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An Intermediate Phase with Slow Decay of Correlations in One Dimensional $1/|x-y|^2$ Percolation, Ising and Potts Models

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Abstract. We rigorously establish the existence of an intermediate ordered phase in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models. The Ising model truncated two-point function has a power law decay exponent θ which ranges from its low (and high) temperature value of two down to zero as the inverse temperature and nearest neighbor coupling vary. Similar results are obtained for percolation and Potts models.

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0. Introduction

The phase transitions of one dimensional Ising ferromagnets (and related models) with inverse square law Hamiltonians,

$$-\tfrac{1}{2}\sum_{x<y} J_{y-x}S_xS_y, \quad \text{where} \quad 0 \leq x^2 J_x \to 1 \quad \text{as} \quad x\to\infty, \tag{0.1}$$

are distinguished by a number of unusual features. Figure 1 is a schematic phase diagram which exhibits some of what has been rigorously proved in recent years. The main contributions of this paper concern the intermediate ordered phase, region II, in which $M \equiv \langle S_0 \rangle_+$ is nonzero while θ (defined by $\langle S_0 S_x \rangle_+ - M^2 \sim |x|^{-\theta}$) varies between 0 and 2.

The intermediate phase of $1/|x-y|^2$ models is of interest for at least two reasons. First, for the general theory of phase transitions, it provides another example of a phase with variable exponent power law decay, in the spirit of the Kosterlitz-Thouless phase of planar spin models in two dimensions [KT]. Second, within the context of $1/|x-y|^2$ models, its properties are closely related to the discontinuity in M. For example, Thouless' original argument for the existence of a discontinuity [T] is inapplicable if $\theta \to 0$ as $\beta \searrow \beta_c$, i.e., at the curve BD in Fig. 1 (see Subsect. 1.ii) below for more details). But precisely this phenomenon was predicted by Bhatacharjee, Chakravarty, Richardson, and Scalapino [BCRS], who used the renormalization group flow equations of Anderson, Yuval, and Hamann [AYH] to first argue for the presence of a phase with temperature-dependent power law exponent.

Although we do not quite prove in this paper that θ vanishes as $\beta \searrow \beta_c$ we do establish the existence of region II, the location of its northeast and northwest corners (points B and A in Fig. 1) and that $\theta \to 0$ at point B. To explain these results and their relation to the percolation-theoretic Aizenman-Newman proof [AN 2]

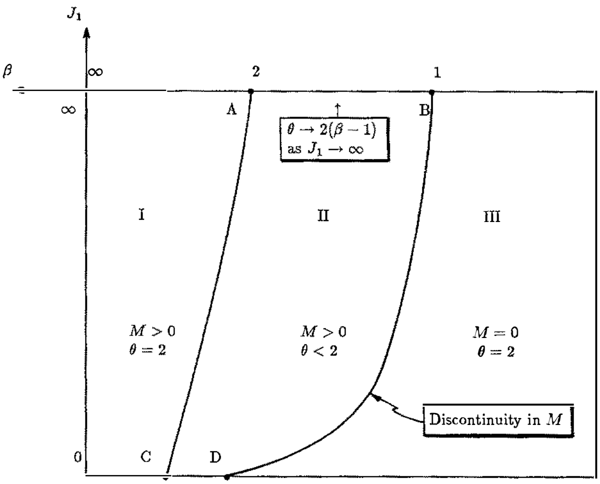

Fig. 1. Schematic phase diagram in the β (inverse temp.)-J_1 (nearest neighbor coupling) plane. M denotes the magnetization (or percolation density) and θ the power law decay exponent for the truncated two-point function. It is conjectured that $\theta = 2(M^2\beta - 1)$ in region II and $\theta = 0$ on the curve BD

of a discontinuity in M (as applied to Ising models by Aizenman, Chayes, Chayes, and Newman [ACCN]), we briefly discuss the following four items. More details may be found in Sect. 1 below.

A. The relation between Ising and percolation models.

B. The notion of dissociated intervals.

C. The significance of β^*, the value of β at the point B in Fig. 1.

D. The renormalization of β to $M^2\beta$ and our conjecture that $\theta = 2(M^2\beta - 1)$ in the intermediate phase.

For given β and J_x's the Fortuin-Kasteleyn random cluster models [KF, FK] are (dependent) bond percolation systems depending on an additional real parameter $q \geqq 1$, which interpolate between the Ising ($q=2$) and Potts ($q=3,4,\ldots$) models. For $q=1$, it reduces to independent bond percolation where the bond $\{x,y\}$ is occupied with probability $1-\exp(-\beta J_{y-x})$. For any q, the order parameter M is defined as the percolation density (the probability that the cluster of the origin is infinite), which for $q=2$ is consistent with the usual magnetization. There is a natural truncated connectivity function, $\tau'(x,y)$, the probability that x and y belong to the same *finite* cluster. For $q=2$, a crucial fact is that

$$\langle S_x S_y \rangle_+ - M^2 \geqq \tau'(x,y). \tag{0.2}$$

An interval of sites $\{x_1, x_1+1, \ldots, x_2\}$, is said to be dissociated if every bond between an x in the interval and a y outside the interval is vacant. This notion was introduced in [AN 2] where it was shown that for $\beta < 1$, M must vanish (regardless of the value of J_1) because dissociated intervals occur (with probability one) on all length scales. In other words, if we let ψ_L denote the probability that an interval of length L is contained within some larger dissociated interval, then $\psi_L = 1$ for all L as long as $\beta < 1$. The result that $\beta < 1$ implies $M=0$ is valid both for $q=1$ [AN 2] and $q>1$ [ACCN]. It may be restated as the inequality $\beta^* \geqq 1$, where $\beta^* = \beta^*(q)$ is defined as the largest β such that $M=0$ for all $J_1 < \infty$.

Now suppose $\beta > 1$. The key calculation which leads us to an intermediate phase is essentially an estimate that as $L \to \infty$, $\psi_L \gtrsim L^{-2(\beta-1)}$. If there is long range order of the right sort, then $\tau'(0,x) \gtrsim \psi_{|x|}$ because the conditional probability that 0 is connected to x given that there is a dissociated interval containing $\{0,1,\ldots,x\}$ will not tend to zero as $|x| \to \infty$. Combining all this [and using (0.2)], we conclude that in the ordered phase $\theta \leqq 2(\beta-1)$.

If $\beta^* < 2$, then this inequality for θ implies the existence of an $M>0$, $\theta<2$ region somewhere in the β, J_1-plane. This phase would necessarily be intermediate since Imbrie proved [I] that $\theta = 2$ for large β (at least for $q=2,3,\ldots$ – see Theorem 1.8 below) and it is also known that $\theta=2$ whenever $M=0$ (at least for $q=1$ [AN 2] and $q=2$ [ACCN]). If $\beta^* \leqq 1$ (hence $\beta^*=1$), then it would further follow from $\theta \leqq 2(\beta-1)$ that $\theta \to 0$ at the northeast corner of region II. For $q=1$, the bound $\beta^* \leqq 1$ was proved by Newman and Schulman [NS], but for $q>1$ the best available result was $\beta^* \leqq q$ [ACCN] which is insufficient to yield an intermediate phase in the Ising case (or for any $q \geqq 2$). A sizable portion of this paper is consequently devoted to proving that $\beta^* \leqq 1$ for all $q>1$. Our basic approach here is not percolation-theoretic but rather attacks the integer q spin models directly (see Sect. 3) with the generalized Peierls arguments invented by Fröhlich and Spencer to prove $\beta^* < \infty$ for $q=2$ [FS].

For both Ising and percolation models, there are natural block variable scaling transformations which lead to a renormalized model in which β is essentially replaced by $M^2\beta$ [AYH, NS]. For example, there is a percolation-theoretic renormalization which replaces the original occupied bonds $\{x, y\}$ by "anchored" bonds in which both x and y must have large scale connections (other than $\{x, y\}$ itself). This scheme was introduced in [AN 2] and used there to renormalize the argument that $\beta<1$ implies $M=0$ (i.e., $\beta^* \leqq 1$) into an argument that $M^2\beta<1$ implies $M=0$. This yielded the discontinuity of M (for $q=1$ in [AN 2] and for $q>1$ in [ACCN]). In this paper, we analogously renormalize the bound $\theta \leqq 2(\beta-1)$ into $\theta \leqq 2(M^2\beta-1)$. This bound has an important consequence if we assume the validity of the conjecture that $M^2\beta=1$ on the critical curve BD – we would have $\theta=0$ on BD as predicted in [FMN]. The improved upper bound on θ, together with some lower bounds valid for large J_1, leads to the natural conjecture that in the intermediate phase, $\theta=2(M^2\beta-1)$. This equality would identify the curve AC with the equation $M^2\beta=2$, and the boundary curve for infinite susceptibility (not drawn in Fig. 1) with the equation $M^2\beta=3/2$.

We conclude this introduction with a list of the main results of this paper (together with previously proven results), valid for all real $q \geqq 1$ except as noted. Precise versions of these results are given in Subsect. 1.iii) below.

a) $\beta^*=1$ (previously proved that $1 \leqq \beta^* \leqq q$).

b) Long range order implies $\theta \leqq \min(2(M^2\beta-1), 2)$.

c) Let $q \geqq 2$ be an integer. Then $\theta=2$ for $\beta>2$ and large J_1 (previously proved for large β) and $\theta \to 2(\beta-1)$ as $J_1 \to \infty$ for $1<\beta \leqq 2$.

1. Main Results

1.i) Setup

Let us briefly define the models we consider and their basic quantities of interest; for more details, see [ACCN].

The spin variables in Ising ($q=2$) or q-state Potts models have two standard representations, σ_x or $\boldsymbol{\sigma}_x$, where σ_x takes values in the set $\{1, \ldots, q\}$ and $\boldsymbol{\sigma}_x$ in the set $\{\hat{e}_1, \ldots, \hat{e}_q\}$ of unit vectors pointing to the vertices of a fixed $(q-1)$ dimensional "tetrahedron." (For $q=2$, $\boldsymbol{\sigma}_x$ reduces to the ± 1 valued variable denoted by S_x in the introduction.) Note that $\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y$ takes on only two values and can be expressed as

$$\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y = [q/(q-1)]\,(\delta_{\sigma_x, \sigma_y} - 1/q)\,.$$

The models are described by a Hamiltonian

$$\mathscr{H} = -\sum_{\{x,y\}} J_{x,y}(\delta_{\sigma_x, \sigma_y} - 1) = -[(q-1)/q] \sum_{\{x,y\}} J_{x,y}(\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y - 1)\,. \tag{1.1}$$

We will only consider translation invariant one-dimensional ferromagnetic models in which x, y are in $\mathbb{Z}$ and $J_{x,y} = J_{y-x} = J_{x-y} \geqq 0$.

The free boundary condition two-point function for the finite interval $[-L, L]$ of lattice points is (for $|x|, |y| \leqq L$)

$$\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_f^L = [q/(q-1)]\,\langle \delta_{\sigma_x, \sigma_y} - 1/q \rangle_f^L, \tag{1.2}$$

where $\langle - \rangle_f^L = \langle - \rangle_f^L(\beta)$ denotes expectation with respect to the free b.c. Gibbs state whose configuration probabilities are proportional to $\exp(-\beta\mathscr{H})$ with the sum in (1.1) restricted to $x, y \in [-L, L]$. We will also consider the 1 or $\hat{e}_1$ boundary condition expectation $\langle - \rangle_1^L$ (which reduces to the + b.c. for $q=2$) in which only x is restricted to $[-L, L]$ while $\boldsymbol{\sigma}_y$ is set to $\hat{e}_1$ for each y outside of $[-L, L]$ $\left(\text{we will always have } |J| \equiv \sum_y J_{y-x} < \infty\right)$. Both free and 1 states have limits as $L \to \infty$ whose expectations are denoted $\langle - \rangle_*$ with $* = f$ or 1. The infinite volume quantities of primary interest to us are the magnetization

$$M = \langle \hat{e}_1 \cdot \boldsymbol{\sigma}_0 \rangle_1 = [q/(q-1)] \langle \delta_{\sigma_0, 1} - 1/q \rangle_1 , \tag{1.3}$$

and the truncated two-point function

$$G^T(y-x) = \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_1 - \langle \boldsymbol{\sigma}_x \rangle_1 \cdot \langle \boldsymbol{\sigma}_y \rangle_1 = \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_1 - M^2 . \tag{1.4}$$

The definition (1.3) is equivalent (see e.g. [ACCN]) to the thermodynamic one, $\beta^{-1} \dfrac{\partial f(h=0+)}{\partial h}$, in which the free energy $f(h)$ is defined by adding to the Hamiltonian a term $-h\Sigma_x \hat{e}_1 \cdot \boldsymbol{\sigma}_x$. Consequently, another natural truncated two-point function G_1^T may be defined as

$$\begin{aligned} G_1^T(y-x) &= \langle (\hat{e}_1 \cdot \boldsymbol{\sigma}_x)(\hat{e}_1 \cdot \boldsymbol{\sigma}_y) \rangle_1 - \langle \hat{e}_1 \cdot \boldsymbol{\sigma}_x \rangle_1 \langle \hat{e}_1 \cdot \boldsymbol{\sigma}_y \rangle_1 \\ &= [q/(q-1)]^2 \cdot [\langle \delta_{\sigma_x, 1} \delta_{\sigma_y, 1} \rangle_1 - \langle \delta_{\sigma_x, 1} \rangle_1 \langle \delta_{\sigma_y, 1} \rangle_1] . \end{aligned} \tag{1.5a}$$

Finally, we define yet a third truncated function

$$G_2^T = [q/(q-1)]^2 \cdot [\langle \delta_{\sigma_x, 2} \delta_{\sigma_y, 2} \rangle_1 - \langle \delta_{\sigma_x, 2} \rangle_1 \langle \delta_{\sigma_y, 2} \rangle_1] . \tag{1.5b}$$

For $q=2$, $G^T = G_1^T = G_2^T$; we shall see below (Proposition 1.1) that for any other q, these three are bounded above and below by multiples of each other and hence have the same long distance decay properties.

Related to the above models are the Fortuin-Kasteleyn random cluster models [KF, FK, F] defined with a real (not necessarily integer) parameter q. These are described by probability measures on the configurations of bond occupation variables $n = \{n_b\}$ which take the values 1 – meaning the bond $b = \{x, y\}$ is occupied, or 0 – meaning b is vacant. For a finite volume $[-L, L]$, the free b.c. measure $\mu_L^f = \mu_L^f(\beta)$ (restricted to bonds with x, y in $[-L, L]$) has configuration probabilities proportional to

$$q^{c(n)} \prod_{b: n_b = 1} (1 - e^{\beta J_b}) \prod_{b: n_b = 0} e^{-\beta J_b} , \tag{1.6}$$

where $c(n)$ denotes the number of distinct clusters (i.e. connected components of the sites in $[-L, L]$) determined by the occupied bonds of n (and for $b = \{x, y\}$, $J_b = J_{x,y}$). For $q=1$, this is just an independent bond percolation model; for $q = 2, 3, \ldots$ one has, for g any function of the spin variables in $[-L, L]$, the identity

$$\langle g(\sigma) \rangle_f^L = \sum_n \mu_L^f(n) E_n^f(g(\sigma)) , \tag{1.7}$$

where for each configuration n of bond variables, $E_n^f(-)$ is a very simple average over the spins σ – the spins constrained to be constant on each cluster with the values for different clusters being independent and symmetric (i.e. with all q values equally likely). A special case of (1.7) is

$$\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_f^L = \mu_L^f(x \leftrightarrow y) \equiv \tau_L^f(x, y), \tag{1.8}$$

where $x \leftrightarrow y$ denotes the event consisting of those configurations in which x and y belong to the same cluster. The analogue of $\langle - \rangle_1^L$ is the "wired" b.c. measure μ_L^w (for bonds with x in $[-L, L]$ and y unrestricted) in which the $c(n)$ is determined after setting n_b to 1 for all b with both x and y outside of $[-L, L]$. In the wired version of (1.7), $E_n^w(-)$ is calculated with $\boldsymbol{\sigma}_x$ set to $\hat{e}_1$ for every x connected by an occupied bond to the outside of $[-L, L]$.

For $q \geqq 1$, infinite volume measures μ^f and μ^w exist [F, ACCN] (these are of course equal for $q=1$) and for $q=2, 3, \ldots$, the following identities are valid:

$$M = \mu^w(x \leftrightarrow \infty), \tag{1.9}$$

$$\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_1 = \tau'(x, y) + \mu^w(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty), \tag{1.10}$$

where τ' is the truncated connectivity function, defined as

$$\tau'(x, y) = \mu^w(x \leftrightarrow y \text{ but } \{x, y\} \not\leftrightarrow \infty), \tag{1.11a}$$

where $x \leftrightarrow \infty$ means that the cluster of x is infinite and $\{x, y\} \not\leftrightarrow \infty$ means that neither x nor y belongs to an infinite cluster. (We note that (1.9) is proved in Theorem 2.3 of [ACCN] and (1.10) is obtained by similar arguments.) The next proposition combines the above formulas with some simple facts about the FK representation; it shows that G^T, G_1^T, and G_2^T are bounded by multiples of each other and each is bounded below by a multiple of τ'. Our strategy in analyzing the decay properties of truncated two-point functions will be to obtain lower bounds for τ' (Sect. 2) and upper bounds for G_2^T (Sect. 4). In the proposition, an important role is played by $\hat{\tau}(x, y)$, another percolation-theoretic truncated two point function, which in certain respects is more analogous to the spin-theoretic G_1^T, G_2^T, and G^T than is τ'. The definition of $\hat{\tau}$ is

$$\hat{\tau}(x, y) = \mu^w(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty) - \mu^w(x \leftrightarrow \infty)\mu^w(y \leftrightarrow \infty). \tag{1.11b}$$

Proposition 1.1. *For* $q = 2, 3, \ldots$

$$G^T(y-x) = \tau'(x, y) + \hat{\tau}(x, y), \tag{1.12a}$$

$$G_1^T(y-x) = (q-1)^{-1}\tau'(x, y) + \hat{\tau}(x, y), \tag{1.12b}$$

$$G_2^T(y-x) = (q-1)^{-1}\tau'(x, y) + (q-1)^{-2}\hat{\tau}(x, y). \tag{1.12c}$$

For any real $q \geqq 1$,

$$\hat{\tau}(x, y) \geqq 0. \tag{1.13}$$

Proof. Equation (1.12a) is a direct combination of the formulas given above for G^T, M and $\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_1$ [i.e. (1.4), (1.9), and (1.10)]. Equation (1.12b) follows similarly from the formulas already given for G_1^T and M along with the identity,

$$\begin{aligned}\langle (\hat{e}_1 \cdot \boldsymbol{\sigma}_x)(\hat{e}_1 \cdot \boldsymbol{\sigma}_y) \rangle_1 &= \mu^w(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty) \\ &\quad + \tau'(x, y) \cdot \left[\frac{1}{q} \cdot 1^2 + \frac{q-1}{q} \cdot \left(-\frac{1}{q-1}\right)^2\right],\end{aligned}$$

which is derived from the wired version of (1.7) and an infinite volume limit. Equation (1.12c) is obtained analogously. Finally, (1.13) follows from the FKG property of μ^w [F, ACCN].

1.ii) Background and Discussion of Results

In this subsection, we review some previous results and discuss their relevance to our work. For more background, see [AN 2, ACCN].

For one-dimensional Ising models with $J_x \approx \text{const}/x^s$, it has been known for a long time that there is no long range order at any temperature if $s>2$ [D, R] but there is at low temperature if $s<2$ [Dy]. Based on the analyses of [T, AYH, AY], it was further believed that long range order does occur for $s=2$ but does not if $x^2 J_x \to 0$; the former was verified in [FS] but until recently the best complementary result [RT] was that $x^2(\log x)^{1/2} J_x \to 0$ implies no long range order.

In fact, by separating out the long and short range couplings, the work of [AYH, AY] yielded a much sharper prediction of the dividing line between long and short range forces in one dimension. This sharper division will be fairly significant for our results about the existence of an intermediate phase. Let us suppose that $\lim\limits_{x\to\infty} x^2 J_x = 1$ and define β^* by the following dichotomy (in which we fix J_x for $x \geqq 2$, but note that the resulting β^* is actually independent of the specific choice):

$$\text{if } \beta > \beta^*, \text{ then } M(\beta, J_1) > 0 \text{ for large } J_1;$$

$$\text{if } \beta < \beta^*, \text{ then } M(\beta, J_1) = 0 \text{ for any } J_1.$$

Although, it is not a priori clear that β^* should have a nontrivial value, it was in fact predicted by [AYH, AY] to be $\beta^* = 1$. In [FS], the long and short range couplings were not explicitly separated, and hence the only result obtained about β^* was $\beta^* < \infty$.

For one-dimensional independent percolation models, a proof of the absence of long range order for $s>2$ appears in [S] with the complementary result for $s \leqq 2$ appearing in [NS]. Let us denote by $\beta^*(q)$ the possibly q-dependent value of β^* for Potts and random cluster models. In [NS] $\beta^*(1)$ was explicitly investigated and it was shown that $\beta^*(1) \leqq 1$. Next, in [AN 2], it was shown that $\beta^*(1) \geqq 1$ (and also that $\beta^* \geqq 1$ for certain dependent percolation models). Thus the [AYH, AY] prediction was verified, but for $q=1$ (independent percolation) rather than for $q=2$ (Ising).

It was then shown in [ACCN] that the upper and lower bounds for $\beta^*(1)$ imply related bounds for all $q \geqq 1$ (and certain bounds for $0<q<1$ as well):

$$1 \leqq \beta^*(q) \leqq q\,. \tag{1.14}$$

These bounds are a direct consequence of the $q=1$ results and the following general comparison principle [F, ACCN]:

$$1 \leqq q \leqq q' \Rightarrow \begin{cases} M_q(\beta, \{J_x\}) \geqq M_{q'}(\beta, \{J_x\}) & \text{(1.15a)} \\ M_q(\beta, \{J_x\}) \leqq M_{q'}(\beta q'/q, \{J_x\})\,. & \text{(1.15b)} \end{cases}$$

$\beta^*(q) \geqq 1$ can also be obtained without use of the comparison principle by applying the dependent percolation result of [AN 2] directly to the random cluster models. We note that the Ising result $\beta^*(2) \geqq \frac{1}{2}$ had been obtained independently by Berbee [B] (without the use of percolation methods). We also remark that for the Ising model with J_x exactly equal to $1/x^2$, there existed numerical estimates for β_c in the vicinity of 1.3 [BCRS, Ma] as well as rigorous lower bounds [Mo], the best of which was 0.882 [V]. This lower bound is improved by the inequality $\beta^*(2) \geqq 1$ which implies that $\beta_c \geqq 1$.

One reason for our concern with the value of β^* is its relevance to the presence of an intermediate phase. As will be seen in Theorem 1.4 and Corollary 1.5 below, to verify the existence, for given q, of an ordered phase with decay of its truncated two-point function slower than const$/|x-y|^2$, we will need to know that $\beta^*(q) < 2$. From this perspective, the upper bound $\beta^*(q) \leqq q$ is fine for independent percolation ($q=1$) but inadequate for Ising models ($q=2$) and for higher q Potts models. Consequently a substantial part of this paper is devoted to improving the upper bound for β^*. In fact our first main result (Theorems 1.2 or 3.1) entirely removes the gap in (1.14) and shows that $\beta^*(q)=1$ for all $q \geqq 1$, thus extending the percolation result of [NS, AN] to $q>1$ and verifying the Ising model prediction of [AY, AYH].

We next turn to background related to the magnetization discontinuity in $1/|x-y|^2$ models. As we shall see, the mechanisms which lead to the discontinuity and to the intermediate phase are intimately related. The existence of a discontinuity in $M(\beta)$ at the critical point of $1/|x-y|^2$ Ising models was first proposed by Thouless [T] on the basis of an elegant energy-entropy argument (see also [SS]). His argument led to the dichotomy that

$$M(\beta)=0 \quad \text{or} \quad M^2(\beta) \cdot \beta \geqq \tfrac{1}{2} \min\{\theta(\beta), 1\}, \tag{1.16}$$

where θ is defined by the assumed power law behavior of the truncated two-point function $G^T(x)$:

$$G^T(x) \sim |x|^{-\theta} \quad \text{as} \quad |x| \to \infty . \tag{1.17}$$

As noted by Thouless, this argument yields a discontinuity *providing* $\theta(\beta) \not\to 0$ as $\beta \searrow \beta_c$.

A different argument for a discontinuity in the $1/|x-y|^2$ Ising model, which did not require any assumptions on $\theta(\beta)$, was given shortly after Thouless by Anderson, Yuval, and Hamann [AYH, AH] based on renormalization group flow equations. Strikingly, further renormalization group analysis by Bhattacharjee, Chakravarty, Richardson, and Scalapino [BCRS] led to the prediction that not only is there a phase below the critical temperature in which θ is temperature dependent (rather than say taking the constant value $\theta=2$ for all $\beta<\beta_c$ as one might naively expect), but also that θ does indeed approach zero as $\beta \searrow \beta_c$. [BCRS] also predicted that in the ordered phase at the critical point, $G^T(x)$ decays as $(\ln x)^{-1}$. Earlier renormalization group analysis [K] had led to the related predictions that the susceptibility χ' is infinite just above β_c and that at β_c, the singular part of M as a function of external field h behaves like $|\ln h|^{-1}$ as $h \searrow 0$. Much of the analysis of [K] was extended to Potts models (and to more general q-state spin systems) in [C].

The fact that the phase studied in [BCRS] must be an *intermediate* one was rigorously shown by Imbrie [I], who proved that $\theta = 2$ for very low temperature. In this paper, we improve that Ising model result and extend it to Potts models (see Theorem 1.8). We also note (Proposition 1.3) that quite generally, $\theta \leqq 2$. Another relevant fact is that $\theta = 2$ whenever $M = 0$, at least for $q = 1$ [AN 2] and for $q = 2$ [ACCN].

On the basis of the [BCRS] analysis, Thouless' original mechanism seems not to account for a magnetization discontinuity. A different mechanism, combining the renormalization group spirit (if not substance) of [AYH, AH] with the focus on a dichotomy as in [T], was used by Aizenman and Newman [AN 2] to derive rigorously for independent (and certain dependent) percolation models the dichotomy

$$M(\beta) = 0 \quad \text{or} \quad M^2(\beta) \cdot \beta \geqq 1, \tag{1.18}$$

which of course yields a discontinuity. It was then shown by Aizenman, Chayes, Chayes and Newman [ACCN] that the FK random cluster models (for $q \geqq 1$) fall within the class of dependent percolation models treated in [AN 2] so that (1.18) is valid for all $q \geqq 1$ including Ising (and Potts) models. It is useful to note that (1.18) may be regarded as a renormalized version of the simpler result of [AN 2] mentioned above that $\beta^* \geqq 1$, or equivalently

$$\beta < 1 \text{ implies } M = 0 \text{ for any } J_1 . \tag{1.19}$$

To obtain (1.18), the β in (1.19) is replaced by its "renormalized value" $M^2\beta$. The renormalization involves the replacement of the notion of occupied bonds by that of occupied "anchor bonds" (see [AN 2] and Subsect. 2.ii). We remark that because of (1.18) the strict inequality in (1.19) can be weakened to $\beta \leqq 1$.

The main purpose of this paper is the rigorous verification of the existence of an intermediate phase with M positive and θ small. Our analysis is closely based on the approach of [AN 2] and its applicability to the $q > 1$ random cluster models as shown in [ACCN]. Roughly speaking, we first apply the arguments which led to (1.19) to obtain an estimate valid when $\beta > 1$ (and there is long range order), namely

$$\theta \leqq 2(\beta - 1), \tag{1.20}$$

and then apply the renormalized arguments of (1.18) to show

$$\theta \leqq 2(M^2\beta - 1). \tag{1.21}$$

In the next subsection we will give precise versions of these two inequalities (Theorem 1.4) and explain in detail the ideas behind the first, unrenormalized, one. Meanwhile, we sketch these ideas.

As mentioned in the introduction, the key notion used to derive (1.19) was that of dissociated interval, i.e. an interval of sites $[\xi_-, \xi_+]$ such that no bond between the interval and its complement is occupied. Let us again denote by ψ_L the probability that an interval of length L is a subset of some larger dissociated interval. The basic estimates of [AN] showed that if $\beta < 1$, then there are so many dissociated intervals that $\psi_L = 1$ for all L; in this paper we show that when $\beta > 1$ similar estimates (see Proposition 1.6) show that $\psi_L \geqq \text{const}/L^{2(\beta-1)+\varepsilon'}$. This will yield (1.20) in the form $G^T(x) \geqq \tau'(0, x) \geqq \text{const}/|x|^{2(\beta-1)+\varepsilon'}$ because if there is long range order (of a sufficiently strong kind), then the conditional probability of 0 and x to be connected, given that $[0,x]$ (or $[-Kx,Kx]$) is a subset of a larger dissociated interval, will not tend to zero as $x\to\infty$.

By virtue of our result that $\beta^*\leqq 1$, we conclude the existence of an intermediate phase from (1.20), since by choosing β close to 1 and then J_1 large, θ may be made arbitrarily small. In fact, by combining (1.20) (and $\theta\leqq 2$) with a complementary upper bound (see Theorem 1.8) obtained from cluster expansion methods, we find that (at least for $q=2,3,\ldots$) $\theta\to\min(2(\beta-1),2)$ as $J_1\to\infty$ (with $\beta>1$). This fact suggests that the renormalized inequality (1.21) may be optimal; i.e. that for all J_1 and β (with $M>0$),

$$\theta=\min(2(M^2\beta-1),2). \tag{1.22}$$

We note that this conjecture is consistent with the predicted behavior as $\beta\searrow\beta_c$ (for fixed J_1) of M [AY] and θ [BCRS]:

$$M(\beta)-M(\beta_c)\sim(\beta-\beta_c)^{1/2},$$

$$\theta(\beta)\sim(\beta-\beta_c)^{1/2}.$$

Even as a one sided inequality, (1.21) is of interest – particularly at the critical point where, coupled with the prediction (see Subsect. 1.iv.a) that $M^2\beta=1$ at β_c, it would yield $\theta(\beta_c)=0$ as conjectured in [FMN]. It would be a first step in confirming the stronger prediction of [BCRS] that at β_c, $G^T(x)\sim(\ln x)^{-1}$. In Subsect. 1.iv) we will discuss further these conjectures and related open problems.

1.iii) Precise Statements of Results

We present in this subsection precise versions of our main results and explain some of the key ideas behind them. We deal throughout with one-dimensional translation invariant parameter-q random cluster models whose couplings J_x are fixed for $x\geqq 2$ and satisfy $\lim\limits_{x\to\infty} x^2J_x=1$; β and J_1 vary as indicated. Except as noted, q may be any real number in $[1,\infty)$. There are three types of results:

a) Sufficient conditions for long range order ($\beta^*\leqq 1$) – Theorem 1.2 and Sect. 3.

b) Lower bounds for τ' ($\theta\leqq\min(2(M^2\beta-1),2)$) – Propositions 1.3 and 1.6, Theorems 1.4 and 1.7, Corollary 1.5, and Sect. 2.

c) Upper bounds for G^T when $q=2,3,\ldots$ $\left(\lim\limits_{J_1\to\infty}\theta\geqq\min(2(\beta-1),2)\right)$ – Theorem 1.8 and Sect. 4.

We begin by defining *long long range order* (LLRO), a concept introduced in [SML], which will be relevant for a) and part of b).

Definition. We shall say that a model has long long range order if there are positive constants v and ε such that for all $L>0$, the free b.c. two point function τ_L^f in the region $[-L,L]$, satisfies

$$\tau_L^f(x,y)\geqq v^2\quad\text{for all}\quad |x|,|y|\leqq\varepsilon L.$$

Remark. Since (for $q\geqq 1$), $\tau^w\geqq\tau^f\geqq\tau_L^f$, and since $M\neq 0$ if $\tau^w(0,x)\not\to 0$, LLRO implies a nonzero magnetization, as is well known for $q=2$ [G]. The converse is presumably true at least for $q=1$ and 2 (with ε arbitrarily close to 1 and v to M for large L); it may not be true for larger values of q at $\beta=\beta_c$, where the infinite volume free state *may* be disordered, as is the case for two dimensional Potts models with large q[KS, M] (see Subsects. 1.iv.c) and 1.iv.d)). The significance of such a converse, in light of Theorem 1.4 below, will be discussed in Subsect. 1.iv.a). A weak converse for $q=1$ and 2 can be obtained from Hammersley-Lieb-Simon type inequalities [H, Si, Li, AN 1, A]; this leads to Theorem 1.7.

Theorem 1.2. *If $\beta>1$, then for all large J_1 there is long long range order.*

Proof. For $q=2,3,\ldots$, this is the essential part of Theorem 3.4 of Sect. 3. For all other q, it then immediately follows from Fortuin's comparison principle [related to (1.15a)] [F, ACCN],

$$\mu_L^f(q) \succcurlyeq \mu_L^f(q') \quad \text{for} \quad q' \geqq q,\ q' \geqq 1\,, \tag{1.23}$$

by choosing $q' \in \{2,3,\ldots\}$; here $\succcurlyeq$ denotes domination in the FKG sense.

Remarks. i) The $q=1$ case of Theorem 1.2 was originally proved in [NS].

ii) Since the inequality (1.23) is valid with $0 \leqq q < 1$, so is the theorem.

iii) The results of Sect. 3 show that the conclusion of the theorem can be strengthened to:

$$\text{as } J_1 \to \infty,\ \tau_L^f(x,y) \geqq 1-e^{-O(J_1)} \text{ for all } L>0 \text{ and } x,y \in [-L,L]\,. \tag{1.24}$$

We turn now to lower bounds for $\tau'(x,y)$ (and by Proposition 1.1 also for G^T, G_1^T, and G_2^T). For $q=1$, one very simple estimate is

$$\begin{aligned}\tau'(x,y) &\geqq \mu^w(\{x,y\} \text{ is occupied, but every other } \{x,z\} \text{ and } \{y,z\} \text{ is vacant})\\ &=(1-e^{-\beta J_{x-y}})\cdot\Bigg(\prod_{z\neq y} e^{-\beta J_{x-z}}\Bigg)\cdot\Bigg(\prod_{z\neq x} e^{-\beta J_{y-z}}\Bigg)\\ &\geqq \mathrm{const}(e^{\beta J_{x-y}}-1)\,,\end{aligned} \tag{1.25}$$

which implies that $\tau'(x,y) \geqq C/|x-y|^2$. A similar argument for $q>1$ yields:

Proposition 1.3. *For some $C>0$,*

$$\tau'(x,y) \geqq C/|x-y|^2 \textit{ for all } \{x,y\}\,. \tag{1.26}$$

Proof. The first inequality of (1.25) remains valid. We express its right-hand side as a product of conditional probabilities for single bonds conditioned on successively more information about the other bonds. It is a consequence of the basic definition (1.6) of the random cluster measures that (see Eq. (2.10) of [ACCN]) for any bond b,

$$\mu^w(n_b=1 \mid \{n_{b'}\}_{b'\neq b}) = 1-e^{-\beta J_b} \quad \text{or} \quad \frac{1-e^{-\beta J_b}}{1-e^{-\beta J_b}+qe^{-\beta J_b}}\,, \tag{1.27}$$

which implies that partially conditioned probabilities satisfy

$$1-e^{-\beta J_b/q} \leqq \mu^w(n_b=1 \mid \{n_{b'}\}_{b'\in B}) \leqq 1-e^{-\beta J_b}\,, \tag{1.28a}$$

$$e^{-\beta J_b} \leqq \mu^w(n_b=0 \mid \{n_{b'}\}_{b'\in B}) \leqq e^{-\beta J_b/q}\,; \tag{1.28b}$$

here B is any set of bonds not including b. Thus the remainder of (1.25) is also valid, providing we change the equality to an inequality and replace the two appearances of J_{x-y} by J_{x-y}/q.

Remark. Proposition 1.1 and the proof of Proposition 1.3 together imply that for general Ising and Potts ferromagnets,

$$\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_1 - \langle \boldsymbol{\sigma}_x \rangle_1 \cdot \langle \boldsymbol{\sigma}_y \rangle_1 \geqq \tilde{C}(\beta) J_{x,y}/q .$$

This is valid in any dimension (even without translation invariance) with

$$\tilde{C}(\beta) = \beta \exp\left(-2\beta \sup_x \sum_y J_{x,y} \right).$$

The next theorem gives a more interesting lower bound on τ'. We state the theorem in two parts; the second part is a stronger result which should be regarded as a renormalized version of the first part.

Theorem 1.4. *If a model exhibits long long range order, then for any $\varepsilon' > 0$, there is some $C > 0$ so that*

$$\text{i)} \qquad \tau'(x,y) \geqq C/|x-y|^{2(\beta-1)+\varepsilon'} \quad \textit{for all } \{x,y\}, \tag{1.29}$$

$$\text{ii)} \qquad \tau'(x,y) \geqq C/|x-y|^{2(\beta M^2-1)+\varepsilon'} \quad \textit{for all } \{x,y\}. \tag{1.30}$$

The following result is an immediate consequence of Theorem 1.2 and the unrenormalized part of the last theorem (and Proposition 1.1). It demonstrates the presence in the (β, J_1) plane of an ordered phase with slow decay of G^T [and infinite susceptibility as defined in (1.32)]; we take (1.12a) as the definition of G^T for $q=1$ and noninteger q. After the corollary we discuss the proof (of the unrenormalized part) of Theorem 1.4 in an important special case; the complete proof is an immediate consequence of Propositions 2.1–2.3 given in Sect. 2.

Corollary 1.5. *If $1 < \beta < 2$, then for all large J_1,*

$$M(\beta, J_1) > 0 \quad \textit{and} \quad G^T(x,y) \geqq \tau'(x,y) \geqq C/|x-y|^{\theta}, \tag{1.31}$$

for some $C > 0$ and $\theta < 2$. If $1 < \beta < 3/2$, then for all large J_1, (1.31) is valid with $\theta < 1$ and hence also

$$\chi'(\beta, J_1) \equiv \sum_{x=-\infty}^{\infty} G^T(0,x) = \infty . \tag{1.32}$$

For any $\varepsilon > 0$, one may choose β sufficiently close to 1 and then J_1 sufficiently large so that (1.31) is valid with $\theta < \varepsilon$.

Most of the basic ideas underlying Theorem 1.4 already are present for the case of independent percolation $(q=1)$, where an important role is played by the self-similar model (introduced in [NS] and used further in [AN 2]), defined by

$$J_{x-y} = \begin{cases} 0, & \text{for } |x-y| = 1 \\ \int_x^{x+1} \int_y^{y+1} |u-v|^{-2} dv\, du, & \text{otherwise}. \end{cases} \tag{1.33}$$

The next proposition, which will be a major ingredient in the proof of Theorem 1.4, helps explain the significance of the exponent $2(\beta-1)$. Immediately after we prove the proposition, we will use it to prove (1.29), the unrenormalized part of Theorem 1.4, (with $\varepsilon'=0$) for self-similar independent percolation. We remark that the extension of (1.29) to other independent percolation models and to $q>1$ is not difficult; on the other hand, the derivation of the renormalized part of Theorem 1.4 (even for $q=1$) will require some of the heavier machinery of [AN 2].

Proposition 1.6. *Define for $k>1$ and positive integer L the event*

$$F_{L,k}=\{\text{for some integer } \xi\in[L,kL),\text{ every bond from } [0,\xi] \text{ to } [\xi+1,\infty) \text{ is vacant}\}.$$

Let P_β denote the probability measure ($=\mu^w=\mu^f$) for the independent percolation ($q=1$) model with J_x given by (1.33). *If $\beta>1$, then for some $C,C'>0$,*

$$P_\beta(F_{L,2})\geqq C/L^{\beta-1} \quad \text{for all } L, \tag{1.34}$$

$$P_\beta([0,L]\nleftrightarrow\infty)\geqq C'/L^{2(\beta-1)} \quad \text{for all } L. \tag{1.35}$$

Proof. We first show that (1.34) implies (1.35). Following [AN 2], we define

$$F^*_{L,k}=\{\text{for some integer } \xi'\in(-(k-1)L,0],\text{ every bond from } [\xi',L] \text{ to } (-\infty,\xi'-1] \text{ is vacant}\}.$$

If the four events $F_{L,2}$, $F^*_{L,2}$, $H_L=\{$every bond from $(-L,0)$ to (L,∞) is vacant$\}$ and $H^*_L=\{$every bond from $(L,2L)$ to $(-\infty,0)$ is vacant$\}$ all occur simultaneously, then there is a dissociated interval $[\xi',\xi]$ (i.e. $[\xi',\xi]\nleftrightarrow[\xi',\xi]^c$) containing $[0,L]$ (in fact with $\xi<2L$ and $\xi'>-L$) and hence $[0,L]\nleftrightarrow\infty$. Hence

$$\begin{aligned}P_\beta([0,L]\nleftrightarrow\infty)&\geqq P_\beta(F_{L,2} \text{ and } F^*_{L,2} \text{ and } H_L \text{ and } H^*_L)\\&\geqq[P_\beta(F_{L,2})\cdot P_\beta(H_L)]^2,\end{aligned} \tag{1.36}$$

where the last inequality is an application of the Harris-FKG inequalities [Ha, FKG] which uses the fact that all four events are decreasing (in the FKG sense). Now

$$\begin{aligned}P_\beta(H_L)&=\prod_{x=-(L-1)}^{-1}\prod_{y=L+1}^{\infty}e^{-\beta J_{y-x}}=\exp\left[-\beta\int_{-(L-1)}^{0}\int_{L+1}^{\infty}(v-u)^{-2}dv\,du\right]\\&=\exp\left[-\beta\int_{-(1-1/L)}^{0}\int_{1+1/L}^{\infty}(v'-u')^{-2}dv'\,du'\right].\end{aligned} \tag{1.37}$$

Since this last expression is bounded away from zero as $L\to\infty$, it follows that

$$P_\beta([0,L]\nleftrightarrow\infty)\geqq\text{const}[P_\beta(F_{L,2})]^2, \tag{1.38}$$

which shows that (1.34) implies (1.35) as claimed.

There are a number of ways to derive (1.34). One of them uses a renewal type argument as follows. Let $\mathcal{N}$ denote the (random) number of ξ's in $[L,2L)$ such that every bond from $[0,\xi]$ to $[\xi+1,\infty)$ is vacant. $F_{L,2}$ is the event that $\mathcal{N}>0$. The expected value of $\mathcal{N}$ (with respect to the measure P_β) is easily calculated as

$$\begin{aligned} E_\beta(\mathcal{N}) &= \sum_{\xi=L}^{2L-1} P_\beta \text{ (every bond from } [0,\xi] \text{ to } [\xi+1,\infty) \text{ is vacant)} \\ &= \sum_{\xi=L}^{2L-1} \exp\left(-\beta \int_{\mathcal{R}(\xi)} (v-u)^{-2} dv\, du\right) \geqq \text{const} \sum_{\xi=L}^{2L-1} \frac{1}{\xi^\beta} \\ &\geqq \text{const}\, L^{1-\beta}, \end{aligned} \tag{1.39}$$

where

$$\mathcal{R}(\xi) = \{0 \leqq u < \xi+1,\ \xi+1 \leqq v < \infty\} \backslash \{\xi \leqq u < \xi+1,\ \xi+1 \leqq v < \xi+2\}. \tag{1.40}$$

To compare $P_\beta(F_{F,2}) = P_\beta(\mathcal{N} > 0)$ to $E_\beta(\mathcal{N})$, we use

$$P_\beta(\mathcal{N} > 0) = \frac{E_\beta(\mathcal{N})}{E_\beta(\mathcal{N} | \mathcal{N} > 0)} \tag{1.41}$$

and note that the conditional expectation can be calculated by conditioning further on the random location X, defined as the first ξ' in $[L, 2L)$ such that every bond from $[0,\xi']$ to $[\xi'+1,\infty)$ is vacant. Thus

$$\begin{aligned} E_\beta(\mathcal{N} | \mathcal{N} > 0) &= \sum_{\xi'=L}^{2L-1} P_\beta(X=\xi' | \mathcal{N} > 0) E_\beta(\mathcal{N} | X=\xi') \\ &= \sum_{\xi'=L}^{2L-1} P_\beta(X=\xi' | \mathcal{N} > 0) \\ &\quad \times \Bigg(1 + \sum_{\xi=\xi'+1}^{2L-1} P_\beta(\text{every bond from } [\xi'+1,\xi] \\ &\qquad \text{to } [\xi+1,\infty) \text{ is vacant})\Bigg) \\ &\leqq \sum_{\xi'=L}^{2L-1} P_\beta(X=\xi' | \mathcal{N} > 0)(1 + E_\beta(\mathcal{N})) = 1 + E_\beta(\mathcal{N}) \\ &\leqq 1 + \text{const} \sum_{\xi=L+1}^{\infty} \frac{1}{\xi^\beta} = \text{const}. \end{aligned} \tag{1.42}$$

Combining all these inequalities yields the desired bound (1.34) on $P_\beta(F_{L,2})$.

Proof of Theorem 1.4 (i) *for Self-Similar Independent Percolation.* Using the ε and v from the definition of LLRO, we take $L = L_x$ as the least integer greater than $\varepsilon^{-1}|x|$ and note that (with P_β replaced by μ^w)

$$\begin{aligned} \tau'(0,x) &= \mu^w(0 \leftrightarrow x, \text{ but } \{0,x\} \not\leftrightarrow \infty) \\ &\geqq \mu^w(x \text{ connected to } 0 \text{ by a path of occupied bonds} \\ &\qquad \text{within } [-L,L], \text{ but } [-L,L] \not\leftrightarrow \infty) \\ &= \tau_L^f(0,x) \cdot \mu^w([-L,L] \not\leftrightarrow \infty) \\ &\geqq v^2 \cdot \mu^w([-L,L] \not\leftrightarrow \infty) \\ &= v^2 \cdot \mu^w([0,2L] \not\leftrightarrow \infty). \end{aligned} \tag{1.43}$$

The proof is completed by applying the second inequality of Proposition 1.6. We remark that (1.43) did not use the self-similarity of the model, and except for its second equality did not use the independent nature of the model; in Sect. 2 (see Proposition 2.1) this equality will be replaced by an inequality for $q>1$.

As mentioned above, it is believed that for $q=1$ or 2 LLRO occurs whenever $M>0$. This would yield a corresponding weakening of the LLRO assumption of Theorem 1.4 and give a result applicable even at the critical point, where $M^2\beta-1$ is believed to vanish (see Subsect. 1.iv.a) for further discussion of this issue). The next theorem gives a somewhat weaker result applicable at the critical point. As explained in Subsect. 2.iii), this result is based on Hammersley-Lieb-Simon type inequalities [H, Si, Li, AN1, A]. Its proof is an immediate consequence of Propositions 2.1, 2.3, and 2.4.

Theorem 1.7. *Suppose $q=1$ (independent percolation) or 2 (Ising model). If $M>0$, then there is an infinite sequence $1<x_1<x_2<\dots$ (with x_{n+1}/x_n bounded as $n\to\infty$) such that for any $\varepsilon'>0$, there is some $C>0$ so that*

$$\tau'(0,x_n)\geqq C/x_n^{2(M^2\beta-1)+\varepsilon'} \quad \textit{for all } n. \tag{1.44}$$

Next, we turn to upper bounds for $G^T(x,y)$; these have so far only been derived for Ising and Potts models.

Theorem 1.8. *Suppose $q\geqq 2$ is an integer. If $\beta>1$ and $\varepsilon'>0$, then for all large J_1, there is some $C>0$ so that*

$$G^T(x-y)\leqq C/|x-y|^{\min(2(\beta-1)-\varepsilon',2)} \quad \textit{for all } \{x,y\}. \tag{1.45}$$

In particular, if $\beta>2$, then for all large J_1, there is some $C>0$ so that

$$G^T(x-y)\leqq C/|x-y|^2 \quad \textit{for all } \{x,y\}. \tag{1.46}$$

Equation (1.46) *is also valid for fixed $J_1>0$ and all large β.*

Proof. Equation (1.45) is an immediate consequence of Theorem 4.1 which gives an upper bound on G_2^T together with Proposition 1.1 which bounds G^T by a multiple of G_2^T. We remark that $J_1=J_1(\varepsilon')$ may be chosen large independently of β as long as β stays away from 1 and similarly in (1.46) if β stays away from 2. The last statement of the theorem was originally proved for $q=2$ by Imbrie [I]; its validity for general Potts models can be shown by minor modifications of the arguments of [I].

Remark. The last statement of the theorem is presumably true without the assumption that $J_1>0$, but we have not checked the details.

1.iv) Open Problems

In this subsection we discuss various open problems for $1/|x-y|^2$ models. These can be loosely grouped according to the following issues: a) saturation of $M^2\beta\geqq 1$ at β_c; b) validity of $\theta=\min(2(M^2\beta-1),2)$; c) critical exponents; and d) number of Gibbs states at β_c. We will discuss the first issue and related items in some detail, the other issues only briefly.

a) Saturation of $M^2\beta > 1$ at β_c

The prediction (for $q=2$) that $M^2\beta=1$ at β_c is probably contained in [AY]. (See their Eqs. (20) and (20′). The situation is somewhat unclear to us because of the comment that their "Eq. (20) is exactly the same, actually, as the Thouless inequality" – presumably referring to $M^2\beta \geqq \frac{1}{2}$ which is Eq. (8) of [T] or (1.16) with the assumption that $\theta \geqq 1$. The saturation of Thouless' inequality is impossible.) We would like to first explore the consequences of such a saturation result and then motivate the result by discussing its relation to the renormalization group approach in [NS] and to various notions of long range order. One point we wish to bring out in that discussion is that (at least for $q=1$), the "interesting" renormalization part of the saturation result has already been solved leaving only some "technical" questions.

The consequences of saturation coupled with (the renormalized part of) Theorem 1.4 and Theorem 1.7 would be to improve our current results about an intermediate phase in the (β, J_1) plane to results about an intermediate phase in β (for any fixed J_1). In particular, we would have (with J_x fixed for *all* x):

i) $\theta < 2$ in some nonempty inverse temperature interval, $(\beta_c, \hat{\beta})$.

ii) $\theta(\beta) \to 0$ as $\beta \searrow \beta_c$.

iii) $\theta(\beta_c) = 0$.

Statements i) and ii) use the fact that $M(\beta)$ is right-continuous since it is the decreasing limit as $L \to \infty$ of the continuous finite volume quantities $\mu_L^w(0 \leftrightarrow [-L, L]^c)$. These results would be true, in the weak sense of Theorem 1.7 for $q=1$ and 2 with no further assumptions; they would be true in the stronger sense of Theorem 1.4 for any q's for which LLRO occurs whenever $M>0$.

Statement iii) would be a first step in solving the open problem of proving the prediction of [BCRS] that at β_c, $G^T(x) \sim (\ln x)^{-1}$. We remark that the proof of iii) would contain, well hidden in its guts (the main gut being the proof of Lemma 5.2 of [AN 2]), a more explicit upper bound on the decay of $G^T(x)$ at β_c, but one which may be far from the $(\ln x)^{-1}$ prediction.

We now restrict attention to $q=1$ and discuss how the renormalization methods of [NS] come "close" to proving saturation. The basic rescaling argument of [NS] considers disjoint blocks of length L as renormalized sites which are alive (with probability λ') if they contain a large cluster (defined using only bonds within the block) containing a specified fraction $\tilde{M}$ of the sites in the block. One then defines a renormalized independent site-bond percolation model in which λ' is the site parameter and the bond parameters $\beta' J'_x$ are defined by (a worst case estimate of) the probability that two living blocks (separated by $|x|-1$ intervening blocks) have an occupied bond between their two large clusters. A key feature of this definition is that percolation of the renormalized model implies percolation of the original model. The proof that $M>0$ for $\beta>1$ and J_1 large is based on an argument which shows that for J_1 large enough one may prescribe a sequence of L's and $\tilde{M}$'s (with the $\tilde{M}$'s close to 1) so that the iterated λ's tend to 1, the β's stay above and bounded away from 1, the J_x's for $x>2$ essentially approach the couplings (1.33) of the self-similar model and the J_1's are driven to ∞.

Although this type of proof that $\beta^* \leqq 1$ is fairly complicated because it involves an infinite sequence of mappings, we point out that a much simpler application of this method, involving a single mapping, can be used to reduce the problem of proving saturation to a technical question about long range order (namely, that $M > 0$ implies uniform long range order, as defined below). The idea is as follows. To prove saturation it suffices to show that if a model has $M^2(\beta) \cdot \beta > 1$, then β can be slightly lowered while leaving $M > 0$. To do this, it suffices to find *some* L and $\tilde{M}$ (with perhaps a stronger requirement for a block to be alive than the one given above) so that the renormalized model is not only percolating, but is far from critical; in that case β can be lowered while the renormalized (and hence the original) model continues to percolate. Because β' is essentially $\tilde{M}^2\beta$ (more accurately $\beta' J'_x$ is asymptotically $\beta\tilde{M}^2/x^2$ for large x), it will be necessary to

1) choose $\tilde{M}$ arbitrarily close to M in order that $\beta' > 1$ be guaranteed by $M^2(\beta) \cdot \beta > 1$,

2) define "living blocks" in such a way that J'_1 is large for large L.

If all this is done properly, then the theorem that $\beta^* \leqq 1$ will imply that (for large L) the renormalized model is percolating and far from critical. The above two requirements suggest consideration of alternative notions of long range order. For the sake of concreteness, we choose one such notion and make the following definition.

Definition. We shall say that a model has *uniform long range order* if it has a positive order parameter M and furthermore for any $\varepsilon' > 0$ and positive integer K, the probability λ_L, defined as

$$\begin{aligned} \lambda_L = \mu^f_{KL} \text{ (there is a cluster in } [-KL, KL] \text{ whose intersection} \\ \text{with each } [jL, (j+1)L) \text{ subinterval contains} \\ \text{at least } (M-\varepsilon')L \text{ points)} \end{aligned} \tag{1.48}$$

satisfies

$$\lim_{L \to \infty} \lambda_L = 1 . \tag{1.49}$$

The next conclusion can be proved by following the above discussion using blocks of size $2KL$ and defining a living block as one with a cluster of the type described in (1.48). By choosing first ε' small [so that $(M-\varepsilon')^2\beta > 1$], then K large and then L large one can construct a renormalized model well inside its percolative regime.

Conclusion. Suppose $q = 1$ and J_1 is fixed. If at $\beta = \beta_c$, there is uniform long range order [we already know $M(\beta_c) > 0$], then $M^2(\beta_c) \cdot \beta_c = 1$.

b) Validity of $\theta = \min(2(M^2\beta - 1), 2)$

The motivation for this conjecture is essentially as follows. We have already proved (at least for $q = 2, 3, \ldots$) that for $\beta > 1$, $\theta \to \min(2(\beta - 1), 2)$ as $J_1 \to \infty$. On the other hand, the renormalization arguments discussed above (for $q = 1$) suggest that on a large enough scale, the model behaves like a renormalized model with $\beta' = M^2\beta$ and J'_1 arbitrarily large. Thus the conjecture.

One weakness of this argument (as indicated by the discrepancy between the q values in its two parts) is that at the moment our upper bounds on G^T [which give

$\theta \geqq \min(2(\beta-1)-\varepsilon', 2)]$ for large J_1 are derived only within the Ising or Potts spin systems while our best renormalization arguments (which allow a replacement of β by $M^2\beta$) are done within the percolation or random cluster systems. An open problem, whose solution would serve as a first step in remedying this situation, is to extend our upper bounds on G^T to the case of independent percolation. While the Peierls argument has been extended to $q=1$ by Schwerer [Sc], it is unclear how to treat the term $\hat{\tau}$ in the decomposition $G^T=\tau'+\hat{\tau}$. The τ' term should be accessible, however.

c) Critical Exponents

There are various predictions of critical behavior extant. Three of these for the Ising case, coming respectively from [AY, K] and [BCRS], are (we use $\tau^f=\tau^w=\tau$ for $\beta<\beta_c$):

$$M(\beta)-M(\beta_c)\sim(\beta-\beta_c)^{1/2} \quad \text{as} \quad \beta\searrow\beta_c,$$

$$\chi(\beta)\equiv\sum_{x=-\infty}^{\infty}\tau(0,x)\sim\exp(\text{const}(\beta_c-\beta)^{-1/2}) \quad \text{as} \quad \beta\nearrow\beta_c,$$

$$\theta(\beta)\sim(\beta-\beta_c)^{1/2} \quad \text{as} \quad \beta\searrow\beta_c.$$

As pointed out previously, the first and third of these are consistent with the conjectured identity for θ just discussed. We note that the predicted behavior of χ has been studied numerically by a high temperature expansion [Ma].

As far as rigorous results are concerned, we can only mention the following. For $q=1$ or 2, χ is known to diverge and the divergence is at least as fast as $(\beta_c-\beta)^{-1}$ [Si, AN 1]; for $q=1$ the discontinuity in M at β_c implies a divergence at least as fast as $(\beta_c-\beta)^{-2}$ [N]. These results are obviously very weak compared to the prediction.

An interesting open problem concerns the critical behavior for $q\neq 2$. For example, does χ diverge for all q or is there for $q>$ some q^* a first-order transition as $\beta\nearrow\beta_c$ like in nearest neighbor models [KS, M] (where q^* depends on the spacial dimension)? The answer according to Cardy [C] should be divergence, since his extension of Kosterlitz' analysis [K] from $q=2$ to other Potts models predicted that the correlation length (and presumably χ as well) behaves as

$$\exp(\text{const}(\beta_c-\beta)^{-p}) \quad \text{as} \quad \beta\nearrow\beta_c$$

with

$$p=\frac{1}{2}\left\{1-\frac{q-2}{[(q-2)^2+8q]^{1/2}}\right\}.$$

We remark that our results on the intermediate phase show that the transition is not first-order from the low temperature side.

d) Number of Gibbs States at β_c

For independent percolation, it is a general fact [AKN] that when $M>0$, there is a unique infinite cluster – even at β_c. The issue of uniqueness is similar to the Ising model question of proving that there are only two translation invariant pure Gibbs states [L]; in particular nonuniqueness of infinite clusters seems analogous to the existence of infinitely many such states [BL]. It is an open problem for every $q>1$ to resolve either the question of uniqueness of the infinite cluster within the random cluster system or (for $q=2,2,\ldots$) the number of Gibbs states within the spin system. This is especially of interest at $\beta=\beta_c$. For example, related to the issue of divergence of χ is the issue of a distinction between μ^f and μ^w at β_c and whether there are exactly q (or perhaps exactly $q+1$ for $q>q^*$) translation invariant pure Gibbs states of the spin system.

2. Lower Bounds for τ'

As in Subsect. 1.iii) above, we will deal throughout this section with one-dimensional translation invariant parameter-q random cluster models with $q\geqq 1$, $\beta>0$, and $\lim\limits_{x\to\infty} x^2 J_x=1$. However, we remark that the natural analogues of all our results are valid for the more general class of (dependent) one-dimensional bond percolation models considered in Lemma 4.2 of [AN 2] (i.e. those which satisfy the strong FKG conditions, are regular and have $\beta^+<\infty$).

2.i) $\theta\leqq 2(\beta-1)$

The next proposition shows why long long range order is useful in obtaining lower bounds for τ'.

Proposition 2.1. *For any $L\geqq|x|$,*

$$\tau'(0,x)\geqq\tau_L^f(0,x)\cdot\mu^w([-L,L]\leftrightarrow\infty)\,. \tag{2.1}$$

Proof. Proceeding as in our previous proof for self-similar independent percolation [see (1.43)], and defining $\tilde{\Lambda}_L$ to be the set of bonds $\{x,y\}$ with both $|x|,|y|\leqq L$, we have

$$\tau'(0,x)\geqq\mu^w(x\text{ connected to 0 by bonds in }\tilde{\Lambda}_L|[-L,L]\leftrightarrow\infty)\mu^w([-L,L]\leftrightarrow\infty)\,. \tag{2.2}$$

Note that the event $[-L,L]\leftrightarrow\infty$ depends only on bonds not in $\tilde{\Lambda}_L$. Thus, by conditioning further on the configuration $n_{\tilde{\Lambda}_L^c}$ of those bonds (i.e. on the σ-algebra generated by these n_b's), we see that it suffices to show that

$$\mu^w(x\text{ connected to 0 by bonds in }\tilde{\Lambda}_L|n_{\tilde{\Lambda}_L^c})\geqq\tau_L^f(0,x)\,. \tag{2.3}$$

As in the proof of Lemma 3.1 of [ACCN], the precise meaning of the left-hand side of (2.3) is

$$\lim_{k\to\infty}\lim_{L'\to\infty}\mu_{L'}^w(x\text{ connected to 0 by bonds in }\tilde{\Lambda}_L|n_{\tilde{\Lambda}_L^c\cap\Lambda_k})\,.$$

Now by the strong FKG property, for an event F (depending only on bonds in $\tilde{\Lambda}_L$),

$$\mu_{L'}^w(F|n_{\tilde{\Lambda}_L^c\cap\Lambda_k})\geqq\mu_{L'}^w(F|n_{\tilde{\Lambda}_L^c}\equiv 0)=\mu_L^f(F)\,.$$

Taking F to be the event that x is connected to 0 by bonds in $\tilde{\Lambda}_L$ completes the proof.

Proposition 2.2. *If $\beta>1$, then for any $\varepsilon'>0$, there is some $C'>0$ so that*

$$\mu^w([0,L]\leftrightarrow\infty)\geqq C'/L^{2(\beta-1)+\varepsilon'} \quad \textit{for all} \quad L\geqq 1\,. \tag{2.4}$$

Proof. We wish to compare our original μ^w with $q\geqq 1$ to the $q=1$ measure $\hat{\mu}$ of a modified self-similar model in which $\hat{J}_x$ is given by (1.33) for $x>R$ and $\hat{J}_x=J_x$ for $x\leqq R$. This is easily done by first using the monotonicity in q of μ^w (analogous to (1.23) – see [F, ACCN]) and then the monotonicity in the J_x's. Given ε', we choose $\hat{\beta}>\beta$ so that

$$2(\hat{\beta}-1)=2(\beta-1)+\varepsilon'\,, \tag{2.5}$$

and then choose R (depending on ε') sufficiently large so that $\hat{\beta}\hat{J}_x\geqq\beta J_x$ for all $x>R$; this is possible because $\lim x^2 J_x=\lim x^2\hat{J}_x=1$. We then have

$$\mu^w([0,L]\leftrightarrow\infty)\geqq\hat{\mu}([0,L]\leftrightarrow\infty)\,. \tag{2.6}$$

Moreover, exactly as in Proposition 1.6,

$$\hat{\mu}([0,L]\leftrightarrow\infty)\geqq\mathrm{const}[\hat{\mu}(F_{L,2})]^2\,. \tag{2.7}$$

Let us define

$$\begin{aligned}\hat{F}_{L,2}=\{&\text{for some integer } \xi\in[L,2L) \text{ every bond of length}\\ &\text{longer than } R \text{ between } [0,\xi] \text{ and } [\xi+1,\infty) \text{ is vacant}\}\,.\end{aligned} \tag{2.8}$$

Since $\hat{F}_{L,2}$ does not involve the short bonds which distinguish between $\hat{\mu}$ and the self-similar model measure $P_{\hat{\beta}}$, we have

$$\hat{\mu}(\hat{F}_{L,2})=P_{\hat{\beta}}(\hat{F}_{L,2})\geqq P_{\hat{\beta}}(F_{L,2})\geqq C/L^{\hat{\beta}-1}\,, \tag{2.9}$$

where we use Proposition 1.6.

On the other hand, by a simple conditioning argument

$$\begin{aligned}\hat{\mu}(F_{L,2})&=\hat{\mu}(\hat{F}_{L,2})\cdot\hat{\mu}(F_{L,2}|\hat{F}_{L,2})\geqq\left(\prod_{x=\xi-R+1}^{\xi}\ \prod_{y=\xi+1}^{\xi+R} e^{-\beta J_{y-x}}\right)\cdot\hat{\mu}(\hat{F}_{L,2})\\ &=\mathrm{const}\,\hat{\mu}(\hat{F}_{L,2})\,.\end{aligned} \tag{2.10}$$

Putting all this together yields (2.4) as desired.

2.ii) $\theta\leqq 2(M^2\beta-1)$

In this subsection, we give the renormalized version of Proposition 2.2. Following Sect. 4 of [AN 2], we first introduce the notion of H-anchored bonds and then point out some facts which suggest how this notion leads to a replacement of β by $M^2\beta$.

Definitions. We shall say that there is an (occupied) *H-anchored bond* $\{x,y\}$ if

a) the bond $\{x,y\}$ is occupied, and

b) x is doubly connected to $[x-H,x+H]^c$, and y is doubly connected to $[y-H,y+H]^c$. (Here we say x is doubly connected to a set S if there are two disjoint paths of occupied bonds leading from x to S.)

We also define M_H by

$$M_H=\sup_{\{m\}} \mu^w(0\leftrightarrow[-H,H]^c|n_b=m_b \text{ for all } b=\{x,y\} \text{ with } |x|,|y|>H). \quad (2.11)$$

Two key facts about these definitions are (see Lemma 3.1 of [ACCN] and Lemma 4.1 of [AN 2]):

i) $$M_H=\mu_H^w(0\leftrightarrow[-H,H]^c)\to M \quad \text{as} \quad H\to\infty, \quad (2.12)$$

ii) for any $\varepsilon''>0$, μ^w (there is an H-anchored bond $\{x,y\}$)

$$\leqq(\beta M_H^2+\varepsilon'')/|x-y|^2, \quad \text{for all large } |x-y|. \quad (2.13)$$

In the next proposition, the role of a dissociated interval is replaced by its H-anchored bond analogue.

Proposition 2.3. *If $M>0$, then for any $\varepsilon'>0$, there is some $C'>0$ so that*

$$\mu^w([0,L]\not\leftrightarrow\infty)\geqq C'/L^{2(M^2\beta-1)+\varepsilon'} \quad \textit{for all} \quad L\geqq 1. \quad (2.14)$$

Proof. Given ε', we choose $\beta'>M^2\beta$ by

$$2(\beta'-1)=2(M^2\beta-1)+\varepsilon', \quad (2.15)$$

and then [by (2.12)] choose H so large that

$$\beta'>M_H^2\beta. \quad (2.16)$$

Now $[0,L]$ will be disconnected from ∞ if there is no occupied (ordinary) bond from $[-H,L+H]$ to $[-L,2L]^c$ and there is no path of occupied H-anchored bonds connecting $[-L,2L]$ to ∞. The latter will be the case if there is an "H-dissociated interval" containing $[-L,2L]$ – i.e. an interval $[\xi',\xi]$ with $\xi'\leqq-L$ and $\xi\geqq L$ with no occupied H-anchored bond connecting $[\xi',\xi]$ to its complement. Thus, by the FKG inequalities,

$$\begin{aligned}\mu^w([0,L]\not\leftrightarrow\infty)\geqq\mu^w &\text{ (there is no occupied bond from}\\ &[-H,L+H] \text{ to } [-L,2L]^c)\\ &\times\mu^w \text{ ([0,3L] is contained in}\\ &\text{some larger } H\text{-dissociated interval)}. \end{aligned} \quad (2.17)$$

We claim that the first factor on the right-hand side of (2.17) is bounded away from zero as $L\to\infty$; this can be seen by combining an argument like that used for Proposition 1.3 with a calculation like that used in Proposition 1.6 to show that $P_\beta(H_L)$ is bounded away from zero. Then, as in Propositions 1.6 and 2.2, the second factor can be handled by (aging using FKG inequalities)

$$\begin{aligned}&\mu^w([0,L] \text{ is contained in some larger } H\text{-dissociated interval})\\ &\geqq\mu^w(\tilde{F}_{L,2}\cap\tilde{F}^*_{L,2}\cap H_L\cap H^*_L)\\ &\geqq[\mu^w(H_L)\mu^w(\tilde{F}_{L,2})]^2\geqq\text{const}[\mu^w(\tilde{F}_{L,2})]^2,\end{aligned}$$

where $\tilde{F}_{L,2}$ is the H-anchored bond analogue of $F_{L,2}$.

It remains to obtain an appropriate lower bond on $\mu^w(\tilde{F}_{L,2})$. But Lemma 4.2 of [AN 2] gives precisely the estimate

$$\mu^w(\tilde{F}_{L,2})\geqq\text{const}\,P_{\beta'}(F_{L,2}) \quad (2.19)$$

comparing μ^w to the self-similar $q=1$ measure $P_{\beta'}$. The desired lower bound now follows from (2.15) and (1.34).

2.iii) Use of Hammersley-Lieb-Simon Type Inequalities

Proposition 2.3 gives a good lower bound on the $L\to\infty$ behavior of $\mu^w([-L,L]\nleftrightarrow\infty)$ whenever $M>0$. The reason it was necessary in our main result, Theorem 1.4, to make the stronger assumption of LLRO, was in order to control the other term, $\tau_L^f(0,x)$, appearing in Proposition 2.1. Since, at the moment there does not exist any rigorous result that $M>0$ implies LLRO, we will give an estimate on $\tau_L^f(0,x)$ which only requires the weaker assumption of $M>0$.

Our estimate on τ_L^f is based on Hammersley-Lieb-Simon type inequalities, which were originally derived in [H, Li, Si] for nearest neighbor models and extended in [FS, AN 1, A] to general long range models. These inequalities are only valid for $q=1$ or 2 and hence our estimates only apply to these two cases. For either $q=1$ or 2, and Λ any finite subset of sites (e.g. $[-L,L]$), the following inequalities are valid for $x\in\Lambda$ and $y\notin\Lambda$:

$$\tau^w(x,y)\leqq\sum_{u\in\Lambda,v\notin\Lambda}\tau_\Lambda^f(x,u)\beta J_{u,v}\tau^w(v,y).\tag{2.20}$$

Here $\tau_\Lambda^f(x,u)$ is the free boundary condition probability that $x\leftrightarrow u$, while

$$\begin{aligned}\tau^w(x,y)&=\lim_{L\to\infty}\mu_L^w(x\leftrightarrow y)=\tau'(x,y)+\mu^w(x\leftrightarrow\infty \text{ and } y\leftrightarrow\infty)\\&=\begin{cases}\mu^w(x\leftrightarrow y), & \text{for}\quad q=1,\\ \langle\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_y\rangle_1(=\langle S_xS_y\rangle_+), & \text{for}\quad q=2.\end{cases}\end{aligned}\tag{2.21}$$

(As stated, these inequalities involve slight variations and special cases of Eqs. (5.17) of [AN 1] and (I.1) of [A].) The inequality (2.20) is very similar to the one analyzed in Sect. 5 of [AN 2] and we will essentially copy some of that analysis here. The next proposition is our basic conclusion; stronger (but messier) statements can be given.

Proposition 2.4. *Suppose $q=1$ or 2. If $M>0$, then for some $\varepsilon>0$ and $C>0$, the following is true for all large L:*

$$\tau_L^f(0,x)>\frac{C}{1+\ln x}\quad \textit{for some } x \textit{ in } [\varepsilon L,L].$$

Proof. If the conclusion of the proposition fails, it must be the case that for every ε and C, and then for infinitely many choices of L,

$$\tau_L^f(0,x)\leqq\frac{C}{1+\ln x}\quad\text{for all } x \text{ in } [\varepsilon L,L].\tag{2.22}$$

Proceeding as in Lemma 5.4 of [AN 2], we first note that if this is the case, then

$$\liminf_{L\to\infty}\alpha_L=0,\tag{2.23}$$

where

$$\alpha_L=\sum_{|u|\leqq L}\sum_{|v|>L}\tau_L^f(0,u)\cdot\beta J_{u,v}.\tag{2.24}$$

To see this, we bound α_L by

$$\begin{aligned}\alpha_L &\leqq \text{const}/L + \text{const} \sum_{0<|u|<L} \sum_{|v|>L} \tau_L^f(0,u)/|v-u|^2 + \text{const}\,\tau_L^f(0,L)\\ &\leqq K\left(L^{-1} + \int_0^{\varepsilon L} [(L-u)^{-1} + (L+u)^{-1}]du\right.\\ &\quad \left. + \int_{\varepsilon L}^{L-1} \frac{C}{1+\ln u}[(L-u)^{-1} + (L+u)^{-1}]du + \frac{C}{1+\ln L}\right)\\ &\to K\left(\int_0^{\varepsilon} [(1-x)^{-1} + (1+x)^{-1}]dx + C\right) \quad \text{as} \quad L\to\infty\,,\end{aligned} \tag{2.25}$$

where K does not depend on the choice of C or ε [and the last limit should be taken through the subsequence of L's for which (2.22) is valid]. Since C and ε may be chosen arbitrarily small, (2.23) follows.

Next we show (proceeding as in Lemma 5.1 of [AN 2]) that (2.23) would imply that for some $\varepsilon'>0$,

$$\sum_x \tau^w(0,x)\,|x|^{\varepsilon'} < \infty\,. \tag{2.26}$$

But this would contradict $M>0$ (and hence complete the proof of Proposition 2.4), since $\tau^w(0,x) \geqq M^2$ by the last statement of Proposition 1.1. To obtain (2.26), we first note (as in [AN 2]) that (2.23) implies that for some L, $\alpha_L<1$ and then for some small $\varepsilon'>0$,

$$\alpha_L(\varepsilon') \equiv \sum_{|u|\leqq L} \sum_{|v|>L} \tau_L^f(0,u)\beta J_{u,v} e^{\varepsilon' d(v)} < 1\,, \tag{2.27}$$

where $d(y-x)=\ln(|x-y|+1)$ is a metric. For x and y with $|y-x|>L$, (2.20) and the triangle inequality for $d(\cdot)$ imply

$$\tau^w(x,y)e^{\varepsilon' d(y-x)} \leqq \sum_{\substack{|u-x|\leqq L\\ |v-x|>L}} \tau^f_{[x-L,x+L]}(x,u)\beta J_{u,v} e^{\varepsilon' d(v-x)} \tau^w(v,y) e^{\varepsilon' d(y-v)}\,.$$

We sum this over y's in $[-M,M]$ and define

$$\chi_{\varepsilon'}^M = \max_{x\in(-\infty,\infty)} \sum_{|y|\leqq M} \tau^w(x,y)e^{\varepsilon'\cdot d(y-x)}\,.$$

(We remark that the max in (5.10) of [AN 2] should have been over x in $(-\infty,\infty)$ and the sums in (5.12)–(5.13) there should have been over $|x|\leqq L$.) We obtain

$$\chi_{\varepsilon'}^M \leqq \sum_{|x|\leqq L} \tau^w(0,x)e^{\varepsilon' d(x)} + \chi_{\varepsilon'}^M \cdot \alpha_L(\varepsilon')\,,$$

or equivalently [where we use (2.27)],

$$\chi_{\varepsilon'}^M \leqq [1-\alpha_L(\varepsilon')]^{-1} \sum_{|x|\leqq L} \tau^w(0,x)e^{\varepsilon' d(x)}\,. \tag{2.28}$$

Since this bound is independent of M, (2.26) follows by letting $M\to\infty$.

3. Long Range Order for Ising and Potts Models

3.i) Spontaneous Magnetization: $\beta^* \leqq 1$

In this section we consider Ising or Potts models with integer $q\geqq 2$, at inverse temperature $\beta>1$. We assume $\lim_{x\to\infty} x^2 J_x = 1$ and take J_1 sufficiently large, depending on q and β, with $J_1 \to \infty$ as $\beta \searrow 1$ or as $q \to \infty$. Our first result concerns the spontaneous magnetization in the 1 state in a finite interval $\Lambda = [-L, L]$. We then consider the existence of long long range order with free boundary conditions.

Theorem 3.1. *For any $\beta > 1$ and integer $q \geqq 2$, let J_1 be sufficiently large. Then the estimate*

$$\langle 1 - \delta_{\sigma_0, 1} \rangle_1^L \leqq e^{-O(J_1)} \tag{3.1}$$

holds uniformly in L. Hence M, as defined in (1.3), *is strictly positive.*

Proof. The proof is closely related to the proof by Fröhlich and Spencer of spontaneous magnetization for large β in the Ising case [FS]. It is a generalized Peierls argument balancing entropy and energy for a carefully defined class of "connected" contours. Entropy and energy estimates have to be done with great care since we are working just below the temperature at which entropy begins to dominate energy.

The reason why $\beta = 1$ is the borderline for the possibility of spontaneous magnetization can be seen already at the level of simple spin flip pairs, or "dipoles." Suppose there is just one such pair with a separation between L and $2L$. The energy can easily be calculated approximately as $2 \ln L + O(J_1)$. Hence the Boltzman weight is approximately $L^{-2\beta} e^{-O(J_1)}$. To account for entropy, we count the number of such pairs that could "surround" the origin-approximately $(q-1)L^2$. Hence for $\beta > 1$ and large J_1, energy dominates entropy, and a Peierls-type argument can succeed.

We begin proving Theorem 3.1 by defining the contours we will be working with. For the convenience of the proof, we use addition modulo q on spin values, with q being identified with 0. In the Potts models, a *contour* is a collection of *spin flips* $\{f_1, \ldots, f_n\}$. Each spin flip specifies a bond in $\mathbb{Z}^*$ (a nearest neighbor pair of sites in $\mathbb{Z}$) and a *charge* in $\mathbb{Z}_q$ associated with that bond. The spin to the right of a flip is equal to the spin to the left plus the charge. In the Ising case two spin flips always cancel, so that spins to the right of such a pair equal the spins to the left. In the Potts case several flips may be necessary. A contour is called *neutral* if the sum of the charges of its spin flips is 0 (mod$q$). If a contour is not neutral it is said to be charged. For any contour γ we let $|\gamma|$ denote the number of spin flips in γ. Geometrically, we think of each flip in γ as the midpoint of the bond it specifies, and this leads to notions of the diameter of γ [denoted by $d(\gamma)$] and the distance between contours [denoted $\mathrm{dist}(\gamma_\mu, \gamma_\nu)$].

Following [I], we say a contour γ is *irreducible* if

A) γ is neutral.

B) There is no decomposition of γ into subsets, $\gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n$, such that each γ_μ is neutral and such that $\mathrm{dist}(\gamma_\mu, \gamma_\nu) \geqq m(\min\{d(\gamma_\mu), d(\gamma_\nu)\})^\kappa$.

Here κ, m are constants to be chosen later; we will require that $1 < \kappa < \beta$ and that $m \to \infty$ as β or $\kappa \to 1$. We will find it convenient to choose a small $\varepsilon > 0$ and let the other constants depend on ε. Thus we make choices in the following order: β, ε, κ, m, J_1. For example, we require $m > D_0$, where D_0 is large enough so that $|x^2 J_x - 1| < \varepsilon/3$ for $|x| \geqq D_0$.

The following two facts can be proven as in [I].

Proposition 3.2. *There is a unique way of partitioning the spin flips of any configuration into irreducible contours* γ_μ *such that*

C) $$\operatorname{dist}(\gamma_\mu, \gamma_\nu) \geqq m(\min\{d(\gamma_\mu), d(\gamma_\nu)\})^\kappa .$$

Proposition 3.3. *If* γ *is an irreducible contour and* $\varrho \subset \gamma$, $\varrho \neq \gamma$, *then*
D) $\operatorname{dist}(\varrho, \gamma \backslash \varrho) \geqq 2m(d(\varrho))^\kappa$ *implies that* ϱ *is charged.*

The Peierls argument proceeds as follows. By Proposition 3.2 any spin configuration determines uniquely a collection of irreducible contours $\gamma_1, \ldots, \gamma_n$ satisfying C. Each γ_i also satisfies A, B, and D. The observable $1-\delta_{\sigma_0, 1}$ in (3.1) is nonzero only if at least one of the γ_i's surrounds the origin. (We say γ surrounds a site if the part of γ to the left of the site is charged, so that the spin at the site is different from what it would have been without γ.) When $\sigma_0 \neq 1$, we can therefore assume that γ_1, say, has the largest diameter of the γ_i's which surround the origin. We write $H(\gamma_1)$ for the energy of γ_1, that is $H(\gamma_1) = \mathscr{H}(\sigma)$, where the spin configuration σ has precisely γ_1 as its collection of spin flips. With $\Gamma = \gamma_2 \cup \ldots \cup \gamma_n$, we define analogously $H(\Gamma)$. We can use the proof of [FS] to show that γ_1 interacts weakly with Γ, in the sense that

$$0 \leqq H(\gamma_1) + H(\Gamma) - H(\gamma_1 \cup \Gamma) \leqq \frac{c}{m}(\ln m)^3 H(\gamma_1) . \tag{3.2}$$

This implies that for m large,

$$\beta H(\gamma_1 \cup \Gamma) \geqq \beta H(\Gamma) + \beta_1 H(\gamma_1) , \tag{3.3}$$

where $\beta_1 = \beta - \varepsilon$. We choose ε so that β_1 is larger than 1 and we are still able to control entropy with energy. We estimate

$$\begin{aligned} \langle 1 - \delta_{\sigma_0, 1} \rangle_1 &\leqq \frac{1}{Z_1(\Lambda)} \sum_{\sigma|_\Lambda, \sigma_0 \neq 1} e^{-\beta H(\Gamma)} e^{-\beta_1 H(\gamma_1)} \\ &\leqq \sum_{\gamma_1 \text{ surrounding } 0} e^{-\beta_1 H(\gamma_1)} \frac{1}{Z_1(\Lambda)} \sum_{\Gamma \text{ compatible with } \gamma_1} e^{-\beta H(\Gamma)} . \end{aligned} \tag{3.4}$$

Dropping the constraints on the sum on Γ, the partition function $Z_1(\Lambda)$ is cancelled, and we are reduced to proving the Peierls estimate,

$$\sum_{\gamma_1 \text{ surrounding } 0} e^{-\beta_1 H(\gamma_1)} \leqq e^{-O(J_1)} . \tag{3.5}$$

Alternatively, we consider γ_1 to "start" at its left-most spin flip f, and show that

$$\sum_{\gamma \text{ starting at } f, d(\gamma) \geqq D} e^{-\beta_1 H(\gamma)} \leqq e^{-O(J_1)} D^{-(2\beta_2 - 1) + \varepsilon} , \tag{3.6}$$

with $\beta_2 = \beta_1(1-\varepsilon) > 1$. Since $d(\gamma_1) \geqq \operatorname{dist}(0, f)$ for γ_1 surrounding 0, we are able to sum over f and obtain (3.5) from (3.6) (again for small enough ε).

In order to organize this estimate properly, we describe the connectedness of γ on a sequence of length scales. The starting scale d_0 must be sufficiently large; $d_0 = m^{2/(\kappa-1)}$ is large enough. Then with $\alpha = \kappa^2$, we define inductively

$$d_k = d_{k-1}^\alpha = (d_0)^{\alpha^k}, \quad k = 1, 2, \ldots .$$

A set of spin flips is called k-connected if the distance between successive spin flips is $\leqq d_k$. The contour γ decomposes into 0-connected components $\{\gamma_\mu^{(0)}\}$. Components $\gamma_\mu^{(0)}$ separated by no more than a distance d_1 are united to form the 1-connected components, $\{\gamma_\mu^{(1)}\}$, of γ. This continues until at some scale γ is a single component.

We sum over γ in (3.6) as follows. We begin by assuming that the first flip of some $\gamma_\mu^{(0)}$ is fixed in space, as we estimate the sum over $|\gamma_\mu^{(0)}|$ and over the positions of the remaining sites in $\gamma_\mu^{(0)}$. We use pairs i,j with $|i-j|<d_0$ to produce the energy needed to control these sums. [This portion of $H(\gamma)$ will be denoted $H^{(0)}(\gamma)$.] Actually, only the nearest neighbor pairs with large coupling J_1 are needed. This is important because only beyond the scale $D_0<d_0$ do the couplings approximate their asymptotic behavior sufficiently well. Proceeding to larger scales of structure, we understand sums over 1-connected components $\gamma_\mu^{(1)}$, again with the first flip fixed. Here we sum over the first 0-connected subcomponent, then sum over the position of the first flip in the next 0-connected subcomponent, and so on through the sum over the last subcomponent. We use $H^{(1)}(\gamma)$, the energy arising from pairs with $d_0 \leqq |i-j| < d_1$, in this part of the estimate. Inductively, the estimates on k-connected components use what has already been proven for $(k-1)$-connected components. Since γ must be K-connected for some K, these estimates give us control over (3.6).

Consider a 0-connected component $\gamma_\mu^{(0)}$. Each spin flip comes with a weight less than $e^{-J_1\beta_1}<e^{-J_1}$ from the nearest-neighbor bonds. There are $q-1$ possible charges for each spin flip. Summation over $|\gamma_\mu^{(0)}|$ is estimated with a combinatoric factor $2^{|\gamma_\mu^{(0)}|}$. (A combinatoric factor is the factor C_T in the estimate $\sum_T f(T) \leqq \sup_T C_T f(T)$, valid when $\sum_T C_T^{-1} \leqq 1$.) There are at most $d_0 = m^{2/(\kappa-1)}$ positions for each successive spin flip. All these factors are controlled by a small power of the "bare activity" e^{-J_1}. Gathering the factors in each $\gamma_\alpha^{(0)}$, we obtain a bound

$$\sum_{\gamma_\mu^{(0)},\,\text{first flip fixed}} \exp(-\beta_1 H^{(0)}(\gamma_\mu^{(0)})) \leqq \sup_{\gamma_\mu^{(0)}} (e^{-J_1|\gamma_\mu^{(0)}|/2} d_0(\gamma_\mu^{(0)})^{-\beta_2}). \tag{3.7}$$

Here we define

$$d_k(\varrho) = d_k \exp(d(\varrho)/d_k) \geqq d(\varrho) \tag{3.8}$$

for any collection of spin flips ϱ. Since $|\gamma_\mu^{(0)}| \geqq d(\gamma_\mu^{(0)})/m$ we have that

$$\exp(-J_1|\gamma_\mu^{(0)}|/4) < d_0(\gamma_\mu^{(0)})^{-\beta_2}$$

for J_1 large. This justifies the final factor in (3.7). At the k^{th} level, we will find activities $d_k(\gamma_\mu^{(k)})^{-\beta_2}$, which essentially give an inverse power of the length scale, d_k. Unusually large components incur an exponential decay on the scale d_k.

Let us assume inductively that an estimate like (3.7) has been proven for $(k-1)$-connected components:

$$\begin{aligned} &\sum_{\gamma_\mu^{(k-1)},\,\text{first flip fixed}} \exp\left[-\beta_1 \sum_{j=0}^{k-1} H^{(j)}(\gamma_\mu^{(k-1)})\right] \\ &\leqq \sup_{\gamma_\mu^{(k-1)}} \exp(-J_1|\gamma_\mu^{(k-1)}|/2) d_{k-1}(\gamma_\mu^{(k-1)})^{-\beta_2}. \end{aligned} \tag{3.9}$$

It is worth noting that in each estimate as in (3.7), (3.9), we use only the part of $H^{(j)}(\gamma)$ that is associated with the component $\gamma_\mu^{(k-1)}$. We use the fact that

$$H^{(j)}(\gamma)=\sum_\mu H^{(j)}(\gamma_\mu^{(j)}),$$

which holds because bonds in $H^{(j)}$ are shorter than d_j, while the subcomponents $\gamma_\mu^{(j)}$ are separated by at least a distance d_j.

To obtain (3.9) with $k-1$ replaced with k, we apply (3.9) successively to each $(k-1)$-connected subcomponent of $\gamma_\sigma^{(k)}$. Once one such subcomponent is fixed, there are no more than d_k choices for the position of the first flip in the next subcomponent. To fix the number of $(k-1)$-connected subcomponents of $\gamma_\sigma^{(k)}$, we need a combinatoric factor 2^N, where N is the number of subcomponents. At this point we must consider several cases.

Case 1. Let $N\geqq 2$. The combinatoric factors are bounded by $(4d_k)^{N-1}$. We bound one of these factors and two factors of $d_{k-1}^{-\beta_2^2}$ from (3.9) by taking $\alpha=\kappa^2$ sufficiently close to 1 and then m sufficiently large:

$$d_{k-1}^{-\beta_2^2}(4d_k)d_{k-1}^{-\beta_2^2}\leqq e^{-\beta_2}d_k^{-(2\beta_2-1)+\varepsilon}\leqq e^{-\beta_2}d_k^{-\beta_2}. \tag{3.10}$$

Here on the right we have the correct power of d_k as required for (3.9), while the center term gives a slightly better bound to be used later. The remaining factors can be paired up, and we use $4d_kd_{k-1}^{-\beta_2^2}\leqq e^{-\beta_2}$. It remains only for us to prove the weak exponential decay. We use the factors of $e^{-\beta_2}$ to obtain a decrease

$$e^{-\beta_2(N-1)}\leqq\exp\left[-\beta_2(d(\gamma_\sigma^{(k)})-\sum_\mu d(\gamma_\mu^{(k-1)}))/d_k\right].$$

When supplemented with the decrease in subcomponent size,

$$\exp(-\beta_2 d(\gamma_\mu^{(k-1)})/d_{k-1}),$$

we obtain the desired decay, $\exp(-\beta_2 d(\gamma_\sigma^{(k)})/d_k)$.

Case 2. Here $N=1$, and furthermore $d(\gamma_\sigma^{(k)})>d_{k-1}^{\alpha\varepsilon/2}$. There is a combinatoric factor of 2, and our bound follows using

$$2d_{k-1}^{-\beta_2^2}\exp(-\tfrac{1}{2}\beta_2 d_{k-1}^{\alpha\varepsilon/2-1})\leqq d_k^{-\beta_2}. \tag{3.11}$$

In both Case 2 and Case 3, the exponential decrease goes through the induction unchanged, since the decay requirement becomes weaker.

Case 3. With $N=1$ and $d(\gamma_\sigma^{(k)})\leqq d_{k-1}^{\alpha\varepsilon/2}$ we need to make use of new energy factors in $H^{(k)}(\gamma_\sigma^{(k)})$. Since $\gamma_\sigma^{(k)}$ is a k-connected component, it satisfies

$$\operatorname{dist}(\gamma_\sigma^{(k)},\gamma\backslash\gamma_\sigma^{(k)})>d_k=d_{k-1}^\alpha.$$

Now with $\alpha=\kappa^2$, $d_0=m^{2/(\kappa-1)}$, and m sufficiently large, we have

$$d_{k-1}^\alpha>2m(d_{k-1})^{\kappa\alpha^{\varepsilon/2}}\geqq 2m(d(\gamma_\sigma^{(k)}))^\kappa,$$

and so by condition D, such isolated subsets of γ must be charged. We extract energy from pairs of sites $i<j$ straddling $\gamma_\sigma^{(k)}$, but limit ourselves to those with $d_{k-1}^{\alpha\varepsilon/2}\leqq|i-j|<d_k$. [We could have used bonds down to d_{k-1}, but not if $d(\gamma_\sigma^{(k)})>d_{k-1}$. We do not use bonds longer than d_k because they may interfere with $\gamma\backslash\gamma_\sigma^{(k)}$.] It is easy to verify that

$$\sum_{i \leqq 0,\, j \geqq d_{k-1}^{\alpha^{\varepsilon/2}};\, |i-j|<d_k} |i-j|^{-2} \geqq \log(d_k/d_{k-1}^{\alpha^{\varepsilon/2}}) - O(1).$$

Note that for these bonds $|x^2 J_x - 1| < \frac{\varepsilon}{3}$ so that

$$\beta_1 J_{i-j} \geqq \frac{\beta_1\left(1-\frac{\varepsilon}{3}\right)}{|i-j|^2} \geqq \frac{\beta_2\left(1+\frac{2\varepsilon}{3}\right)}{|i-j|^2}.$$

Thus we have a Gibbs factor

$$e^{O(\beta_1)} \exp\left(-\beta_2\left(1+\frac{2\varepsilon}{3}\right)(1-\alpha^{\varepsilon/2-1})\log d_k\right) \leqq \frac{1}{2}\exp(-\beta_2(1-\alpha^{-1})\log d_k)$$
$$= \frac{1}{2}(d_k/d_{k-1})^{-\beta_2}. \tag{3.12}$$

In the first inequality we use the fact that $(1-\alpha^{\varepsilon/2-1})/(1-\alpha^{-1}) \sim 1-\varepsilon/2$ for small $\alpha-1$. The Gibbs factor in (3.12) boosts the inductive estimate from $d_{k-1}^{-\beta_2}$ to $d_k^{-\beta_2}$.

Altogether, we have in each case an estimate as in the right-hand side of (3.9), and the induction step is complete. To obtain (3.6), let K be the smallest integer such that γ is K-connected. Using (3.9) we can estimate

$$\sum_{\gamma \text{ starting at } f,\, d(\gamma) \geqq D} e^{-\beta_1 H(\gamma)} \leqq \sum_K \sup_{\gamma : d(\gamma) \geqq D} e^{-J_1|\gamma|/2} d_K(\gamma)^{-\beta_2}$$
$$\leqq \sum_K e^{-O(J_1)} d_K^{-\beta_2} \exp(-\beta_2 D/d_K).$$

We can improve this a bit using the fact that in the last step we must have had $N \geqq 2$, so we can use the improved center estimate in (3.10). Thus $d_K^{-\beta_2}$ can be replaced with $d_K^{-(2\beta_2-1)+\varepsilon}$. The sum over K with $d_K \geqq D$ is controlled by this factor, giving a bound $O(1)e^{-O(J_1)}D^{-(2\beta_2-1)+\varepsilon}$. The sum over K with $d_{K+1} < D$ is controlled by $\exp(-\beta_2 D/d_K)$, which decreases very rapidly as K decreases. It is easy to see that the one remaining term is similarly bounded and (3.6) then follows. This completes the proof of Theorem 3.1.

3.ii) Long Long Range Order

We now consider free boundary conditions, dropping all bonds between $\Lambda = [-L, L]$ and Λ^c. We establish the following theorem:

Theorem 3.4. *For any $\beta > 1$ and integer $q \geqq 2$, let J_1 be sufficiently large. Then for all L and all $x, y \in \Lambda$,*

$$\langle 1 - \delta_{\sigma_x, \sigma_y} \rangle_f^L \leqq e^{-O(J_1)}. \tag{3.13}$$

Hence $\tau_L^f(x, y) = \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle_f^L$, as given in (1.2), *is uniformly strictly positive, and long long range order holds.*

Proof. This result is closely related to Theorem 3.1, and we obtain (3.13) by making the needed modifications in our arguments above. The new difficulties arise from the loss of energy from bonds between Λ and Λ^c.

First of all, we fix the first spin in Λ to be 1, say, so that spin configurations are again in one-to-one correspondence with collections of spin flips. The numerator and denominator in $\langle\delta_{\sigma_x,\sigma_y}\rangle_f^L$ acquire factors of q, which cancel. Secondly, before defining the irreducible contours of each collection of spin flips in Λ, we append the two bonds in $\partial\Lambda$ to the collection of spin flips. These bonds are included as geometrical aids only, and we do not assign any charge to them. Accordingly, when referring to a contour as neutral or charged, it is implicit that the contour does not contain boundary bonds. If the contour γ contains a bond in $\partial\Lambda$, then its diameter $d(\gamma)$ reflects the presence of that bond. However, since $|\gamma|$ should measure the energy from nearest neighbor pairs in γ, we define $|\gamma|$ without including bonds in $\partial\Lambda$.

We must now consider two types of irreducible contours – those containing bonds in $\partial\Lambda$ and those that do not. If a contour does not contain boundary bonds, then irreducibility is defined as before using conditions A and B. When boundary bonds are in a contour γ we use the following conditions:

B′. A contour γ containing bonds in $\partial\Lambda$ is *irreducible* if there is no neutral subset γ_1 of γ such that

$$\operatorname{dist}(\gamma_1,\gamma\backslash\gamma_1)\geqq m(d(\gamma_1))^\kappa .$$

In addition, if γ contains both the left and right boundary bonds b_L, b_R, then there should be no decomposition $\gamma=\gamma_L\cup\gamma_R$ with $b_L\in\gamma_L$, $b_R\in\gamma_R$ and such that

$$\operatorname{dist}(\gamma_L,\gamma_R)\geqq m(\max\{d(\gamma_L),d(\gamma_R)\})^\kappa .$$

Now any spin configurations can be decomposed uniquely into irreducible contours γ_μ satisfying C above, as well as

C′. Let γ_μ contain at least one boundary bond, and let γ_ν be any other irreducible contour. Then

$$\operatorname{dist}(\gamma_\mu,\gamma_\nu)\geqq m(d(\gamma_\nu))^\kappa .$$

This is accomplished by successively breaking apart contours until they are irreducible; conditions B, B' guarantee that this can always be done. Furthermore, Proposition 3.3 has to be modified by replacing the conclusion with

D′. $\operatorname{dist}(\varrho,\gamma\backslash\varrho)\geqq 2m(d(\varrho))^\kappa$ implies that either ϱ is charged or that ϱ contains a bond in $\partial\Lambda$.

Isolated subcontours not containing boundary bonds must be charged, or else they would violate B or B'.

Proceeding to the Peierls argument, we need to show that $\langle 1-\delta_{\sigma_x,\sigma_y}\rangle_f^L$ is small. But in order for σ_x to differ from σ_y, there must be at least one irreducible contour that surrounds x or y. We let γ_1 be the longest such contour.

We need to check (3.2) for γ_1 (weak interaction of γ_1 with the other irreducible contours) in the case of free boundary conditions. We defer this analysis for the moment. The Peierls argument can be applied as before to reduce the problem to the following estimate:

$$\sum_{\gamma_1 \text{ surrounding } x \text{ or } y} e^{-\beta_1 H(\gamma_1)} \leqq e^{-O(J_1)}, \tag{3.14}$$

where $H(\gamma_1)$ refers to the energy of γ_1 with free boundary conditions on Λ.

For contours γ_1 not involving $\partial\Lambda$, this estimate is the same as (3.5), applied separately to x and y. Note that in proving (3.5), we used only bonds of length $\leqq d_k$, where d_k is the scale at which γ_1 becomes k-connected. None of these bonds cross $\partial\Lambda$ (if one did, then part of $\partial\Lambda$ would have become part of γ_1). Thus the proof of (3.5) applies here as well.

We now prove (3.14) for γ_1 containing one or the other bond in $\partial\Lambda$. Let us take the left boundary bond, b_L; the other case is identical. For $(k-1)$-connected components containing b_L, we can prove the following inductive estimate, instead of (3.9):

$$\begin{aligned}\sum_{\gamma_\mu^{(k-1)}\ni b_L} &\exp\left[-\beta_1\sum_{j=0}^{k-1} H^{(j)}(\gamma_\mu^{(k-1)})\right] \\ &\leqq \sup_{\gamma_\mu^{(k-1)}\ni b_L}[2^k\exp(-J_1|\gamma_\mu^{(k-1)}|/2)d(\gamma_\mu^{(k-1)})^{-\beta_2(1-\alpha^{-1})} \\ &\quad\times\exp(-\beta_2 d(\gamma_\mu^{(k-1)})/d_{k-1})]\,.\end{aligned} \tag{3.15}$$

Here the exponential decay on the scale d_{k-1} is as before, but we find only a small inverse power of $d(\gamma_\mu^{(k-1)})$. This diameter may in fact be much smaller than d_{k-1}. We easily obtain (3.15) for 0-connected contours $\gamma_\mu^{(0)}$. [If γ consists only of b_L, then we interpret $d(\gamma)=1$.] The factor of 2 reflects the combinatoric factor $2^{|\gamma_\mu^{(0)}|+1}$ needed for the sum over $|\gamma_\mu^{(0)}|$.

To obtain (3.15) with $k-1$ replaced by k, we consider three cases as before. In case 1 ($N\geqq 2$) we adjust (3.10) for the loss of one factor of $(d_{k-1})^{-\beta_2}$; the other is still present because each subcomponent after the first obeys the stronger estimate (3.9). The exponential decay comes out as before, and we obtain an overall estimate.

$$(d_k/d_{k-1})^{-\beta_2}\exp(-2\beta_2 d(\gamma_\sigma^{(k)})/d_k)\leqq d(\gamma_\sigma^{(k)})^{-\beta_2(1-\alpha^{-1})}\exp(-\beta_2 d(\gamma_\sigma^{(k)})/d_k)\,,$$

which is sufficiient to obtain (3.15) in this case. Cases 2 and 3 ($N=1$) are easier than before, because the bound (3.15) remains essentially the same. We have allowed for a combinatoric factor of 2 for the sum over N.

To obtain (3.14) we can now estimate

$$\sum_{\gamma_1\ni b_L} e^{-\beta_1 H(\gamma_1)}\leqq\sum_{k=0}^{\infty} 2^{k+1}e^{-O(J_1)}d_{k-1}^{-\beta_2(1-\alpha^{-1})}\leqq e^{-O(J_1)}\,.$$

Here we have used (3.15) on the scale where $d_{k-1}<d(\gamma_1)\leqq d_k$, at which point γ_1 is certainly k-connected. We may of course assume $|\gamma_1|\geqq 1$.

If γ_1 contains both bonds in $\partial\Lambda$, then since each of these bonds can be regarded as fixed, we can work from both ends of Λ using arguments as above. We obtain the bound (3.14) also in this case.

We return to proving (3.2). First, note that x', y' interaction terms in $H(\gamma_1)+H(\Gamma)-H(\gamma_1\cup\Gamma)$ arise when y' is flipped relative to x' by both γ_1 and Γ. The corresponding term in the Hamiltonian appears at most once in $H(\gamma_1\cup\Gamma)$, so there is a nonvanishing contribution to $H(\gamma_1)+H(\Gamma)-H(\gamma_1\cup\Gamma)$. The structure here is the same as in the Ising case, considered in [FS]. There, using the distance condition C to ensure that contours other than γ_1 occupy only a small fraction of space and to keep γ_1 far from contours surrounding it, the interaction energy, was estimated as

$$|H(\gamma_1)+H(\Gamma)-H(\gamma_1\cup\Gamma)|\leqq cm^{-1}(\ln m)L(\gamma_1)\,. \tag{3.16}$$

Here $L(\gamma_1)$ is the logarithmic length of γ_1; it is defined as

$$2^{L(\gamma_1)} = \prod_{i=1}^{N-1} [2\,\mathrm{dist}(f_i, f_{i+1})], \tag{3.17}$$

where there are N flips or bonds f_i in γ_1. This part of the interaction energy estimate uses the distance condition C (which holds in the present situation also) and neutrality. While contours containing boundary bonds are not actually neutral, the absence of interaction terms with $x' \in \Lambda$, $y' \notin \Lambda$ (free boundary conditions) makes them behave like neutral contours as far as this estimate is concerned.

The remainder of the anaylsis of [FS] leading to (3.2) uses the energy of isolated (charged) parts of γ_1 to estimate $L(\gamma_1)$ in terms of $H(\gamma_1)$. We do not have this energy available for parts involving $\partial\Lambda$. However, we notice that the right-hand side of (3.17) contains precisely the set of combinatoric factors that we used in summing over γ_1. In fact we used for each flip the length scale d_k such that $d_k \geqq \mathrm{dist}(f_i, f_{i+1}) > d_{k-1}$. Thus implicit in our entropy-energy estimates is a bound

$$\beta_1 H(\gamma_1) \geqq (\ln 2) L(\gamma_1), \tag{3.18}$$

and the required estimate (3.2) then follows from (3.16).

An additional argument is needed in the case where γ_1 contains both boundary bonds, for in this case we started summing from both ends of Λ, leaving one factor of $\mathrm{dist}(f_i, f_{i+1})$ from (3.17) uncontrolled. However, this last factor cannot be too large in comparison to the others, or γ_1 would have been split into two contours. The gap cannot be larger than the κ^{th} power of the diameter of either side of γ_1. Thus $3H(\gamma_1)$ will easily control all factors in (3.17), and we must replace (3.18) with

$$\beta_1 H(\gamma_1) \geqq \tfrac{1}{3}(\ln 2) L(\gamma_1).$$

This does not affect (3.2), and so the proof of Theorem 3.4 is complete.

4. Upper Bounds on Truncated Correlations

In this section we prove that truncated correlations in the long-range Potts models with $\beta > 1$ and large J_1 decay as a small power of the separation. This result complements our lower bounds on the two-point function. The method of proof is completely different, however – we use the cluster expansion of [I] rather than estimates in the percolation language. The two methods nicely cover each other's weaknesses, and together provide a very precise picture of the variable power law regime for large J_1. We return to the state $\langle \cdot \rangle_1^L$ used in the proof of Theorem 3.1, and derive estimates uniform in L.

We now state our main estimate on

$$G_2^T = \left(\frac{q}{q-1}\right)^2 [\langle \delta_{\sigma_x,2} \delta_{\sigma_y,2} \rangle_1 - \langle \delta_{\sigma_x,2} \rangle_1 \langle \delta_{\sigma_y,2} \rangle_1].$$

By Proposition 1.1, a similar bound applies as well to G_1^T and G^T.

Theorem 4.1. *Let $\beta>1$ be given, and choose any $\varepsilon>0$. Then for J_1 sufficiently large (depending on q,β,ε) the following bounds holds for all x,y and uniformly in L:*

$$|\langle\delta_{\sigma_x,2}\delta_{\sigma_y,2}\rangle_1^L-\langle\delta_{\sigma_x,2}\rangle_1^L\langle\delta_{\sigma_y,2}\rangle_1^L|\leqq e^{-O(J_1)}|x-y|^{-2(\beta-1)(1-\varepsilon)}. \tag{4.1}$$

If $2(\beta-1)(1-\varepsilon)>2$, then the final factor must be replaced with $|x-y|^{-2}$.

We will be brief, sketching only a few points on how the estimates of [I] need to be modified. This is possible because the expansion can be used precisely as described in [I] for $q=2$ (the Ising model). For $q>2$ the form of the interaction between irreducible contours is a little more complicated because of the dependence on how the q states are explored by the contours. However, there is no essential difference in the structure of the expansion.

It is worth noting that for β close to 1, the correlation function at separation $|x-y|$ is dominated by terms involving contours of diameter $\gtrsim|x-y|$. The power $2(\beta-1)$ can naively be understood as arising from the excess of energy over entropy for simple spin flip pairs, as in the discussion at the start of Sect. 3. For larger values of β, this decay is faster than the decrease in the couplings, so the dominant effect comes from the couplings between small contours.

In brief, the expansion is organized as follows. After the Peierls expansion, there is a Mayer expansion in the interaction bonds $\langle x,y\rangle$ coupling different contours. These bonds must be flipped by both of the contours involved. Finally, the polymer formalism is used to expand in the hard core exclusions associated with the distance condition C.

These estimates of [I] rely on the possibility of summing over chains of contours and interaction bonds, with each bond connecting two contours, and each contour connecting two interaction bonds. When there is a possibility of several bonds connected to one fixed contour γ, we use the fact that the sum of $|x-y|^{-2}$ over allowed x,y is bounded by $cL(\gamma)$. Using the activities $e^{-O(J_1)}$ at the end of each such bond, the combinatorics produce an overall factor of $\exp(e^{-O(J_1)}L(\gamma))$. By (3.18), this can be absorbed into a small decrease in β_1. Summations over contours surrounding a given point have been estimated already in (3.5), (3.6).

A subtler case occurs in estimates for chains bridging the gap between sites. In detail, the organization of these estimates must differ from [I], where the large inverse power law allows some simplification. What is needed here is a decay as a small power of the separation between the sites.

We use a slightly different form of (3.6) to control sums over contours:

$$\sum_{\gamma\,\text{surrounding}\,x,y} e^{-\beta_1 H(\gamma)}\leqq e^{-O(J_1)}|x-y|^{-2(\beta-1)(1-\varepsilon/3)}. \tag{4.2}$$

To obtain this, we apply (3.6) (with a smaller ε) to contours with $d(\gamma)\in[D,2D)$. For these a factor $2D$ is necessary to sum over f. Of course $\beta-\beta_2$ can be made arbitrarily small for large J_1. We obtain

$$\sum_{\gamma\,\text{surrounding}\,x,\,d(\gamma)\in[D,2D)} e^{-\beta_1 H(\gamma)}\leqq e^{-O(J_1)}D^{-2(\beta-1)(1-\varepsilon/4)}.$$

By using a small power of D to sum over $D=|x-y|2^0,|x-y|2^1,\ldots$, we obtain (4.2). This controls the sum over the first link in a chain of contours and interaction bonds.

Now let us assume control over chains of n contours linked with $n-1$ interaction bonds from x_1 to x_{2n}:

$$\sum_{x_2<x_3} |x_2-x_3|^{-2} \dots \sum_{x_{2n-2}<x_{2n-1}} |x_{2n-2}-x_{2n-1}|^{-2} \sum_{\gamma_1 \text{ surrounding } x_1, x_2} e^{-\beta_1 H(\gamma_1)}$$
$$\dots \sum_{\gamma_n \text{ surrounding } x_{2n-1}, x_{2n}} e^{-\beta_1 H(\gamma_n)} \leqq e^{-O(J_1)n} |x_1-x_{2n}|^{-\delta}. \tag{4.3}$$

Here we use $\delta=\min\{2, 2(\beta-1)(1-\varepsilon/3)\}$. The case $n=1$ is just (4.2). To obtain (4.3) for $n+1$ contours, fix x_1 and perform the following sum:

$$\sum_{x_{2n}} |x_1-x_{2n}|^{-\delta} |x_{2n}-x_{2n+1}|^{-2} \leqq c|x_1-x_{2n+1}|^{-\delta}. \tag{4.4}$$

This bound is easily obtained by considering separately the terms with x_{2n} closer to x_1. These are bounded by $c|x_1-x_{2n+1}|^{-1-\delta}$, while the rest are bounded as in (4.4). Next we sum over γ_{n+1} surrounding x_{2n+1}, x_{2n+2}. By (4.2) this leads to an additional factor $e^{-O(J_1)}|x_{2n+1}-x_{2n+2}|^{-\delta}$, and together with the decrease in (4.4), we obtain (4.3).

Finally, we need to understand how to control the full expansion, where chains as above can be linked together through the expansion of the hard core exclusions from condition C and other constraints. Here it is necessary to sum over contours γ which violate the distance condition with respect to a fixed contour γ_0. There are $m(d(\gamma))^\kappa$ choices for the first flip in γ. Therefore our basic estimate (4.2) would be replaced by one with a slightly smaller power – as always for κ close to 1. However, we can still proceed as above to obtain decay as $|x-y|^{-\delta'}$ for chains covering sites x, y, with

$$\delta'=\min\{2, 2(\beta-1)(1-\varepsilon/2)\}.$$

Furthermore, the gaps between chains are no larger than the κ^{th} power of the diameter of a chain, so we have sufficient convergence factors to prove decay as $|x-y|^{-\min\{2,2(\beta-1)(1-\varepsilon)\}}$ for a continuous sequence of links (contours, interaction bonds, or hard core interactions) bridging the distance from x to y.

When there are several hard-core attachments to a single chain, we note that there are only $O(|\Gamma|)$ places to attach, where Γ is the union of the contours in the chain. From the above, we see that each attachment sums up to something small, so the combinatorics of the hard-core expansion produces factors $\exp(e^{-O(J_1)}|\Gamma|)$, which are easily absorbed by a slight decrease in β_1 in (4.2).

Of course the full expansion for $\langle \delta_{\sigma_x,2}, \delta_{\sigma_y,2}\rangle_1^L - \langle \delta_{\sigma_x,2}\rangle_1^L \langle \delta_{\sigma_y,2}\rangle_1^L$ will involve only terms bridging between x and y, and so we obtain the decay claimed in Theorem 4.1.

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