Lower Critical Dimension of the Random-Field Ising Model

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A new argument is given for a lower critical dimension $d_l=2$ for the Ising model in a random magnetic field. It forms the basis for a proof that the three-dimensional model exhibits long-range order at zero temperature and small disorder. This settles the controversy between the values $d_l=2$ and $d_l=3$.

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In this paper I present a new argument that the lower critical dimension, d_l, for the Ising model in a random magnetic field is 2. Previous heuristic proposals for $d_l=2$ and also for $d_l=3$ have been given. Both cases have adherents. The question hinges on whether or not long-range order occurs in three dimensions, at low or zero temperature in the presence of a small, random magnetic field.

This physics issue has been resolved by the finding of a new, exact formula for the ground state energy—see (2) below. This formula is used elsewhere[1] to prove that $d_l \leq 2$. The formula for the energy has two important properties: (1) The energy is expressed as a sum of local functions of the magnetic fields in various regions, so that it is amenable to statistical analysis. (2) The sizes of the regions vary through a succession of increasing length scales (associated with an inductive analysis of the ground state), and the probability distributions of the functions scale accordingly.

Specifically, it is shown that if the disorder is small, the model in dimension $d=3$ exhibits long-range order at zero temperature. At the conceptual level, this argument leads to the same conclusion for low temperatures. Indeed, I expect that a proof for low temperatures will be possible by combining the methods described here with the expansion methods developed for disordered systems by Fröhlich and Imbrie.[2]

The model is defined by the Hamiltonian for a finite subset $\Lambda \subset Z^3$ with plus boundary conditions:

$$H^+(\Lambda) = \sum_{\langle i,j \rangle} \tfrac{1}{2}(1-\sigma_i\sigma_j) - \sum_{i \in \Lambda} \tfrac{1}{2} h_i \sigma_i.$$

Here $\sigma_i = \pm 1$ for $i \in Z^3$, $\sigma_i = 1$ for $i \notin \Lambda$, and $\langle i,j \rangle$ denotes a nearest-neighbor pair. The magnetic fields h_i are taken to be independent random variables with a common Gaussian distribution with mean zero and width $\langle h_i^2 \rangle^{1/2} = \epsilon$ (a measure of the disorder). The angular brackets indicate an average over the magnetic fields. We write $P(E)$ for the probability of the event E. Let us write $\sigma^{\min}(\Lambda^+)$ for the spin configuration of minimum energy $H^+(\Lambda)$. It is unique, with probability 1.

At temperature $T=0$, the question of long-range order reduces to properties of $\sigma^{\min}(\Lambda^+)$ as Λ increases to Z^3. We have long-range order if $\sigma_i^{\min}(\Lambda^+)$ is more often $+1$ than -1 for some fixed $i \in Z^3$, and if the disparity is uniform as Λ increases to Z^3. This is the content of the following theorem, proved in Ref. 1.

Theorem.—Let Λ_n be a sequence of cubes centered at the origin $0 \in Z^3$. There exists a constant $C>0$ such that for any $i \in Z^3$ and any n,

$$P(\sigma_i^{\min}(\Lambda_n^+) = -1) \leq \exp(C/\epsilon^2). \qquad (1)$$

The limit $\lim_{n\to\infty} \sigma_i^{\min}(\Lambda_n^+) \equiv \sigma_i^{\min}$ exists with probability 1 and satisfies the same bound.

Other results of Ref. 1 include a proof of near-exponential decay of correlations between ground-state spins:

$$|\langle \sigma_0^{\min}\sigma_j^{\min}\rangle - \langle \sigma_0^{\min}\rangle\langle \sigma_j^{\min}\rangle| \leq \exp\{-cj\exp[-c'(\ln\ln j)^2]\epsilon^{-2}\}.$$

In Ref. 2 it is shown that the model has no long-range order for large ϵ. Hence there is a $T=0$ transition from long-range order to absence of long-range order as the disorder parameter ϵ increases.

I expect that my methods will be useful in other problems, for example in studying the interface in random-field models. A reasonable conjecture is that the interface in d dimensions is rigid for $d>3$, as a result of the similarity with the $(d-1)$-dimensional bulk problems studied here. The continuum interface may of course be much rougher.[3,4]

We recall that the lower critical dimension is defined as the dimension above which long-range order occurs. Recent numerical work[5] has indicated ordering in three dimensions, which would imply $d_l=2$. However, neither the domain-wall argument for $d_l=2$ nor the dimensional-reduction argument[6] for $d_l=3$ has been universally accepted. Domain walls are defined as surfaces separating regions of constant σ. According to the domain-wall argument,[7] the typical fluctuation of the magnetic field energy in a domain of linear dimension L is of the order of $L^{d/2}$. This should be smaller than the domain-wall energy L^{d-1} in an ordered state; hence $d > 2$ is necessary for long-range order. As pointed out by Imry,[8] this is an argument for $d_l \geqslant 2$, rather than for $d_l = 2$, because other mechanisms could destroy long-range order in three or more dimensions. This is in fact what was argued for in some of the interface work.[3] The most obvious mechanism is the entropy of domain walls. There are $\exp[O(L^{d-1})]$ domain walls of area L^{d-1}; this has the potential for swamping the small probability $\exp[-O(L^{d-2})]$ that the magnetic field energy inside any particular domain wall is larger than the surface energy.

The entropy problem was solved by Fisher, Fröhlich, and Spencer[9] and by Chalker.[10] They used coarse-grained domain walls to exploit the large degree of dependence amongst the field energies for different domain walls. Their result is that for $d > 2$ it is unlikely that any domain wall surrounding the origin encloses a total field exceeding the area of the wall.

The second problem with the domain-wall argument lies with the assumption that the energy shift resulting from forming a domain wall is essentially the sum of the magnetic fields inside the domain wall. This assumption can only be valid if the system is known to be ordered, so that domain walls within the given domain wall are unimportant. For example, it almost certainly fails for $d = 2$, where the model is generally believed not to have long-range order. It has recently been argued[11] that domain walls within domain walls raise d_l to 3, in contrast to the present results.

The problem is circumvented in my argument because I use an exact formula for the energy shift, valid independently of the behavior of the system (ordered or not). However, in three or more dimensions, there is sufficient control over the random variables appearing in the formula to show that there is long-range order.

One might ask what when wrong with the dimensional reduction, especially in light of the nonperturbative[12] and rigorous[13] versions that now exist. The Parisi-Sourlas correspondence is exact only in the case of unique solutions to the equations of motion, as was pointed out by Parisi and Sourlas.[14] This excludes the case of most interest for the Ising model at low temperature, since the desired interaction potential is nonconvex. It is still possible that the correspondence is of relevance in a disordered phase, where it would be more reasonable to assume a convex potential.

I now explain why the three-dimensional random-field Ising model should be ordered. Let us take $\Lambda = Z^3$ in this discussion. The main point is to expand the ground-state energy in terms of local random variables,

$$H^*(\sigma^{\min}(\Lambda^+)) = -\tfrac{1}{2}\sum_{i\in\Lambda} h_i + \sum_{\gamma: V(\gamma)\subseteq\Lambda} r_\gamma(h). \tag{2}$$

Here γ runs over all possible connected domain walls, and $V(\gamma)$ denotes the volume enclosed by γ. Each random variable $r_\gamma(h)$ depends only on h_i for $i \in B(\gamma)$, where $B(\gamma)$ is a cube of diameter $2^{k(\gamma)+3}$ with $\operatorname{diam}(\gamma) \in [2^{k(\gamma)}, 2^{k(\gamma)+1})$. This formula is expression of the fact that the ground state can be obtained through a sequence of local ground states in cubes $B(\gamma)$ on increasing length scales.

To see how this is accomplished, let us say that γ is favored $(+)$ if it is an outer domain wall of the configuration $\sigma^{\min}(B(\gamma)^+)$, the local ground state in $B(\gamma)$. [Outer means $V(\gamma) \not\subset V(\gamma')$ for any other domain wall γ' of the configuration.] Furthermore, let us say that γ is maximal $(\subseteq V^+)$ if it is favored $(+)$, if $V(\gamma) \subseteq V$, and if there is no other favored $(+)$ domain wall γ' with $V(\gamma) \subseteq V(\gamma') \subseteq V$. It can be shown[1] that the maximal $(\subseteq \Lambda^+)$ domain walls are the outer domain walls of the ground state $\sigma^{\min}(\Lambda^+)$. This fact, together with the fact that favored $(+)$ domain walls have nonintersecting interiors, allows us to derive (2) as a kind of telescoping expansion. For any $V \subset Z^3$, we put

$$H^+_{\min}(V) = H^+(\sigma^{\min}(V^+)) + \sum_{i\in V} \tfrac{1}{2} h_i, \tag{3}$$

$$E^+(V) = \sum_{\gamma \text{ maximal } (\subsetneq V^+)} H^+_{\min}(V(\gamma)), \tag{4}$$

$$r_\gamma(h) = \begin{cases} 0, & \text{if } \gamma \text{ is not favored } (+), \\ H^+_{\min}(V(\gamma)) - E^+(V(\gamma)), & \text{otherwise.} \end{cases} \tag{5}$$

Here maximal $(\subsetneq V^+)$ is defined in the same way as maximal$(\subseteq V^+)$, only replacing $\subseteq$ with $\subsetneq$ everywhere. It is easy to see that $E^+(V) - \sum_{i\in V}\frac{1}{2}h_i$ is the energy of a comparison configuration in V equaling $\sigma^{\min}(V(\gamma)^+)$ in each $V(\gamma)$, γ maximal$(\subsetneq V)$, and equaling 1 elsewhere. Thus r_γ measures the amount that the energy can be lowered by permitting γ to occur in $V(\gamma)^+$. This implies that $r_\gamma \leqslant 0$, with $r_\gamma < 0$ if γ is favored $(+)$. As a consequence of all these definitions, we have the following expansions:

$$H^{+}_{\min}(V)=\sum_{\gamma:V(\gamma)\subseteq V} r_{\gamma}, \tag{6}$$

$$E^{+}_{\min}(V)=\sum_{\gamma:V(\gamma)\subsetneq V} r_{\gamma}. \tag{7}$$

[We assume $V=\Lambda$ or $V=\Lambda(\gamma)$ with γ favored (+).] Working by induction on diam(V), we note that (6) implies (7) for larger V's by substitution into (4). Furthermore, (7) implies (6) by the definition of $r_{\partial V}$ (∂V is the boundary of V). We obtain finally the desired expansion (2) by combining (3) and (6).

We now obtain the long-range order from an estimate on the distribution of the r_γ's. Consider all domain walls γ^* with $B(\gamma^*)=B_x$, a cube centered at x, and with $k(\gamma^*)=k$, $\text{area}(\gamma^*)\in[a,2a)$, $|V(\gamma^*)|\in[w,2w)$. If the number of such domain walls is in the range $[\nu,2\nu)$, then we define

$$r_{k,a,w,\nu,x}(h)=\sum_{\gamma^*} r_{\gamma^*}(h).$$

Otherwise, we put $r_{k,a,w,\nu,x}(h)=0$. Putting $N(a,w)=\frac{2}{3}\log_2(w/a)$, we formulate our main estimate:

$$P(r_{k,a,w,\nu,x}<-R)\leqslant\exp\left\{-\frac{(\nu a+R)^2}{\epsilon^2\nu w}\exp(-c_0[\ln N(a,w)]^2)\right\}. \tag{8}$$

Here c_0 is a constant, and $R\geqslant 0$. In the simplest case, $R=0$, $\nu=1$, and the right-hand side is bounded by $\exp[-(a^2/\epsilon^2 w)^{1-\eta}]$ for any fixed small η. This estimate exhibits the basic scaling (area)^2/volume in the exponent, up to logarithmic corrections involving $N(a,w)$. Since $a^2/w\geqslant 2^k$ in three dimensions, we easily see that with probability $1-\exp(-C/\epsilon^2)$, there is no domain wall γ surrounding the origin with $r_\gamma<0$. However, if $\sigma_0=-1$ in the ground state, then there must exist a favored (+) domain wall surrounding the origin. Since such contours satisfy $r_\gamma<0$, we obtain the long-range order (1).

The main estimate (8) is proven by induction on $N(a,w)$. This means that information about how likely it is for domain walls of a certain size to appear is used to estimate probabilities for larger domain walls. In order to see why an estimate such as (8) should hold, we express r_γ in terms of $r_{\gamma'}$'s with $V(\gamma')\subsetneq V(\gamma)$. This is done by noticing that

$$H^{+}_{\min}(V(\gamma))=\text{area}(\gamma)+\sum_{i\in V(\gamma)} h_i+H^{-}_{\min}(\tilde V(\gamma)),$$

where $\tilde V(\gamma)$ is obtained by deleting from $V(\gamma)$ sites adjacent to γ, and where $H^{-}_{\min}$ is defined as in (4) but with fields and boundary conditions flipped. This identity is just a reflection of the fact that if γ is favored (+), having plus boundary conditions on $V(\gamma)$ is the same as having minus boundary conditions on $\tilde V(\gamma)$. With use of (5)–(7), this implies that

$$r_\gamma(h)=\text{area}(\gamma)+\sum_{i\in V(\gamma)} h_i+\sum_{\gamma':V(\gamma')\subseteq\tilde V(\gamma)} r_{\gamma'}(-h)-\sum_{\gamma':V(\gamma')\subsetneq V(\gamma)} r_{\gamma'}(h). \tag{9}$$

The field term can be treated as in Ref. 9, yielding an estimate like (8) (without the inside exponential). The other terms form almost a sum of symmetrized random variables. To deal with the entropy problem (too many domain walls to permit individual treatment) we reformulate (9) in terms of the aggregate variables $r_{k,a,w,\nu,x}$. Except for some harmless positive terms on the right-hand side, we obtain a sum of symmetrized, essentially independent, random variables for each scale k and each a,w,ν. (Independence is a consequence of locality; $r_{k,a,w,\nu,x}$ depends only on h_i for $|i-x|\leqslant 2^{k+2}$.) These properties and the bound (8) allow us to show that it is unlikely for the symmetrized terms to exceed area(γ) in magnitude (as would be necessary for $r_\gamma<0$), for any γ contributing to $r_{k,a,w,\nu,x}$. Again, coarse-graining methods[9] are needed. In this way we recover (8) for larger values of $N(a,w)$. The factor $\exp\{-c_0[\ln N(a,w)]^2\}$ controls the deterioration of the estimates (due to sums over k,a,w,ν) as the induction proceeds.

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[1]J. Imbrie, Commun. Math. Phys. (to be published).

[2]J. Fröhlich and J. Imbrie, Commun. Math. Phys. (to be published).

[3]E. Pytte, Y. Imry, and D. Mukamel, Phys. Rev. Lett. **46**, 1173 (1981); D. Mukamel and E. Pytte, Phys. Rev. B

25, 4779 (1982).
[4]G. Grinstein and S. Ma, Phys. Rev. Lett. **49**, 685 (1982), and Phys. Rev. B **28**, 2588 (1983); J. Villain, J. Phys. (Paris) **43**, L551 (1982).
[5]D. Andelman, H. Orland, and L. Wijcwardhana, Phys. Rev. Lett. **52**, 145 (1984).
[6]G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[7]Y. Imry and S. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[8]Y. Imry, J. Stat. Phys. **34**, 849 (1984).
[9]D. Fisher, J. Fröhlich, and T. Spencer, J. Stat. Phys. **34**, 863 (1984).
[10]J. Chalker, J. Phys. C **16**, 6615 (1983).
[11]U. Krey, J. Phys. C (to be published).
[12]J. Cardy, Phys. Lett. **125B**, 470 (1983); A. Klein and J. Perez, Phys. Lett. **125B**, 473 (1983).
[13]A. Klein, L. Landau, and J. Perez, Commun. Math. Phys. **94**, 459 (1984).
[14]G. Parisi and N. Sourlas, Nucl. Phys. **B206**, 321 (1982).