ITERATED MAYER EXPANSIONS AND THEIR APPLICATION TO COULOMB GASES

John Z. Imbrie

1. Introduction

There are many problems in statistical mechanics and field theory where systems of interacting particles (or particle-like defects, contours, molecules, clusters, etc.) arise. Understanding these systems would be simpler if the particles were noninteracting. If the particles are dilute and weakly interacting, then the corrections to the independent particle approximation are small. These can be isolated and regarded as new kinds of (nonlocal) particles by a procedure known as the Mayer expansion. The nonlocal particles (or clusters) carry all the information about the long-distance behavior of the system.

There are many variants of Mayer expansions and cluster expansions that can be used depending on the type of system under consideration. The following grand canonical partition function is a representative example:

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int dx_1 \dots dx_N \exp(-\beta \sum_{i<j} v(x_i - x_j)). \tag{1.1}$$

Here N is the number of particles, z is the activity of each particle, β is the inverse temperature, $x_i \in \Lambda \subseteq \mathbb{R}^d$ is the position of the i^{th} particle, and $v(x_i - x_j)$ is a two-body interaction potential. The basic expansion step is to use the fundamental theorem of calculus to write

$$ze^{-\beta v(x_i - x_j)} = z + z\int_0^1 ds(-\beta)v(x_i - x_j)e^{-s\beta v(x_i - x_j)} \tag{1.2}$$

The first term is an isolated particle, the second is a two-particle cluster. When identities like (1.2) have been applied many times, we find (after some combinatorics) an expansion for $|\Lambda|^{-1}\log Z$ roughly speaking in powers of $z\beta \int|v(x)|dx e^{-\beta E}$, where E is a lower bound for the energy per particle in any N-particle configuration. Thus we obtain convergence for small activities or high temperatures, depending on the strength of the interaction (in the L^1-sense) and on the stability bound E.

A simple Mayer expansion as outlined above is not well suited to interactions that are strong ($\beta E << -1$) only in a region that contributes little to the integral $\int|v(x)|dx$. In this situation we should split the interaction into two or more parts $v^\ell, \ell = 0,1,\ldots$, and expand in the part with the worst stability estimate E^ℓ and shortest range first. The clusters from this expansion are the particles of the next expansion. One can proceed in this way through all the parts of v without ever encountering a very large product $\int|v^\ell(x)|dx\ e^{-\beta E^\ell}$ Then one obtains an expansion that converges with relatively mild conditions on activities.

This procedure of iterated Mayer expansions was formalized and applied to the lattice Yukawa gas by Göpfert and Mack [5]. It fits into the renormalization group idea that the behavior of systems is best understood by considering what happens at successively larger length scales. In its current form, however, the procedure works only when there are just a finite number of length scales to be considered. This limitation doesn't matter for Coulomb systems where screening (exponential clustering) sets in at some length scale to break the scale invariance of the interaction. For truly critical systems, one needs more refined techniques.

To illustrate the technique, we shall consider a soft core continuum Yukawa gas in three dimensions. For the Coulomb gas application we have in mind, it is sufficient to split the interaction into two parts. Let $e_i = \pm 1$ be the charge of the i^{th} particle. Write $\xi_i = (e_i, x_i)$, $\int d\xi_i = \sum_{e_i} \int_\Lambda dx_i$, and put $\ell_D = (2z\beta)^{-1/2}$. Define a two-body interaction

$$v(\xi_i,\xi_j) = e_i\left[\frac{e^{-|x_i-x_j|/(\lambda\ell_D)} - e^{-|x_i-x_j|/R}}{4\pi|x_i - x_j|}\right]e_j. \tag{1.3}$$

Here λ parametrizes the range of the Yukawa potential, and R is a short distance cutoff, necessary for stability. For simplicity we assume $R \le \beta \le \lambda\ell_D$. The partition function is

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int d\xi_1 \ldots d\xi_N \exp(-\beta \sum_{i<j} v(\xi_i,\xi_j)). \tag{1.4}$$

If we put $\lambda = \infty$ in (1.3), then we have a Coulomb potential with soft core and ℓ_D, the Debye length, is the length for exponential decay of correlations in the Debye-Hückel theory. Our goal here is to write an expansion valid for λ small, but without putting restrictions on λ that depend on the other parameters β, R, z. This will ultimately lead to a large region for screening for the corresponding Coulomb gas [6,7]. In Sect. 4 we discuss applications of these expansions to Coulomb gases.

A convenient splitup of v is

$$v = v^0 + v^1 \tag{1.5}$$

$$v^0(\xi_i,\xi_j) = e_i\left[\frac{e^{-|x_i-x_j|/\beta} - e^{-|x_i-x_j|/R}}{4\pi|x_i - x_j|}\right]e_j \tag{1.6}$$

$$v^1(\xi_i,\xi_j) = e_i\left[\frac{e^{-|x_i-x_j|/(\lambda\ell_D)} - e^{-|x_i-x_j|/\beta}}{4\pi|x_i - x_j|}\right]e_j. \tag{1.7}$$

Notice that v^0 can be written as

$$v^0(\xi_i,\xi_j) = e_i[(-\Delta + \beta^{-2})^{-1} - (-\Delta + R^{-2})^{-1}](x_i,x_j)e_j. \tag{1.8}$$

This form is useful for stability and two-body estimates. We have

$$\sum_{i,j} v^0(\xi_i,\xi_j) = \langle\rho,[(-\Delta + \beta^{-2})^{-1} - (-\Delta + R^{-2})^{-1}]\rho\rangle \geq 0, \tag{1.9}$$

where $\rho(x) = \sum_i e_i\delta(x - x_i)$. Thus

$$\sum_{1\leq i<j\leq N} v^0(\xi_i,\xi_j) \geq -\frac{1}{2}\sum_i v^0(\xi_i,\xi_i) = -cN/R, \tag{1.10}$$

and E^0, the lower bound for the energy per particle in a system with interaction v^0, is equal to $-c/R$ for some constant c. Similarly $E^1 = -c/\beta$; and E, the lower bound for the full system, is equal to $-c/R$. Note that $\int|v^0(\xi_i,\xi_j)|dx_j$ is just the Fourier transform of $(-\Delta + \beta^{-2})^{-1} - (-\Delta + R^{-2})^{-1}$ at $p = 0$. Thus we have

$$\int|v^0(\xi_i,\xi_j)|dx_j = \beta^2 - R^2 \leq \beta^2 \tag{1.11}$$

$$\int|v^1(\xi_i,\xi_j)|dx_j \leq \lambda^2\ell_D^2 \tag{1.12}$$

$$\int |v(\xi_i,\xi_j)|\,dx_j \le \lambda^2 \ell_D^2. \tag{1.13}$$

We can now apply the conditions $z\beta\int|v^*(\xi_i,\xi_j)|dx_j e^{-\beta E^*} << 1$ to see for what ranges of R,β,z the expansions should converge. For the v^0 expansion we require $z\beta^3 e^{c\beta/R} << 1$. For the v^1 expansion we need $z\beta\lambda^2\ell_D^2 e^c = c\lambda^2 << 1$. Thus expanding in v^0 and v^1 separately put conditions on λ independent of z,β,R as desired. In contrast, expanding in v all at once would require $z\beta\lambda^2\ell_D^2 e^{c\beta/R} = c\lambda^2 e^{c\beta/R} << 1$, or $\lambda^2 << e^{-c\beta/R}$. One is forced to apply the weak stability estimate E over the whole range of v -- a much less efficient procedure than the iterated expansion we describe below.

2. An Iterated Mayer Expansion

We want to develop an expansion for

$$Z^N(\xi_1,\ldots,\xi_N) = \frac{z^N}{N!}\exp(-\beta\sum_{i<j} v(\xi_i,\xi_j)). \tag{2.1}$$

At first we consider only v^0 and expand

$$Q_0^N(\xi_1,\ldots,\xi_N) = z^N \exp(-\beta\sum_{i<j} v^0(\xi_i,\xi_j)). \tag{2.2}$$

One could write

$$e^{\sum v_{ij}} = \prod_{i<j}[1 + (e^{v_{ij}} - 1)] = \sum_G \prod_{\{i,j\}\in G}(e^{v_{ij}} - 1), \tag{2.3}$$

where G runs over all sets of unordered pairs $\{i,j\}$, but for estimates it is better to have an interpolation formula. The formula we give is based on [2]. For α a subset of $\{1,\ldots,N\}$ we write

$$V^0(\alpha) = \sum_{i<j;\, i,j\in\alpha} v^0(\xi_i,\xi_j). \tag{2.4}$$

The interpolated interaction for the first step is

$$V^0_{s_1}(1;\alpha) = V^0(\{1\}\cup\alpha)s_1 + V^0(\alpha)(1 - s_1), \tag{2.5}$$

where we take $\alpha = \{2,\ldots,N\}$. This yields

$$Q_0^N(\xi_1,\ldots,\xi_N) = zQ_0^{N-1}(\xi_2,\ldots,\xi_N) + z^N \sum_{1<i} \int_0^1 ds_1(-\beta)v^0(\xi_1,\xi_i)\cdot$$

$$\cdot\exp(-\beta V^0_{s_1}(1,\alpha)). \tag{2.6}$$

In the first term particle 1 is isolated from the other particles, and in the second it is connected to particle i through a "line" $v^0(\xi_1,\xi_i)$. The next interpolation depends on i. In an attempt to remove interactions between {1,i} and the other particles, we put

$$V^0_{s_1 s_2}(1,i;\alpha) = V^0_{s_1}(1;\{i\} \cup \alpha)s_2 + (V^0_{s_1}(1;\{i\}) + V^0(\alpha))(1 - s_2)$$

$$= s_1 v^0(\xi_1,\xi_i) + \sum_{j\in\alpha} (s_1 s_2 v^0(\xi_1,\xi_j) + s_2 v^0(\xi_i,\xi_j)) + \sum_{j<k;j,k\in\alpha} v^0(\xi_j,\xi_k), \tag{2.7}$$

where α contains all particles but 1,i. Thus

$$\frac{d}{ds_2} V^0_{s_1 s_2}(1,i;\alpha) = \sum_{j\in\alpha} (s_1 v^0(\xi_1,\xi_j) + v^0(\xi_i,\xi_j)), \tag{2.8}$$

and the i^{th} term in (2.6) interpolates to

$$z^2 \int ds_1(-\beta)v^0(\xi_1,\xi_i)\exp(-\beta V^0_{s_1}(1,\{i\}))Q^{N-2}(\alpha) +$$

$$+ z^N \sum_{1<j\neq i} \int ds_1 ds_2(-\beta)^2 v^0(\xi_1,\xi_i)(s_1 v^0(\xi_1,\xi_j) + v^0(\xi_i,\xi_j))\cdot$$

$$\cdot\exp(-\beta V^0_{s_1 s_2}(1,i;\alpha)). \tag{2.9}$$

We continue in this fashion, always attempting to isolate the group of particles connected to particle 1 in the remainder terms [the second group in (2.9)].

To describe the general step, let $i_1, i_2, \ldots$ be the sequence of particles differentiated down in some term of the expansion ($i_1 = 1, i_2 = i, i_3 = j$ above). We define inductively

$$V^0_{s_1\ldots s_n}(i_1,\ldots,i_n;\alpha) = V^0_{s_1\ldots s_{n-1}}(i_1,\ldots,i_{n-1};\{i_n\} \cup \alpha)s_n +$$

$$+ (V^0_{s_1\ldots s_{n-1}}(i_1,\ldots,i_{n-1};\{i_n\}) + V^0(\alpha))(1 - s_n). \tag{2.10}$$

It can easily be checked that

$$V^0_{s_1\ldots s_n}(i_1,\ldots,i_n;\{i_{n+1},\ldots,i_m\}) = \sum_{1\le\mu<\nu\le m} s_\mu\cdots s_{\min\{\nu-1,n\}}\cdot$$
$$\cdot v^0(\xi_{i_\mu},\xi_{i_\nu})$$

satisfies the recursion (2.10). Thus the general formula analogous to (2.8) is

$$\frac{d}{ds_n} V^0_{s_1\ldots s_n}(i_1,\ldots,i_n;\alpha) = \sum_{i_{n+1}\in\alpha}\ \sum_{\eta(n+1)=1}^{n} s_{\eta(n+1)}\cdots s_{n-1}\cdot$$
$$\cdot v^0(\xi_{i_{n+1}},\xi_{i_{\eta(n+1)}}). \qquad (2.11)$$

We use the convention that $s_n\cdots s_{n-1} = 1$. We see that i_{n+1} has been determined for the next interpolation. The procedure also generates a tree graph η, which is a function from $\{1,\ldots,k\}$ to itself satisfying $\eta(n) < n$. The interpolations stop when all N particles have been used, so that all terms have a form analogous to that of the first term in (2.9). The expansion then reads

$$Q^N(\xi_1,\ldots,\xi_N) = \sum_{k=1}^{N}\ \sum_{(i_2,\ldots,i_k)} z^k(-\beta)^{k-1}\int ds_1\ldots ds_{k-1}\sum_\eta$$
$$\prod_{\ell=2}^{k}[s_{\eta(\ell)}\cdots s_{\ell-2}v^0(\xi_{\eta(\ell)},\xi_\ell)]\ \exp(-\beta V^0_{s_1\ldots s_{k-1}}(1,i_2,\ldots,i_{k-1};\{i_k\}))\cdot$$
$$\cdot Q^{N-k}(\xi_{\alpha^c}). \qquad (2.12)$$

Here $(i_2,\ldots,i_k)$ is any ordered subset of $\{2,\ldots,N\}$, α^c is the complementary subset, $\xi_\alpha = (\xi_i)_{i\in\alpha}$, and η is a tree on k vertices.

We have isolated connected parts of Q^N which will become the particles for the second expansion. In place of activities z, these nonlocal particles (or 1-vertices, or subsets α of $\{1,\ldots,N\}$) have vertex functions

$$\sigma^1_k(\xi_\alpha) = \mathbb{S}\ z^k\ \frac{(-\beta)^{k-1}}{k}\int ds_1\ldots ds_{k-1}\sum_\eta\prod_{\ell=2}^{k}[s_{\eta(\ell)}\cdots s_{\ell-2}v^0(\xi_{\eta(\ell)},\xi_\ell)]$$
$$\exp(-\beta V^0_{s_1\ldots s_{k-1}}(1,\ldots,k-1;\{k\})). \qquad (2.13)$$

We have labeled the particles in α as $1,\ldots,k$. The operator $\mathbb{S}$ symmetrizes the expression following in $\xi_1,\ldots,\xi_k$. Permutations involving

ξ_1 are not present in (2.12), hence the factor 1/k. With these vertex functions (2.12) can be written as

$$Q_0^N(\xi_1,\ldots,\xi_N) = \sum_{k=1}^{N} \sum_{\alpha\subseteq\{1,\ldots,N\},1\in\alpha,|\alpha|=k} k!\sigma_k^1(\xi_\alpha)Q_0^{N-k}(\xi_{\alpha^c}), \quad (2.14)$$

where now we sum over unordered subsets $\alpha \subseteq \{1,\ldots,N\}$.

Equation (2.14) can be inserted into itself repeatedly to yield a complete decomposition of Q_0^N into products of 1-vertex functions:

$$Q_0^N(\xi_1,\ldots,\xi_N) = \sum_{\underline{\pi}\in P(1,\ldots,N)} \prod_{\alpha\in\underline{\pi}} [|\alpha|!\sigma_{|\alpha|}^1(\xi_\alpha)]. \quad (2.15)$$

Here $P(1,\ldots,N)$ is the set of partitions of $\{1,\ldots,N\}$. We would like to write (2.15) in a form more like a grand canonical partition function. To this end we change from summing over partitions to summing over multiplicities N_k for 1-vertices of size $k = |\alpha|$. Given a set of multiplicities N_1, $N_2,\ldots$ and some corresponding partition of $\{1,\ldots,N\}$, a new partition can be obtained by permuting $1,\ldots,N$. However, permutations of particles within any element of the partition do not change the partition, and permutations of elements of the partition of a given size do not change the partition. Thus a combinatoric factor $N!(\prod_{\alpha\in\underline{\pi}} |\alpha|!)^{-1}(\prod_k N_k!)^{-1}$ should be included, and we obtain

$$Q_0^N(\xi_1,\ldots,\xi_N) = \sum_{N_1,N_2,\ldots;\Sigma kN_k=N} N!(\prod_k N_k!)^{-1} \mathbb{S} \prod_{\alpha\in\underline{\pi}} \sigma_{|\alpha|}^1(\xi_\alpha). \quad (2.16)$$

Here $\underline{\pi}$ is an arbitrary partition corresponding to N_1, $N_2,\ldots$, and $\mathbb{S}$ symmetrizes in $\xi_1,\ldots,\xi_N$.

Finally we can apply this expansion to $Z^N(\xi_1,\ldots,\xi_N)$ in (2.1) by reintroducing the v^1-interactions as interactions between 1-vertices. Let us write for two 1-vertices α,β

$$v^1(\alpha,\beta) = \sum_{i\in\alpha,j\in\beta,i\neq j} v^1(\xi_i,\xi_j). \quad (2.17)$$

Then we have

$$Z^N(\xi_1,\ldots,\xi_N) = \sum_{N_1,N_2,\ldots;\Sigma kN_k=N} (\prod_k N_k!)^{-1} \mathbb{S} \prod_{\alpha\in\underline{\pi}} \sigma_{|\alpha|}^1(\xi_\alpha)\cdot \cdot\exp(-\frac{\beta}{2}\sum_{\alpha,\beta} v^1(\alpha,\beta)). \quad (2.18)$$

This expression is similar in structure to (2.1), only here there are more kinds of particles and the particles have more structure in their activities $\sigma^1_{|\alpha|}(\xi_\alpha)$. Thus it is possible to formulate these expansions inductively (see [5]). This would be especially useful if v had been decomposed into many pieces. Since we have only one expansion more to do, we will simply write it with equations (2.10) - (2.16) as a guide.

Interpolating interactions are defined as before by taking convex sums. Letting $\underline{\pi}$ denote a collection of 1-vertices $\{\alpha_1,\ldots,\alpha_M\}$ we define

$$V^1(\underline{\pi}) = \frac{1}{2} \sum_{\alpha,\beta\in\underline{\pi}} v^1(\alpha,\beta) \tag{2.19}$$

$$V^1_{s_1\ldots s_n}(\alpha_{i_1},\ldots,\alpha_{i_n};\underline{\pi}) = V^1_{s_1\ldots s_{n-1}}(\alpha_{i_1},\ldots,\alpha_{i_{n-1}};\{\alpha_{i_n}\}\cup\underline{\pi})s_n + $$

$$+ (V^1_{s_1\ldots s_{n-1}}(\alpha_{i_1},\ldots,\alpha_{i_{n-1}};\{\alpha_{i_n}\}) + V^1(\underline{\pi}))(1 - s_n). \tag{2.20}$$

Then the differentiated interaction is

$$\frac{d}{ds_n} V^1_{s_1\ldots s_n}(\alpha_{i_1},\ldots,\alpha_{i_n};\underline{\pi})$$

$$= \sum_{i_{n+1}:\alpha_{i_{n+1}}\in\underline{\pi}} \sum_{\eta(n+1)=1}^{n} s_{\eta(n+1)}\cdots s_{n-1} v^1(\alpha_{i_{n+1}},\alpha_{i_{\eta(n+1)}}). \tag{2.21}$$

With

$$Q^M_1(\underline{\pi}) = \prod_{\alpha\in\underline{\pi}} \sigma^1_{|\alpha|}(\xi_\alpha)\exp(-\beta V^1(\underline{\pi})), \tag{2.22}$$

The first interpolation yields

$$Q^M_1(\underline{\pi}) = \sigma^1_{|\alpha_1|}(\xi_{\alpha_1})Q^{M-1}_1(\underline{\pi} \setminus \{\alpha_1\})$$

$$+ \sum_{j\neq 1} \int ds_1 \prod_{\alpha\in\underline{\pi}} \sigma^1_{|\alpha|}(\xi_\alpha)(-\beta)v^1(\alpha_1,\alpha_j)\exp(-\beta V^1_{s_1}(\alpha_1;\underline{\pi} \setminus \{\alpha_1\})), \tag{2.23}$$

and in general

$$Q_1^M(\underline{\pi}) = \sum_{k=1}^{M} \sum_{(i_2,\dots,i_k)} \prod_{\nu=1}^{k} \sigma^1_{|\alpha_{i_\nu}|}(\xi_{\alpha_{i_\nu}})(-\beta)^{k-1}\int ds_1\dots ds_{k-1} \sum_{\eta}$$

$$\prod_{\ell=2}^{k} [s_{\eta(\ell)}\cdots s_{\ell-2} v^1(\alpha_{i_{\eta(\ell)}},\alpha_{i_\ell})]\cdot$$

$$\cdot \exp(-\beta V^1_{s_1\dots s_{k-1}}(\alpha_{i_1},\dots,\alpha_{i_{k-1}};\{\alpha_{i_k}\}))Q_1^{M-k}(\underline{\pi}^c). \tag{2.24}$$

Just as a 1-vertex α is a collection of particles or 0-vertices, we define a 2-vertex $\underline{\alpha}$ to be a collection of 1-vertices $\{\alpha_1,\dots,\alpha_k\}$, and give it coordinates $\xi_{\underline{\alpha}} = (\xi_i)_{i\in\alpha_j\in\underline{\alpha}}$. Let $N_t^{\underline{\alpha}}$ be the number of 1-vertices in $\underline{\alpha}$ of size t. Then we define 2-vertex functions

$$\sigma^2_{\underline{\alpha}}(\xi_{\underline{\alpha}}) = k!(\prod_t N_t^{\underline{\alpha}}!)^{-1}\,\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})$$

$$\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}}) = \mathbb{S} \prod_{j=1}^{k} \sigma^1_{|\alpha_j|}(\xi_{\alpha_j}) \frac{(-\beta)^{k-1}}{k}\int ds_1\dots ds_{k-1}\sum_{\eta}$$

$$\prod_{\ell=2}^{k} [s_{\eta(\ell)}\cdots s_{\ell-2} v^1(\alpha_{\eta(\ell)},\alpha_\ell)]\cdot$$

$$\cdot\exp(-\beta V^1_{s_1\dots s_{k-1}}(\alpha_1,\dots,\alpha_{k-1};\{\alpha_k\})), \tag{2.25}$$

where $\mathbb{S}$ symmetrizes in $\alpha_1,\dots,\alpha_k$. In terms of these, (2.24) becomes

$$Q_1^M(\underline{\pi}) = \sum_{k=1}^{M} \sum_{\underline{\alpha}\subseteq\underline{\pi},\alpha_1\in\underline{\alpha},|\underline{\alpha}|=k} k!\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})Q_1^{M-k}(\underline{\pi}\setminus\underline{\alpha}). \tag{2.26}$$

Just as in (2.15) we insert (2.26) into itself to obtain

$$Q_1^M(\underline{\pi}) = \sum_{p\in P(\underline{\pi})} \prod_{\underline{\alpha}\in p} [|\underline{\alpha}|!\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})]. \tag{2.27}$$

This expansion can be inserted into (2.18) to yield

$$Z^N(\xi_1,\dots,\xi_N) = \sum_{N_1,N_2,\dots;\Sigma kN_k=N} (\prod_k N_k!)^{-1}\,\mathbb{S} \sum_{p\in P(\underline{\pi})} \prod_{\underline{\alpha}\in p} [|\underline{\alpha}|!\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})]. \tag{2.28}$$

As before, we would rather sum over multiplicities of types of 2-vertices. The type $[\underline{\alpha}]$ of a 2-vertex is just the set of multiplicities $(N_k^{\underline{\alpha}})_{k=1,2,\dots}$ of 1-vertices of each size in $\underline{\alpha}$. If there are $N_{[\underline{\alpha}]}$ vertices of type $[\underline{\alpha}]$, let p be a corresponding partition in (2.28).

A combinatoric analysis as in the first expansion shows that there are $\prod_k N_k!(\prod_{[\underline{\alpha}]} N_{[\underline{\alpha}]}!)^{-1}(\prod_{\underline{\alpha}\in p} \prod_k N_k^{\underline{\alpha}}!)^{-1}$ terms in (2.28) corresponding to $\{N_{[\underline{\alpha}]}\}$. The third group of factorials and each $|\underline{\alpha}|!$ in (2.28) converts $\hat{\sigma}^2$ to σ^2, and we obtain

$$Z^N(\xi_1,\ldots,\xi_N) = \sum_{\{N_{[\underline{\alpha}]}\},\ \sum_{[\underline{\alpha}]} N_{[\underline{\alpha}]}\sum_k kN_k^{\underline{\alpha}}=N} \left(\prod_{[\underline{\alpha}]} N_{[\underline{\alpha}]}!\right)^{-1} \mathbb{S} \prod_{\underline{\alpha}\in p} \sigma_{\underline{\alpha}}^2(\xi_{\underline{\alpha}}). \quad (2.29)$$

We can now obtain the full partition function by summing over N and integrating over $\xi_1,\ldots,\xi_N$. Symmetrization is no longer necessary, and furthermore the sums over $N_{[\underline{\alpha}]}$ factor into a product of sums

$$\sum_{N_{[\underline{\alpha}]}=0}^{\infty} (N_{[\underline{\alpha}]}!)^{-1} \int d\xi_{\underline{\alpha}} \sigma_{\underline{\alpha}}^2(\xi_{\underline{\alpha}}) = \exp\left(\int d\xi_{\underline{\alpha}} \sigma_{\underline{\alpha}}^2(\xi_{\underline{\alpha}})\right).$$

The expansion reaches its final form:

$$Z = \sum_{N=0}^{\infty} \int d\xi_1 \ldots d\xi_N Z^N(\xi_1,\ldots,\xi_N) = \exp\left(\sum_{[\underline{\alpha}]} \int d\xi_{\underline{\alpha}} \sigma_{\underline{\alpha}}^2(\xi_{\underline{\alpha}})\right). \quad (2.30)$$

We have achieved our goal of writing the partition function as an ensemble of noninteracting 2-vertices. Equation (2.30) also gives us an expansion for the pressure:

$$\lim_{\Lambda\to\mathbb{R}^3} |\Lambda|^{-1} \log Z = \sum_{[\underline{\alpha}]} \int_{x_1=0} d\xi_{\underline{\alpha}} \sigma_{\underline{\alpha}}^2(\xi_{\underline{\alpha}}), \quad (2.31)$$

where the x-integrals on the right extend over $\mathbb{R}^3$. Of course the validity and usefulness of (2.30), (2.31) depend on some convergence estimates; these will be discussed in the next section.

3. Convergence of the Expansion

We estimate the vertex functions σ^ℓ in (2.13) and (2.25). The restrictions derived in Sect. 1,

$$z\beta^3 e^{c\beta/R} << 1, \quad \lambda << 1, \quad (3.1)$$

will guarantee convergence of the first and second expansions, respectively. We claim that

$$\int \prod_{i\in\alpha, i\neq i_0} d\xi_i |\sigma_k^1(\xi_\alpha)| \leq z(z\beta^3 e^{c\beta/R})^{k-1}. \tag{3.2}$$

The definition of $V^0_{s_1\ldots s_{k-1}}$ through successive convex combinations (2.10) preserves the stability estimate derived in Sect. 1. Thus

$$\exp(-\beta V^0_{s_1\ldots s_{k-1}}(1,\ldots,k-1;\{k\})) \leq e^{ck\beta/R}. \tag{3.3}$$

The ξ-integrals are handled using (1.11); we integrate out sequentially the positions of extremal vertices of the tree η, leaving in the end only the i_0 vertex, which is fixed. Each integration produces a factor β^2. The remaining sums and integrals in (2.13) are handled using

$$\int ds_1\ldots ds_{k-1} \sum_\eta \prod_{\ell=2}^{k} [s_{\eta(\ell)}\cdots s_{\ell-2}] \leq e^{k-1}. \tag{3.4}$$

This is a standard estimate that appears in many kinds of cluster expansions. It is quite important as it controls the sum over the $(k - 1)!$ trees η. We prove it as a special case of (3.7) below. Putting all these estimates together, we obtain (3.2). Note that when $k = 1$ we have $\sigma_k^1(\xi_\alpha) = \pm z$.

The estimate for σ^2 proceeds similarly, and we can see an inductive structure emerging. Let $\underline{\alpha} = \{\alpha_1,\ldots,\alpha_k\}$ with $\sum_i |\alpha_i| = t$. We prove that

$$\int \prod_{i\neq i_0} d\xi_i |\hat{\sigma}^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})| \leq z(cz\beta^3 e^{c\beta/R})^{t-k}(c\lambda^2)^{k-1}. \tag{3.5}$$

Again the stability estimate

$$\exp(-\beta V^1_{s_1\ldots s_{k-1}}(\alpha_1,\ldots,\alpha_{k-1};\{\alpha_k\})) \leq c^k \tag{3.6}$$

is preserved in the interpolation process. If we expand out the sums over particles in $\alpha_{\eta(\ell)},\alpha_\ell$ in $v^1(\alpha_{\eta(\ell)},\alpha_\ell)$, then (2.25) is represented as a sum of $\prod_{\ell=2}^{k}[|\alpha_{\eta(\ell)}||\alpha_\ell|]$ terms. In each term, and for each $\ell = 2,\ldots,k$, a particle in α_ℓ is connected to a particle in $\alpha_{\eta(\ell)}$ through a line $v^1(\xi_i,\xi_j)$. We use (3.2) to integrate over the coordinates of all but one particle in each α_ℓ and we use (1.12) to integrate out the remaining particle. Extremal 1-vertices are integrated out first. Estimate (3.4) generalizes to

$$\int ds_1 \ldots ds_{k-1} \sum_{\eta} \prod_{\ell=2}^{k} [s_{\eta(\ell)} \cdots s_{\ell-2} |\alpha_{\eta(\ell)}|] \leq \exp(\sum_{\ell=1}^{k-1} |\alpha_\ell|), \tag{3.7}$$

and the remaining $|\alpha_\ell|$ factors are controlled using

$$\prod_{\ell=2}^{k} |\alpha_\ell| \exp(\sum_{\mu=1}^{k-1} |\alpha_\mu|) \leq \exp(2 \sum_{\ell=1}^{k} |\alpha_\ell| - 2) = e^{2(t-1)}. \tag{3.8}$$

Altogether we have factors $(cz\beta^3 e^{c\beta/R})^{|\alpha_\ell|-1}$ at each α_ℓ and factors $cz\beta\lambda^2 \ell_D^2 = c\lambda^2$ for each v^1-line. This proves (3.5).

To prove (3.7) write the sum over η as $k - 1$ sums over $\eta(j) \in [1, j-1]$ for $2 \leq j \leq k$, and notice that the left-hand side is less than

$$\int ds_1 \ldots ds_{k-1} \sum_{\eta(2)\ldots\eta(k)} \prod_{\ell=2}^{k} [s_{\eta(\ell)} \cdots s_{\ell-2} |\alpha_{\eta(\ell)}|] \cdot$$

$$\cdot \exp(\sum_{\mu=1}^{k-1} s_\mu \cdots s_{k-1} |\alpha_\mu|)$$

$$= \int ds_1 \ldots ds_{k-1} \sum_{\eta(2)\ldots\eta(k-1)} \prod_{\ell=2}^{k-1} [s_{\eta(\ell)} \cdots s_{\ell-2} |\alpha_{\eta(\ell)}|] \frac{d}{ds_{k-1}} \cdot$$

$$\cdot \exp(\sum_{\mu=1}^{k-1} s_\mu \cdots s_{k-1} |\alpha_\mu|)$$

$$\leq \int ds_1 \ldots ds_{k-2} \sum_{\eta(2)\ldots\eta(k-1)} \prod_{\ell=2}^{k-1} [s_{\eta(\ell)} \cdots s_{\ell-2} |\alpha_{\eta(\ell)}|] \cdot$$

$$\cdot \exp(\sum_{\mu=1}^{k-2} s_\mu \cdots s_{k-2} |\alpha_\mu| + |\alpha_{k-1}|). \tag{3.9}$$

These two steps can be repeated for each s_ℓ-integration, and in the end we obtain the right-hand side of (3.7).

We now use (3.5) to control the full expansion (2.31) for the pressure. We have

$$\sum_{[\underline{\alpha}]} \int_{x_1=0} d\xi_{\underline{\alpha}} |\sigma^2_{\underline{\alpha}}(\xi_{\underline{\alpha}})| \leq 2 \sum_{N_1^{\underline{\alpha}}, N_2^{\underline{\alpha}}, \ldots} z(cz\beta^3 e^{c\beta/R})^{t-k} (c\lambda^2)^{k-1}, \tag{3.10}$$

where the 2 accounts for the sum over e_1, and where

$t = \sum_j jN_j^{\alpha}$, $k = \sum_j N_j^{\alpha} = |\underline{\alpha}|$. Let us write $y = cz\beta^3 e^{c\beta/R}$, $w = c\lambda^2$; then

$$\sum_{N_1^{\alpha}, N_2^{\alpha}, \ldots} zy^{t-k}w^{k-1} = z \sum_{k=1}^{\infty} w^{k-1} \left(\sum_{m=1}^{\infty} y^{m-1}\right)^k$$

$$= zw^{-1} \sum_{k=1}^{\infty} \left(\frac{w}{1-y}\right)^k = \frac{z}{1-y-w}. \qquad (3.11)$$

Thus the series for the pressure is absolutely convergent and bounded by $2z(1 - cz\beta^3 e^{c\beta/R} - c\lambda^2)^{-1}$, provided the two expansion parameters y and w are small. We can see that after many iterated expansions we would lose control of the sums; if $\lambda_0, \lambda_1, \ldots, \lambda_R$ were the expansion parameters, then a bound of the form $2z(1 - \lambda_0 - \lambda_1 \cdots - \lambda_R)^{-1}$ would result. The λ_ℓ would also have a tendency to grow beyond their naive values $\lambda_\ell = z\beta\int |v^\ell(x)| dx e^{-\beta E^\ell}$ because of the factors arising from estimates like (3.7),(3.8).

4. Coulomb Gas Applications

In this section we describe how iterated Mayer expansions can be used in studying screening in the three dimensional Coulomb gas [6,7]. Let us consider charges ±1 again, with a soft-core Coulomb potential

$$v_c(\xi_i, \xi_j) = e_i \left[\frac{1 - e^{-|x_i - x_j|/R}}{4\pi |x_i - x_j|}\right] e_j. \qquad (4.1)$$

At activity z and inverse temperature β, the partition function is given by (1.4) with v_c replacing v. To see screening, we wish to perform a sine-Gordon transformation whereby the system becomes a scalar field theory with cosine interaction. The curvature at the minimum of the cosine should act like a mass and lead to exponential decay of correlation functions, or screening. The problem with going directly to the sine-Gordon form is that the short distance cutoff length R is too small to control the analysis of the sine-Gordon theory, particularly if we are interested in the region where the expected screening length $\ell_D = (2z\beta)^{-1/2}$ is much larger then R.

The solution to this problem lies in the use of a mixed gas-sine-Gordon representation. The Mayer expansion of Sects. 1 - 3 is well suited to the short distance analysis, but reaches its limit at a length scale $\lambda\ell_D$. Constructive field theory techniques applied to the sine-Gordon representation work well at length scales above $\lambda\ell_D$, but break down if λ is too small. Thus the two approaches complement each other. The splitting of the problem according to length scale fits well with the renormalization group philosophy that one should never attempt to treat at one time a greater range of lengths than can easily be accomodated in a given procedure.

The splitting we take consists in writing

$$v_c(\xi_i,\xi_j) = v(\xi_i,\xi_j) + v^2(\xi_i,\xi_j), \tag{4.2}$$

where $v = v^0 + v^1$ is the interaction considered in Sects. 1 - 3 and

$$v^2(\xi_i,\xi_j) = e_i\left[\frac{1 - e^{-|x_i-x_j|/(\lambda\ell_D)}}{4\pi|x_i - x_j|}\right]e_j. \tag{4.3}$$

Let $\rho(x) = \sum_i e_i\delta(x - x_i)$ be the charge density. The interaction can be written as

$$V = \sum_{i<j} v_c(\xi_i,\xi_j) = \frac{1}{2}\int dxdy\rho(x)u(x,y)\rho(y) - \sum_i (8\pi\lambda\ell_D)^{-1} + \sum_{i<j} v(\xi_i,\xi_j), \tag{4.4}$$

where

$$u(x,y) = \frac{1 - e^{-|x-y|/(\lambda\ell_D)}}{4\pi|x - y|} = [(-\Delta)^{-1} - (-\Delta + \lambda^{-2}\ell_D^{-2})^{-1}](x,y). \tag{4.5}$$

The second term on the right-hand side of (4.4) cancels the self-energies that were included in the first term. It merely changes the effective activity to $\tilde{z} = ze^{\beta/(8\pi\lambda\ell_D)}$, and for $(\beta/\ell_D)^2 = 2z\beta^3$ small this is only a slight modification. The sine-Gordon transformation treats the first term by using the identity

$$e^{-\beta\langle\rho,u\rho\rangle/2} = \int e^{i\beta^{1/2}\langle\phi,\rho\rangle}d\mu_u(\phi), \tag{4.6}$$

where $d\mu_u(\phi)$ is the Gaussian measure with covariance u. We see that every particle i has a phase factor $\exp(i\beta^{1/2}e_i\phi(x_i))$ associated to it. The third term in (4.4) is the interaction we expanded in (2.30). The same expansion applies here, only the phase factors must be carried along. Thus the Coulomb gas partition function can be represented as

$$Z = \int \exp\Big(\sum_{[\underline{\alpha}]} \int d\xi_{\underline{\alpha}} \sigma^2_{\underline{\alpha}}(\xi_{\underline{\alpha}}) \prod_{i\in\alpha\in\underline{\alpha}} e^{i\beta^{1/2} e_i \phi(x_i)}\Big) d\mu_u(\phi). \tag{4.7}$$

(Actually to proceed with the screening proof we modify u near the boundary by replacing Δ in (4.5) with Δ_V, the Laplacian with Dirichlet boundary conditions at ∂V, $V \subset \Lambda$.)

A final transformation of (4.7) writes $e^{i\beta^{1/2} e_i \phi(x_i)} = 1 + \varepsilon(\xi_i)$ and expands the resulting product. This yields

$$Z = \int \exp\Big(\rho_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int d\xi_1 \ldots d\xi_s \rho_s(\xi_1, \ldots, \xi_s) \varepsilon(\xi_1) \ldots \varepsilon(\xi_s)\Big) d\mu_u(\phi), \tag{4.8}$$

where

$$\rho_s(\xi_1, \ldots, \xi_s) = s! \sum_{t\geq s} \binom{t}{s} \int d\xi_{s+1} \ldots d\xi_t \sum_{[\underline{\alpha}] : \sum_k k N_k^{\underline{\alpha}} = t} \sigma^2_{\underline{\alpha}}(\xi_{\underline{\alpha}}) \tag{4.9}$$

is the s-point truncated correlation function of the Yukawa gas. To understand this action, look at the leading term s = 1 in (4.8) and t = s in (4.9). It is

$$\int dx\, \tilde{z}\big(e^{i\beta^{1/2}\phi(x)} + e^{-i\beta^{1/2}\phi(x)} - 2\big) = \int dx\, 2\tilde{z}(\cos \beta^{1/2}\phi(x) - 1). \tag{4.10}$$

Thus we have a generalized sine-Gordon theory, with interaction $2\tilde{z} \cos \beta^{1/2}\phi$ plus higher order nonlocal corrections. The nonlocal corrections are the price we pay for having a good short distance cutoff (at length $\lambda \ell_D$). They are well controlled by estimates like the ones in Sect. 3. A strong form of exponential decay of the functions $\rho_s(\xi_1, \ldots, \xi_s)$ results, see [3,7]. The usefulness of the Mayer expansion is now clear. It expressed the partition function of the Yukawa gas in exponential form, with exponentially localized vertex functions. These properties are basic to the subsequent sine-Gordon analysis.

If we expand the cosine about $\phi = 0$ we obtain

$$2\tilde{z}(\cos \beta^{1/2}\phi - 1) = -\tilde{z}\beta\phi^2 + \frac{1}{12} \tilde{z}\beta^2\phi^4 + \ldots . \tag{4.11}$$

The leading term adds $\tilde{\ell}_D^{-2} = 2\tilde{z}\beta \approx \ell_D^{-2}$ to the inverse covariance of $d\mu_u(\phi)$. The new covariance is

$$(u^{-1} + \tilde{\ell}_D^{-2})^{-1}(x,y) = (\lambda^2 \ell_D^2 (-\Delta_V)^2 - \Delta_V + \tilde{\ell}_D^{-2})^{-1}(x,y), \tag{4.12}$$

which decays exponentially like $e^{-|x-y|/\tilde{\ell}_D}$ and is bounded by $c\lambda^{-1}$ on the diagonal. We can begin to see how the screening arises. To see what is necessary to control the corrections to the Gaussian, choose units where $\tilde{\ell}_D = 1$. Then the quartic term in (4.11) is proportional to β. Thus we need $(\beta/\ell_D)^2 = 2z\beta^3$ small -- it turns out that $z\beta^3 \ll e^{-c/\lambda}$ is small enough. Note that with no Mayer expansion we would have $\lambda = R/\ell_D$ and thus we would need both $z\beta^3$ and β/R small to satisfy $z\beta^3 \ll \exp(-c(z\beta^3)^{-1/2}\beta/R)$.

Space does not allow a discussion of the expansion of this sine-Gordon model about the massive Gaussian; the reader should consult [1,3,4,7]. However we can see what conditions are needed on z,β,R to obtain convergent expansions. The iterated Mayer expansion of Sects. 1 - 3 required $z\beta^3 \ll e^{-c\beta/R}$, $\lambda \ll 1$. The cluster expansion for the sine-Gordon model requires $z\beta^3 < e^{-c/\lambda}$, $\lambda \ll 1$. Altogether $z\beta^3 \ll e^{-c\beta/R}$ is sufficient to prove screening. This is a considerable improvement over the region obtained with a single Mayer expansion with the requirement $\lambda^2 \ll e^{-c\beta/R}$. The β/R-dependent restriction on λ entails that $z\beta^3 \ll \exp(-ce^{c\beta/R})$, which is much worse for $\beta \gg R$.

An even more striking improvement in screening regions was made by Göpfert and Mack [6]. They studied the lattice Coulomb gas dual to the three dimensional U(1) lattice gauge theory. In this model the activity is linked to β according to $z = e^{-\beta v_c(0)/2}$, where v_c is now the lattice Coulomb potential. [In other words, the self-energy terms with $i = j$ are included in V whereas we have omitted them -- see (4.4).] The lattice spacing plays the role of our short distance cutoff R, and we set both to 1 here by a choice of units. Using very precise stability estimates in the first Mayer expansion, they were able to sharpen the requirement $z\beta^3 \ll e^{-c\beta}$ to one like $z\beta^3 \ll e^{-(1-\varepsilon)\beta v_c(0)/2}$ with $\varepsilon > 0$. Thus with z as above, the condition on β becomes $\beta^3 \ll e^{\varepsilon\beta v_c(0)/2}$, and screening can be proven for large β. The expansion also proves confinement in the U(1) model at large β, and confinement then follows at all temperatures by correlation inequalities.

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