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Improved Perturbation Expansion for Disordered Systems: Beating Griffiths Singularities*

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Abstract. We introduce a new expansion to prove exponential clustering of connected correlations in a large class of disordered systems. Our expansion converges for values of the temperature and magnetic field where standard cluster expansions diverge, due to the presence of Griffiths type singularities. It is organized inductively over an infinite sequence of increasing distance scales. In each induction step one redefines what is means by the "unperturbed system", a procedure somewhat reminiscent of K.A.M. theory. Our techniques may be useful in dealing with the so-called large-field problem in real-space renormalization group schemes.

1. Introduction

1.1. Overview

In this paper we introduce a new method to partially resum high-temperature, or low-activity expansions in situations where they actually diverge. Our method can be used, for example, to analyze spin glasses, the random-field Ising model and other disordered systems at temperatures and activities where straight high-temperature, or low-activity expansions diverge, due to the presence of so-called Griffiths singularities [1]. We think that our results and methods are a prerequisite for understanding critical behavior in disordered systems.

Among the mathematical problems that one encounters in the study of disordered systems are:

A. Certain random couplings, such as the spin-spin couplings, J_{ij}, in a spin glass or the inverse of the magnetic field, h_j, in a random field model, can have anomalously large values over large regions of the lattice with very small, but positive probability. In the vicinity of such regions the correlation length is

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anomalously large. As a consequence, one may find singularities in the free energy and/or in certain correlations as functions of e.g. the inverse temperature, β, arbitrarily close to the origin, and ordinary cluster expansions (such as the high-temperature expansion) are bound to diverge. In a special model singularities of this type have been proven to exist [1]. They are called Griffiths singularities.

Consider, for example, a nearest-neighbor Ising spin glass model on a lattice $\mathbb{Z}^\nu$ with Hamilton function

$$\mathscr{H} = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - h\sum_j \sigma_j,$$

and suppose that the couplings J_{ij} are random variables that can become arbitrarily large with non-zero probability. Then there exist, for any $n = 1, 2, 3, \ldots$, arbitrarily large (hyper-) cubes, Λ_n, in $\mathbb{Z}^\nu$ such that $J_{ij} \geqq n$, for all i, j in Λ_n. In the vicinity of Λ_n, "correlation lengths" are expected to be large, for large n. More precisely, if one performs a high-temperature expansion for expectations of spin variables which couple to spins in Λ_n, one expects that it diverges for $|\beta| > \text{const} \cdot 1/n$. Since this argument can be used for all values of n, one concludes that the radius of convergence of the ordinary high-temperature expansion is presumably zero. If the support of the J_{ij}'s is bounded, but their variance is non-zero one still expects that the high-temperature expansion starts to diverge *inside* the single phase region, well *before* any transition point is reached. If all J_{ij}'s are required to be positive then the high-temperature expansion is likely to diverge when

$$\beta > [\max J_{ij}]^{-1} \beta_{\text{crit}}(\text{Ising}),$$

but long-range ordering only sets in for

$$\beta \gtrsim \overline{[J_{ij}]}^{-1} \beta_{\text{crit}}(\text{Ising}).$$

In such models one can use the *Lee-Yang circle theorem* to actually prove that, in some cases, the magnetization, $M(h)$, has a singularity at $h = 0$, without there being spontaneous magnetization, i.e. $M(h=0) = 0$, provided β is sufficiently large (but not so large as to cause spontaneous magnetization). In this situation one expects that correlations have no analytic continuation from $\text{Re}\,h > 0$ to $\text{Re}\,h < 0$, and expansions in powers of h around $h = 0$ diverge, [1].

Difficulties which are closely related, mathematically, to the ones described above are also encountered when one tries to carry out *block-spin transformatins* in a real-space renormalization group calculation: The purpose of such calculations is to construct an effective Hamilton function (or effective action) as a functional of the block-spins with the help of e.g. cluster expansions [2–4]. However, the block-spins can be anomalously large over fairly extended regions of the coarser lattice, albeit with small a priori probability. Such events obstruct the convergence of the high-temperature expansion that one would like to use to integrate out the fluctuation field. This difficulty is known as the *large-field problem* [2–4]. The techniques introduced in this paper might provide a rather efficient way of dealing with large-field problems in real-space renormalization group calculations.

B. Another mathematical problem arises if the random couplings, J_{ij}, between spins may have *anomalously long range*, e.g. in the sense that $\sum_{j \in \mathbb{Z}^\nu} |J_{ij}|$ diverges with probability one. In this situation one expects that if $\sum_{j \in \mathbb{Z}^\nu} |J_{ij}|^2 < \infty$ and the phases of J_{ij} are sufficiently random, then no long range order, or sensitive dependence on boundary conditions, appears at high temperatures. Indeed, a suitable form of the cluster expansion should converge, in an L_p-sense with respect to $\{J_{ij}\}$, with $p \geqq 2$.

In this paper we develop analytical methods to deal with problem A. The basic philosophy underlying our methods is inspired by previous work on Anderson localization [5] and shares various ideas and concepts with K.A.M. theory in classical Hamiltonian dynamics [6]. Our methods are inductive; the induction being indexed by an infinite sequence of increasing distance scales. In the course of our inductive construction we redefine successively what we mean by the "unperturbed system". Each induction step involves a cluster expansion about a new "unperturbed system", incorporating more and more, larger and larger lattice regions, where the random couplings are large. The cluster expansions are done using the techniques in [7–9]. The goal of our construction is to establish, *with probability one*, *uniqueness* of the equilibrium state and *exponential cluster properties* for connected correlations in disordered systems at fairly high temperatures or in fairly strong magnetic fields. We obtain an expansion for the logarithm of the partition function in terms of quantities that depend locally on the random fields. This corresponds to calculating the effective action in a block-spin renormalization group setting. Thus we develop tools which we believe may be useful to control the large-field problem.

1.2. Models

The physical systems which we propose to study are spin glasses, disordered ferromagnets, and ferromagnets in random magnetic field. A typical mathematical model of such a system is an Ising-spin model with Hamilton function

$$\mathscr{H} = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_j h_j\sigma_j, \tag{1.1}$$

where i and j range over the lattice $\mathbb{Z}^\nu$, the spins σ_j take values ± 1 with equal a priori probability, for all $j \in \mathbb{Z}^\nu$, and the couplings

and
$$\left.\begin{array}{l} J : \mathbb{Z}^\nu \times \mathbb{Z}^\nu \ni (i,j) \mapsto J_{ij}, \\ h : \mathbb{Z}^\nu \ni j \mapsto h_j \end{array}\right\} \tag{1.2}$$

are real-valued, independent, identically distributed random variables. Typical distributions for these variables are the following ones:

(1) *Random Field Ising Model (RFIM), Large Disorder*

$$\left.\begin{array}{l} J_{ij} = 0, \quad \text{for} \quad |i-j| \neq 1, \\ J_{ij} = 1, \quad \text{for} \quad |i-j| = 1, \\ d\lambda(h_j) = (\sqrt{2\pi}\,H)^{-1} \exp(-h_j^2/2H^2)dh_j, \\ e^{-\beta(H-\text{const})} \ll 1. \end{array}\right\} \tag{1.3}$$

(2) *High Temperature Spin Glass*

$$\left.\begin{array}{l} J_{ij}, \quad \text{for} \quad |i-j| \neq 1, \\ d\varrho(J_{ij}) = (\sqrt{2\pi}\,\Delta)^{-1} \exp(-(J_{ij}-\bar{J})^2/2\Delta^2) dJ_{ij}, \\ d\lambda(h_j)\text{: an arbitrary probability measure,}, \quad \beta \ll 1. \end{array}\right\} \tag{1.4}$$

(3) *High Temperature Spin Glass, Slowly Decaying Distribution for* J_{ij}

$$\left.\begin{array}{l} J_{ij}=0, \quad \text{for} \quad |i-j| \neq 1, \\ d\varrho(J_{ij}) = (\Delta/\pi)(J_{ij}^2+\Delta^2)^{-1} dJ_{ij}, \\ d\lambda(h_j) \text{ arbitrary}, \quad \beta \ll 1. \end{array}\right\} \tag{1.5}$$

We define

$$\overline{F(J,h)} \equiv \int \prod_{(i,j)} d\varrho(J_{ij}) \prod_j d\lambda(h_j) F(J,h). \tag{1.6}$$

(4) *Low Temperature, Predominantly Ferromagnetic Spin Glass*

$$\begin{array}{l} J_{ij} \text{ as in (2)}, \\ \bar{J}=1, \quad \Delta \ll 1, \quad \beta \gg 1, \quad h_j \equiv 0. \end{array} \tag{1.7}$$

(4′) *Random Field Ising Model, Small Disorder*

$$J_{ij} \text{ as in (1)}, \quad \overline{h_j}=0, \quad \overline{h_j^2} \ll 1, \quad \beta \gg 1. \tag{1.8}$$

(5) *High Temperature Spin Glass, Long Range Interactions*

$$\text{with} \quad \left.\begin{array}{l} |\overline{J_{ij}^p}| \leqq D_{i-j}, \quad p=1,2,3,\ldots, \\ \sum_{j\in\mathbb{Z}^\nu} D_j \leqq 1; \\ d\lambda(h_j) \text{ arbitrary}, \quad \beta \ll 1. \end{array}\right\} \tag{1.9}$$

The point of this example is that J_{ij} may have *very long range* with non-zero probability, but that the *phase* of J_{ij} is sufficiently random to wipe out correlations over very far distances. It can happen that $\overline{|J_{ij}|}$ is *not* summable, but that, nevertheless, a standard cluster expansion converges in L^p with respect to $\{J_{ij}\}$, $p \geqq 2$.

All these examples offer different challenges of varying difficulty.

To start with the analysis of these models one first studies finite subsystems. In the definition (1.1) of $\mathscr{H}$ one restricts the summations over j to a finite subset Λ of $\mathbb{Z}^\nu$. The corresponding Hamilton function is denoted $\mathscr{H}_\Lambda$. Moreover, one chooses a probability measure, dP_Λ, on the space of configurations $\{\sigma_j\}_{j\in\mathbb{Z}^\nu\setminus\Lambda}$. The equilibrium state of the system in Λ with boundary conditions given by dP_Λ is then defined by

$$d\mu_{\beta,P_\Lambda}(\sigma) = Z_{\beta,P_\Lambda}^{-1} e^{-\beta\mathscr{H}_\Lambda(\sigma)} dP_\Lambda(\sigma), \tag{1.10}$$

where Z_{β,P_Λ} is the partition function chosen so that $\int d\mu_{\beta,P_\Lambda}(\sigma)=1$. (The integral sign stands for summation over all possible configurations $\{\sigma_j\}_{j\in\mathbb{Z}^\nu}$.)

In this paper we set out to analyze existence and uniqueness of the thermodynamic limit, $\Lambda \nearrow \mathbb{Z}^\nu$, of the states $d\mu_{\beta,P_\Lambda}$ and the cluster properties of $d\mu_{\beta,P_\Lambda}$, uniformly in Λ, under suitable hypotheses on β, H, Δ, $\bar{J}$. We seek to prove results which hold with probability one with respect to $(J,h) \equiv \{J_{ij}, h_j\}_{i,j\in\mathbb{Z}^\nu}$.

1.3. Results

We now sketch our main results. We define the expectation $\langle(\cdot)\rangle_{\beta,\Lambda}(J,h)$ by

$$\langle F\rangle_{\beta,\Lambda}(J,h) \equiv \int F(\sigma) d\mu_{\beta,P_\Lambda}(\sigma). \tag{1.11}$$

If F is some function of $\{\sigma_j\}_{j\in\mathbb{Z}^\nu}$ we let $\operatorname{supp} F$ denote the set of sites $x \in \mathbb{Z}^\nu$ with the property that F depends non-trivially on σ_x. Let A and B be functions depending only on finitely many spins, σ_j. Let $a\in\mathbb{Z}^\nu$, and define B_a by the equation $B_a(\{\sigma_j\}) = B(\{\sigma_{j-a}\})$. We are interested in the behavior of connected correlations

$$\langle A; B_a\rangle_{\beta,\Lambda}(J,h) \equiv \langle A\cdot B_a\rangle_{\beta,\Lambda}(J,h) - \langle A\rangle_{\beta,\Lambda}(J,h)\cdot\langle B_a\rangle_{\beta,\Lambda}(J,h), \tag{1.12}$$

for large values of $|a|$, where $|a|$ denotes the Euclidean length of a. More generally, we propose to study the asymptotic behavior of connected correlations of n observables

$$\langle A^1_{a_1}; A^2_{a_2}; \ldots; A^n_{a_n}\rangle_{\beta,\Lambda}(J,h), \tag{1.13}$$

as $|a_i - a_j| \to \infty$, for $i \neq j$. We say that the connected correlation function (1.13) has tree decay with decay rate M if, for arbitrary but fixed sites $x_1, \ldots, x_n$ in $\mathbb{Z}^\nu$, $a_i \equiv \theta x_i$, $i = 1, \ldots, n$, for all $\theta = 1, 2, 3, \ldots$,

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu} \langle A^1_{a_1}; A^2_{a_2}; \ldots; A^n_{a_n}\rangle_{\beta,\Lambda}(J,h) \leqq C(J,h; x_1, \ldots, x_n)\exp(-M|T(a_1, \ldots, a_n)|). \tag{1.14}$$

Here $T(a_1, \ldots, a_n)$ is the shortest tree with end-points in the sites $a_1, \ldots, a_n$, and $|T(a_1, \ldots, a_n)|$ denotes its Euclidean length. While M is almost surely independent of the sample (J,h) one has chosen, $C(J,h; x_1, \ldots, x_n)$ is a random variable. An alternative formulation of (1.14) is to set $a_1 = 0$ and consider the limit $|a_i| \to \infty$, $|a_i - a_j| \to \infty$, for $i,j = 2, \ldots, n$.

Our main results are as follows.

Let ν be the dimension of the lattice, $H^2 = \overline{h_j^2}$ and $\Delta^2 = \overline{J_{ij}^2}$. In the following, F, $A_1, \ldots, A_n$ are always arbitrary bounded functions of the spins of finite support.

(i) Consider the model (1), the RFIM, and suppose that β and/or H are so large that

$$e^{(4\nu - \varepsilon_0 H)\beta} \ll 1, \tag{1.15}$$

for some small $\varepsilon_0 < 1$. Then

$$\lim_{\Lambda\nearrow\mathbb{Z}^\nu} \langle F\rangle_{\beta,\Lambda}(J,h) \equiv \langle F\rangle_\beta(J,h) \tag{1.16}$$

exists and is independent of the boundary conditions (b.c.) P_Λ, almost surely with respect to J and h. Moreover, there exists some constant $M(\beta) > 0$, independent of

J and h, such that

$$\langle A^1_{a_1}; \ldots; A^n_{a_n}\rangle_{\beta,\Lambda}(J,h)$$

has almost surely tree decay with decay rate $M \geqq M(\beta)$.

(i) Consider models (2) or (3) and suppose that $\beta\Delta$ is sufficiently small. Then

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \langle F\rangle_{\beta,\Lambda}(J,h) \equiv \langle F\rangle_\beta(J,h)$$

exists and is independent of the b.c. and $\langle A^1_{a_1}; A^2_{a_2}; \ldots; A^n_{a_n}\rangle_{\beta,\Lambda}(J,h)$ has tree decay, with decay rate $M>0$, almost surely with respect to J and h.

The proof of these results for model (3) is considerably more difficult than the proof for model (2). The methods developed in this paper cover both cases on an equal footing, but are somewhat complicated. Berretti [10] has found a simple way of proving some of the results described above for model (2), based on the Glimm-Jaffe-Spencer form of cluster expansions [11]. However, his techniques appear to fail in model (3), the reason being the slow decay of $d\varrho(J_{ij})$, as $|J_{ij}| \to \infty$. His method also applies to model (1), although since he expands in βJ_{ij}, he requires H to be large, depending on β ($H\to\infty$ as $\beta\to\infty$). Our method is uniform as $\beta\to\infty$; in fact, large β improves convergence, see (1.15).

In order to develop some perspective and for convenience we briefly paraphrase Berretti's ideas; for details see [10]. Let $\langle\!\langle \cdot \rangle\!\rangle_{\beta,\Lambda}$ denote the expectation with respect to the equilibrium state

$$d\mu_{\beta,P_\Lambda}(\sigma) \otimes d\mu_{\beta,P_\Lambda}(\sigma') \tag{1.17}$$

of a duplicate system. (The spins $\{\sigma_j\}$ and $\{\sigma'_j\}$ are duplicates of one another with identical distributions.) If A is a function of $\{\sigma_j\}$ we let A' denote the same function of $\{\sigma'_j\}$. Then

$$\langle A; B\rangle_{\beta,\Lambda}(J,h) = \tfrac{1}{2}\langle\!\langle (A-A')(B-B')\rangle\!\rangle_{\beta,\Lambda}. \tag{1.18}$$

Next, we rewrite the expectation $\langle\!\langle \cdot \rangle\!\rangle_{\beta,\Lambda}$ by using the simple identity

$$\exp(\beta J_{ij}\sigma_i\sigma_j)\exp(\beta J_{ij}\sigma'_i\sigma'_j) = 1 + E_\beta(i,j), \tag{1.19}$$

where

$$E_\beta(i,j) = \exp(\beta J_{ij}[\sigma_i\sigma_j + \sigma'_i\sigma'_j]) - 1, \tag{1.20}$$

and expanding in E_β. We start the expansion by choosing a nearest-neighbor pair $\langle i,j\rangle$ such that $\langle i,j\rangle \cap (\operatorab{supp} A \cup \operatorname{supp} B) \neq \emptyset$. In $\langle\!\langle (A-A')(B-B')\rangle\!\rangle_{\beta,\Lambda}$ we now replace $\exp(\beta J_{ij}\sigma_i\sigma_j)\exp(\beta J_{ij}\sigma'_i\sigma'_j)$ by the right side of (1.19) and expand in a sum of two terms. Next, we choose a pair $\langle i_1,j_1\rangle$ such that $\langle i_1,j_1\rangle \cap (\operatorname{supp} A \cup \operatorname{supp} B \cup \langle i,j\rangle) \neq \emptyset$ and repeat the expansion step described above. We continue until we have generated a graph G of nearest-neighbor pairs, $\langle i,j\rangle$, in Λ with the property that the set $X \equiv G \cup \operatorname{supp} A \cup \operatorname{supp} B$ is connected. With each such graph G we associate a number

$$K_{A,B}(G) = \sum_{\{\sigma_j,\sigma'_j\}_{j\in X}} \left[(A(\sigma)-A'(\sigma'))(B(\sigma)-B'(\sigma')) \prod_{\langle i,j\rangle \subset G} E_\beta(i,j)\right]. \tag{1.21}$$

We then resum the expansion in $\Lambda\backslash X$ and obtain

$$\langle A; B\rangle_{\beta,\Lambda}(J,h)=\tfrac{1}{2}\sum_{G} K_{A,B}(G)\frac{Z_{\Lambda\backslash X}}{Z_\Lambda}, \tag{1.22}$$

where $Z_\Lambda \equiv Z_{\beta,P_\Lambda}$, and $Z_{\Lambda\backslash X}$ is the partition function of the system in $\Lambda\backslash X$ with b.c. P_Λ at $\partial\Lambda$ and zero b.c. at $\partial X\backslash(\partial X\cap\partial\Lambda)$. By rather straightforward estimates [10]

$$\begin{aligned}\overline{|K_{A,B}(G)(Z_{\Lambda\backslash X}/Z_\Lambda)|}&\leqq \overline{[K_{A,B}(G)^2]}^{1/2}\overline{[(Z_{\Lambda\backslash X}/Z_\Lambda)^2]}^{1/2}\\ &\leqq C_{A,B}\exp[-C'|X|],\end{aligned} \tag{1.23}$$

where $|X|$ is the cardinality of X, and $C'\to\infty$ when $\beta\Delta\to 0$. This estimate suffices to show that, for sufficiently small $\beta\Delta$, the expansion (1.22) converges *absolutely*, *in mean*. This proves exponential decay of $\langle A; B_a\rangle_\beta(J,h)$ in a, for almost all J and h, when $\beta\Delta$ is small.

Unfortunately, $|K_{A,B}(G)(Z_{\Lambda\backslash X}/Z_\Lambda)|$ need *not* be integrable with respect to $\prod_{\langle i,j\rangle} d\varrho(J_{ij})$ if $d\varrho(J)/dJ$ has slow decay, as $|J|\to\infty$, e.g. $d\varrho(J)=\Delta\pi^{-1}(J^2+\Delta^2)^{-1}dJ$, and if $d\lambda(h_j)$ is chosen appropriately.

In this paper we propose an expansion which avoids the problem just described, which is entirely constructive, in the sense that the expansion terms depend only locally on the random fields, and which converges absolutely, almost surely. It can be used to prove result (ii) for models (2) and (3).

(iii) For model (4) we can prove that, with + boundary conditions and at sufficiently low temperature, there is spontaneous magnetization, and connected correlations have exponential decay properties, with probability one. The fact that the system exhibits spontaneous magnetization at low temperature, almost surely, can be shown, *in certain cases*, with the help of a *Peierls argument*: Let $\langle\ \cdot\ \rangle_{\beta,+}(J)$ be the equilibrium state with + b.c. Then

$$\langle\sigma_0\rangle_{\beta,+}(J)\geqq 1-\sum_{\gamma|0}\ \prod_{\langle i,j\rangle\in\gamma} e^{-2\beta J_{ij}}, \tag{1.24}$$

where $\gamma|0$ means that γ surrounds the origin. Now if

$$\int d\varrho(J)e^{-2\beta J}\to 0 \quad \text{as} \quad \beta\to\infty\,^{1}, \tag{1.25}$$

then for sufficiently large β,

$$\overline{\langle\sigma_0\rangle_{\beta,+}(J)}>0, \tag{1.26}$$

as follows from (1.24) and (1.25), by standard arguments. The ergodic theorem tells us that $\overline{\langle\sigma_0\rangle_{\beta,+}(J)}$ is the spontaneous magnetization, almost surely. For closely related arguments see [12]. However, when $\mathrm{supp}\, d\varrho \not\subset [0,\infty)$, e.g. when $e^{-2\beta J}$ is not $d\varrho(J)$-integrable, as the case may be, then more detailed probability estimates are needed before Peierls inequality (1.24) can be used to establish positivity of $\overline{\langle\sigma_0\rangle_{\beta,+}(J)}$. The methods developed in this paper circumvent such difficulties and are constructive.

1 If $\mathrm{supp}\, d\varrho\not\subset[0,\infty)$, then it is still possible that there is an interval $[\beta_0,\beta_1]$, $\beta_1<\infty$, such that for $\beta\in[\beta_0,\beta_1]$, $\int d\varrho(J)e^{-2\beta J}$ is so small that (1.24) implies that $\langle\sigma_0\rangle_{\beta,+}(J)>0$

Model (4′) is the *random field Ising model*. It has been in the focus of much recent theoretical and experimental attention; see references quoted in [13]. At high temperatures, the equilibrium state of this model is unique, and connected correlations have exponential decay properties. The conjecture is that, in three or more dimensions and at sufficiently small temperatures, the system exhibits spontaneous magnetization with probability one, while in one and two dimensions, there is never any spontaneous magnetization. It turns out that it is surprisingly difficult to prove this conjecture. For some partial results, see [13]. The methods developed in this paper may be useful to extend a proof of the conjecture for $d \geqq 3$ and $T=0$, i.e. in the ground state, to small, but positive temperatures, T.

(iv) For model (5) a cluster expansion can presumably be derived which converges in $L_p\left(\{J_{ij}\}, \prod_{i,j} d\varrho(J_{ij})\right)$, for $p \geqq 2$. As a corollary one would get clustering $(\propto \sqrt{D_{i-j}})$ of connected correlations at sufficiently high temperatures. If, in this example, large J_{ij} or small h_j problems are eliminated [by requiring (1.9), for example] the proofs appear to be quite standard.

All these results follow from detailed estimates on correlation functions as functions of (J, h). These should be of interest in their own right and are stated in Sects. 7, 8.

Our paper is organized as follows. In most of the paper (Sects. 2–7) we develop the method in the case of model (1). Then, since the majority of the work is model-independent, we discuss only the modifications needed to handle models (2)–(4) (Sect. 8). No applications of our techniques to renormalization (block-spin) transformations are studied.

2. Singular Sets and Entropy Bounds

We consider model (1), with $\sigma|\Lambda^c$ specified arbitrarily. We define an unnormalized expectation value

$$[A]_h = \sum_{\{\sigma_i\}_{i\in\Lambda}} A \exp\left(\beta \sum_{\langle i,j\rangle \subset \Lambda\cup\partial\Lambda} \sigma_i\sigma_j + \beta \sum_{i\in\Lambda} h_i\sigma_i\right),$$

where the sum runs over all spin configurations in the volume $\Lambda \subset \mathbb{Z}^\nu$. The observable is

$$A = \prod_{i\in\mathscr{A}} e^{s_i\sigma_i},$$

with all s_i real and $\sum_{i\in\mathscr{A}} |s_i| \leqq \delta$, some constant of order unity. Normalized expectation values and truncated expectations will be obtained by differentiating our expansion for $\log[A]_h$ term by term with respect to the s_i.

The measure for the magnetic field at each site is

$$d\lambda(h) = (\sqrt{2\pi}\,H)^{-1} e^{-h^2/2H^2} dh.$$

We choose a cutoff H' on h. If $|h_i|>H'$ then σ_i is strongly favored to equal $\operatorname{sgn} h_i$. The probability that $|h_i|<H'$ is

$$\varepsilon=(\sqrt{2\pi}H)^{-1}\int_{-H'}^{H'} e^{-h^2/2H^2}dh<\sqrt{\frac{2}{\pi}}\frac{H'}{H}.$$

We choose H' such that ε is small. We also demand that $e^{4\nu\beta-2H'\beta}\ll 1$.

The main difficulties in the problem arise from regions of singular sites. These are sites where the random field is weak ($|h_i|<H'$) so that it is impossible to expand about a known, most probable, spin configuration. Only relatively crude methods can be applied near singular sites, and while a cluster expansion can be devised, the resulting clusters have an exponential divergence in the volume of the singular region (see Sect. 3). Such divergences make it difficult to take the logarithm of the partition function or to divide numerator by denominator to compute normalized expectations in the presence of the random field. The iterative expansion of Sect. 4 is designed to overcome these difficulties. The iteration proceeds through a hierarchy of more and more extended singular regions, which we now define.

Definition. We choose a sequence of distances $d_0, d_1, d_2, \ldots$ as follows:

$$d_0=1\,,\qquad d_k=2^{\alpha^{k+k_0}}\quad \text{for } k>0\,.$$

Here k_0 is a fairly small integer, and $1<\alpha<2$. We have $d_{k+1}=d_k^\alpha$ for $k\geqq 1$.

A set $D\subset\mathbb{Z}^\nu$ is called k-connected iff every site, j, in D can be connected to any other site, i, in D by a sequence of jumps $(x_l, x_{l+1})\subset D$ with $|x_l-x_{l+1}|\leqq d_k$, $l=0,\ldots,m$, $x_0=j$, $x_{m+1}=i$.

Definition. $S_0=\{j:|h_j|<H'\}$. We decompose S_0 into its 0-connected subsets. Every 0-connected subset whose volume is at most $\sqrt{d_0}=1$, whose diameter is at most $d_0^{(1+\alpha)/2}=1$ and which is separated from its complement in S_0 by a distance $\geqq d_1$ is defined to be a 0-component, $C_\alpha^{(0)}$, $\alpha=1,2,3,\ldots$. We set

$$S_0^g=\bigcup_{\alpha=1,\ldots} C_\alpha^{(0)}\,,$$

$$S_1=S_0\backslash S_0^g\,.$$

Note that S_0^g consists of $C_\alpha^{(0)}$'s which are single sites separated by distances $d_1\geqq 2$. Next, we decompose S_1 into its 1-connected subsets, and so on. Suppose now that S_{k-1} has been constructed. Decompose S_{k-1} into $(k-1)$-connected subsets. Every such subset, whose diameter is at most $d_{k-1}^{(1+\alpha)/2}$, whose volume is not larger than $\sqrt{d_{k-1}}$ and which is separated from its complement in S_{k-1} by a distance $\geqq d_k$ is defined to be a $(k-1)$-component, $C_\alpha^{(k-1)}$, $\alpha=1,2,\ldots$. We set

$$S_{k-1}^g=\bigcup_{\alpha=1,2,\ldots} C_\alpha^{(k-1)}\,,\quad\text{and}\quad S_k=S_{k-1}\backslash S_{k-1}^g\,.$$

Note that, by this construction, a k-component, $C_\alpha^{(k)}$, has the properties

(a) $$\operatorname{diam} C_\alpha^{(k)}\leqq d_k^{(1+\alpha)/2}\,, \tag{2.1}$$

(b) $$\operatorname{vol}(C_\alpha^{(k)})\leqq\sqrt{d_k}\,, \tag{2.2}$$

(c) $$\operatorname{dist}(C_\alpha^{(k)}, S_k\backslash C_\alpha^{(k)})\geqq d_{k+1}\,, \tag{2.3}$$

and that S_k^g is a *maximal* union of k-components.

Entropy of Components

The following proposition shows that the "entropy" of singular components is at most linear in their volume.

Proposition 2.1. *As the magnetic fields $\{h_j\}$ vary, the number of possible components $C_\alpha^{(k)}$ of volume v containing the origin is bounded by $2^{\kappa v}$, for some constant κ. The minimal volume of a component, $C_\alpha^{(k)}$, of S_k^g is 2^k.*

Proof. Proposition 2.1 is clearly true for $k=0$. Thus we suppose that $k \geqq 1$. We start by proving the minimal-volume property of the components, $C_\alpha^{(k)}$, of S_k^g. We decompose $C_\alpha^{(k)}$ into its $(k-1)$-connected components, $D_1^{(k-1)}, \ldots, D_{N_{k-1}}^{(k-1)}$. If $N_{k-1}=1$ then, by (2.1) and (2.2), either

$$\text{I)} \qquad \operatorname{diam}(D_1^{(k-1)}) > d_{k-1}^{(1+\alpha)/2},$$

or

$$\text{II)} \qquad \operatorname{vol}(D_1^{(k-1)}) > \sqrt{d_{k-1}}\,.$$

(For, otherwise, $C_\alpha^{(k)}$ would really be a component of S_{k-1}^g!) In case II), $\operatorname{vol}(C_\alpha^{(k)}) = \operatorname{vol}(D_1^{(k-1)}) \geqq 2^k$, since $1+\sqrt{d_{k-1}} \geqq 2^k$, for $k \geqq 1$. In case I), $C_\alpha^{(k)}$ contains at least

$$1+(d_{k-1}^{(1+\alpha)/2}/d_{k-1}) = 1 + d_{k-1}^{(\alpha-1)/2} \geqq 2^k$$

sites, for $k \geqq 1$. [This follows immediately from the facts that $C_\alpha^{(k)}$ is $(k-1)$-connected and that $\operatorname{diam}(C_\alpha^{(k)}) \geqq d_{k-1}^{(1+\alpha)/2}$.] If $N_{k-1} \geqq 2$, then

$$\operatorname{vol}(C_\alpha^{(k)}) \geqq 2 \min_\beta (\operatorname{vol}(D_\beta^{(k-1)})) \geqq 2^k, \tag{2.4}$$

provided $\operatorname{vol}(D_\beta^{(k-1)}) \geqq 2^{k-1}$.

We now prove by induction that the volume of each l-connected component, $D_\gamma^{(l)}$, of $C_\alpha^{(k)}$ is at least 2^l, for all $l \leqq k-1$. This is clearly true for $l=0$. By conditions (2.1) and (2.2), either $\operatorname{diam}(D_\gamma^{(l)}) > d_{l-1}^{(1+\alpha)/2}$, or $\operatorname{vol}(D_\gamma^{(l)}) > \sqrt{d_{l-1}}$. (Otherwise, $D_\gamma^{(l)}$ would be a component of S_{l-1}^g!) Since $\sqrt{d_{l-1}}+1 \geqq 2^l$, it suffices to consider the case where $\operatorname{diam}(D_\gamma^{(l)}) > d_{l-1}^{(1+\alpha)/2}$. If $D_\gamma^{(l)}$ is composed of at least two $(l-1)$-connected subsets, $D_\gamma^{(l-1)}$, we may apply the induction hypothesis. Otherwise, $D_\gamma^{(l)}$ is $(l-1)$-connected and hence consists of at least $1+d_{l-1}^{(\alpha-1)/2} \geqq 2^l$ sites. This proves the minimal-volume property asserted in Proposition 2.1.

Next, we wish to estimate the total number, $n(N_{k-1})$, of possible components, $C_\alpha^{(k)}$ (as the magnetic fields $\{h_j\}$ are varied), which contain the origin and consist of N_{k-1} $(k-1)$-connected constituents, $D_\beta^{(k-1)}$, $\beta=1,\ldots,N_{k-1}$. We let $n(N_{k-2,\beta})$ denote the total number of possible $(k-1)$-connected constituents, $D_\beta^{(k-1)}$, containing the origin and composed of $N_{k-2,\beta}$ $(k-2)$-connected sub-constituents, $D_\gamma^{(k-2)}$.

By the minimal-volume property,

$$N_{k-1} \leqq v/2^{k-1}, \qquad \sum_\beta N_{k-2,\beta} \leqq v/2^{k-2}, \ldots, \tag{2.5}$$

where $v \equiv \operatorname{vol}(C_\alpha^{(k)})$. By a simple geometrical consideration,

$$\begin{aligned} n(N_{k-1}) &\leqq [(2d_k)^\nu]^{2N_{k-1}} \prod_{\beta=1}^{N_{k-1}} n(N_{k-2,\beta}) \\ &\leqq [(2d_k)^\nu]^{2v/2^{k-1}} \prod_{\beta=1}^{N_{k-1}} n(N_{k-2,\beta}), \end{aligned} \tag{2.6}$$

and we have used (2.5) in the second inequality. By iterating this inequality we obtain

$$n(N_{k-1}) \leqq \prod_{l=1}^{k} [(2d_l)^{\nu}]^{v/2^{l-2}}$$

$$\leqq 2^{\left[1+\sum_{l=1}^{k} 2\alpha^{1+\kappa_0(\alpha/2)^{l-1}}\right] \nu v} \leqq 2^{\kappa' v},$$

for some constant κ' which is finite if $\alpha < 2$.

From this analysis we conclude that the total number, $n^{(k)}$, of possible components $C_\alpha^{(k)}$ (as the h_j's vary) with volume v which contain the origin is bounded by

$$n^{(k)} \leqq 2^{(1+\kappa')v} \equiv 2^{\kappa v}, \tag{2.7}$$

since the number of possible sequences $(N_{l,\beta})$ is bounded by 2^v. This ends the proof.

The entropy factor, $2^{\kappa v}$, is balanced by the small probability, ε^v, that a given set of v points lie in S_k^g. Thus if $\varepsilon 2^\kappa \ll 1$, then the $C_\alpha^{(k)}$'s are rather rare. More precisely,

$$\operatorname{prob}(\exists C_\alpha^{(k)} \ni 0) \leqq \sum_{v=2^k}^{\infty} (\varepsilon 2^\kappa)^v \leqq c(\varepsilon 2^\kappa)^{2^k}. \tag{2.8}$$

Given a component $C_\alpha^{(k)}$, we choose a simply connected set $\bar{C}_\alpha^{(k)}$ with the following properties:

$$\bar{C}_\alpha^{(k)} \supset C_\alpha^{(k)},$$

$$d_k \leqq \min_{b \in \partial \bar{C}_\alpha^{(k)}} \operatorname{dist}(b, C_\alpha^{(k)}) \leqq \max_{b \in \partial \bar{C}_\alpha^{(k)}} \operatorname{dist}(b, C_\alpha^{(k)}) \leqq 2d_k,$$

$$\partial \bar{C}_\alpha^{(k)} \cap C_{\alpha'}^{(k')} = \emptyset, \quad \text{for all } k' \text{ and } \alpha'.$$

It is not hard to see [5] that such a set can always be chosen. If $k=0$ we define $\bar{C}_\alpha^{(0)} = C_\alpha^{(0)}$.

3. An Expansion in the Nonsingular Region

Let $\sigma^{\min} = \sigma^{\min}(h)$ denote a spin configuration of minimal energy $E^{\min}$. This configuration satisfies $\sigma_i^{\min} = \operatorname{sgn} h_i$ for $i \in S_0^c$, because $|h_i| > H'$ implies that it is favorable for σ_i to align with h_i no matter what the neighboring spins are. (We choose H' such that $e^{4\nu\beta - 2\beta H'} < 1$.) The minimal configuration in a component $C_\alpha^{(k)}$ depends only on $h|C_\alpha^{(k)}$. It is nonunique only on a set of measure zero for $h|C_\alpha^{(k)}$; we choose it arbitrarily but depending only on $h|C_\alpha^{(k)}$. We have dependence of $\sigma^{\min}|C_\alpha^{(k)}$ on boundary conditions only for $C_\alpha^{(k)}$ adjacent to Λ^c.

We write our unnormalized expectation as

$$[A]_h = e^{-\beta E^{\min}} A(\sigma^{\min}) \sum_{\{\sigma_i\}} \exp\left[\beta \sum_{\langle i,j \rangle} (\sigma_i \sigma_j - \sigma_i^{\min} \sigma_j^{\min}) + \sum_i (\beta h_i + s_i)(\sigma_i - \sigma_i^{\min})\right]. \tag{3.1}$$

Each spin configuration $\{\sigma_i\}$ defines a collection of clusters $\{X_\beta^{(0)}, Y_\gamma^{(0)}\}$. These are the C-connected components of the set $\{i : \sigma_i \neq \sigma_i^{\min} \text{ or } i \in S_0\}$.

Definition. Two lattice sites are C-connected if they are nearest neighbors or if they lie in the same component $C_\alpha^{(k)}$ of singular sites. A subset X of $\mathbb{Z}^\nu$ is C-connected if there is a sequence of C-connected sites whose union is X.

The clusters that contain sites in S_0 are denoted $\{X_\beta^{(0)}\}_{\beta=1,2,\ldots}$; the others are denoted $\{Y_\gamma^{(0)}\}_{\gamma=1,2,\ldots}$. The basic "cluster expansion" for $[A]_h$ is

$$[A]_h = e^{-\beta E^{\min}} A(\sigma^{\min}) \sum_{\{X_\beta^{(0)}, Y_\gamma^{(0)}\}} \prod_\beta \varrho(X_\beta^{(0)}) \prod_\gamma \varrho(Y_\gamma^{(0)}), \tag{3.2}$$

where ϱ is defined as follows.

Definition. For clusters $X^{(0)}$, $Y^{(0)}$, spin configurations $\sigma(Y^{(0)})$ and $\sigma(X^{(0)})$ are defined as

$$\sigma(Y^{(0)})_i = \begin{cases} -\sigma_i^{\min}, & i \in Y, \\ \sigma_i^{\min}, & i \in Y^c, \end{cases} \tag{3.3}$$

$$\sigma(X^{(0)})_i = \begin{cases} -\sigma_i^{\min}, & i \in X^{(0)} \backslash S_0, \\ \sigma_i, & i \in S_0 \cap X^{(0)}, \\ \sigma_i^{\min}, & i \in X^c. \end{cases} \tag{3.4}$$

The configuration $\sigma(X^{(0)})$ depends on $\sigma|X^{(0)}$.

In terms of $\sigma(X^{(0)})$ and $\sigma(Y^{(0)})$ we have

$$\varrho(Y^{(0)}) = \exp\left\{ \sum_{\langle i,j \rangle} \beta(\sigma(Y^{(0)})_i \sigma(Y^{(0)})_j - \sigma_i^{\min}\sigma_j^{\min}) + \sum_i (\beta h_i + s_i)(\sigma(Y^{(0)})_i - \sigma_i^{\min}) \right\}, \tag{3.5}$$

$$\begin{aligned} \varrho(X^{(0)}) = \sum_{\sigma|S_0 \cap X^{(0)}} \exp\Bigg\{ & \sum_{\langle i,j \rangle} \beta(\sigma(X^{(0)})_i \sigma(X^{(0)})_j - \sigma_i^{\min}\sigma_j^{\min}) \\ & + \sum_i (\beta h_i + s_i)(\sigma(X^{(0)})_i - \sigma_i^{\min}) \Bigg\}. \end{aligned} \tag{3.6}$$

Note that the sums over $\langle i,j \rangle$ may as well be restricted to $\partial Y^{(0)}$ or to bonds contained in $X^{(0)} \cup \partial X^{(0)}$. Similarly, the sum over i may as well be restricted to $Y^{(0)}$ or $X^{(0)}$.

We prove estimates on these cluster activities and on their s-derivatives. Let $\mathscr{B} \subset \mathscr{A}$ be a (possibly empty) collection of $|\mathscr{B}|$ sites in the support of the observable A. Let $(\partial/\partial s)_{\mathscr{B}} = \prod_{i \in \mathscr{B}} \partial/\partial s_i$.

Proposition 3.1. *The bounds*

$$\left| \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \varrho(Y^{(0)}) \right| \leqq 2^{|\mathscr{B}|} e^{2\delta} e^{-m|Y^{(0)}|}, \tag{3.7}$$

$$\left| \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \varrho(X^{(0)}) \right| \leqq 2^{|\mathscr{B}|} e^{2\delta} 2^{|X^{(0)} \cap S_0|} e^{-m|X^{(0)} \backslash S_0|} \tag{3.8}$$

hold with $m = (2H' - 4\nu)\beta$.

Proof. Let us consider $\varrho(Y^{(0)})$ first. We have that

$$\beta h_i(\sigma(Y_i^{(0)}) - \sigma_i^{\min}) = -2\beta |h_i| < -2\beta H',$$

since $i \in S_0^c$. There are at most $2\nu|Y^{(0)}|$ bonds in the sum over $\langle i,j\rangle$; each term is bounded by 2. This yields the decay $e^{-m|Y^{(0)}|}$. We have $\sum_{i\in\mathcal{A}} |s_i| \leqq \delta$, so that $\sum_i s_i(\sigma_i - \sigma_i^{\min}) \leqq 2\delta$. Each s-derivative brings down a factor $\sigma - \sigma^{\min}$, which we bound by 2.

With the factor $2^{|X^{(0)}\cap S_0|}$ we can consider a fixed σ in the sum defining $\varrho(X^{(0)})$. Define $\tilde{\sigma}(X^{(0)}) = \sigma$ in $X^{(0)} \cap S_0$ and $\tilde{\sigma}(X^{(0)}) = \sigma^{\min}$ elsewhere. The energy of $\sigma(X^{(0)})$ is at least $(2H' - 4\nu)|X^{(0)} \backslash S_0|$ larger than the energy of $\tilde{\sigma}(X^{(0)})$, since we have aligned $|X^{(0)}\backslash S_0|$ spins with the field in S_0^c. The energy of $\tilde{\sigma}(X^{(0)})$ is larger than or equal to the energy of $\sigma^{\min}(X^{(0)})$ because $\sigma^{\min}$ has minimum energy. Hence the energy difference is at least $(2H' - 4\nu)|X^{(0)}\backslash S_0|$. The s-derivatives and the terms involving s in the exponent are estimated as before, and the proposition is proven.

The First Step

We want to apply the polymer formalism to exponentiate the expansion away from S_0. The unexponentiated part can be used to define activities of the mildly singular regions in S_0^g. A second cluster expansion (Sect. 4) can be derived about an unperturbed partition function which is the product of these activities. This can be exponentiated away from the next most singular regions; exponential decays coming from large distances between the rare singular regions beat the exponential divergences in the volumes of singular regions. Activities for the next most singular regions can then be defined, and so on. In the end (Sect. 7), the logarithm of $[A]_h$ is expressed as a sum of logarithms of activities of singular regions, plus a sum of small, localized interactions from all the exponentiated expansions. This yields the desired estimates on correlation functions.

The reader should consult [8] for more details on the version of the polymer expansion used here. We write

$$\sum_{\{Y_\gamma^{(0)}\}} = \sum_{(Y_1^{(0)},\ldots,Y_l^{(0)})} \frac{1}{l!},$$

where the second sum is over *ordered* collections of $Y^{(0)}$'s. Next we extend the sum over $(Y_1^{(0)}, \ldots, Y_l^{(0)})$ to conclude $Y^{(0)}$'s overlapping each other or overlapping $X^{(0)}$'s.

Definition. Two clusters *overlap* if they contain common sites or if a site in one is a nearest neighbor of a site in the other.

Similarly, we extend the sum over $\{X_\beta^{(0)}\}$ to include overlapping $X_\beta^{(0)}$'s, but we maintain a constraint $X_{\beta_1}^{(0)} \cap X_{\beta_2}^{(0)} \cap S_0 = \emptyset$. The newly added terms are removed with factors u:

$$u(Z_1^{(0)}, Z_2^{(0)}) = \begin{cases} 0 & \text{if } Z_1^{(0)}, Z_2^{(0)} \text{ overlap}, \\ 1 & \text{otherwise}. \end{cases}$$

Here $Z_i^{(0)}$ is either an $X_\beta^{(0)}$ or a $Y_\gamma^{(0)}$. The expansion is now

$$[A]_h = e^{-\beta E^{\min}} A(\sigma^{\min}) \sum_{\{X_\beta^{(0)}\}} \sum_{(Y_1^{(0)},\ldots,Y_l^{(0)})} \frac{1}{l!} \prod_{\mathcal{L}} u(\mathcal{L}) \prod_\beta \varrho(X_\beta^{(0)}) \prod_{\gamma=1}^{l} \varrho(Y_\gamma^{(0)}). \tag{3.9}$$

Here $\mathcal{L}$ is any $\{X_\beta^{(0)}, Y_\gamma^{(0)}\}$, $\{Y_{\gamma_1}^{(0)}, Y_{\gamma_2}^{(0)}\}$, or $\{X_{\beta_1}^{(0)}, X_{\beta_2}^{(0)}\}$.

Note that $Y^{(0)}$'s containing sites adjacent to S_0 did not occur in the original expansion, though they can occur here. However, there is no dependence on $\sigma|S_0$ because σ is set to $\sigma^{\min}$ in $Y^{(0)c}$ in the definition of $\varrho(Y^{(0)})$. Similar considerations apply to the new $X^{(0)}$'s.

Standard manipulations [8] lead to the following form for the expansion [we put $u(\mathscr{L})=1+a(\mathscr{L})$]:

$$[A]_h=e^{-\beta E^{\min}}A(\sigma^{\min})\sum_{\{X_\beta^{(0)}\}}\sum_{(Y_1^{(0)},\ldots,Y_l^{(0)})}\frac{1}{l!}\sum_{G_c(\{X_\beta^{(0)}\})}\prod_{\mathscr{L}\in G_c(\{X_\beta^{(0)}\})}a(\mathscr{L})$$

$$\cdot\prod_\beta \varrho(X_\beta^{(0)})\prod_{\gamma=1}^{l}\varrho(Y_\gamma^{(0)})\exp\Big(\sum_{\bar{Y}^{(0)}}V^{(0)}(\bar{Y}^{(0)})\Big),\tag{3.10}$$

where

$$V^{(0)}(\bar{Y}^{(0)})=\sum_{(Y_1^{(0)},\ldots,Y_m^{(0)})\text{ filling }\bar{Y}^{(0)}}\frac{1}{m!}\sum_{G_c}\prod_{\mathscr{L}\in G_c}a(\mathscr{L})\prod_{\gamma=1}^{m}\varrho(Y_\gamma^{(0)}).\tag{3.11}$$

Here $G_c(\{X_\beta^{(0)}\})$ is a graph of lines $\mathscr{L}$ which is connected in the sense that each $Y_\gamma^{(0)}$ is connected directly or indirectly to some $X_\beta^{(0)}$. The graph G_c must be connected. In the expansion for $[A]_h$ above, we unite into a single cluster $\bar{X}^{(0)}$ all clusters $X^{(0)}$ or $Y^{(0)}$ involved in a connected component of the graph $G_c(\{X^{(0)}\})$. Grouping together all terms in $\bar{X}_\delta^{(0)}$, we obtain

$$[A]_h=e^{-\beta E^{\min}}A(\sigma^{\min})\sum_{\{\bar{X}_\delta^{(0)}\}}\prod_\delta\bar{\varrho}_1(\bar{X}_\delta^{(0)})\exp\Big(\sum_{\bar{Y}^{(0)}}V^{(0)}(\bar{Y}^{(0)})\Big).\tag{3.12}$$

Here $\bar{X}_{\delta_1}^{(0)}\cap\bar{X}_{\delta_2}^{(0)}\cap S_0=\emptyset$, the $\{\bar{X}_\delta^{(0)}\}$ cover all of S_0, and

$$\bar{\varrho}_1(\bar{X}_\delta^{(0)})=\sum_{\{X_\beta^{(0)}\},(Y_1^{(0)},\ldots,Y_l^{(0)})\text{ filling }\bar{X}^{(0)}}\sum_{G_c}\prod_{\mathscr{L}\in G_c}a(\mathscr{L})\prod_\beta\varrho(X_\beta^{(0)})\prod_{\gamma=1}^{l}\varrho(Y_\gamma^{(0)}).\tag{3.13}$$

The graph G_c is now an arbitrary connected graph.

We would like to eliminate the constraint that the $\{\bar{X}_\delta^{(0)}\}$ cover S_0^g. This is accomplished by regarding the expansion as an expansion about a new unperturbed system, $\prod_\alpha z(C_\alpha^{(0)})$, a product of activities associated with the components, $C_\alpha^{(0)}$, of S_0^g. The activities, $z(C_\alpha^{(0)})$, are defined as follows. Since $\operatorname{diam}(C_\alpha^{(0)})\leqq d_0=1$, each $C_\alpha^{(0)}$ is just one site, i. We set

$$z(C_\alpha^{(0)})=\bar{\varrho}_1(C_\alpha^{(0)})=\sum_{\sigma=\pm1}\exp\Big[\sum_{\langle i,j\rangle}\beta(\sigma-\sigma_i^{\min})\sigma_j^{\min}+(\beta h_i+s_i)(\sigma-\sigma_i^{\min})\Big];\tag{3.14}$$

see Eq. (3.6). The term with $\sigma=\sigma_i^{\min}$ on the right side of (3.14) is equal to 1. The other term is between 0 and $e^{2\delta}$. Thus

$$1<z(C_\alpha^{(0)})\leqq 1+e^{2\delta}.\tag{3.15}$$

We can now write the expansion (3.12) as

$$[A]_h=Z_1\sum_{\{\bar{X}^{(0)}\}}{}'\prod_\delta\varrho_1(\bar{X}_\delta^{(0)}),\tag{3.16}$$

where

$$Z_1 = e^{-\beta E^{\min}} A(\sigma^{\min}) \exp\Big(\sum_{\bar{Y}^{(0)}} V^{(0)}(\bar{Y}^{(0)})\Big) \prod_\alpha z(C_\alpha^{(0)}), \tag{3.17}$$

$$\varrho_1(\bar{X}_\delta^{(0)}) = \bar{\varrho}_1(\bar{X}_\delta^{(0)}) \prod_{\alpha: C_\alpha^{(0)} \subset \bar{X}_\delta^{(0)}} z(C_\alpha^{(0)})^{-1}. \tag{3.18}$$

The sum $\sum'$ does not contain $\bar{X}_\delta^{(0)}$'s covering just one site in S_0^g, because $\varrho_1(\bar{X}_\delta^{(0)}) = 1$ for such $\bar{X}_\delta^{(0)}$'s. The point of this operation is to make all cluster activities $\varrho_1(\bar{X}_\delta^{(0)})$ small, as long as $\bar{X}_\delta^{(0)} \cap S_1 = \emptyset$. Such clusters can now be exponentiated. The process can be continued; we take (3.16) as the starting point for the induction step.

4. The Induction Step

After k steps, we have the following expansion for our unnormalized expectation:

$$[A]_h = Z_k \sum_{\{\bar{X}_\delta^{(k-1)}\}}' \prod_\delta \varrho_k(\bar{X}_\delta^{(k-1)}), \tag{4.1}$$

where

$$Z_k = e^{-\beta E^{\min}} A(\sigma^{\min}) \prod_{j=0}^{k-1} Z^{(j)}, \tag{4.2}$$

$$Z^{(j)} = \exp\Big(\sum_{\bar{Y}^{(j)}} V^{(j)}(\bar{Y}^{(j)})\Big) \prod_{\alpha: C_\alpha^{(j)} \subset S_j^g} z(C_\alpha^{(j)}). \tag{4.3}$$

The sum over $\{\bar{X}_\delta^{(k-1)}\}$ satisfies the following constraints:

(a) The $\bar{X}_\delta^{(k-1)}$'s are C-connected; if $\bar{X}_\delta^{(k-1)} \cap C_\alpha^{(k')} \neq \emptyset$, then $C_\alpha^{(k')} \subset \bar{X}_\delta^{(k-1)}$.

(b) $\bar{X}_{\delta_1}^{(k-1)} \cap \bar{X}_{\delta_2}^{(k-1)} \cap S_{k-1} = \emptyset$.

(c) All $\bar{X}_\delta^{(k-1)}$'s intersect S_{k-1}; their union covers all of S_k.

(d) If $\bar{X}_\delta^{(k-1)}$ intersects some $C_\alpha^{(k-1)} \subset S_{k-1}^g$, then it intersects $[\bar{C}_\alpha^{(k-1)}]^c$.

The last condition results from a division by appropriate $z(C_\alpha^{(k-1)})$'s, so that small $\bar{X}_\delta^{(k-1)}$'s intersecting S_{k-1}^g do not occur. The cluster activities $\varrho_k(\bar{X}_\delta^{(k-1)})$, the $z(C_\alpha^{(k-1)})$'s, and the $V^{(j)}(\bar{Y}^{(j)})$'s will be defined below.

Given a term on the right-hand side of (4.1), indexed by a collection of $\bar{X}^{(k-1)}$'s, we denote those $\bar{X}^{(k-1)}$'s that do not intersect S_k by $\{Y_\gamma^{(k)}\}$; those that do intersect S_k are denoted $\{X_\beta^{(k)}\}$. As in the first step, described in Sect. 3, we wish to exponentiate the part of the expansion involving the $Y^{(k)}$'s. Thus we extend the sums over $X^{(k)}$'s and $Y^{(k)}$'s to include clusters violating (b) above, subject to the constraints $Y_\gamma^{(k)} \cap S_k = \emptyset$, $X_{\gamma_1}^{(k)} \cap X_{\gamma_2}^{(k)} \cap S_k = \emptyset$. The additional terms are removed with factors u:

$$u(\bar{X}_1^{(k-1)}, \bar{X}_2^{(k-1)}) = \begin{cases} 0 & \text{if} \quad \bar{X}_1^{(k-1)} \cap \bar{X}_2^{(k-1)} \cap S_{k-1}^g \neq \emptyset, \\ 1 & \text{otherwise}. \end{cases}$$

The expansion (4.1) becomes

$$[A]_h = Z_k \sum_{\{X_\beta^{(k)}\}} \prod_{\mathscr{L}'} u_0(\mathscr{L}') \sum_{(Y_1^{(k)}, \ldots, Y_l^{(k)})} \frac{1}{l!} \prod_{\mathscr{L}} u(\mathscr{L}) \prod_\beta \varrho_k(X_\beta^{(k)}) \prod_{\gamma=1}^{l} \varrho_k(Y_\gamma^{(k)}), \tag{4.4}$$

where we have made explicit the constraints $X_{\delta_1}^{(k)} \cap X_{\delta_2}^{(k)} \cap S_k = \emptyset$ with factors $u_0(\mathscr{L}') = u_0(X_{\delta_1}^{(k)}, X_{\delta_2}^{(k)})$, which vanish when the constraint is not satisfied.

The $Y^{(k)}$-part of the expansion is now exponentiated, yielding

$$[A]_h = Z_k \sum_{\{X_\beta^{(k)}\}} \prod_{\mathscr{L}'} u_0(\mathscr{L}') \sum_{(Y_1^{(k)}, \ldots, Y_l^{(k)})} \frac{1}{l!} \sum_{G_c(\{X_\beta^{(k)}\})} \prod_{\mathscr{L} \in G_c(\{X^{(k)}\})} a(\mathscr{L})$$
$$\cdot \prod_\beta \varrho_k(X_\beta^{(k)}) \prod_{\gamma=1}^{l} \varrho_k(Y_\gamma^{(k)}) \exp\Big(\sum_{\bar{Y}^{(k)}} V^{(k)}(\bar{Y}^{(k)})\Big), \tag{4.5}$$

with

$$V^{(k)}(\bar{Y}^{(k)}) = \sum_{(Y_1^{(k)}, \ldots, Y_m^{(k)}) \text{ filling } \bar{Y}^{(k)}} \frac{1}{m!} \sum_{G_c} \prod_{\mathscr{L} \in G_c} a(\mathscr{L}) \prod_{\gamma=1}^{m} \varrho_k(Y_\gamma^{(k)}). \tag{4.6}$$

As before, $G_c(\{X_\beta^{(k)}\})$ is a graph such that each $Y_\gamma^{(k)}$ is ultimately connected to some $X_\beta^{(k)}$, and G_c is any connected graph on the $Y^{(k)}$'s. We now unite into a single cluster $\bar{X}^{(k)}$ all clusters $X^{(k)}$ or $Y^{(k)}$ involved in a connected component of the graph $G_c(\{X_\beta^{(k)}\})$. We obtain for (4.5),

$$[A]_h = Z_k \sum_{\{\bar{X}_\delta^{(k)}\}} \prod_\delta \bar{\varrho}_{k+1}(\bar{X}_\delta^{(k)}) \exp\Big(\sum_{\bar{Y}^{(k)}} V^{(k)}(\bar{Y}^{(k)})\Big), \tag{4.7}$$

with

$$\bar{\varrho}_{k+1}(\bar{X}^{(k)}) = \sum_{\{X^{(k)}\}, (Y_1^{(k)}, \ldots, Y_l^{(k)}) \text{ filling } \bar{X}^{(k)}} \frac{1}{l!} \prod_{\mathscr{L}'} u_0(\mathscr{L}')$$
$$\cdot \sum_{G_c} \prod_{\mathscr{L} \in G_c} a(\mathscr{L}) \prod_\beta \varrho_k(X_\beta^{(k)}) \prod_{\gamma=1}^{l} \varrho_k(Y_\gamma^{(k)}). \tag{4.8}$$

The $\bar{X}_\delta^{(k)}$'s in (4.7) of course satisfy $\bar{X}_{\delta_1}^{(k)} \cap \bar{X}_{\delta_2}^{(k)} \cap S_k = \emptyset$.

We now define activities for components $C_\alpha^{(k)} \subset S_k^g$ by summing over $\bar{X}^{(k)}$'s contained in $\bar{C}_\alpha^{(k)}$, the neighborhood of $C_\alpha^{(k)}$ defined in Sect. 2:

$$z(C_\alpha^{(k)}) = \sum_{\bar{X}^{(k)} \subset \bar{C}_\alpha^{(k)}} \bar{\varrho}_{k+1}(\bar{X}^{(k)}). \tag{4.9}$$

Dividing through by the product of the $z(C_\alpha^{(k)})$'s, we obtain the expansion (4.1) with $k+1$ replacing k:

$$[A]_h = Z_{k+1} \sum_{\{\bar{X}_\delta^{(k)}\}}' \prod_\delta \varrho_{k+1}(\bar{X}_\delta^{(k)}). \tag{4.10}$$

Here

$$\varrho_{k+1}(\bar{X}_\delta^{(k)}) = \bar{\varrho}_{k+1}(\bar{X}_\delta^{(k)}) \prod_{\alpha : C_\alpha^{(k)} \subset \bar{X}_\delta^{(k)}} z(C_\alpha^{(k)})^{-1}, \tag{4.11}$$

and we have included the product

$$\prod_{\alpha : C_\alpha^{(k)} \subset S_k^g} z(C_\alpha^{(k)}) \exp\Big(\sum_{\bar{Y}^{(k)}} V^{(k)}(\bar{Y}^{(k)})\Big) = Z^{(k)} \tag{4.12}$$

with Z_k to form Z_{k+1}. The prime indicates that the restrictions (a)–(d) stated after (4.3) are in effect.

We continue the process until all singular components have been incorporated into the expansion. In a finite volume there is a largest k for a singular component

$C_\alpha^{(k)}$, so $S_k=\emptyset$ for k sufficiently large. There are no $\bar{X}^{(k)}$'s at that point, since $\bar{X}^{(k)}\cap S_k \neq \emptyset$. Thus (4.10) reduces to $[A]_h = Z_{k+1}$, and using (4.2), (4.3) we can immediately calculate the logarithm, for $A=\exp\left(\sum_{i\in\mathscr{A}} s_i\sigma_i\right)$:

$$\log[A]_h = -\beta E^{\min} + \sum_{i\in\mathscr{A}} s_i\sigma_i^{\min} + \sum_{j=0}^{\infty}\left[\sum_{\alpha:C_\alpha^{(j)}\subset S_j^g} \log z(C_\alpha^{(j)}) + \sum_{\bar{Y}^{(j)}} V^{(j)}(\bar{Y}^{(j)})\right]. \tag{4.13}$$

Truncated expectations of spins can now be generated by differentiating with respect to the s_i.

Inductive Bounds

Bounds on polymer activities $\bar{\varrho}_k$, ϱ_k will be proven inductively in the next section. They guarantee that the procedure can be continued indefinitely. First, we define a sequence of decay rates, m_k, which decrease to some positive m_∞:

$$m_0 = m - c\,, \qquad m_{k+1} = m_k(1+c_0 d_{k-1}^{-1/2})^{-1}\,, \qquad k=0,1,2,\ldots,$$

with $d_{-1}\equiv 1$. We will also need numbers

$$n_i(k) = 1 + \mathrm{Card}\{j : j<k \text{ and } i\in \bar{C}_\alpha^{(j)}, \text{ for some } \alpha\}$$

to measure combinatoric effects of observables (or, equivalently, derivatives with respect to s). We also define the notion of "generalized covering": $\bar{X}^{(k)}$ *G-covers* a site i if $i\in\bar{X}^{(k)}$ or if $i\in\bar{C}_\alpha^{(j)}$ with $C_\alpha^{(j)}\subset\bar{X}^{(k)}$, $0\leqq j<\infty$. A collection of clusters $\{\bar{X}_\beta^{(k)}\}$ G-covers $\mathscr{A}\subset\mathbb{Z}^\nu$ if each $i\in\mathscr{A}$ is G-covered by some $\bar{X}_\beta^{(k)}$ and each $\bar{X}_\beta^{(k)}$ G-covers some $i\in\mathscr{A}$. The cluster activities $\bar{\varrho}_{k+1}(\bar{X}^{(k)})$, $\varrho_{k+1}(\bar{X}^{(k)})$ depend only on s_i for i G-covered by $\bar{X}^{(k)}$.

Proposition 4.1. *Let $\mathscr{B}\subset\mathscr{A}$ be a lattice subset. The polymer activities $\bar{\varrho}_{k+1}$ satisfy*

$$\sum_{\{\bar{X}^{(k)}\}\ G\text{-covering }\mathscr{A}} \left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_\beta \bar{\varrho}_{k+1}(\bar{X}_\beta^{(k)})\right| \prod_\beta \left[e^{m_{k+1}|\bar{X}_\beta^{(k)}\setminus S_k|} e^{-c_S|\bar{X}_\beta^{(k)}\cap S_k|}\right]$$
$$\leqq \left[\prod_{i\in\mathscr{B}} n_i(k)\right] 2^{|\mathscr{B}|}|\mathscr{B}|!\,. \tag{4.14}$$

Here c_S is a constant independent of $k\geqq 0$, and $(\partial/\partial s)_{\mathscr{B}} = \prod_{i\in\mathscr{B}} \partial/\partial s_i$. The $\bar{X}_\beta^{(k)}$'s satisfy $\bar{X}_\beta^{(k)}\cap\bar{X}_{\beta'}^{(k)}\cap S_k=\emptyset$, $\bar{X}_\beta^{(k)}\cap S_k\neq\emptyset$, and if $C_\alpha^{(k)}\subset\bar{X}_\beta^{(k)}$, then $\bar{X}_\beta^{(k)}\setminus\bar{C}_\alpha^{(k)}\neq\emptyset$. The exponentiated terms, $V^{(k)}$, satisfy

$$\sum_{\bar{Y}^{(k)}\ G\text{-covering }\mathscr{A}} \left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} V^{(k)}(\bar{Y}^{(k)})\right| e^{m_{k+1}|\bar{Y}^{(k)}|} \leqq \left[\prod_{i\in\mathscr{B}} n_i(k)\right] 2^{|\mathscr{B}|}|\mathscr{B}|!\,. \tag{4.15}$$

We need some estimates on the activities of singular components, in order to obtain bounds on ϱ_{k+1} from (4.14). These are contained in the following proposition, proven in Sect. 6.

Proposition 4.2. *Let $\mathscr{B}\subset\bar{C}_\alpha^{(k)}$. Then*

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} z(C_\alpha^{(k)})^{-1}\right| \leqq 2^{|\mathscr{B}|}|\mathscr{B}|!\,. \tag{4.16}$$

This implies, in particular, that

$$z(C_\alpha^{(k)})^{-1} \leqq 1 . \tag{4.17}$$

Bounds on $z(C_\alpha^{(k)})$ and its derivatives follow from (4.14), since in (4.9) we have $\bar{X}^{(k)} \cap S_k = C_\alpha^{(k)}$.

Corollary 4.3.

$$\left| \left(\frac{\partial}{\partial s} \right)_{\mathscr{B}} z(C_\alpha^{(k)}) \right| \leqq e^{cs|C_\alpha^{(k)}|} \left[\prod_{i \in \mathscr{B}} n_i(k) \right] 2^{|\mathscr{B}|} |\mathscr{B}|! . \tag{4.18}$$

Next, we formulate bounds on cluster activities ϱ_{k+1}.

Proposition 4.4.

$$\begin{aligned} &\sum_{\{\bar{X}_\beta^{(k)}\}\ G\text{-covering}\ \mathscr{A}} \left| \left(\frac{\partial}{\partial s} \right)_{\mathscr{B}} \prod_\beta \varrho_{k+1}(\bar{X}_\beta^{(k)}) \right| \prod_\beta \left[e^{m_{k+1}|\bar{X}_\beta^{(k)} \setminus S_k|} e^{-cs|\bar{X}_\beta^{(k)} \cap S_k|} \right] \\ &\quad \leqq \left[\prod_{i \in \mathscr{B}} n_i(k+1) \right] 2^{|\mathscr{B}|} |\mathscr{B}|! . \end{aligned} \tag{4.19}$$

The sum over $\{\bar{X}_\beta^{(k)}\}$ *is as in Proposition 4.1.*

Proof. We combine Propositions 4.1 and 4.2, since (4.11) expresses ϱ_{k+1} in terms of $\bar{\varrho}_{k+1}$ and $z(C_\alpha^{(k)})^{-1}$. We sum over $\mathscr{B}' \subset \mathscr{B}$, the set of derivatives that act on $\prod_\beta \bar{\varrho}_{k+1}(\bar{X}_\beta^{(k)})$. The rest, $\mathscr{B} \setminus \mathscr{B}'$, act on the inverse activities. We obtain

$$\begin{aligned} \text{(LHS of 4.19)} \leqq &\sum_{\mathscr{B}' \subset \mathscr{B}} \ \sum_{\{\bar{X}_\beta^{(k)}\}\ G\text{-covering}\ \mathscr{A}} \left| \left(\frac{\partial}{\partial s} \right)_{\mathscr{B}'} \prod_\beta \bar{\varrho}_{k+1}(\bar{X}_\beta^{(k)}) \right| \\ &\cdot \left| \left(\frac{\partial}{\partial s} \right)_{\mathscr{B} \setminus \mathscr{B}'} \prod_{\alpha : C_\alpha^{(k)} \subset \bigcup_\beta \bar{X}_\beta^{(k)}} z(C_\alpha^{(k)})^{-1} \right| \prod_\beta \left[e^{m_{k+1}|\bar{X}_\beta^{(k)} \setminus S_k|} e^{-cs|\bar{X}_\beta^{(k)} \cap S_k|} \right] . \end{aligned} \tag{4.20}$$

Note that each $C_\alpha^{(k)}$ appears only once, because $\bar{X}_\beta^{(k)} \cap \bar{X}_{\beta'}^{(k)} \cap S_k = \emptyset$. Furthermore, each derivative in $\mathscr{B} \setminus \mathscr{B}'$ can act on only one $z(C_\alpha^{(k)})^{-1}$ because the $\bar{C}_\alpha^{(k)}$'s are nonoverlapping, and $(\partial/\partial s_i) z(C_\alpha^{(k)}) = 0$ for $i \notin \bar{C}_\alpha^{(k)}$. Thus the derivatives of the inverse activities are bounded by

$$\left[\prod_{i \in \mathscr{B} \setminus \mathscr{B}'} (n_i(k+1) - n_i(k)) \right] 2^{|\mathscr{B} \setminus \mathscr{B}'|} |\mathscr{B} \setminus \mathscr{B}'|! .$$

Here we have used the fact that $n_i(k+1) - n_i(k) = 1$ if i is in some $\bar{C}_\alpha^{(k)}$, 0 otherwise. Applying (4.14) we obtain

$$\begin{aligned} \text{(LHS of 4.19)} &\leqq \sum_{\mathscr{B}' \subset \mathscr{B}} \prod_{i \in \mathscr{B}'} n_i(k) \prod_{i \in \mathscr{B} \setminus \mathscr{B}'} (n_i(k+1) - n_i(k)) 2^{|\mathscr{B}|} |\mathscr{B}|! \\ &\leqq \left[\prod_{i \in \mathscr{B}} n_i(k+1) \right] 2^{|\mathscr{B}|} |\mathscr{B}|! , \end{aligned} \tag{4.21}$$

which completes the proof.

5. Estimates for the Induction Step

This section is devoted to the proof of Proposition 4.1. To make an induction, we first prove (4.19) for $\varrho_0 \equiv \varrho$ (case $k=-1$). We use $\bar{X}^{(-1)}$ to denote either $X^{(0)}$ or $Y^{(0)}$. Using the estimates (3.7), (3.8) we wish to prove that

$$\sum_{\{\bar{X}_\beta^{(-1)}\}\ G\text{-covering}\ \mathscr{A}} \left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_\beta \varrho(\bar{X}_\beta^{(-1)})\right| \prod_\beta [e^{m_0|\bar{X}_\beta^{(-1)}\setminus S_0|} e^{-c|\bar{X}_\beta^{(-1)}\cap S_0|}] \leqq 2^{|\mathscr{B}|}|\mathscr{B}|!\,, \tag{5.1}$$

where $\bar{X}_\beta^{(-1)}$'s are nonoverlapping. Here the constant c is somewhat smaller than c_S, the constant in (4.14). No overlap implies that each s-derivative must act on a particular term in the product over β. Thus (3.7) and (3.8) yield

$$(\text{LHS of } 5.1) \leqq \sum_{\{\bar{X}_\beta^{(-1)}\}\ G\text{-covering}\ \mathscr{A}} 2^{|\mathscr{B}|} \prod_\beta [e^{2\delta} e^{-(m-m_0)|\bar{X}_\beta^{(-1)}\setminus S_0|} e^{-(c-\log 2)|\bar{X}_\beta^{(-1)}\cap S_0|}]. \tag{5.2}$$

We estimate (5.2) by repeatedly using the bound $\sum_t f(t) \leqq \sup_t c_t f(t)$, which is valid when $\sum_t c_t^{-1} \leqq 1$. The c_t's are called combinatoric factors. We allow a factor $2^{|\mathscr{A}|}$ in order to fix a subset $\{a_1, \ldots, a_m\} \subset \mathscr{A}$, and then we sum over clusters $\bar{X}_1^{(-1)}, \ldots, \bar{X}_m^{(-1)}$ such that a_j is G-covered by $\bar{X}_j^{(-1)}$. Since $|\mathscr{A}| \leqq \sum_\beta |\bar{X}_\beta^{(-1)}|$, we have

$$(\text{LHS of } 5.1) \leqq 2^{|\mathscr{B}|} \prod_{j=1}^m \left[\sum_{\bar{X}_j^{(-1)}\ G\text{-covering}\ a_j} 2^{-(2\nu+2)|\bar{X}_j^{(-1)}\cap S_0|} e^{-c'|\bar{X}_j^{(-1)}\setminus S_0|}\right], \tag{5.3}$$

where we assume $m_0 \leqq m - 2\delta - \log 2 - c'$, $c - 2\log 2 - 2\delta \geqq (2\nu+2)\log 2$. We now prove that the sum inside the brackets is less than 1, for c' a sufficiently large constant. We first sum over the ways that $\bar{X}_j^{(-1)}$ can G-cover a_j. Either $a_j \in \bar{X}_j^{(-1)}\setminus S_0$, or else $a_j \in \bar{C}_\alpha^{(k)}$ with $C_\alpha^{(k)} \subset \bar{X}_j^{(-1)}$. The first case is given a combinatoric factor $2 \leqq 2^{|X_j^{(-1)}\setminus S_0|}$; the second is given $4^{(k+1)} \leqq 2^{2|\bar{X}_j^{(-1)}\cap S_0|}$. This suffices, since $1/2 + \sum_{k=0}^{\infty} 4^{-(k+1)} \leqq 1$. In the first case we proceed by summing over X, the component of $\bar{X}_j^{(-1)}\setminus S_0$ containing a_j, using the standard estimate

$$\sum_{X \ni 0,\ X\ \text{connected}} e^{-c''|X|} \leqq 1\,. \tag{5.4}$$

A factor $2^{|X|}$ allows us to choose which $C_\alpha^{(k)}$'s neighboring X are contained in $\bar{X}_j^{(-1)}$. Then if $c' = c'' + 2\log 2$, all three combinatoric factors are cancelled by $e^{-c'|\bar{X}_j^{(-1)}\setminus S_0|}$ in (5.3). We are now in a situation like the second case, where a_j is in a fixed $\bar{C}_\alpha^{(k)}$. Let X' denote the union of the chosen $C_\alpha^{(k)}$'s. We must allow a combinatoric factor $2^{2\nu|X'|}$ for the choice of a subset of the (at most) $2\nu|X'|$ sites neighboring X'. These sites are the "starting points" for additional components of $\bar{X}_j^{(-1)}\setminus S_0$. We use (5.4) for these components, and continue the process until the choice is made not to add new $C_\alpha^{(k)}$'s or components of $\bar{X}_j^{(-1)}\setminus S_0$. All combinatoric factors are cancelled by terms in (5.3); thus the whole sum is bounded by 1. This completes the proof of (5.1); the $|\mathscr{B}|!$ is unnecessary here.

We now prove (4.14) for $k \geqq 0$, assuming (4.19) for $k-1$. Let us rewrite the definition of $\bar{\varrho}_{k+1}$ [Eq. (4.8)] in a form treating $X^{(k)}$'s and $Y^{(k)}$'s on the same footing. Both $X^{(k)}$ and $Y^{(k)}$ are types of $\bar{X}^{(k-1)}$, depending on whether the set intersects S_k or not. We write

$$\sum_{\{X_\beta^{(k)}\}} = \sum_{(X_1^{(k)},\ldots,X_{l'}^{(k)})} \frac{1}{l'!},$$

using the fact that the u_0-factors in (4.8) remove duplicated $X^{(k)}$'s. Since $\frac{1}{l'!}\frac{1}{l!} = \frac{1}{(l+l')!}\binom{l+l'}{l}$, the sum over $X^{(k)}$'s and $Y^{(k)}$'s can be rewritten as follows:

$$\bar{\varrho}_{k+1}(\bar{X}^{(k)}) = \sum_{(\bar{X}_1^{(k-1)},\ldots,\bar{X}_p^{(k-1)}) \text{ filling } \bar{X}^{(k)}} \frac{1}{p!} \prod_{\mathscr{L}'} u_0(\mathscr{L}') \sum_{G_c} \prod_{\mathscr{L}\in G_c} a(\mathscr{L}) \prod_{\delta=1}^{p} \varrho_k(\bar{X}_\delta^{(k-1)}). \tag{5.5}$$

In (5.5) it is assumed that $\bar{X}^{(k)} \cap S_k \neq \emptyset$, but it will be helpful to avoid this constraint later on. So we unify notations by defining $\bar{\varrho}_{k+1}(\bar{Y}^{(k)})$ by the same formula (5.5), where $\bar{Y}^{(k)}$ is a union of $Y^{(k)}$'s, $\bar{Y}^{(k)} \cap S_k \neq \emptyset$. Thus $\bar{\varrho}_{k+1}(\bar{Y}^{(k)}) \equiv V^{(k)}(\bar{Y}^{(k)})$, by (4.6). We let $\bar{Z}^{(k)}$ denote either an $\bar{X}^{(k)}$ or a $\bar{Y}^{(k)}$. We prove a stronger form of (4.14) allowing arbitrary $\bar{Z}^{(k)}$'s. As a bonus we obtain (4.15) as a special case by restricting to a single $\bar{Z}^{(k)} = \bar{Y}^{(k)}$ in the collection $\{\bar{Z}_\beta^{(k)}\}$.

We insert our formula for $\bar{\varrho}_{k+1}(\bar{Z}^{(k)})$ into the left-hand side of (4.14), allowing arbitrary $\bar{Z}^{(k)}$'s. Let us note that $\bar{Z}_{\beta_1}^{(k)} \cap \bar{Z}_{\beta_2}^{(k)} \cap S_k = \emptyset$, and in (5.5), $\bar{X}_{\delta_1}^{(k-1)} \cap \bar{X}_{\delta_2}^{(k-1)} \cap S_k = \emptyset$. Thus the entire collection of $\bar{X}^{(k-1)}$'s satisfies $\bar{X}_\delta^{(k-1)} \cap \bar{X}_{\delta'}^{(k-1)} \cap S_k = \emptyset$, and we make this explicit by extending the product over $\mathscr{L}'$ to include all pairs of $\bar{X}^{(k-1)}$'s. We combine all the sums over collections of $\bar{X}^{(k-1)}$'s using the multinomial theorem. Thus

$$\begin{aligned}
&\sum_{\{\bar{Z}_\beta^{(k)}\}\ G\text{-covering } \mathscr{A}} \left| \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_\beta \bar{\varrho}_{k+1}(\bar{Z}^{(k)}) \right| \prod_\beta \left[e^{m_{k+1}|\bar{Z}_\beta^{(k)} \setminus S_k|} e^{-cs|\bar{Z}_\beta^{(k)} \cap S_k|} \right] \\
&\leqq \sum_{(\bar{X}_1^{(k-1)},\ldots,\bar{X}_p^{(k-1)}):\ \mathscr{A} \text{ is } G\text{-covered}} \frac{1}{p!} \prod_{\mathscr{L}'} u_0(\mathscr{L}') \\
&\quad \cdot \left| \sum_G \prod_{\mathscr{L}\in G} a(\mathscr{L}) \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_{\delta=1}^{p} \varrho_k(\bar{X}_\delta^{(k-1)}) \right| \prod_\delta \left[e^{m_{k+1}|\bar{X}_\delta^{(k-1)} \setminus S_k|} e^{-cs|\bar{X}_\delta^{(k-1)} \cap S_k|} \right].
\end{aligned} \tag{5.6}$$

On the right-hand side, *not* every $\bar{X}_\delta^{(k-1)}$ need G-cover part of $\mathscr{A}$, but each $i \in \mathscr{A}$ is G-covered by some $\bar{X}_\delta^{(k-1)}$. The graph G is the union of the connected graphs G_c corresponding to each $\bar{Z}_\beta^{(k)}$. It need not be connected but each connected component involves a cluster $\bar{X}_\delta^{(k-1)}$ that G-covers some $i \in \mathscr{A}$. (Note that by considering arbitrary $\bar{Z}^{(k)}$'s we lose the constraint that there be an $X^{(k)}$ in each connected component of G.) We have distributed the exponential factors amongst the $\bar{X}_\delta^{(k-1)}$'s, but the conditions $\bar{X}_\delta^{(k-1)} \cap \bar{X}_{\delta'}^{(k-1)} \cap S_k = \emptyset$ imply that each portion of $\bar{X}_\beta^{(k)} \cap S_k$ is used only once.

To estimate the right-hand side of (5.6), we need to isolate those clusters G-covering $\mathscr{A}$. These will be held fixed while summing over the others. Call these the $\mathscr{A}$-clusters, and denote by $G_{\mathscr{A}}$ the subgraph of G of lines connecting $\mathscr{A}$-clusters. Let $G_c = G \backslash G_{\mathscr{A}}$. The graph $G_{\mathscr{A}}$ is an arbitrary graph on the $\mathscr{A}$-clusters (each cluster is connected, directly or indirectly, to some $\mathscr{A}$-cluster). Using this decomposition we have

$$\begin{aligned}(5.6) \leqq & \sum_{(\bar{X}_1^{(k-1)}, \ldots, \bar{X}_n^{(k-1)}) \text{ G-covering } \mathscr{A}} \frac{1}{n!} \sum_{(\bar{X}_{n+1}^{(k-1)}, \ldots, \bar{X}_{n+p}^{(k-1)})} \frac{1}{p!} \prod_{\mathscr{L}'} u_0(\mathscr{L}') \\ & \cdot \left| \sum_{G_{\mathscr{A}}} \prod_{\mathscr{L} \in G_{\mathscr{A}}} a(\mathscr{L}) \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_{\delta=1}^{n} \varrho_k(\bar{X}_\delta^{(k-1)}) \right| \left| \sum_{G_c} \prod_{\mathscr{L} \in G_c} a(\mathscr{L}) \prod_{\delta=n+1}^{n+p} \varrho_k(\bar{X}_\delta^{(k-1)}) \right| \\ & \cdot \prod_{\delta=1}^{n+p} [e^{m_{k+1}|X \backslash S_k|} e^{-c_S |X \cap S_k|}] . \end{aligned} \tag{5.7}$$

Since $G_{\mathscr{A}}$ is arbitrary, we have $\sum_{G_{\mathscr{A}}} \prod_{\mathscr{L} \in G_{\mathscr{A}}} a(\mathscr{L}) = \prod_{\mathscr{L}} u(\mathscr{L})$, where the product runs over pairs of $\mathscr{A}$-clusters. The u-factors enforce the constraints $\bar{X}_{\beta_1}^{(k-1)} \cap \bar{X}_{\beta_2}^{(k-1)} \cap S_{k-1}^g = \emptyset$, and the u_0-factors imply that $\bar{X}_{\beta_1}^{(k-1)} \cap \bar{X}_{\beta_2}^{(k-1)} \cap S_k = \emptyset$. Thus we have $\bar{X}_{\beta_1}^{(k-1)} \cap \bar{X}_{\beta_2}^{(k-1)} \cap S_{k-1} = \emptyset$, which is one of the conditions that were assumed in Proposition 4.1 (the inductive hypothesis). If $k=0$ the u-factors imply that $\bar{X}_{\beta_1}^{(k-1)}, \bar{X}_{\beta_2}^{(k-1)}$ do not overlap, which was assumed when we verified the $k=-1$ case. We can drop all the other constraints $u_0(\mathscr{L}')$ – this only makes the right-hand side of (5.7) bigger.

The following lemma controls the sums over $\bar{X}_{n+1}^{(k-1)}, \ldots, \bar{X}_{n+p}^{(k-1)}$.

Lemma 5.1. *Let $\bar{X}_1^{(k-1)}, \ldots, \bar{X}_n^{(k-1)}$ be fixed. Then for $k>0$*

$$\begin{aligned} & \sum_{p \geqq P} \sum_{(\bar{X}_{n+1}^{(k-1)}, \ldots, \bar{X}_{n+p}^{(k-1)})} \frac{1}{p!} \left| \sum_{G_c} \prod_{\mathscr{L} \in G_c} a(\mathscr{L}) \right| \prod_{\delta=n+1}^{n+p} |\varrho_k(\bar{X}_\delta^{(k-1)}) e^{m_{k+1}|\bar{X}_\delta^{(k-1)} \backslash S_k|} e^{-c_S |\bar{X}_\delta^{(k-1)} \cap S_k|}| \\ & \leqq \exp\left[\sum_{\delta=1}^{n} c_1 d_{k-1}^{-1/2} |\bar{X}_\delta^{(k-1)} \backslash S_k| \right] . \end{aligned} \tag{5.8}$$

In G_c, each $\bar{X}_{n+j}^{(k-1)}$, $1 \leqq j \leqq p$, is ultimately connected to some $\bar{X}_l^{(k-1)}$, $1 \leqq l \leqq n$. If $k=0$ the estimate holds with $|\bar{X}_\delta^{(k-1)}|$ replacing $|\bar{X}_\delta^{(k-1)} \backslash S_k|$ on the right-hand side ($d_{-1} \equiv 1$).

The basic content of the lemma is the fact that the clusters connected to a fixed set of clusters forms a gas with a small activity, $c_1 d_{k-1}^{-1/2}$.

We use the lemma with $P = \infty$ to estimate the right-hand of (5.7) by

$$\begin{aligned} & \sum_{\{\bar{X}_1^{(k-1)}, \ldots, \bar{X}_n^{(k-1)}\} \text{ G-covering } \mathscr{A}} \left| \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \prod_{\delta=1}^{n} \varrho_k(\bar{X}_\delta^{(k-1)}) \right| \\ & \cdot \prod_{\delta=1}^{n} \exp[(m_{k+1} + c_1 d_{k-1}^{-1/2}) |\bar{X}_\delta^{(k-1)} \backslash S_k| - c_S |\bar{X}^{(k-1)} \cap S_k|] , \quad k>0 . \end{aligned} \tag{5.9}$$

For $k=0$, c_S is replaced by $c = c_S - c_1$, to allow for the modification in Lemma 5.1. Now we use the inequality

$$|\bar{X}_\delta^{(k-1)} \cap S_{k-1}^g| \leqq c_2 d_{k-1}^{-1/2} |\bar{X}_\delta^{(k-1)} \backslash S_{k-1}| , \quad k>0 , \tag{5.10}$$

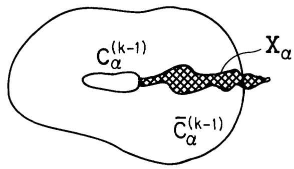

Fig. 1

to estimate the argument of the exponential as follows:

$$(m_{k+1}+c_1 d_{k-1}^{-1/2})|\bar{X}_\delta^{(k-1)}\backslash S_k| - c_S|\bar{X}_\delta^{(k-1)}\cap S_k|$$

$$\leqq [(m_{k+1}+c_1 d_{k-1}^{-1/2})(1+c_2 d_{k-1}^{-1/2})+c_S c_2 d_{k-1}^{-1/2}]|\bar{X}_\delta^{(k-1)}\backslash S_{k-1}| - c_S|\bar{X}_\delta^{(k-1)}\cap S_{k-1}|$$

$$\leqq m_k|\bar{X}_\delta^{(k-1)}\backslash S_{k-1}| - c_S|\bar{X}_\delta^{(k-1)}\cap S_{k-1}|\,. \tag{5.11}$$

Here $m_{k+1}=m_k(1+c_0 d_{k-1}^{-1/2})^{-1}$, with c_0 chosen appropriately. Now we can use (4.19) to obtain

$$(5.9) \leqq \prod_{i\in\mathscr{B}} n_i(k)2^{|\mathscr{B}|}|\mathscr{B}|!\,, \tag{5.12}$$

which completes the proof of Proposition 4.1, assuming (5.10) and Lemma 5.1. [For the case $k=0$, we have $m_{k+1}+c_1 d_{k-1}^{-1/2}\leqq m_k$, so by (5.1) we estimate (5.9) (with c replacing c_S) by $2^{|\mathscr{B}|}|\mathscr{B}|!$ to complete the proof.]

To verify (5.10), note that to each singular component $C_\alpha^{(k-1)}\subset S_{k-1}^g$ contained in $\bar{X}_\delta^{(k-1)}$ we can attach a large C-connected subset, X_α, of $\bar{X}^{(k-1)}\backslash S_{k-1}$. (See Fig. 1.) If $k=1$ then $|X_\alpha|\geqq 1$, otherwise we have only that $\operatorname{diam}(X_\alpha)\geqq d_{k-1}$. The fact that $\operatorname{diam}(X_\alpha)\geqq d_{k-1}$ follows by noticing that $(\bar{X}_\delta^{(k-1)}\backslash S_{k-1})\cap[\bar{C}_\alpha^{(k-1)}]^c\neq\emptyset$. This is because all $\varrho(\bar{X}^{(k-1)})$'s with $C_\alpha^{(k-1)}\subset\bar{X}^{(k-1)}$, $\bar{X}^{(k-1)}\subset\bar{C}_\alpha^{(k-1)}$, were removed when we divided by $z(C_\alpha^{(k-1)})$ – see condition (d) after (4.3). All the X_α's are disjoint because $\operatorname{dist}(C_\alpha^{(k-1)}, S_{k-1}\backslash C_\alpha^{(k-1)})\geqq d_k$.

Lemma 5.2. *If a C-connected set X does not intersect S_{k-1} and it has diameter at least $d_{k-1}/2$, then*

$$|X|\geqq(1-3d_1^{-(\alpha-1)/2})\operatorname{diam}X\,. \tag{5.13}$$

Proof. The set X can lose volume because of empty space in $C_\alpha^{(j)}\subset X$. However, by (2.3), $\operatorname{dist}(C_\alpha^{(j)}, C_{\alpha'}^{(j)})\geqq d_{j+1}=d_j^\alpha$ and by (2.1), $\operatorname{diam}C_\alpha^{(j)}\leqq d_j^{(1+\alpha)/2}$. Thus the fraction of $\operatorname{diam}X$ lost due to $C_\alpha^{(j)}$'s is less than $2d_j^{(1+\alpha)/2}/d_j^\alpha=2d_j^{-(\alpha-1)/2}$, $j\geqq 1$. Similarly, the fraction lost due to $C_\alpha^{(0)}$'s is less than d_1^{-1}, and the total loss is

$$\sum_{j=1}^{k-2} 2d_j^{-(\alpha-1)/2}+d_1^{-1}\leqq 3d_1^{-(\alpha-1)/2}\,, \tag{5.14}$$

which yields (5.13).

Using the lemma we find that

$$|X_\alpha|\geqq\tfrac{1}{2}d_{k-1}\geqq\tfrac{1}{2}d_{k-1}^{1/2}|C_\alpha^{(k-1)}|\,, \tag{5.15}$$

since by (2.2), $|C_\alpha^{(k-1)}|\leqq\sqrt{d_{k-1}}$. The bound (5.10) now follows by summing over $C_\alpha^{(k-1)}\subset\bar{X}_\delta^{(k-1)}\cap S_{k-1}^g$.

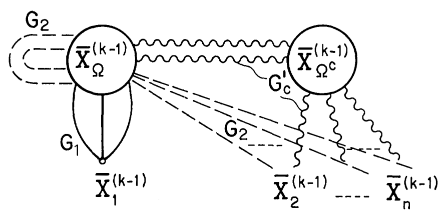

Fig. 2

Proof of Lemma 5.1. We use the method of [7]. Assume the lemma for smaller $n+P$. For $n+P=1$ the lemma is trivial. We decompose G_c into $G_1 \cup G_2 \cup G_c'$: G_1 contains lines $\{\bar{X}_1^{(k-1)}, \bar{X}_{n+j}^{(k-1)}\}_{j\in\Omega}$, $\Omega \subset \{1, \ldots, p\}$; G_2 contains lines $\{\bar{X}_l^{(k-1)}, \bar{X}_{n+j}^{(k-1)}\}_{2\leq l\leq n+p,\, j\in\Omega}$; and G_c' contains lines $\{\bar{X}_{n+j}^{(k-1)}, \bar{X}_l^{(k-1)}\}_{j\in\Omega^c,\, 2\leq l\leq n+p}$.

It is easy to see that G_2 is arbitrary, while G_c' is a graph in which each $\bar{X}_{n+j}^{(k-1)}$, $j\in\Omega^c$ is ultimately connected to $\bar{X}_2^{(k-1)}, \ldots, \bar{X}_n^{(k-1)}$ or to $\bar{X}_{n+j'}^{(k-1)}, j'\in\Omega$. Summing up $1+a(\mathcal{L})=u(\mathcal{L})$ for $\mathcal{L}\in G_2$, we obtain (see Fig. 2)

$$\sum_{G_c}\prod_{\mathcal{L}\in G_c} a(\mathcal{L}) = \sum_{\Omega}\prod_{\mathcal{L}\in G_1} a(\mathcal{L}) \prod_{\substack{\mathcal{L}=\{\bar{X}_l^{(k-1)},\bar{X}_{n+j}^{(k-1)}\}_{2\leq l\leq n,\ \text{or}\ l=n+m} \\ m\in\Omega;\ j\in\Omega}} u(\mathcal{L}) \sum_{G_c'}\prod_{\mathcal{L}\in G_c'} a(\mathcal{L}). \tag{5.16}$$

We bound $u(\mathcal{L})$ by 1 and $a(\mathcal{L})$ by 1 for $\mathcal{L}\in G_1$; however we enforce the condition that $\bar{X}_{n+j}^{(k-1)}$ intersect $\bar{X}_1^{(k-1)}\cap S_{k-1}^g$ (overlap $\bar{X}_1^{(k-1)}$ if $k=0$) for all $j\in\Omega$, since $a(\mathcal{L})=0$ otherwise. We also write

$$\begin{aligned} &\sum_{p\leq P}\sum_{\Omega}\sum_{(\bar{X}_{n+1}^{(k-1)},\ldots,\bar{X}_{n+p}^{(k-1)})}\frac{1}{p!} \\ &= \sum_{|\Omega|=0}^{p}\frac{1}{|\Omega|!}\sum_{(\bar{X}_{n+1}^{(k-1)},\ldots,\bar{X}_{n+|\Omega|}^{(k-1)})}\sum_{p'=0}^{P-|\Omega|}\sum_{(\bar{X}_{n+|\Omega|+1}^{(k-1)},\ldots,\bar{X}_{n+|\Omega|+p'}^{(k-1)})}\frac{1}{(p'-|\Omega|)!}. \end{aligned} \tag{5.17}$$

For $k>0$ the left-hand side of (5.8) is now bounded by

$$\begin{aligned} &\sum_{|\Omega|=0}^{P}\frac{1}{|\Omega|!}\sum_{(\bar{X}_{n+1}^{(k-1)},\ldots,\bar{X}_{n+|\Omega|}^{(k-1)})}\sum_{p'=0}^{P-|\Omega|}\frac{1}{p'!}\sum_{(\bar{X}_{n+|\Omega|+1}^{(k-1)},\ldots,\bar{X}_{n+|\Omega|+p'}^{(k-1)})} \\ &\quad\cdot\left|\sum_{G_c'}\prod_{\mathcal{L}\in G_c'} a(\mathcal{L})\right|\prod_{l=n+1}^{n+|\Omega|+p'}\left|\varrho_k(\bar{X}_l^{(k-1)})e^{m_{k+1}|\bar{X}_l^{(k-1)}\setminus S_k|}e^{-c_S|\bar{X}_l^{(k-1)}\cap S_k|}\right| \\ &\leq \sum_{|\Omega|=0}^{P}\frac{1}{|\Omega|!}\left\{\sum_{\bar{X}^{(k-1)}:a(\bar{X}^{(k-1)},\bar{X}_1^{(k-1)})\neq 0}|\varrho_k(\bar{X}^{(k-1)})|\right. \\ &\quad\left.\cdot\exp[(m_{k+1}+c_1d_{k-1}^{-1/2})|\bar{X}^{(k-1)}\setminus S_k| - c_S|\bar{X}^{(k-1)}\cap S_k|]\right\}^{|\Omega|} \\ &\quad\cdot\exp\left[\sum_{l=2}^{n}c_1d_{k-1}^{-1/2}|\bar{X}_l^{(k-1)}\setminus S_k|\right]. \end{aligned} \tag{5.18}$$

We have used the induction hypothesis, (5.8) with $P\to P-|\Omega|$, $n\to n+|\Omega|-1$, $n+P\to n+P-1$. The lemma clearly follows from (5.18), provided that

$$\begin{aligned} &\sum_{\bar{X}^{(k-1)}:a(\bar{X}^{(k-1)},\bar{X}_1^{(k-1)})\neq 0}|\varrho_k(\bar{X}^{(k-1)})| \\ &\quad\cdot\exp[(m_{k+1}+c_1d_{k-1}^{-1/2})|\bar{X}^{(k-1)}\setminus S_k| - c_S|\bar{X}^{(k-1)}\cap S_k|] \leq c_1d_{k-1}^{-1/2}|\bar{X}_1^{(k-1)}\setminus S_k|. \end{aligned} \tag{5.19}$$

We prove (5.19) by noting that $a(\bar{X}^{(k-1)}, \bar{X}_1^{(k-1)}) \neq 0$ implies that $\bar{X}^{(k-1)} \cap \bar{X}_1^{(k-1)} \cap S_{k-1}^g \neq \emptyset$. Now there are at most $c_1 d_{k-1}^{-1/2} |\bar{X}_1^{(k-1)} \backslash S_k|$ sites in $\bar{X}_1^{(k-1)} \cap S_{k-1}^g$, by (5.10). Furthermore, $m_{k+1} + c_1 d_{k-1}^{-1/2} \leqq m_k$, so (5.19) follows from

$$\sum_{\bar{X}^{(k-1)} \ni i} |\varrho_k(\bar{X}^{(k-1)})| e^{m_k |\bar{X}^{(k-1)} \backslash S_k| - c_S |\bar{X}^{(k-1)} \cap S_k|} \leqq 1. \tag{5.20}$$

This is a special case of Proposition 4.4, which was shown in Sect. 4 to follow from our induction hypothesis, Proposition 4.1. Thus Lemma 5.1 is proven for $k > 0$.

The case $k=0$ is only slightly different. On the right-hand side of (5.18) $|\bar{X}_l^{(k-1)} \backslash S_k|$ must be replaced with $|\bar{X}_l^{(k-1)}|$, and we have an additional term $c_k d_{k-1}^{-1/2} |\bar{X}^{(k-1)} \cap S_k|$ inside the first exponential. Since $c_S - c_1 d_{-1}^{-1/2}$ is equal to the constant c in (5.1), the lemma follows from

$$\sum_{\bar{X}^{(-1)} : a(\bar{X}^{(-1)}, \bar{X}_1^{(-1)}) \neq 0} |\varrho_0(\bar{X}^{(-1)})| \exp[(m_1 + c_1 d_{-1}^{-1/2}) |\bar{X}^{(-1)} \backslash S_0| - c |\bar{X}^{(-1)} \cap S_0|]$$
$$\leqq c_1 d_{-1}^{-1/2} |\bar{X}_1^{(-1)}|. \tag{5.21}$$

This follows from (5.1), because $m_1 + c_1 d_{-1}^{-1/2} \leqq m_0$ and because there are at most $c_1 |\bar{X}_1^{(-1)}|$ sites overlapping $\bar{X}_1^{(-1)}$. [Only $\bar{X}^{(-1)}$ overlapping $\bar{X}_1^{(-1)}$ have $a(\bar{X}^{(-1)}, \bar{X}_1^{(-1)}) \neq 0$.]

6. Estimates on Activities of Singular Sets

In this section we prove Proposition 4.2. It is crucial to obtain lower bounds on the activities $z(C_\alpha^{(k)})$ because we obtain ϱ_{k+1} from $\bar{\varrho}_{k+1}$ by dividing by these activities. The lower bound is not obvious from the expansion (4.9) defining $z(C_\alpha^{(k)})$, but by resuming the expansion we can express $z(C_\alpha^{(k)})$ as a ratio of certain partition functions, and the lower bound $z(C_\alpha^{(k)}) \geqq 1$ follows.

Let $\langle \cdot \rangle_{h, \bar{C}_\alpha^{(k)}}$ denote the normalized measure on spin configurations in $\bar{C}_\alpha^{(k)}$ obtained by restricting the original measure, $A e^{-\beta \mathscr{H}_\Lambda}$, to configurations with $\sigma = \sigma^{\min}$ in $[\bar{C}_\alpha^{(k)}]^c$. Thus

$$\langle B \rangle_{h, \bar{C}_\alpha^{(k)}} = [B]_{h, \bar{C}_\alpha^{(k)}} / [1]_{h, \bar{C}_\alpha^{(k)}},$$

with

$$[B]_{h, \bar{C}_\alpha^{(k)}} = \sum_{\{\sigma_i\}_{i \in \bar{C}_\alpha^{(k)}}} B(\sigma) A(\sigma) \exp\Big(\beta \sum_{\langle i,j \rangle \subset \bar{C}_\alpha^{(k)} \cup \partial \bar{C}_\alpha^{(k)}} \sigma_i \sigma_j + \beta \sum_{i \in \bar{C}_\alpha^{(k)}} h_i \sigma_i\Big), \tag{6.1}$$

and $\sigma = \sigma^{\min}$ outside $\bar{C}_\alpha^{(k)}$. We claim that

$$z(C_\alpha^{(k)})^{-1} = \langle \chi^{\min}(C_\alpha^{(k)}) \rangle_{h, \bar{C}_\alpha^{(k)}}, \tag{6.2}$$

where $\chi^{\min}(C_\alpha^{(k)})$ is the characteristic function of the event $\sigma = \sigma^{\min}$ in $C_\alpha^{(k)}$.

We prove (6.2) by comparing the expansions for $[\chi^{\min}(C_\alpha^{(k)})]_{h, C_\alpha^{(k)}}$ and $[1]_{h, C_\alpha^{(k)}}$. By (4.10) we have

$$[1]_{h, \bar{C}_\alpha^{(k)}} = Z_{k+1} = Z_k \exp\Big(\sum_{\bar{Y}^{(k)}} V^{(k)}(\bar{Y}^{(k)})\Big) z(C_\alpha^{(k)}), \tag{6.3}$$

where Z_{k+1} is given by (4.2), (4.3), except that everything is computed in $\bar{C}_\alpha^{(k)}$. Thus the sum over $\bar{Y}^{(j)}$ is restricted to $\bar{Y}^{(j)} \subset \bar{C}_\alpha^{(k)}$; the product over α is restricted to α such that $C_\alpha^{(j)} \subset S_j^g \cap \bar{C}_\alpha^{(k)}$. When we expand $[\chi^{\min}(C_\alpha^{(k)})]_{h,\bar{C}_\alpha^{(k)}}$, we pretend that $C_\alpha^{(k)}$ is not part of the interaction region. This is possible because σ is fixed to $\sigma^{\min}$ there. Then we do not obtain any clusters intersecting $C_\alpha^{(k)}$. Clusters otherwise vary arbitrarily in $\bar{C}_\alpha^{(k)}$ – in the first expansion, analogous to (3.2), there are even $Y^{(0)}$'s containing sites neighboring $C_\alpha^{(k)}$. Each $Z^{(j)}$ produced for $j<k$ is exactly as in the expansion for $[1]_{h,\bar{C}_\alpha^{(k)}}$, because they do not involve clusters intersecting $C_\alpha^{(k)}$. Similarly, we have the same factor $\exp\left(\sum_{\bar{Y}^{(k)} \subset \bar{C}_\alpha^{(k)}} V^{(k)}(\bar{Y}^{(k)})\right)$, but there is no $z(C_\alpha^{(k)})$ in $Z^{(k)}$. Thus

$$[\chi^{\min}(C_\alpha^{(k)})]_{h,\bar{C}_\alpha^{(k)}} = Z_k \exp\left(\sum_{\bar{Y}^{(k)} \subset \bar{C}_\alpha^{(k)}} V^{(k)}(\bar{Y}^{(k)})\right), \tag{6.4}$$

and (6.2) follows immediately by dividing by (6.3).

Since a characteristic function is bounded by 1, we have $z(C_\alpha^{(k)})^{-1} \leqq 1$, by (6.2). More generally, we obtain simple estimates on derivatives of $z(C_\alpha^{(k)})^{-1}$, since

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} z(C_\alpha^{(k)})^{-1} &= \left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \langle \chi^{\min}(C_\alpha^{(k)})\rangle_{h,\bar{C}_\alpha^{(k)}} \\ &= \left\langle \chi^{\min}(C_\alpha^{(k)}) \prod_{i\in\mathscr{B}} [\,;\sigma_i]\right\rangle_{h,\bar{C}_\alpha^{(k)}}.\end{aligned} \tag{6.5}$$

$\left[\text{Recall that a factor } A(\sigma) = \exp\left(\sum_{i\in\mathscr{A}} s_i\sigma_i\right) \text{ is present in the measure.}\right]$ After this truncated expectation is expanded into sums of products of ordinary expectations, we estimate each ordinary expectation by 1, since $\left|\chi^{\min}(C_\alpha^{(k)}) \prod_{i\in\mathscr{B}'} \sigma_i\right| \leqq 1$, for any $\mathscr{B}' \subset \mathscr{B}$. Thus

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} z(C_\alpha^{(k)})^{-1}\right| \leqq 2^{|\mathscr{B}|}|\mathscr{B}|!\,, \tag{6.6}$$

because there are at most $2^{|\mathscr{B}|}|\mathscr{B}|!$ terms in the expansion. (This is easily seen because after applying $r-1$ derivatives, each term has at most r expectations multiplied together. Each expectation can be differentiated in numerator or denominator, yielding at most $2r$ new terms for each existing term. Taking a product over $r=1,\ldots,|\mathscr{B}|$, we obtain at most $2^{|\mathscr{B}|}|\mathscr{B}|!$ terms.) This completes the proof of Proposition 4.2.

Corollary 6.1.

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log z(C_\alpha^{(k)})\right| \leqq 2^{|\mathscr{A}|}(|\mathscr{A}|-1)!\,. \tag{6.7}$$

Proof. By (6.2) we have

$$\left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log z(C_\alpha^{(k)}) = \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log[1]_{h,\bar{C}_\alpha^{(k)}} - \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log[\chi^{\min}(C_\alpha^{(k)})]_{h,\bar{C}_\alpha^{(k)}}.$$

Each of these terms is a truncated expectation of the σ_i, $i\in\mathscr{A}$. Hence it can be estimated as above by $2^{|\mathscr{A}|-1}(|\mathscr{A}|-1)!$.

7. Exponential Clustering of Correlations in the RFIM for Large H

In this section we harvest the fruits of our efforts in Sects. 2–6. We prove the results described in Sect. 1.3(i), i.e. we show that, in the RFIM, model (1), the thermodynamic limit of arbitrary correlations exists and is independent of boundary conditions, and connected correlations have tree decay, provided β and H are so large that

$$e^{(4\nu-\varepsilon_0 H)\beta} \ll 1\,, \tag{7.1}$$

where ε_0 is a small number. Thus we consider the regime where the external field is so strong that, most of the time, the orientation of the spin, σ_i, follows $\operatorname{sgn} h_i$.

The starting point of our considerations is Eq. (4.13), i.e.

$$\log[A]_h = -\beta E^{\min} + \sum_{i\in\mathscr{A}} s_i\sigma_i^{\min} + \sum_{j=0}^{\infty}\left[\sum_{\alpha:C_\alpha^{(j)}\subset S_j^g}^{\Lambda} \log z(C_\alpha^{(j)}) + \sum_{\bar{Y}^{(j)}}^{\Lambda} V^{(j)}(\bar{Y}^{(j)})\right], \tag{7.2}$$

where $\sum^{\Lambda}$ indicates that all the sets, $C_\alpha^{(j)}$ and $\bar{Y}^{(j)}$, to be summed over must be contained in Λ, and

$$A = \prod_{i\in\mathscr{A}} e^{s_i\sigma_i}\,. \tag{7.3}$$

Clearly,

$$\begin{aligned}&\langle\sigma_{i_1};\ldots;\sigma_{i_n}\rangle_{\beta,\Lambda}(h)\\ &= \sum_{j=0}^{\infty}\left[\sum_{\alpha:C_\alpha^{(j)}\supset\mathscr{A}}^{\Lambda}\left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log z(C_\alpha^{(j)}) + \sum_{\bar{Y}^{(j)}\ G\text{-covering}\ \mathscr{A}}^{\Lambda}\left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} V^{(j)}(\bar{Y}^{(j)})\right],\end{aligned} \tag{7.4}$$

where we have set $\mathscr{A}=\{i_1,\ldots,i_n\}$. The notion of "$G$-covering" of a set in $\mathbb{Z}^\nu$ has been defined in Sect. 4. A site i is G-covered by $\bar{Y}^{(j)}$ if $i\in\bar{Y}^{(j)}$, or if $i\in\bar{C}_\alpha^{(l)}$, with $C_\alpha^{(l)}\subset\bar{Y}^{(j)}$. For $n=1$, a term $\sigma_{i_1}^{\min}$ must be included on the right-hand side of (7.4). It comes from the second term on the right-hand side of (7.2) which is linear in $\{s_i\}$. Reflecting on (7.4) it is clear what the thermodynamic limit of $\langle\sigma_{i_1};\ldots;\sigma_{i_n}\rangle_{\beta,\Lambda}(h)$ is: For all $j<\infty$, the restriction that $\bar{C}_\alpha^{(j)}$ and $\bar{Y}^{(j)}$ lie in Λ is simply dropped on the right-hand side of (7.4). All terms then become independent of Λ, as soon as Λ is sufficiently large (depending on j and on $\bar{Y}^{(j)}$).

We now propose to study, with the help of the estimates proven in Sects. 5 and 6, whether the limits $j\to\infty$ and $\Lambda\nearrow\mathbb{Z}^\nu$ (thermodynamic limit) really exist, and how fast the thermodynamic limit is approached. We suppose that

$$\operatorname{dist}(\mathscr{A},\Lambda^c) \geqq d_k\,,\quad \text{for some } k\,, \tag{7.5}$$

where d_k is as in Sect. 2, i.e. $d_k = 2^{\alpha^{k+k_0}}$. By (2.8), we may estimate the probability that $\mathscr{A}\subset\bar{C}_\alpha^{(j)}$, for some α and $j\geqq k$, by

$$\operatorname{Prob}(\mathscr{A}\subset\bar{C}_\alpha^{(j)},\, j\geqq k) \leqq \sum_{j\geqq k} d_j^\nu (c'\varepsilon)^{2^j} \leqq (c\varepsilon)^{2^k}\,, \tag{7.6}$$

for finite constants c' and c. Here we use once again the fact that α must be chosen to be <2. We now assume that the magnetic fields $\{h_i\}$ are chosen such that

$$\mathscr{A} \not\subset \bar{C}_\gamma^{(j)}, \quad \text{for all } \gamma \text{ and all } \quad j \geqq k. \tag{7.7}$$

Since

$$\operatorname{diam}(\bar{C}_\gamma^{(l)}) \leqq d_{k-1}^{(1+\alpha)/2} + 4d_{k-1} \leqq \tfrac{1}{2} d_k,$$

for $l<k$, those components $C_\gamma^{(l)}$ for which $\mathscr{A} \subset \bar{C}_\gamma^{(l)}$ are, by (7.5), properly contained in the interior of Λ. We conclude that under these conditions the terms involving derivatives of $\log z(C_\gamma^{(j)})$ on the right-hand side of (7.4) are *independent* of Λ. By (7.6) we find therefore that, on sets of magnetic fields $\{h_i\}$ of probability $\geqq 1-(c\varepsilon)^{2^k}$, and for arbitrary $\Lambda' \supset \Lambda$,

$$\langle \sigma_{i_1}; \dots; \sigma_{i_n} \rangle_{\beta,\Lambda}(h) - \langle \sigma_{i_1}; \dots; \sigma_{i_n} \rangle_{\beta,\Lambda'}(h) = \sum_{j=0}^{\infty} \sum_{\bar{Y}^{(j)}}^{*} \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \{V_\Lambda^{(j)}(\bar{Y}^{(j)}) - V_{\Lambda'}^{(j)}(\bar{Y}^{(j)})\},$$

where $\sum^*$ ranges over all $\bar{Y}^{(j)}$ which G-cover $\mathscr{A}$ and at least one site in $\Lambda' \backslash \Lambda$ or in $\partial\Lambda$. Since $\bar{Y}^{(j)}$ G-covers $\mathscr{A}$ and at least one $i \in \Lambda' \backslash \Lambda$, and since $\bar{Y}^{(j)} \cap S_j = \emptyset$,

$$\operatorname{diam} \bar{Y}^{(j)} \geqq \operatorname{dist}(\mathscr{A}, \partial\Lambda)/2. \tag{7.8}$$

From condition (2.3) and the definition of $\bar{C}_\alpha^{(j)}$, more precisely

$$\operatorname{dist}(\partial \bar{C}_\alpha^{(j)}, C_\alpha^{(j)}) \geqq d_j,$$

and from our construction of $\bar{Y}^{(j)}$ in Sect. 4, (4.4)–(4.6), it follows that

$$\operatorname{diam} \bar{Y}^{(j)} \geqq d_j.$$

We may therefore use Lemma 5.2 and (7.8) to conclude that

$$|\bar{Y}^{(j)}| \geqq \max\{d_j/2, \operatorname{dist}(\mathscr{A}, \partial\Lambda)/4\}. \tag{7.9}$$

The bounds in Proposition 4.1, namely (4.15), and the simple fact that

$$m_j \geqq m_\infty \geqq cm, \quad \text{for some} \quad c>0,$$

for all j, show that

$$\sum_{\bar{Y}^{(j)}}^{*} \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} V^{(j)}(\bar{Y}^{(j)}) \leqq (j+1)^{|\mathscr{A}|} 2^{|\mathscr{A}|} |\mathscr{A}|! e^{-cm \max\{d_j/2, \operatorname{dist}(\mathscr{A}, \partial\Lambda)/4\}}. \tag{7.10}$$

From (7.10) we obtain the bound

$$\begin{aligned} &|\langle \sigma_{i_1}; \dots; \sigma_{i_n} \rangle_{\beta,\Lambda}(h) - \langle \sigma_{i_1}; \dots; \sigma_{i_n} \rangle_{\beta,\Lambda'}(h)| \\ &\leqq 2^n n! \exp(-cm \operatorname{dist}(\mathscr{A}, \partial\Lambda)/8) \left(\sum_{j=0}^{\infty} 2(j+1)^n e^{-cmd_j/4} \right) \\ &\leqq c_1^n (n!)^{1+\varepsilon'} \exp(-c_2 m \operatorname{dist}(\mathscr{A}, \partial\Lambda)), \end{aligned} \tag{7.11}$$

for constants c_1, c_2, and ε', which is valid with probability $\geqq 1-(c\varepsilon)^{2^k}$. The decay rate m is given by

$$m = (2H' - 4v) \sim c_3 \varepsilon H\beta, \quad \text{for large } H\beta. \tag{7.12}$$

We have thus proven

Theorem 7.1. *With probability* 1 *the limit*

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_{\beta,\Lambda}(h) \equiv \langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h)$$

exists and is independent of boundary conditions, for arbitrary $\{i_1, \ldots, i_n\}$. *The limit is approached exponentially fast and is given by the formula*

$$\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta = \sum_{j=0}^{\infty} \left[\sum_{\alpha: C_\alpha^{(j)} \supset \mathscr{A}} \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} \log z(C_\alpha^{(j)}) + \sum_{\bar{Y}^{(j)}\ G\text{-covering}\ \mathscr{A}} \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}} V^{(j)}(\bar{Y}^{(j)}) \right],$$

where $\mathscr{A} \equiv \{i_1, \ldots, i_n\}$, *and* $V^{(j)}(\bar{Y}^{(j)})$ *is calculated in any volume* Λ *that is somewhat larger than* $\bar{Y}^{(j)}$. *These terms satisfy bounds* (4.15), (6.7).

Next, we turn to our proof of *tree decay* of the correlations $\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h)$. We fix n arbitrary sites $x_1, \ldots, x_n$ in $\mathbb{Z}^\nu$ and set $i_j = \theta x_j$, $j = 1, \ldots, n$, with $\theta = 1, 2, 3, \ldots$. Let $\mathscr{A}_\theta = \{i_1, \ldots, i_n\}$. We begin by estimating the probability, $P(\theta_0)$, of the event that

$$\mathscr{A}_{\theta_l} \cap \bar{C}_\alpha^{(j)} = \emptyset, \quad \text{for any } \alpha \text{ and } \quad j \geqq p_l, \quad \text{and for all} \quad l = 0, 1, 2, \ldots. \tag{7.14}$$

Here $\theta_l = \theta_0 + l$, p_l is the largest integer with the property that $\theta_l > d_{p_l}$. Clearly,

$$P(\theta_0) \geqq 1 - \sum_{l=0}^{\infty} \mathrm{Prob}(\mathscr{A}_{\theta_l} \cap \bar{C}_\alpha^{(j)} \neq \emptyset, \text{ for some } \alpha \text{ and } j \geqq p_l). \tag{7.15}$$

By (7.6),

$$\mathrm{Prob}(\mathscr{A}_{\theta_l} \cap \bar{C}_\alpha^{(j)} \neq \emptyset, j \geqq p_l) \leqq n(c\varepsilon)^{2^{p_l}}.$$

Hence

$$\sum_{l=0}^{\infty} \mathrm{Prob}(\mathscr{A}_{\theta_l} \cap \bar{C}_\alpha^{(j)} \neq \emptyset, j \geqq p_l) \leqq \sum_{p=p_0}^{\infty} 2^{\alpha p + p_0} n(c'\varepsilon)^{2^p} \leqq n(c''\varepsilon)^{2^{p_0}}, \tag{7.16}$$

when $\theta_0 \to \infty$, $p_0 \to \infty$. Hence

$$1 - n(c''\varepsilon)^{2^{p_0}} \leqq P(\theta_0) \nearrow 1 \quad \text{as} \quad \theta_0 \to \infty, \tag{7.17}$$

if ε is small enough.

Let us now assume (7.14), and prove an estimate on $\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h)$, with $\{i_1, \ldots, i_n\} = \mathscr{A}_{\theta_l}$. Since

$$\mathrm{diam}\, \bar{C}_\alpha^{(j)} \leqq d_j^{(1+\alpha)/2} + 4d_j \ll d_{j+1}, \tag{7.18}$$

we have that $\mathscr{A}_{\theta_l} \subset \bar{C}_\alpha^{(j)}$ for $j < p_l$, because $\mathrm{diam}\, \mathscr{A}_{\theta_l} \geqq \theta_l > d_{p_l}$. By (7.14), $\mathscr{A}_{\theta_l} \not\subset \bar{C}_\alpha^{(j)}$ for $j \geqq p_l$. Therefore, we do not have to worry about derivatives of $\log z(C_\alpha^{(j)})$ in Theorem 7.1, and we conclude that

$$\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h) = \sum_{j=0}^{\infty} \sum_{\bar{Y}^{(j)}\ G\text{-covering}\ \mathscr{A}_{\theta_l}} \left(\frac{\partial}{\partial s}\right)_{\mathscr{A}_{\theta_l}} V^{(j)}(\bar{Y}^{(j)}). \tag{7.19}$$

Every $\bar{Y}^{(j)}$ contributing to the right-hand side of (7.19) G-covers $\mathscr{A}_{\theta_l}$. Thus $\bar{Y}^{(j)}$ contains a C-connected tree passing through $\mathscr{A}$. (The statement that a C-connected set X passes through $\mathscr{A}$ means that each site $i \in \mathscr{A}$ is either contained in X or in a set $\bar{C}_\alpha^{(l)}$ with the property that $C_\alpha^{(l)} \subset X$.) As in the proof of Lemma 5.2 it then follows that

$$|\bar{Y}^{(j)}| > \tfrac{1}{4}|T(i_1, \ldots, i_n)| + \tfrac{1}{4}\mathrm{diam}\, \bar{Y}^{(j)}, \tag{7.20}$$

where $T(i_1, \ldots, i_n)$ is the shortest tree containing $\mathscr{A}$. The details of the argument leading to (7.20) are a little lengthy, but straightforward. We only remark that (7.14) is used to insure that θ_l, the minimum distance between points in $\mathscr{A}_{\theta_l}$, is much larger than $d_{p_l-1}^{(1+\alpha)/2} + 4d_{p_l-1}$, the largest possible diameter of a $\bar{C}_\alpha^{(j)}$ intersecting $\mathscr{A}_{\theta_l}$. Repeating the arguments leading to (7.10) and (7.11) and using (7.17) we get the following result.

Theorem 7.2. *Let* $i_j = \theta x_j$, $j = 1, 2, \ldots, n$, *with* $\theta = \theta_0 + l$, $l = 0, 1, 2, \ldots$. *With probability* $P(\theta_0) \geqq 1 - n(c''\varepsilon)^{2^{p_0}}$,

$$|\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h)| \leqq c_1^n (n!)^{1+\varepsilon'} \exp[-c_2 m |T(i_1, \ldots, i_n)|],$$

for some finite constants c_1 *(depending on* $x_1, \ldots, x_n$ *and* θ_0*) and* c_2, *and some small* ε'.

It follows that, with probability 1, $\langle \sigma_{i_1}; \ldots; \sigma_{i_n} \rangle_\beta(h)$ has tree decay with decay rate $M \geqq \text{const} \cdot m > 0$, provided $H\beta$ is sufficiently large.

A simple application of the ergodic theorem shows that M is almost surely independent of the chosen sample, h.

8. Extensions to Other Models

Many of the features of our expansion method are model-independent, and we show here how to treat some other disordered systems. We consider two random Ising models. The first has randomness in the nearest-neighbor couplings J_{ij} as well as (possibly) a random magnetic field, and we consider the high-temperature regime. The second has a random J_{ij} but typically $J_{ij} \approx \bar{J} > 0$; we take zero magnetic field and low temperatures. Altogether we will have treated the three basic types of expansions for Ising systems – large magnetic field, high temperature, and low temperature expansions.

We remark that a much simpler high temperature expansion can be given if $|J_{ij}|$ is never larger than a fixed constant. For then the standard high temperature expansion $\exp(\sum \beta J_{ij} \sigma_i \sigma_j) = \sum \prod (e^{\beta J_{ij} \sigma_i \sigma_j} - 1)$ converges for β sufficiently small (depending on the range of J_{ij}). However, the temperature domain for convergence is unnecessarily restrictive, and deteriorates as the range of J_{ij} increases. We consider J_{ij}'s taking arbitrary real values, yet obtain a convergent expansion for $0 \leqq \beta \leqq \beta_0$, with $\beta_0 > 0$.

Similar remarks apply to the low temperature case, where a standard Peierls expansion can be given if $0 < C < J_{ij}$. The domain of convergence disappears as $C \to 0$, whereas we consider J_{ij}'s taking arbitrary real values.

High Temperature Spin Glass

This model is defined as follows:

$$[A]_{J,h} = \sum_{\{\sigma_i\}_{i \in \Lambda}} A(\sigma) \exp \beta \Big[\sum_{\langle i,j \rangle \subset \Lambda \cup \partial\Lambda} J_{ij} \sigma_i \sigma_j + \sum_{i \in \Lambda} h_i \sigma_i \Big],$$

$$\langle A \rangle_{J,h} = [A]_{J,h} / [1]_{J,h} \,. \tag{8.1}$$

The observable A is as before, σ is specified arbitrarily in Λ^c. The measure for the h_i's is an arbitrary probability measure, except that the h_i's are independent and identically distributed. We take independent, identically distributed J_{ij}'s also, Gaussian, with $\langle J_{ij}\rangle = \bar{J}$, $\langle (J_{ij}-\bar{J})^2\rangle = \Delta^2$.

We expand only the J_{ij}'s that are not too large, and we require that large J_{ij}'s are unlikely. By rescaling β, we can change the scale of the J_{ij}'s. Thus it is no loss of generality to choose the cutoff at $|J_{ij}|=1$, and we require

$$\frac{1}{\sqrt{2\pi}\Delta}\int_{|J|>1} e^{-(J-\bar{J})^2/2\Delta^2}dJ \equiv \varepsilon \ll 1\,. \tag{8.2}$$

For a good high temperature expansion, we need $\beta \ll 1$ also.

The singular sets S_k, S_k^q and the singular components $C_\alpha^{(k)}$ are constructed exactly as in Sect. 2, starting with

$$S_0 = \{j : |J_{ij}| > 1 \text{ for some nearest neighbor } i \text{ of } j\}\,.$$

The entropy estimates are unchanged.

As in the model considered in the body of the paper, it pays to be careful about what one perturbs about. In each component $C_\alpha^{(k)}$ we define $\sigma^{\min}|C_\alpha^{(k)}$ as the configuration minimizing the energy in $C_\alpha^{(k)}$,

$$E(\sigma, C_\alpha^{(k)}) = -\sum_{\langle i,j\rangle \subset C_\alpha^{(k)}} J_{ij}\sigma_i\sigma_j - \sum_{i\in C_\alpha^{(k)}} h_i\sigma_i\,. \tag{8.3}$$

Then we put $E^{\min}(C_\alpha^{(k)}) = E(\sigma^{\min}, C_\alpha^{(k)})$. We do not define $\sigma^{\min}$ in S_0^c because we are perturbing about fully disordered spins. However, we need the free energy of a site with a fully disordered spin,

$$e^{-\beta E^{\min}(i)} \equiv e^{\beta h_i} + e^{-\beta h_i}, \qquad i \in S_0^c\,. \tag{8.4}$$

Finally, we define $E^{\min} = \sum_{i\in S_0^c} E^{\min}(i) + \sum_{\alpha,k} E^{\min}(C_\alpha^{(k)})$. A similar process yields the analog of $A(\sigma^{\min})$. We put

$$\begin{aligned} A^{\min}(i) &\equiv \frac{e^{\beta h_i + s_i} + e^{-\beta h_i - s_i}}{e^{\beta h_i} + e^{-\beta h_i}}, \\ A^{\min}(C_\alpha^{(k)}) &\equiv \exp\Big(\sum_{i\in C_\alpha^{(k)}} s_i\sigma_i^{\min}\Big), \\ A(\sigma^{\min}) &\equiv \prod_{i\in S_0^c} A^{\min}(i) \prod_{\alpha,k} A^{\min}(C_\alpha^{(k)})\,. \end{aligned} \tag{8.5}$$

The cluster expansion consists in writing

$$e^{\beta J_{ij}\sigma_i\sigma_j} = 1 + (e^{\beta J_{ij}\sigma_i\sigma_j} - 1)$$

for bonds $\langle i,j\rangle \subset \Lambda \cup \partial\Lambda$ such that i or j is in S_0^c ($|J_{ij}|<1$). Expanding the product over such $\langle i,j\rangle$, we obtain a sum of terms. In each term we define the clusters $\{X_\beta^{(0)}, Y_\gamma^{(0)}\}$ as the connected components of $S_0 \cup \{i \in \Lambda : \langle i,j\rangle$ is a bond where $e^{\beta J_{ij}\sigma_i\sigma_j} - 1$ is selected for some $j\}$. As before, we let the $X_\beta^{(0)}$'s be the components intersecting S_0. Resuming all terms leading to the same clusters, we obtain the cluster expansion (3.2), with

$$\varrho(\bar{X}^{(-1)}) = \sum_{\sigma|\bar{X}^{(-1)}} \sum_{g_c} \prod_{\langle i,j\rangle \in g_c} (e^{\beta J_{ij}\sigma_i\sigma_j} - 1) \exp\left[\sum_{i\in \bar{X}^{(-1)}\setminus S_0} (\beta h_i + s_i)\sigma_i\right]$$
$$\cdot \exp\left[\sum_{\langle i,j\rangle \subset S_0\cap \bar{X}^{(-1)}} \beta J_{ij}(\sigma_i\sigma_j - \sigma_i^{\min}\sigma_j^{\min}) + \sum_{i\in S_0\cap\bar{X}^{(-1)}} (\beta h_i + s_i)(\sigma_i - \sigma_i^{\min})\right]$$
$$\cdot \prod_{i\in\bar{X}^{(-1)}\setminus S_0} [e^{-\beta E^{\min}(i)} A^{\min}(i)]^{-1}. \tag{8.6}$$

Here $\bar{X}^{(-1)}$ is either an $X_\beta^{(0)}$ or a $Y_\gamma^{(0)}$, and g_c is summed over all subsets of bonds $\langle i,j\rangle$ with i or j in S_0^c, with $i\in\bar{X}^{(-1)}, j\in\bar{X}^{(-1)}\cup\Lambda^c$, and such that all sites in $\bar{X}^{(-1)}\setminus S_0$ are in some bond in g_c.

We prove estimates (3.7), (3.8) for these cluster activities, with $m = 1/2|\log 2\beta| - 2\nu\log 2$, and with an extra $|\mathscr{B}|!$ on the right-hand sides. First, consider $\mathscr{B}=\emptyset$. Note that $\varrho(\bar{X}^{(-1)})$ is an expectation of

$$\sum_{g_c} \prod_{\langle i,j\rangle\in g_c} (e^{\beta J_{ij}\sigma_i\sigma_j} - 1)\exp\left[\sum_{\langle i,j\rangle\subset S_0\cap\bar{X}^{(-1)}} \beta J_{ij}(\sigma_i\sigma_j - \sigma_i^{\min}\sigma_j^{\min})\right.$$
$$\left. + \sum_{i\in S_0\cap\bar{X}^{(-1)}} (\beta h_i + s_i)(\sigma_i - \sigma_i^{\min})\right] \tag{8.7}$$

in a normalized measure $\frac{1}{N}\exp\left[\sum_{i\in\bar{X}^{(-1)}\setminus S_0} (\beta h_i + s_i)\sigma_i\right]$ on $\sigma|\bar{X}^{(-1)}\setminus S_0$. [The last factor in (8.6) is the normalization.] A factor $2^{|\bar{X}^{(-1)}\cap S_0|}$ provides the normalization for the uniform measure on $\sigma|\bar{X}^{(-1)}\cap S_0$, thus we take the supremum of (8.7) over all $\sigma|\bar{X}^{(-1)}$. The exponential is bounded by $\exp\left(\sum_{i\in S_0\cap\bar{X}^{(-1)}} s_i(\sigma_i - \sigma_i^{\min})\right) \leqq e^{2\delta}$ by definition of $\sigma^{\min}$. There are at most $2\nu|\bar{X}^{(-1)}\setminus S_0|$ bonds that can appear in g_c, hence at most $2^{2\nu|\bar{X}^{(-1)}\setminus S_0|}$ possible graphs g_c. We bound $e^{\beta J_{ij}\sigma_i\sigma_j} - 1$ by 2β, for small β, and there are at least $1/2|\bar{X}^{(-1)}\setminus S_0|$ such factors. This proves (3.7), (3.8) for $\mathscr{B}=\emptyset$. In general, taking s-derivatives generates truncated expectations in the measure on $\sigma|\bar{X}^{(-1)}\setminus S_0$. As in Sect. 6, these are expanded into at most $2^{|\mathscr{B}\setminus S_0|}|\mathscr{B}\setminus S_0|!$ terms, each of which is a product of ordinary expectations involving derivatives of (8.7) and σ_i, $i\in\mathscr{B}\setminus S_0$. We use the bounds

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}\cap S_0} (8.7)\right| \leqq 2^{|\mathscr{B}\cap S_0|} 2^{|\bar{X}^{(-1)}\cap S_0|} e^{2\delta} e^{-m|\bar{X}^{(-1)}\setminus S_0|}$$

and $|\sigma_i| \leqq 1$. Thus we obtain (3.7), (3.8) with an extra $|\mathscr{B}|!$.

We proceed from the basic expansion (3.2) as in Sects. 3–4 – these steps are essentially model-independent, as long as (3.7), (3.8) hold. As in the discussion after (3.9), we consider new clusters which overlap, but do not intersect, some $C_\alpha^{(k)}$ (they contain nearest neighbor sites to $C_\alpha^{(k)}$). These cluster activities are defined by (8.6), but we pretend that the $C_\alpha^{(k)}$'s overlapping but not intersecting $\bar{X}^{(-1)}$ are in Λ^c and we fix $\sigma = \sigma^{\min}$ there. [Note that if $\bar{X}^{(-1)}$ abuts Λ^c, then $\varrho(\bar{X}^{(-1)})$ depends on J_{ij} for $i\in\bar{X}^{(-1)}$, $j\in\bar{X}^{(-1)}\cup\Lambda^c$.] These new cluster activities are precisely the ones that would arise from the expansion of unnormalized expectations of products of characteristic functions $\chi^{\min}(C_\alpha^{(k)})$ setting $\sigma = \sigma^{\min}$ in $C_\alpha^{(k)}$. This is important so that the formula (6.2) for $z(C_\alpha^{(k)})^{-1}$ will hold.

The exponentiation of clusters avoiding singular regions and extraction of activities of singular regions are as in Sects. 3, 4. Even the inductive bounds are proven as before. We actually proved (5.1) without the $|\mathscr{B}|!$ on the right-hand side; now with $|\mathscr{B}|!$ in (3.7), (3.8) we obtain precisely (5.1), and no further modifications are necessary in the estimates.

Estimates and formulas for $z(C_\alpha^{(k)})$ are proven as in Sect. 6. We need only remark on how the measure $\langle \cdot \rangle_{J,h,\bar{C}_\alpha^{(k)}}$, which plays the role of $\langle \cdot \rangle_{h,\bar{C}_\alpha^{(k)}}$ in (6.2), is defined. We simply take the original measure, set $J_{ij}=0$ for $\langle i,j\rangle \in \partial\bar{C}_\alpha^{(k)}$, and restrict it to spin configurations in $\bar{C}_\alpha^{(k)}$. Then when expanding $[1]_{J,h,\bar{C}_\alpha^{(k)}}$ or $[\chi^{\min}(C_\alpha^{(k)})]_{J,h,\bar{C}_\alpha^{(k)}}$, we produce only clusters contained in $\bar{C}_\alpha^{(k)}$, as required by the definition of $z(\bar{C}_\alpha^{(k)})$.

The final result is a formula analogous to (4.13):

$$\log[A]_{J,h} = \sum_{i\in S_0^c} \log(e^{\beta h_i + s_i} + e^{-\beta h_i - s_i}) + \sum_{j=0}^{\infty} \left[\sum_{\alpha: C_\alpha^{(j)} \subset S_j^g} \log z(C_\alpha^{(j)}) + \sum_{\bar{Y}^{(j)}} V^{(j)}(\bar{Y}^{(j)}) \right]. \tag{8.8}$$

We apply s-derivatives to obtain expectations, and estimate the result as in Sect. 7. Derivatives at $i\in S_0^c$ produce expectations of σ_i in the single-site measure $\frac{1}{N}e^{\beta h_i \sigma_i}$, instead of $\sigma_i^{\min}$, which was produced in Sect. 7. Still, the term is bounded by 1, and the remaining estimates are identical. Thus we obtain the following theorem on decay and boundary condition independence of expectations.

Theorem 8.1. *Let β be sufficiently small. Then, with probability* 1, *the limit*

$$\lim_{A\nearrow \mathbb{Z}^\nu} \langle \sigma_{i_1}; \ldots; \sigma_{i_n}\rangle_{\beta,A}(J,h) \equiv \langle \sigma_{i_1}; \ldots; \sigma_{i_n}\rangle_\beta(J,h)$$

exists and is independent of boundary conditions, for arbitrary $\{i_1, \ldots, i_n\}$. *The approach to the limit is exponential. Furthermore, the correlations* $\langle \sigma_{i_1}; \ldots; \sigma_{i_n}\rangle_\beta(J,h)$ *have almost surely tree decay, with decay rate* $M \geqq \text{const}\,|\log\beta| > 0$, *in the sense explained in Sect. 1.3,* (1.14), *and in Theorem 7.2.*

It is worth noting that the expansion described in this section extends the domain of convergent expansions for the fixed J_{ij}, random h_i case. We can obtain the model considered in Sects. 1–7 by putting $\bar{J}=1$, $\Delta=0$. Then $S_0=\emptyset$ and we have an expansion no matter what $H=(\overline{h_i^2})^{1/2}$ is, for sufficiently small β. The expansion before worked for large H, in particular H such that $e^{2\nu\beta - \sqrt{2\pi}\varepsilon H\beta} \ll 1$, with ε small. Altogether the hatched region in Fig. 3 is covered.

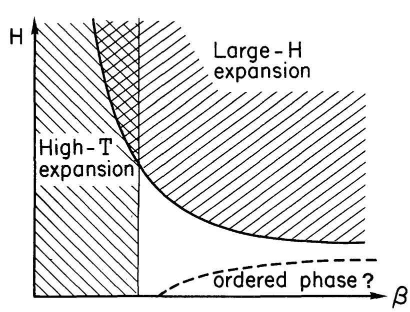

Fig. 3

Near the $H=0$ axis, and at low temperatures, there is a possible ordered phase [model (4′) in the Introduction]. Of course, when $H=0$ there are convergent low temperature (Peierls) expansions for the ordered phase.

Low Temperature, Predominantly Ferromagnetic Spin Glass

This model is defined in terms of

$$[A]_J = \sum_{\{\sigma_i\}_{i\in\Lambda}} A(\sigma)\exp\beta\left[\sum_{\langle i,j\rangle\subset\Lambda\cup\partial\Lambda} J_{ij}\sigma_i\sigma_j\right], \tag{8.9}$$

and we put $\sigma_i\equiv 1$ for $i\in\Lambda^c$ (symmetry breaking boundary conditions). We assume $\nu\geqq 2$ so that a Peierls argument can work. We again take independent Gaussian J_{ij}'s, with $\overline{J_{ij}}=\bar{J}$, $\overline{(J_{ij}-\bar{J})^2}=\Delta^2$. By scaling β we can assume $\bar{J}=1$. We require that Δ be small, in particular $\Delta^2<(\kappa\beta)^{-1}$ for some constant κ. Thus small or negative J_{ij}'s are unlikely. For a good low temperature expansion, we need $\beta\gg 1$, too.

We must make some modifications in the constructions of singular sets. Starting with $S_0=\{j\colon J_{ij}<1/2$ for some nearest neighbor i of $j\}$, we go through the constructions as before, but with a modified notion of volume. For $X\subset\mathbb{Z}^\nu$ we define

$$\mathrm{vol}(X)=\mathrm{Card}\,X+\left[\sum_{\langle i,j\rangle\subset X, J_{ij}<0}\beta|J_{ij}|\right], \tag{8.10}$$

where the brackets denote integer part. This allows for the especially poor estimates we expect for clusters containing antiferromagnetic bonds. Thus in (2.2) or in other appearances of $\mathrm{vol}(X)$ or $|X|$, we use (8.10). The proof of Proposition 2.1 (entropy bound and minimal volume bound) goes through with this new definition of volume.

As before, we must verify that the volume factors (from entropy bounds or estimates on cluster activities) can be dominated by the small probability of occurrence of singular sets. Let $C\subset\mathbb{Z}^\nu$. The probability that C is a component, $C_\alpha^{(k)}$, of S_k^q is bounded by

$$3^{\nu\,\mathrm{Card}C}e^{-N_2/8\Delta^2}e^{-N_1/2\Delta^2}\exp\left(-\frac{1}{2\Delta^2}\sum_{\langle i,j\rangle\subset X, J_{ij}<0}|J_{ij}|\right)\leqq\varepsilon^{\mathrm{vol}(C)}, \tag{8.11}$$

where $\varepsilon\to 0$ as $\Delta\to 0$. The first factor allows us to choose which of the at most $\nu\,\mathrm{Card}\,C$ bonds in C have $J_{ij}<0$, which have $0\leqq J_{ij}<1/2$, and which have $1/2\leqq J_{ij}$. On the second class (N_2 bonds) we estimate $\frac{1}{\sqrt{2\pi}\Delta}\int_0^{1/2}e^{-(J-1)^2/2\Delta^2}dJ$ by $e^{-1/2\Delta^2}$; on the first class (N_1 bonds) we estimate $\frac{1}{\sqrt{2\pi}\Delta}e^{-(J-1)^2/2\Delta^2}$ by $e^{-(1+2|J|)/2\Delta^2}$. Allowing a combinatoric factor $e^{|J|/2\Delta^2}$ to control the J-integral, we obtain the left-hand side of (8.11). This is bounded by $\varepsilon^{\mathrm{vol}(C)}$ because $N_1+N_2\geqq 1/2\,\mathrm{Card}\,C$.

The next step is to produce an expansion analogous to (3.2). We start by writing a contour expansion without regard to singular regions with poor bounds. We define $\sigma^{\min}\equiv 1$, $E^{\min}=-\sum_{\langle i,j\rangle\subset\Lambda\cup\partial\Lambda}J_{ij}$, and $A(\sigma^{\min})=\exp\left(\sum_{i\in\Lambda}s_i\right)$. Given a spin configuration, let Γ denote the set of sites in frustrated bonds (bonds $\langle i,j\rangle$ such that $\sigma_i\sigma_j = -1$). Let $\{\gamma_\mu\}$ denote the connected components of Γ – connectedness in the nearest neighbor sense. These are the contours of the configuration. We label each component of γ^c with a boundary condition, $+$ or $-$, to indicate the sign of spins in γ adjacent to those components.

In order to avoid unpleasant constraints arising from matching of boundary conditions of contours, and to treat the effect of contours on the observable, we use collapsing expansions, emphasized in [8]. The spin configuration is resummed in minus components of γ^c to produce a partition function with minus boundary conditions. We multiply and divide by the corresponding partition function with plus boundary conditions, and contour-expand the one in the numerator. Continuing the process, we obtain a contour expansion, where each contour γ has $+$ boundary conditions on the component of γ^c containing Λ^c, and where for each minus component of γ^c there is a corresponding factor of a ratio of partition functions, minus over plus.

The result is the following expansion:

$$[A]_J = A(\sigma^{\min})e^{-\beta E^{\min}} \sum_{\{\gamma_\mu\}} \prod_\mu r(\gamma_\mu), \tag{8.12}$$

where

$$r(\gamma) = \sum_{\sigma|\gamma \text{ compatible with } \gamma} \exp\left[\sum_{i\in\gamma} s_i(\sigma_i - 1) + \sum_{\langle i,j\rangle\subset\gamma} \beta J_{ij}(\sigma_i\sigma_j - 1)\right] \cdot \prod_\nu \frac{[A(V_\nu)]^-_{J,V_\nu}}{[A(V_\nu)]^+_{J,V_\nu}}. \tag{8.13}$$

For σ to be compatible with γ_c, it must equal 1 on the boundary of the component of γ^c containing Λ^c, and it must be constant on the boundary of every component of γ^c. Furthermore, each $i\in\gamma$ must be in a frustrated bond $\langle i,j\rangle$ with $\sigma_i\sigma_j = -1$. The components of γ^c with minus boundary conditions are denoted $\{V_\nu\}$, and we have defined

$$[A(V_\nu)]^\pm_{J,V_\nu} = \sum_{\{\sigma_i\}_{i\in V_\nu} : \sigma = \pm 1 \text{ for } i\in\partial V_\nu} \exp\left[\sum_{i\in V_\nu} s_i\sigma_i + \sum_{\langle i,j\rangle\subset V_\nu} \beta J_{ij}\sigma_i\sigma_j\right]. \tag{8.14}$$

The contour activities obey the bound

$$|r(\gamma)| \leqq e^{c\,\mathrm{vol}(\gamma)} e^{-\frac{1}{2}\beta|\gamma\setminus S_0|}. \tag{8.15}$$

Each frustrated bond $\langle i,j\rangle$, $i\in\gamma\setminus S_0$ yields a factor $e^{-2\beta J_{ij}} \leqq e^{-\beta J_{ij}}$. Changing variables, $\sigma\to-\sigma$, in $[A(V_\nu)]^-_{J,V_\nu}$, we see that each ratio is really an expectation of $\exp\left[-2\sum_{i\in V_\nu} s_i\sigma_i\right]$ in an appropriate measure. Hence it is bounded by $\exp\left[\sum_{i\in V_\nu} 2|s_i|\right]$, and all the ratios and s-factors are bounded by $e^{2\delta}$. The spin sum produces a factor $2^{\mathrm{Card}\,\gamma}$. Each frustrated bond $\langle i,j\rangle\subset\gamma\cap S_0$ produces a factor $e^{-2\beta J_{ij}}$, which is less than 1 if $J_{ij}\geqq 0$, less than $e^{2\beta|J_{ij}|}$ if $J_{ij}\leqq 0$. Hence (8.15) holds with our modified definition of volume.

Differentiation with respect to s-parameters are treated in a manner analogous to that in the high temperature spin glass, since derivatives of the ratios in V_ν produce truncated expectations of $\exp\left[-2\sum_{i\in V_\nu} s_i\sigma_i\right]$ with σ_i's. The only new feature is an extra factor $2^{|\mathscr{B}|}$ to allow for the choice of differentiating the observable $\exp\left[-2\sum_{i\in V_\nu} s_i\sigma_i\right]$ or the measure

$$([A(V_\nu)]^+_{J,V_\nu})^{-1}\exp\left[\sum_{i\in V_\nu} s_i\sigma_i+\sum_{\langle i,j\rangle\subset V_\nu}\beta J_{ij}\sigma_i\sigma_j\right].$$

Thus we have

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} r(\gamma)\right| \leqq e^{c\,\mathrm{vol}(\gamma)}e^{-\frac{1}{2}\beta|\gamma\backslash S_0|}4^{|\mathscr{B}|}|\mathscr{B}|!\,. \tag{8.16}$$

We now obtain the standard form (3.2) for our expansion by summing (8.12) within C-connected sets $X^{(0)}$ or $Y^{(0)}$. We have

$$\begin{aligned}\varrho(Y^{(0)})&=r(Y^{(0)}),\\ \varrho(X^{(0)})&=\sum_{\{\gamma_\mu\}\text{ filling }X^{(0)}}\prod_\mu r(\gamma_\mu).\end{aligned} \tag{8.17}$$

We obtain (3.7), (3.8) for these cluster activities, with $2^{|\mathscr{B}|}$ replaced by $4^{|\mathscr{B}|}|\mathscr{B}|!$ and with $2^{|X^{(0)}\cap S_0|}$ replaced by $e^{c\,\mathrm{vol}(X^{(0)}\cap S_0)}$. Thus

$$\left|\left(\frac{\partial}{\partial s}\right)_{\mathscr{B}} \varrho(\bar{X}^{(-1)})\right| \leqq 4^{|\mathscr{B}|}|\mathscr{B}|!\,e^{c\,\mathrm{vol}(\bar{X}^{(-1)}\cap S_0)}e^{-m|\bar{X}^{(-1)}\backslash S_0|}, \tag{8.18}$$

with $m=\beta/2-c$. The combinatorics of the sum over $\{\gamma_\mu\}$ in (8.17) is by now trivial; the combinatoric factors are bounded by $e^{c\,\mathrm{Card}(X^{(0)})}\leqq e^{c\,\mathrm{vol}(X^{(0)})}$. The large decay $\frac{1}{2}\beta|X^{(0)}\backslash S_0|$ allows the replacement of $\mathrm{vol}(X^{(0)})$ with $\mathrm{vol}\,(X^{(0)}\backslash S_0)$. The modifications in (3.7), (3.8) do not affect the estimates in Sect. 5 much; in particular, we still obtain (5.1) with $4^{|\mathscr{B}|}$ replacing $2^{|\mathscr{B}|}$.

We can proceed with the expansion as in Sects. 3 and 4. Some new clusters are considered which as always are the ones resulting from an expansion of unnormalized expectations of products of characteristic functions in $C_\alpha^{(k)}$'s. In this case we have $\chi^{\min}(C_\alpha^{(k)})$ setting $\sigma=1$ in $C_\alpha^{(k)}$, and the new clusters can contain sites in $\partial C_\alpha^{(k)}$ (just as if $C_\alpha^{(k)}$ were part of Λ^c). We obtain then the formula (6.2) for $z(C_\alpha^{(k)})^{-1}$, only $\langle\cdot\rangle_{h,\bar{C}_\alpha^{(k)}}$ is replaced with $\langle\cdot\rangle_{J,\bar{C}_\alpha^{(k)}}$. The latter is defined by restricting the original measure to configurations with $\sigma=1$ in $\partial\bar{C}_\alpha^{(k)}\cup\bar{C}_\alpha^{(k)c}$, so that only clusters contained in $\bar{C}_\alpha^{(k)}$ arise in the reconstruction of $z(C_\alpha^{(k)})^{-1}$.

The end result is the expansion

$$\log[A]_J=\sum_i s_i+\sum_{\langle i,j\rangle\subset\Lambda\cup\partial\Lambda}\beta J_{ij}+\sum_{j=0}^{\infty}\left[\sum_{\alpha:C_\alpha^{(j)}\subset S_j^g}\log z(C_\alpha^{(j)})+\sum_{\bar{Y}^{(j)}}V^{(j)}(\bar{Y}^{(j)})\right], \tag{8.19}$$

from which we easily obtain expectations by differentiating with respect to the s_i's. We find that the expectation of a single spin is strictly positive, with high probability (symmetry breaking), as well as our usual results about decay of truncated functions.

Theorem 8.2. *Choose β sufficiently large. Then, with probability* 1 *with respect to J, there is nonzero spontaneous magnetization, and connected correlations have tree decay, with decay rate $M\geqq\mathrm{const}\cdot\beta>0$, in the sense of Sect. 1.3,* (1.14).

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