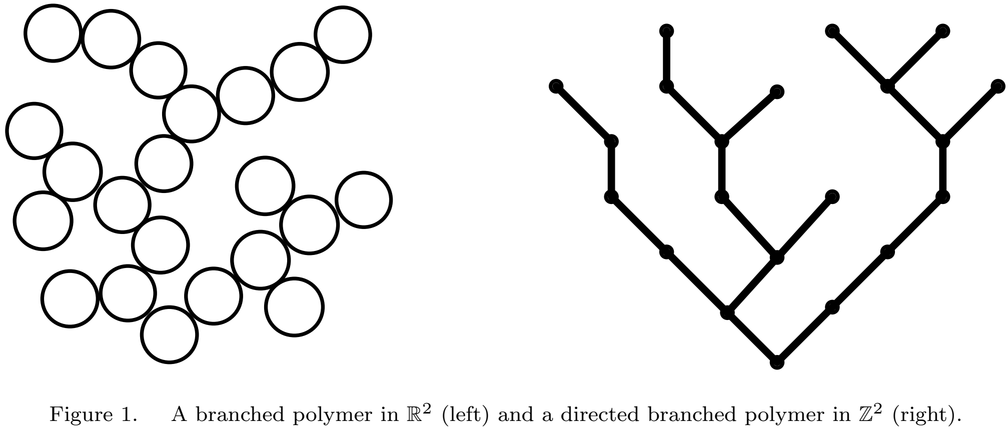

Figure 1. A branched polymer in $\mathbb{R}^2$ (left) and a directed branched polymer in $\mathbb{Z}^2$ (right).

For DBP, one can obtain a hard-sphere model by taking $U(t,x) = \vartheta(t^2+|x|^2-1)$, but note that the angular distribution of links is no longer uniform; it favors the preferred direction (the t axis). Alternatively, if we let $U(t,x) = \vartheta(t+|x|-1)$, then we obtain models of hard diamonds (when $|x| = \sum_{\alpha=1}^{D} |x_\alpha|$) or hard double-cones (when $|x|^2 = \sum_{\alpha=1}^{D} x_\alpha^2$) distributed uniformly in contact with the positive surface of the monomer it is linked to (subject to the constraint of nonoverlap with other monomers).

In $D = 1$, the pressure of the hard-sphere gas is computable, it is

$$p(z) = \texttt{LambertW}(z) = -T(-z), \tag{11}$$

where $T(z) = \sum_{N=1}^{\infty} z^N N^{N-1}/N!$ is the tree generating function [2]. So Theorem 1.1 implies that

$$Z_{\mathrm{BP}}(z) = -\frac{1}{2\pi}\rho_{\mathrm{HC}}(-2\pi z) = \sum_{N=1}^{\infty} \frac{(2\pi z)^N N^N}{2\pi N!},$$
$$Z_{\mathrm{DBP}}(z) = -\rho_{\mathrm{HC}}(-z) = \sum_{N=1}^{\infty} \frac{z^N N^N}{N!}. \tag{12}$$

Thus we have exact expressions for the volume available to BP and DBP of size N.

One can also consider lattice DBP examples by taking $U(t,x) = 1 - I(x)\vartheta(1-t)$, where $I(x)$ is the indicator function of a set of "neighbors" in the lattice, such as $\{x : |x| \le 1\}$. Since $V(t,x) = I(x)\delta(t-1)$, the set determines which sites a link can jump to, with t always increasing by 1. This model is closest to the standard examples of DBP. See Fig. 1 for a configuration in two dimensions. The factors $\prod_{ji\notin T} U(y_{ji})$ enforce nearest-neighbor exclusion for monomers on the same level (same value of t). But there is a subtlety when a monomer at level t has n neighbors at level $t-1$ in the polymer. In this case, we can write the U-factors as $\vartheta^{n-1}(0)$, which should be interpreted as $\frac{1}{n} = \int \vartheta^{n-1} d\vartheta$ (this can be seen by approximating ϑ with a smooth function and integrating over t). Monomers which are separated by more than one unit in the t direction do not interact, since $U(t,x) = 1$ for $t > 1$.

By Theorem 1.1, the generating functions of these models equate to the density of the nearest-neighbor exclusion models in D dimensions associated with the weights $U(0,x) = 1 - I(x)$. In $D = 1$, the pressure can be calculated explicitly [16, Eqn. 2.16]

$$p(z) = \ln(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4z}), \tag{13}$$

so the generating function is

$$Z_{\mathrm{DBP}}(z) = -z\frac{d}{dz}p(-z) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4z}} - 1\right) = \sum_{N=1}^{\infty} \frac{[2N-1]!!2^{N-1}z^N}{N!}, \tag{14}$$

which gives an explicit enumeration of the number of DBP with N monomers.

3. Critical Exponents

As these reduction formulas are valid out to the edge of convergence of the generating functions, one can deduce the values of the BP and DBP critical exponents by looking at solvable repulsive gas models in low dimension. Let us define an exponent α_{HC} from the singularity of the pressure of the hard-core gas, $p(z) \sim (z-z_c)^{2-\alpha_{\mathrm{HC}}}$. Likewise, susceptibility exponents γ_{BP} and γ_{DBP} can be defined from

$$Z_{\mathrm{BP}}\left(-\tfrac{z}{2\pi}\right) \sim (z-z_c)^{1-\gamma_{\mathrm{BP}}}, \qquad Z_{\mathrm{DBP}}(-z) \sim (z-z_c)^{1-\gamma_{\mathrm{DBP}}}. \tag{15}$$

Note that z_c is negative. By Theorem 1.1, the singularities must be the same, so

$$\alpha_{\mathrm{HC}}(D) = \gamma_{\mathrm{BP}}(D+2) = \gamma_{\mathrm{DBP}}(D+1). \tag{16}$$

Closely related to the susceptibility exponents are the counting exponents. If one defines c_N, d_N from

$$Z_{\mathrm{BP}}(z) = z\frac{d}{dz}\sum_{N=1}^{\infty} c_N z^N, \qquad Z_{\mathrm{DBP}}(z) = \sum_{N=1}^{\infty} d_N z^N, \tag{17}$$

then θ_{BP}, θ_{DBP} are determined from the asymptotic behaviors

$$c_N \sim \left(-\frac{z_c}{2\pi}\right)^{-N} N^{-\theta_{\mathrm{BP}}}, \qquad d_N \sim (-z_c)^{-N} N^{-\theta_{\mathrm{DBP}}}. \tag{18}$$

The exponent θ_{BP} is usually defined from counting unrooted BP, the difference being a factor N which is produced by $z\frac{d}{dz}$. We see that the unrooted generating function $\sum c_N z^N$ is related to the pressure $p(z)$ of the repulsive gas [1].

From (16) and the relations

$$\theta_{\mathrm{BP}} = 3 - \gamma_{\mathrm{BP}}, \quad \theta_{\mathrm{DBP}} = 2 - \gamma_{\mathrm{DBP}}, \tag{19}$$

one can determine θ_{BP}, θ_{DBP} from α_{HC}.

In $D=0$, we have $\alpha_{\mathrm{HC}} = 2$ because $Z_{\mathrm{HC}}(z) = 1+z$, so the "pressure" has a logarithmic singularity. For $D=1$, one can compute $\alpha_{\mathrm{HC}} = \frac{3}{2}$ from the solutions for the pressure (11),(13), which have a square root singularity. For $D=2$ there is an exact solution for the hard-hexagon model [17], which has $\alpha_{\mathrm{HC}} = \frac{7}{6}$ [5, 18]. This model works as the starting point for one of our lattice DBP models (with $D+1=3$). Our construction for BP does not work as described above for the hard hexagon model. Nevertheless, one expects the same value of α_{HC} for hard spheres. Hence the exponents $\theta_{\mathrm{BP}}(D+2), \theta_{\mathrm{DBP}}(D+1)$ are determined for $D=0,1,2$ (rigorously for $D=0,1$). See the table below.

Theorem 1.2 connects the Green's functions for the three models, so it is clear that the exponents for the divergence of the correlation length are all equal. For DBP this gives information only on the transverse correlation exponent $\nu_{\mathrm{DBP}}^{\perp}$, which describes the vanishing of the rate of decay in the x directions. The exponent ν_{HC} can be computed in $D=1$ or more generally determined from α_{HC} via hyperscaling ($D\nu_{\mathrm{HC}} = 2 - \alpha_{\mathrm{HC}}$). Theorem 1.2 also implies that $\eta_{\mathrm{HC}} = \eta_{\mathrm{BP}}$ [2], and one can compute η_{HC} in $D=1$, or more generally determine it from Fisher's relation ($\eta = 2 - \gamma/\nu$). In $D=2$ it can be obtained from the conformal field theory of the Yang-Lee edge [19].

The repulsive gas singularity at negative activity is believed to be in the same universality class as the Yang-Lee edge [16, 20]. One way of seeing this is by writing down the sine-Gordon representation for the repulsive gas (see (10) for the $D=0$ case). The interaction is $e^{i\varphi}$, whose lowest order term at the critical point is $i\varphi^3$, the Yang-Lee edge interaction. The Yang-Lee edge exponent σ can be equated with $1-\alpha_{\mathrm{HC}}$. This leads to the Parisi-Sourlas relation $\theta_{\mathrm{BP}}(D+2) = 2 + \sigma(D)$ [3] and its analogue for DBP: $\theta_{\mathrm{DBP}}(D+1) = 1 + \sigma(D)$

[6, 7, 21]. The table below summarizes what we know about exponents for these models (the last line gives the presumed mean-field exponents and upper critical dimensions; see [22, 23] for results on BP in high dimensions).

HC dim D	DBP dim $D+1$	BP dim $D+2$	$\alpha_{\rm HC} = \gamma_{\rm BP} = \gamma_{\rm DBP}$	$\theta_{\rm DBP}$	$\theta_{\rm BP}$	$\nu_{\rm HC} = \nu_{\rm BP} = \nu^{\perp}_{\rm DBP}$	$\eta_{\rm HC} = \eta_{\rm BP}$	$\sigma = 1-\alpha_{\rm HC}$
0	1	2	2	0	1			-1
1	2	3	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$
2	3	4	$\frac{7}{6}$	$\frac{5}{6}$	$\frac{11}{6}$	$\frac{5}{12}$	$-\frac{4}{5}$	$-\frac{1}{6}$
MFT $D>6$	$D>7$	$D>8$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{4}$	0	$\frac{1}{2}$

4. Forest-Root Formulas

The key ingredients for proving Theorems 1 and 2 are a pair of *forest-root formulas*, which we use to interpolate in the extra dimensions. Let $f(\mathbf{t})$ depend on a collection of non-negative real variables $\mathbf{t} = \{t_{ij}\}_{1\le i<j\le N}$, $\{t_i\}_{1\le i\le N}$. Assume $f \to 0$ when any $t_i \to \infty$. In the BP case, the t's are functions of another set of variables: $t_{ij} = |w_i - w_j|^2$, $t_i = |w_i|^2$, with each $w_i \in \mathbb{C}$. The forest-root formula is

$$f(0) = \sum_{(F,R)} \int_{\mathbb{C}^N} \prod_{i=1}^{N} \frac{d^2 w_i}{-\pi} f^{(F,R)}(t). \tag{20}$$

The sum is over *forests* F and *roots* R (R is any subset of $\{1,\ldots,N\}$ and F is a loop-free graph on $\{1,\ldots,N\}$ such that each connected component or *tree* of F has exactly one root in R). See Fig. 2. The expression $f^{(F,R)}$ denotes the $N^{\rm th}$ partial derivative of f with respect to t_{ij}, $ij \in F$ and t_i, $i \in R$. The simplest example is when $N = 1$, $F = \emptyset$, $R = \{1\}$, in which case (20) reduces to the fundamental theorem of calculus:

$$f(\mathbf{0}) = \int_{\mathbb{C}} f'(t) \frac{d^2 w}{-\pi} = -\int_0^\infty f'(t) dt. \tag{21}$$

In the DBP case the t's are the underlying variables, and $t_{ij} = |t_i - t_j|$. Our second forest-root formula is

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{R}_+^N} \prod_{r\in R} [-dt_r] \prod_{ji\in F} [-d(t_j - t_i)] f^{(F,R)}(\mathbf{t}). \tag{22}$$

As in (2), each link in a tree of F connects a vertex j to a vertex i which is one step closer to the root for that tree. The integration region is $\{t_r \ge 0,\ r \in R \text{ and } t_j \ge t_i,\ ji \in F\}$. For $N = 1$, (22) is just $f(0) = -\int_0^\infty f'(t)dt$.

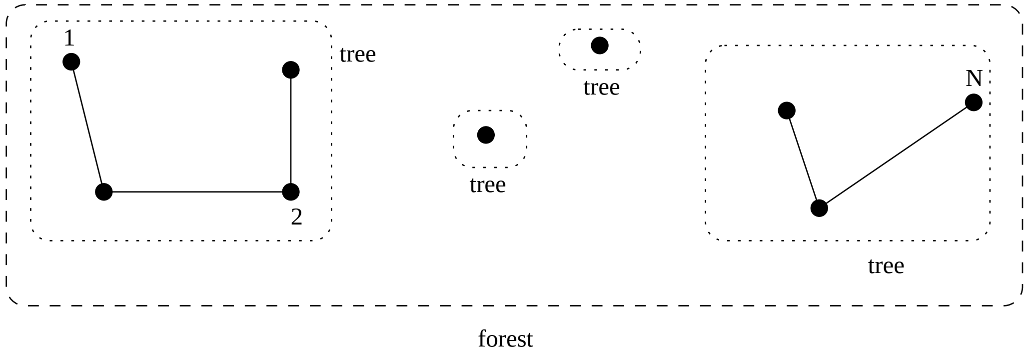

Figure 2. Example of a forest

4.1. *Proof of the Main Results*

Let us see how these forest-root formulas lead to a proof of Theorem 1.1. Let

$$f(\mathbf{t}) = g(t_1/\epsilon) \prod_{i=2}^{N} g(\epsilon t_i) \prod_{1 \le i < j \le N} U_{ij}, \tag{23}$$

where U_{ij} is either $U(|y_{ij}|^2) = U(t_{ij} + |x_{ij}|^2)$ for BP or $U(t_{ij}, x_{ij})$ for DBP (we suppress the dependence on $\{x_{ij}\}$ in $f(\mathbf{t})$). Here g is any smooth function which decreases to 0 and satisfies $g(0) = 1$. For each $ji \in F$, U_{ji} is differentiated and becomes the linking weight V_{ji} ($V_{ji}/2$ for BP). For each $r \in R$ a g is differentiated. One finds that (up to terms which vanish as $\epsilon \to 0$) all the trees of F decouple due to the large separation in the t-direction (the distribution of t_r for an n-vertex tree is essentially $-(g(\epsilon t_r)^n)' dt_r$ or $-(g(\epsilon t_r)^n)' \frac{d^2 w}{\pi}$, which are very spread-out probability measures). One tree has its root fixed at 0 by a factor $-(g(t_r/\epsilon))' dt_r$ or $-(g(t_r/\epsilon))' \frac{d^2 w}{\pi}$, which converges to a δ-measure at 0. The others cancel with the normalization $Z_{\mathrm{HC}}(z)$, so that

$$\begin{aligned} \rho_{\mathrm{HC}}(z) &= \lim_{\Lambda \nearrow \mathbb{R}^D} \lim_{\epsilon \searrow 0} Z_{\mathrm{HC}}(z)^{-1} \sum_{N=1}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \prod_{i=1}^{N} dx_i f(\mathbf{0}) \\ &= \lim_{\Lambda \nearrow \mathbb{R}^D} \lim_{\epsilon \searrow 0} Z_{\mathrm{HC}}(z)^{-1} \sum_{N=1}^{\infty} \frac{z^N}{N!} \int_{(\mathbb{C} \times \Lambda)^N} \prod_{i=1}^{N} \frac{dy_i}{-\pi} f^{(F,R)}(\mathbf{t}) \\ &= -2\pi Z_{\mathrm{BP}}(-\tfrac{z}{2\pi}). \end{aligned} \tag{24}$$

Similarly, one can show that $\rho_{\mathrm{HC}}(z) = -Z_{\mathrm{DBP}}(-z)$. For each link there is a factor $-\frac{1}{2\pi}$ (BP) or -1 (DBP), and this leads to the scaling of Z_{DBP} and Z_{BP} and their arguments in Theorem 1.1. Further details of this argument are in [2]. Theorem 1.2 can be obtained by differentiating (6) with respect to a source at x (see [1] for details).

4.2. *Decoupling Expansions*

It is interesting to look at these forest-root formulas from the point of view of decoupling expansions. Complete decoupling occurs only at $t = \infty$, so it is important to note that the trees of F still interact in (20),(22). However, if we start with a function h which depends only on $\{t_{ij}\}$, and let $f(\mathbf{t}) = h(\mathbf{t})\prod_{i=1}^{N} g(\epsilon t_i)$ as above, then in the limit $\epsilon \searrow 0$ we obtain (from (22), for example) a forest-root formula where all the trees are decoupled:

$$h(\mathbf{0}) = \sum_F \int_{\mathbb{R}_+^N} \prod_{ji\in F} [-d(t_j - t_i)] h^{(F)}(\mathbf{t}). \tag{25}$$

Note that if a tree has n vertices, the sum over its root gives a factor n. This appears in $-(g(\epsilon t_r)^n)' dt_r$ which, as noted above, is a spread out probability measure. Thus the root sums disappear (one still needs to select a root for each tree, however—take the one with the smallest label in $\{1,\ldots,N\}$, for example). One is tempted to change variables here, to $s_{ji} = t_j - t_i$ for $ji \in F$. Then $h^{(F)}$ has to be evaluated at $t_{kl} = |t_k - t_l|$, where t_k is a height variable obtained as the sum of s parameters along the tree joining the root to k ($t_{kl} = \infty$ if k and l are not in the same tree). The result is a new decoupling formula which has some similarity with Theorem III.2 in [12]. In fact, the rooted Taylor forest formula of [12] can be obtained from (25) by a limiting procedure similar to what we did in making a lattice model of DBP.

Let $H(\mathbf{w})$ depend on $\{w_{ij}\}$, a set of decoupling parameters in $[0,1]$ (no assumption that $w_{ij} = w_i - w_j$). Make a change of variable $w_{ij} = \vartheta(1 - s_{ij})$, where ϑ is a smooth, monotone approximation to the step function. Apply (25) with $s_{ji} \equiv t_j - t_i$ for $ji \in F$. The result of this is that for each $kl \notin F$, s_{kl} is determined as the height difference t_{kl}. As $s_{ji} \in [1-\varepsilon, 1+\varepsilon]$ for $ji \in F$, the height variables t_k become discretized and actually measure the number of steps from the root to k. Then if the height difference is 2 or more, $w_{kl} = 0$, and if the difference is 1, all w_{jk} linking j to the next level down are equal to w_{ji}, where $ji \in F$. Thus we obtain

$$H(\mathbf{1}) = \sum_F \int_{[0,1]^{N-1}} \prod_{ji\in F} dw_{ji} H^{(F)}(\mathbf{w}), \tag{26}$$

which is Theorem III.2 of [12]. We have switched from s-derivatives and integrals to w-derivatives and integrals, noting that under a change of variables $\frac{\partial H}{\partial s} \cdot ds$ is invariant (except for the loss of the minus sign, because $\frac{dw}{ds} < 0$).

4.3. *The Two-Dimensional Forest-Root Formula*

The two-dimensional forest-root formula (20) is proven using a supersymmetry argument in [1]. Replace each variable t_i in $f(\mathbf{t})$ with

$$\tau_i = w_i \bar{w}_i + \frac{dw_i \wedge d\bar{w}_i}{2\pi i}, \tag{27}$$

and each t_{ij} with

$$\tau_{ij} = w_{ij} \bar{w}_{ij} + \frac{dw_i \wedge d\bar{w}_i}{2\pi i}. \tag{28}$$

Then $f(\underline{\tau})$ is defined by its Taylor series. A "localization" formula

$$\int_{\mathbb{C}^N} f(\underline{\tau}) = f(\mathbf{0}) \tag{29}$$

holds, which becomes the forest-root formula when expanded out. This can be proven by deforming the problem to the independent case (21), using ideas from [24]. See [14] for an alternative argument, which uses the linearity of (29) to reduce to the case where f is an exponential (a Gaussian calculation) [25]. See also [26].

4.4. *Proof of the One-Dimensional Forest-Root Formula*

As mentioned above, the $N = 1$ case is just the fundamental theorem of calculus. For $N = 2$, consider $f(t_1, t_2, t_{12})$ and use subscripts 1, 2, 12 to denote partial derivatives. Then

$$f(\mathbf{0}) = -\int_0^\infty ds(f_1(s,s,0) + f_2(s,s,0)). \tag{30}$$

Apply the $N = 1$ formula to the f_1 term (integrating with respect to $x_2 - x_1$), and to the f_2 term (integrating with respect to $x_1 - x_2$). The result is

$$f(\mathbf{0}) = \int_0^\infty dt_1 \int_0^\infty d(t_2 - t_1)(f_{1,2} + f_{1,12}) + \int_0^\infty dt_2 \int_0^\infty d(t_2 - t_1)(f_{2,2} + f_{2,12}), \tag{31}$$

since $\frac{dt_{12}}{dt_2} = 1$ for $t_2 > t$, and $\frac{dt_{12}}{dt_1} = 1$ for $t_1 > t_2$. The two $f_{1,2}$ terms combine to form $\int_{\mathbb{R}_+^2} dt_1 dt_2 f_{1,2}$, which is the term $R = \{1,2\}$ of (22). The other two integrals are the terms $R = \{1\}, \{2\}$.

We prove the general case by indication on N. Begin as above with

$$f(\mathbf{0}) = -\int_0^\infty ds \sum_{k=1}^{N} f_k(s, \ldots, s, 0, \ldots, 0), \tag{32}$$

where the integral is along the diagonal, $t_1 = t_2 = \cdots = t_N$. Consider one of these terms, say $k = N$, and apply (22) in the variables $\tilde{t}_i = t_i - t_N$, $i = 1, \ldots, N-1$, keeping $t_N = s$ fixed:

$$f_N(t_N, \ldots, t_N, 0, \ldots, 0) = \sum_{(\tilde{F},\tilde{R})} \int_{\mathbb{R}_+^{N-1}} \prod_{r\in\tilde{R}} [-d\tilde{t}_r] \prod_{ji\in\tilde{F}} [-d(\tilde{t}_j - \tilde{t}_i)] f_N^{(\tilde{F},\tilde{R})}(\mathbf{t}). \tag{33}$$

Note that when computing the derivative of f_N with respect to $\tilde{t}_r$, there will be a term $f_{N,r}$ and also a term $f_{N,rN}$ (coming from the dependence on $t_{rN} = t_r - t_N = \tilde{t}_r$). Thus each $(\tilde{F}, \tilde{R})$ on $\{1, \ldots, N-1\}$ gives rise to $2^{|\tilde{R}|}$ terms, each of which can be assigned a unique (F, R) on $\{1, \ldots, N\}$. R consists of N, together with each $r \in \tilde{R}$ with an $f_{N,r}$ term. F consists of $\tilde{F}$, together with rN, $r \in \tilde{R}$ when r gives rise to an $f_{N,rN}$ term. Observe that each root in $\tilde{R}$ ceases to be a root in R if it is connected by a bond rN in F. We obtain in this way all (F, R) with $N \in R$, and each satisfies the condition that each tree of F contains exactly one root. As a result,

$$f(\mathbf{0}) = \sum_{k=1}^{N} \sum_{(F,R):k\in R} \int_{\mathbb{R}_+^N} [-dt_k] \prod_{i\in R\setminus\{k\}} [-d(t_i - t_k)] \prod_{ji\in F} [-d(t_j - t_i)] f^{(F,R)}(\mathbf{t}). \tag{34}$$

It is evident that if we take the term (F, R) of (22), and consider the subset of the integration region for which $t_k = \min_{r\in R} t_r$, we obtain the term $k, (F, R)$ of (34). This completes the proof.

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