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EXACT RENORMALIZATION GROUP FOR GAUGE THEORIES+

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1. INTRODUCTION

Renormalization group ideas have been extremely important to progress in our understanding of gauge field theory. Particularly the idea of asymptotic freedom leads us to hope that nonabelian gauge theories exist in four dimensions and yet are capable of producing the physics we observe--quarks confined in meson and baryon states. For a thorough understanding of the ultraviolet behavior of gauge theories, we need to go beyond the approximation of the theory at some momentum scale by theories with one or a small number of coupling constants. In other words, we need a method of performing exact renormalization group transformations, keeping control of higher order effects, nonlocal effects, and large field effects that are usually ignored.

Rigorous renormalization group methods have been described or proposed in the lectures of Gawedzki, Kupiainen, Mack, and Mitter. Earlier work of Glimm and Jaffe[1] and Gallavotti et al.[2] on the ϕ^4 model in three dimensions were quite important to later developments in this area.

We present here a block spin procedure which works for gauge theories, at least in the superrenormalizable case. It should be enlightening for the reader to compare the various methods described in these proceedings--especially from the point of view of how each method is suited to the physics of the problem it is used to study.

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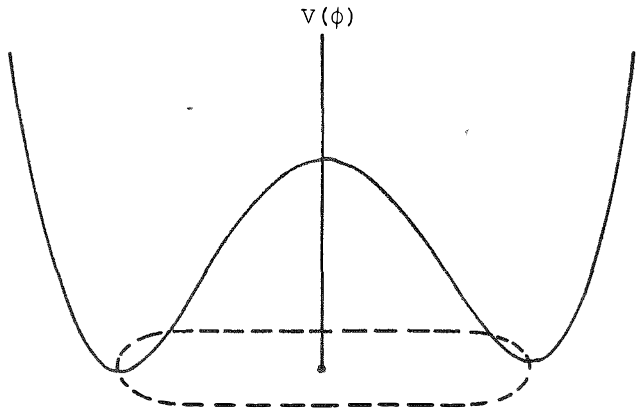
We believe that our approach is advantageous for the study of gauge theories.

The problem we present in some detail is the abelian Higgs model in two or three dimensions, with a "wine bottle" potential for the Higgs field. The continuum action density is

$$\frac{1}{2}\,|(\partial_\mu - ieA_\mu)\phi|^2 + \frac{1}{4}\,F_{\mu\nu}^2 + \lambda|\phi|^4 - \frac{1}{4}\,m^2|\phi|^2 + \text{counterterms},$$



where ϕ is a complex scalar field. We use a lattice approximation and renormalization transformations in the form of block spins to study two problems.

1. Ultraviolet stability. This phrase describes the bounds necessary to control the continuum limit, i.e. estimates uniform in the lattice spacing ε for finite volume partition functions and correlation functions.

2. The Higgs mechanism. Here infrared problems complementary to ultraviolet stability are considered. We seek bounds on correlations that have exponential decay in the separation between observables. The rate of decay should be uniform in the lattice spacing and in the volume, and it should be independent of the observables which are considered. Such a uniform decay rate ensures a gap in the mass spectrum of the model, and this gap is also known as the Higgs effect. The mass gap at first appears surprising, since one might expect a zero-mass scalar boson (Higgs particle) to arise from transverse motion in the wine bottle potential. One might also expect a massless gauge particle (photon) because the gauge field has no explicit mass term.

The uniformity of the estimates as $\varepsilon \to 0$ and as the volume goes to infinity means that bounds carry over to the corresponding limits, if they exist. The exponential decay should allow us to take the infinite volume limit, because distant regions contribute only exponentially small effects to a given correlation function. The $\varepsilon \to 0$ limit is more difficult, but we expect that our methods will apply.

The standard heuristic physics explanation of the Higgs mechanism arises from consideration of the model in the "unitary" gauge. In this point of view, the scalar field is written in polar form, $\phi = re^{i\theta}$, with $r \geqslant 0$, and one chooses the gauge $\theta = 0$. In this manner, the transverse (angular) fluctuations of ϕ are eliminated, while radial fluctuations in a neighborhood of the potential minimum at $r = r_0 = m(8\lambda)^{-1/2}$ are characterized by a mass m. (In the semiclassical limit, the mass squared is twice the sectional curvature of the potential.) Hence one expects a Higgs particle with a mass m_H which is close to m. We choose $m = O(1)$.

Furthermore, in the unitary gauge $1/2|(\partial_\mu - ieA_\mu)\phi|^2 = 1/2|\partial_\mu\phi|^2 + 1/2\, e^2A_\mu^2|\phi|^2$. Assuming that $\phi - r_0$ is small, this yields a semiclassical photon mass equal to $er_0 = me/(8\lambda)^{1/2}$. Thus in order to obtain semiclassical masses which are $O(1)$, we also also require that $e^2/\lambda = O(1)$. The quantum effects for small e,λ will modify the Higgs mass m_H and the photon mass m_{ph} from their classical values,

$$m_H = m(1 + O(e,\lambda)), \qquad m_{ph} = me(8\lambda)^{-1/2}(1 + O(e,\lambda)) \ .$$

The modifications are expected to be small for $0 \leqslant e^2, \lambda \ll 1$, with m, $e^2/\lambda = O(1)$, and this is the region in which we work.

What is wrong with this picture of mass generation and the Higgs effect? The problem is that the unitary $(\theta = 0)$ gauge used above is a nonrenormalizable gauge. Propagators which occur in perturbation theory are very badly behaved for large momenta, i.e. they have bad local regularity properties for small ε. This points to the fact that the $\theta = 0$ gauge is physically a poor way of understanding the Higgs model. It intertwines the infrared and the ultraviolet problems, making each less clear. Furthermore, there are unanswered questions about the role of configurations where $\phi \approx 0$, namely singular gauge configurations or vortex configurations.

Our answer to these difficulties is to treat high-momentum degrees of freedom differently from low-momentum degrees of freedom. The high-momentum degrees of freedom are studied using gauges which have well-behaved ultraviolet properties. The function space integral over these degrees of freedom can be carried out and analyzed. The result is an effective low momentum theory. For this resulting model there are no problems with the $\theta = 0$ gauge, and we use this unitary gauge to exhibit the generation of a positive mass.

The first problem is to find an appropriate gauge-invariant splitting of the model into its high and low momentum parts. We choose a block spin method to achieve this splitting. Thus while the Higgs effect is not generally considered as a renormalization group problem, we do so and perform the momentum space splitting in a series of steps. Each step will transform a model on a lattice with spacing δ into another model on a lattice with spacing δL, where L is a small positive integer. After a finite number $k = k(\varepsilon)$ steps, $\varepsilon L^k = O(1)$, we say that we have arrived at the "unit lattice" model. We analyze this model using the unitary gauge and find a mass gap $\bar{m} > 0$. Here $\bar{m}$ is uniform in ε and $|\Lambda|$ as $\varepsilon \to 0$ and $|\Lambda| \to \infty$.

In order to control the errors at each stage of renormalization, it is necessary to eliminate portions of the effective action which couple distant space-time regions with very small probability; in other words we "localize" terms in the effective action. Furthermore we stop the renormalization transformation procedure in spacial neighborhoods of places where the fields produce a large action. In such regions we find it impossible to define a useful definition of background field configurations and fluctuations about such configurations. However these events have a small probability and contribute little to expectations. All these complications are handled at each length scale, and hence lead us to a "multi-scale cluster expansion." This is the technical tool we use for estimates.

This work on Higgs models is built upon years of experience with superrenormalizable quantum field theories--many models with bosons and fermions in two and three dimensions have been constructed and properties such as particle structure and phase structure have been analyzed--see Glimm and Jaffe's book[3] for references. The most popular approach (which we pursue here) has been to work with Euclidean functional integrals, verifying the Osterwalder-Schrader axioms[4] which guarantee the existence of a Minkowski theory satisfying the basic axioms of quantum field theory.

The work on gauge theories to date is more limited. The most complete construction is of the two dimensional abelian Higgs model in two dimensions, by Brydges, Fröhlich, and Seiler[5]. For this model all the Osterwalder-Schrader axioms were verified except clustering. In particular, the existence of a mass gap is open, even for the two dimensional model. More limited results were obtained by Challifour and Weingarten[6] and by Ito[7] for two dimensional quantum electrodynamics, see also Seiler.[8]

The work described here is partly published and partly in progress. Balaban's papers[9] prove ultraviolet stability for the somewhat simpler case of the abelian Higgs model with a massive vector field. Our work on the Higgs mechanism for the model without an explicit mass for the gauge field (Wilson action) is not yet complete, but is sufficiently far along for us to present the main ideas here. Balaban is extending the stability result to the nonabelian case in two and three dimensions--see Ref. 10 and subsequent papers in preparation. Readers particularly interested in the nonabelian model should consult these papers as they appear, see also Ref. 11. However, the overall strategy and many of the details are common to the abelian and nonabelian case, so our discussion here of the abelian model should serve as an introduction to the methods used in all of these works.

Of course, the most interesting case is four dimensional nonabelian Yang Mills. We are still far from understanding this model (we will see below some features of the method which seem to be particularly troublesome for four dimensions). However, an understanding of how to do exact renormalization group theory for nonabelian models in three dimensions is surely a prerequisite to results in four dimensions. We refer the reader to the Cargèse lectures for the year 2003 for a construction of $(YM)_4$. (We don't know the authors yet)!

2. BLOCK SPINS FOR GAUGE THEORIES

2.1 The Lattice Approximation

We work with a lattice regularization because it is extremely important to preserve gauge invariance--not just at the start but also for all effective theories obtained after applying some number of renormalization transformations. Of course there are disadvantages of the lattice approximation: loss of Euclidean invariance, for example, or loss of an obvious geometric interpretation. However, for us (and many other authors) the advantages outweigh the disadvantages.

We also find it convenient to work with a compact action (the Wilson action). We hope that noncompact actions, or actions where the gauge field has a larger period than the Higgs field can be treated with some additional work. One can convert a noncompact model to a compact one at the price of introducing vortices. This device was used in Ref. 12 to study the noncompact model with a fixed (not arbitrarily small) lattice spacing.

On the ε-lattice we use fields $u_b \in U(1)$, $\phi(x) \in \mathbb{C}$, where b is a bond and x is a site on a periodic lattice T_ε. If we write

$u_{<x,x+\varepsilon e_\mu>} = e^{ie\varepsilon A_\mu(x)}$, then $A_\mu(x)$ corresponds to the usual gauge field for continuum formulations. By convention, bonds are oriented, and $u_b = u_{-b}^{-1}$, where $-b$ denotes the reverse of b. If Γ is any oriented contour composed of bonds, then we define $u(\Gamma) = \Pi_{b\in\Gamma} u_b$. In particular, $u(p)$ is the usual plaquette variable formed by taking the product of u_b's around the plaquette p. We define the covariant derivative of ϕ as $(D_u\phi)_b = \varepsilon^{-1}(u_b\phi_{b_+} - \phi_{b_-})$, where b_+, b_- are the forward, backward sites of b.

The action on the ε-lattice is

$$S^\varepsilon(u,\phi) = \sum_p \varepsilon^d \frac{1}{e^2\varepsilon^4}(1-\mathrm{Re}\, u(p)) + \frac{1}{2}\sum_b \varepsilon^d |(D_u^\varepsilon\phi)_b|^2$$

$$+ \sum_x \varepsilon^d(\lambda|\phi(x)|^4 - \frac{1}{4}m^2|\phi(x)|^2 - \frac{1}{2}\delta m^2|\phi(x)|^2) + E .$$

Here $\delta m^2 = \delta m^2(e,\lambda,\varepsilon)$ is a mass counterterm for the Higgs field. It is given by a diagrammatic expansion to some order in coupling constants, and is divergent as $\varepsilon \to 0$. The constant E contains vacuum energy subtractions, including divergent counterterms, again given by a diagrammatic expansion. From this action we obtain the partition function

$$Z = \int du d\phi\, e^{-S^\varepsilon(u,\phi)}$$

and expectations

$$\langle F\rangle = Z^{-1}\int du d\phi\, F e^{-S^\varepsilon(u,\phi)} .$$

We consider only gauge invariant observables. (Non-invariant observables like $\phi(x)$ are easily seen to have vanishing expectation--no gauge fix is needed above because of the compactness of the u-integration.) The gauge transformations for these fields are

$$\phi(x) \to \phi(x)e^{i\lambda(x)}$$

$$u_b \to u_b e^{-i(\lambda(b_+)-\lambda(b_-))}$$

where λ is a real-valued function on sites. The action is easily seen to be invariant, as are observables such as $|\phi(x)|^2$, $\phi(b_-)u_b\phi(b_+)$, $u(p)$.

The goal of this work is to prove bounds on expectation of appropriately renormalized observables that are uniform in the lattice spacing and in the volume. Furthermore, we wish to take the infinite volume limit and establish exponential clustering of truncated correlations $\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle$ at a rate uniform in ε (the Higgs mechanism). The existence and Euclidean in variance of the limit $\varepsilon \to 0$ is open for the moment. We expect that our methods will eventually be used to understand these questions, thereby yielding a complete construction of the model and verification of the axioms. Partial results in this direction have been obtained by C. King.

2.2 Rescaling to the Unit Lattice

It is convenient to work with unit lattice spacing for each effective theory. Thus we rescale the ε-lattice to the unit lattice, performing canonical scalings on the fields $(\phi^{\varepsilon} = \varepsilon^{-(d-2)/2}\phi^1$, etc.). The partition function becomes

$$\int du d\phi \; e^{-S(u,\phi)}$$

with

$$S(u,\phi) = \sum_p \frac{1}{e(\varepsilon)^2}(1-\text{Re } u(p)) + \frac{1}{2}\sum_b |(D_u\phi)_b|^2$$

$$+ \sum_x (\lambda(\varepsilon)|\phi(x)|^4 - \frac{1}{4} m^2\varepsilon^2|\phi(x)|^2 - \frac{1}{2}\delta m^2\varepsilon^2|\phi(x)|^2) + E.$$

We are using rescaled coupling constants

$$\lambda(\varepsilon) = \lambda\varepsilon^{4-d}, \qquad e(\varepsilon) = e\varepsilon^{(4-d)/2},$$

and D_u is the unit lattice covariant derivative, $D_u\phi = u_b\phi_{b_+} - \phi_{b_-}$.

Note that all nonquadratic pieces of the action have acquired coefficients that are positive powers of ε, for $d < 4$. This becomes more apparent when we expand $u(p)$ in terms of correctly normalized field strength variables. With

$$f_p = \frac{1}{ie(\varepsilon)} \log u(p),$$

we have

$$\frac{1}{e(\varepsilon)^2}(1-\text{Re } u(p)) = \frac{1}{2} f_p^2 - \frac{1}{4!} e(\varepsilon)^2 f_p^4 + \cdots .$$

The mass counterterm δm^2 diverges like ε^{-1} in three dimensions, logarithmically in ε in two dimensions. Thus $|\phi|^2$ still has a positive power of ε in its coefficient. The fact that the model becomes extremely weakly coupled when viewed on this scale is just a manifestation of its superrenormalizability.

The price we pay for the rescaling, however, is that the model appears almost massless. Mass terms in the basic Gaussian have acquired powers of ε (at any rate they only appear in the Higgs part of the action, and then only with the wrong sign). The masslessness and the associated long-range correlations are circumvented by the method of block spin renormalization transformations. If we fix the block averages of the fields and integrate only over the fluctuations, then the gradient terms in the action will act like a mass. Thus we will obtain an integral which is a small perturbation of a massive Gaussian measure. Such integrals are relatively easy to control--they have well behaved perturbative expansions with remainder terms which can be controlled by means of convergent cluster expansions.

2.3 Average Fields and the Renormalization Transformation

We divide the lattice into blocks of L^d sites each, where L is a small positive integer like 2 or 3. A naive definition of the average of ϕ on a block would be $\Sigma_{x\in \text{Block}} L^{-d}\,\phi(x)$. However this definition of average is not gauge covariant--each term in the sum transforms according to the gauge transformation at a different point. In order to have an average that transforms according to the gauge function at a single point (say at the corner of the block) we use "parallel transport operators" $u(\Gamma_{y,x})$. Here $\Gamma_{y,x}$ is some contour from y (the corner of the block) to x (an arbitrary point in the block. For example we can take $\Gamma_{y,x}$ as in figure 1.

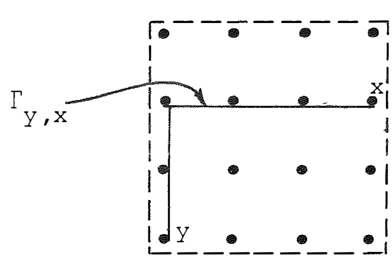


Figure 1.

We denote the gauge covariant average of ϕ by

$$(Q(u)\phi)_y = \sum_{x\in B(y)} L^{-d} u(\Gamma_{y,x})\phi(x) ,$$

where $B(y)$ is the block containing y, a point of the L-lattice. Under the gauge transformation

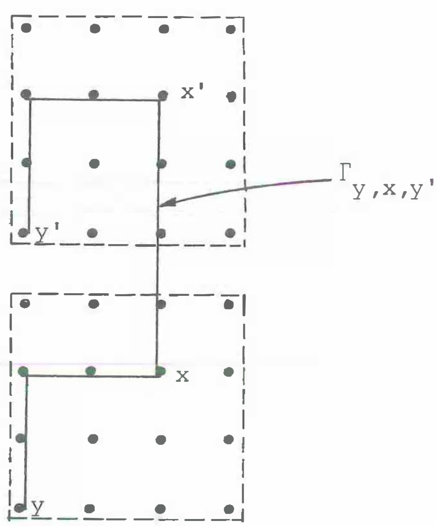
$$\phi(x) \to \phi(x)e^{i\lambda(x)} , \qquad u_b \to u_b e^{-i(\lambda(b_+)-\lambda(b_-))}$$

we have

$$(Q(u)\phi)_y \to (Q(u)\phi)_y \, e^{i\lambda(y)} .$$

For the gauge field the average should be defined for bonds of the block lattice $L\mathbb{Z}^d$, i.e. for $b' = \langle y,y'\rangle$, y,y' corners of nearest neighbor blocks. For gauge covariance we consider a collection of contours each starting at y and ending at y'. A convenient choice is the following: For each $x \in B(y)$ we define $\Gamma_{y,x,y'} = \Gamma_{y,x} \cup \langle x,x'\rangle \cup \Gamma_{x'y'}$, where $x' = x + y' - y$ and $\Gamma_{x',y}$ is the reverse of $\Gamma_{y,x'}$.

Figure 2.



We form the collection of group elements $\{u(\Gamma_{y,x,y'})\}$, as x ranges over $B(y)$. An average of these is defined as follows. We regard the $u(\Gamma_{y,x,y'})$ as points on the unit circle. If all the points lie inside some half-circle, the average is the point in the half-circle such that the sum of the difference angles vanishes. In other words, the argument of the average is the average of the arguments, as long as the jump in the definition of the argument occurs outside the half-circle. If not all points lie inside a half-circle, we add the group elements as complex numbers and divide by the modulus of the result to get back to the unit circle.

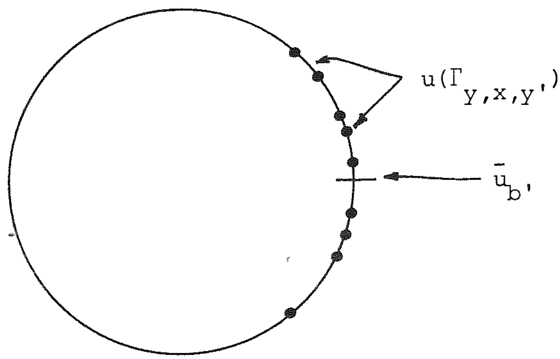


Figure 3.

The second case only occurs when there are large plaquette variables ($u(p)-1$ not small). As these occurrences are suppressed strongly by the action, almost any definition of average would do. The average of $\{u(\Gamma_{y,x,y'})\}$ is denoted $\bar{u}_{b'}$, and it transforms as follows:

$$\bar{u}_{b'} \to \bar{u}_{b'}\, e^{-i(\lambda(b'_+)-\lambda(b'_-))}$$

We now define the renormalization transformation, which is a transformation from densities $\rho(u,\phi)$ to densities $(T\rho)(v,\psi)$. Here $v \in U(1)$, $\psi \in \mathbb{C}$ are block fields defined on bonds, sites of $L\mathbb{Z}^d$. Let us take $\rho(u,\phi) = e^{-S(u,\phi)}$, and define

$$\begin{aligned}\rho_1(v,\psi) &= (T\rho)(v,\psi) \\ &= c\int du d\phi\ \delta(v\bar{u}^{-1}) \exp[-1/2 <\psi-Q(u)\phi, \psi-Q(u)\phi>]\ e^{-S(u,\phi)},\end{aligned}$$

where

$$\delta(v\bar{u}^{-1}) = \prod_{b'} \delta((v\bar{u}^{-1})_{b'})$$

and δ is the δ-function at the identity of $U(1)$. The constant c is chosen so that $T1 = 1$. Of course ρ_1 satisfies the basic property

$$\int dv d\psi\ \rho_1(v,\psi) = \int du d\phi\ e^{-S(u,\phi)}$$

The transformation for the gauge fields consists simply of integrating out all u with fixed averages $\bar{u} = v$. For the scalar field there is no δ-function but the approximate δ-function $\exp[-1/2\langle\psi - Q(u)\phi, \psi - Q(u)\phi\rangle]$ makes it highly probable that $Q(u)\phi$ is close to ψ.

2.4 Gauge Invariance and Gauge Fixing

It is extremely important that we preserve gauge invariance for the block fields v, ψ. We will exploit this in several ways throughout the sequence of renormalization transformations; also at the end when we analyze the effective unit lattice theory, we need gauge invariance to go to the unitary gauge $(\theta = 0)$. Let us verify that $\rho_1(v,\psi)$ is invariant under

$$\psi(y) \to \psi(y)e^{i\lambda(y)} \qquad v_{b'} \to v_{b'}\, e^{-i(\lambda(b'_+)-\lambda(b'_-))}$$

Extend λ in an arbitrary way to a function on all sites (only the values at corners of blocks are used in the above transformation). We can replace u, ϕ by their gauge transforms because $dud\phi$ is gauge invariant. The action is unchanged, while the gauge transformations of ψ, $Q(u)\phi$ and v, $\bar{u}$ match and drop out of the integrand, leaving $\rho_1(v,\psi)$ in its original form.

From this calculation we see that the integrand is still invariant under gauge transformations λ that vanish at corners of blocks. We wish to calculate the effective action for v, ψ by using a Gaussian approximation for the u,ϕ integral, but the invariance will lead to zero modes and spoil the approximation. Hence we need to fix the gauge for the u,ϕ integral.

We use the simplest possible gauge fix, a kind of axial gauge on blocks. The choice of gauge fix is not very important at this stage, since we are doing an integral that corresponds to one slice of momenta only. We set $u_b = 1$ for each bond in a maximal tree of bonds in each block. The tree (Fig. 4), composed of the contours $\Gamma_{y,x}$, is convenient.

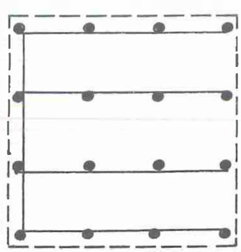


Figure 4.

We thus define

$$\delta_{axial}(u) = \prod_{blocks} \prod_{b\in tree} \delta(u_b)$$

and insert it under the integral. The density $\rho_1(v,\psi)$ is unchanged, as we take $\int du = 1$ by convention.

2.5 Effective Masses

The most important effect of the renormalization transformation is to introduce effective masses into the quadratic forms. In the density

$$\rho_1(v,\psi) = c\int du d\phi\, \delta(v\bar{u}^{-1})\delta_{axial}(u)\exp\Big[-\frac{1}{2}<\psi-Q(u)\phi,\psi-Q(u)\phi>$$
$$-\frac{1}{2}<D_u\phi,D_u\phi> - \frac{1}{2}<f,f> - \sum_x(\lambda(\varepsilon)|\phi(x)|^4 - \frac{1}{4}m^2\varepsilon^2|\phi(x)|^2$$
$$-\frac{1}{2}\delta m^2\varepsilon^2|\phi(x)|^2) - \sum_p\sum_{n=1}^{\infty}\frac{(-1)^n e(\varepsilon)^{2n}}{(2n+2)!}f_p^{2n+2} - E\Big] \quad (1)$$

we have quadratic forms $D_u^*D_u + Q(u)^*Q(u)$ for ϕ, and $\partial^*\partial$ for A (writing $u = e^{ie(\varepsilon)A}$, $f = \partial A$, with $(\partial A)(p) = \Sigma_{b\in p} A_b$). We can prove strictly positive lower bounds on these forms, at least in the small field region, where f is not too large (see the next section). Thus we have, for example

$$<\phi,(D_u^*D_u + Q(u)^*Q(u))\phi> \geqslant \sum_x c|\phi(x)|^2$$

for some $c > 0$, and we have a lower bound given by a mass term. The term $Q(u)^*Q(u)$ selects out the "constant" mode which would have been a zero mode for $D_u^*D_u$ alone. The form $\partial^*\partial$ has many zero modes, but the δ-functions $\delta(v\bar{u}^{-1})$ and $\delta_{axial}(u)$ reduce the integration over A to a subspace, on which the desired lower bound holds.

The lower bounds on the quadratic forms allow us to prove exponentially decaying bounds on the corresponding inverse operators, giving the correlations between fields at different sites in the Gaussian approximation. Thus distant fields are almost independent, and we can make localized calculations involving only small portions of the lattice. Of course we have not lost the long range correlations of the original model--they are carried by the external fields v,ψ. The effective masses also allow us to take advantage of the extremely small rescaled coupling constants $\lambda(\varepsilon)$, $e(\varepsilon)$ describing the corrections to the Gaussian. In what follows we show how these simplifying features can be used to compute an effective action for v,ψ. The price to pay for all the simplicity is the need to iterate the renormalization transformation many times ($\log_L \varepsilon^{-1}$ times). However the main features of the calculation are independent of the iteration step.

2.6 Large and Small Fields

As mentioned above, we lack some basic estimates on quadratic forms and propagators when the gauge field is rough. In addition, the fact that A is really a periodic variable makes the Gaussian approximation break down completely wherever there are large fields. Therefore we wish to separate large and small field regions, do perturbative calculations in the small field region, and use only the basic positivity of the action to control the large field integrations.

We define the large field region to be some neighborhood of places where any of the following inequalities hold:

$$\begin{aligned}
&|\phi| > \lambda(\varepsilon)^{-1/4}|\log\varepsilon|^p \\
&|D_u\phi| > |\log\varepsilon|^p \\
&|\psi-Q(u)\phi| > |\log\varepsilon|^p \\
&\frac{1}{e(\varepsilon)}|u(p)-1| > |\log\varepsilon|^p
\end{aligned} \tag{2}$$

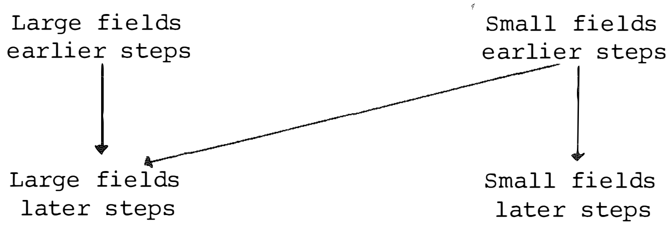
In each case, one of the terms $[\lambda(\varepsilon)|\phi|^4$, $|D_u\phi|^2$, $|\psi-Q(u)\phi|^2$, $\frac{1}{e(\varepsilon)^2}\mathrm{Re}(1-u(p))]$ in the exponential in (1) is larger than $|\log\varepsilon|^{2p}$. Thus we can expect to obtain small factors $\exp(-|\log\varepsilon|^{2p}) \leqslant \varepsilon^{\kappa}$, κ arbitrary, from integrals in the large field region. Thus large field regions are rare, and contribute very little to the partition function.

We implement these ideas in our integral (1) by inserting a partition of unity under the integral. Thus we write

$$1 = \sum_{\Lambda} \chi_{\Lambda} \zeta_{\Lambda^c} \quad ,$$

where χ_Λ is an approximate characteristic function restricting all fields to be small in Λ, in the sense of (2), and where ζ_{Λ^c} forces some fields to be large throughout Λ^c.

At this point we do almost nothing in Λ^c except extract the factors $(\varepsilon^\kappa)^{|\Lambda^c|}$. Even in later steps, no calculations are performed there. We always treat Λ^c as a large field region. Thus the flow of information is always along the following lines:



The missing arrow from large to small fields contrasts the procedure described in the lectures of Gawedzki and Kupiainen.

While our treatment seems to be forced by the nonlinearities in our model--the action does not split neatly into a Gaussian piece plus an interaction piece--it is the source of troubles with generalizations to four dimensions. This is seen when we estimate the contribution of large field regions to the partition function. Taking into account the entropy coming from the sum over Λ, we allow for a factor $1+\varepsilon^\kappa$ at each site. At the k-th step, we allow for a factor $1+(L^k\varepsilon)^\kappa$ at each L^k-block. The product over all k and all sites or blocks is bounded by

$$\exp\left[\sum_{k=0}^{\log_L \varepsilon^{-1}} (L^k\varepsilon)^\kappa (L^k\varepsilon)^{-d} |T_\varepsilon|\right] \leqslant \exp(c|T_\varepsilon|) \quad .$$

Here $|T_\varepsilon|$ is the volume of the lattice, which is ε^{-d} times the number of sites in the lattice. Then $|T_\varepsilon|(L^k\varepsilon)^{-d}$ is the number of blocks on the scale k. As long as $\kappa > d$ the sum over k converges, yielding a finite contribution to the vacuum energy. The method breaks down in four dimensional Yang-Mills theory, because the effective coupling constant is expected to be only logarithmically small in ε. This reduces the size of allowable small fields, and so yields less convergence from large fields. In particular, κ is at best a small positive constant, and the sum over k would diverge. The basic problem is that we are not cancelling the vacuum energy contribution from large fields at each step. This works in the superrenormalizable case but apparently not in the borderline case of logarithmically asymptotically free models.

2.7 Translations and the Background Fields

In the small field region the integral in

$$\rho_1(v,\psi) = \sum_{\Lambda} c \int du d\phi \; \chi_\Lambda \zeta_{\Lambda^c} \delta(v\bar{u}^{-1}) \delta_{axial}(u)$$

$$\exp\left[-\frac{1}{2}\langle\psi - Q(u)\phi, \psi - Q(u)\phi\rangle - \frac{1}{2}\langle D_u\phi, D_u\phi\rangle - \frac{1}{2}\langle f,f\rangle - \ldots\right]$$

is a small perturbation of a Gaussian measure. The best way to treat the integral is to translate the fields to the (v,ψ-dependent) minimum of the Gaussian part of the action. The minimum is the background field, and we will do perturbation theory and cluster expansions about this configuration.

More precisely, we do this in two steps--one for u and one for ϕ--since the quadratic form for ϕ depends on the external gauge field. We write $u = u_1 e^{ie(\varepsilon)A'}$, where $u_1 = u_1(v)$ is the unique minimum of $\langle f,f\rangle$ under the constraints $\delta(v\bar{u}^{-1})\delta_{axial}(u)$. The background field u_1 can be written explicitly in terms of v. The fluctuation field A' is small ($|A'| \leqslant |\log\varepsilon|^p$) in the small field region, and the integral over u is rewritten as an integral over A'.

At this point we expand all the terms in the exponential in powers of A'. This means that the scalar field forms produce new interactions:

$$\frac{1}{2}\langle\psi - Q(u)\phi, \psi - Q(u)\phi\rangle + \frac{1}{2}\langle D_u\phi, D_u\phi\rangle$$

$$= \frac{1}{2}\langle\psi - Q(u_1)\phi, \psi - Q(u_1)\phi\rangle + \frac{1}{2}\langle D_{u_1}\phi, D_{u_1}\phi\rangle + \text{interaction terms} .$$

The terms coming from the expansion of $\langle D_u\phi, D_u\phi\rangle$ are the usual scalar field-vector field interaction vertices, and the others are new vertices from the renormalization transformation. All terms are small, as each power of A' comes with a coupling constant $e(\varepsilon)$.

The δ-functions, when written in terms of A', become linear constraints on the A'-integral:

$$\delta(v\bar{u}^{-1})\,\delta_{axial}(u) = \delta(QA')\delta_{axial}(A') \quad .$$

Here Q is a kind of averaging operator for functions on bonds:

$$(QA)_{b'} = \sum_{x\in B(b'_-)} L^{-d} A(\langle x,x'\rangle) \quad .$$

We put $A(\Gamma) = \Sigma_{b\in\Gamma} A_b$, and as in the definition of $\bar{u}_{b'}$, $x-x' = b'_- - b'_+$. In $\delta_{axial}(A')$, all A'_b are set to zero for b in any of the maximal trees in blocks. The averaging procedure for A' is not much different from the simplest procedure for scalar fields (averaging over blocks). It yields the same kind of result: The form $\langle f,f\rangle$ under the constraints $\delta(QA')\delta_{axial}(A')$ has an effective mass and well-behaved, exponentially decaying propagators. Note that under a gauge transformation $A' \to A' - \delta\lambda$, QA' transforms into $QA' - \bar{\lambda}(b'_+) + \bar{\lambda}(b'_-)$, where $\bar{\lambda}(y) = \Sigma_{x\in B(y)} L^{-d}\lambda(x)$. This contrasts with the transformation law for $\bar{u}$, which involved λ restricted to corners of blocks. Of course the axial gauge conditions broke the invariance under transformations λ that are not constant on blocks, and the two laws are equivalent if λ is constant on blocks.

We next compute the minimum of the quadratic action for ϕ (external field u_1). It is a linear function of ψ, and we denote it $\phi_1 = K_1(u_1)\psi$. Thus we write $\phi = \phi_1 + \phi'$, with ϕ_1 the background field, ϕ' the fluctuation field. The fluctuation field is small: $|\phi'| \leqslant c|\log\varepsilon|^p$.

2.8 Calculation of the Effective Action

To simplify the discussion, let us consider only the term $\Lambda = T_\varepsilon$, i.e., the whole lattice is the small field region. The effect of the translations on the quadratic terms in the action is to split each quadratic form into two forms, one for the block field and one for the fluctuation field. So we write

$$\langle f,f\rangle = \langle f^{(1)},\sigma_1 f^{(1)}\rangle + \langle\partial A';\partial A'\rangle$$

$$\langle D_{u_1}\phi, D_{u_1}\phi\rangle + \langle\psi - Q(u_1)\phi, \psi - Q(u_1)\phi\rangle$$

$$= \langle\psi,\Delta_1(u_1)\psi\rangle + \langle\phi',(D^*_{u_1}D_{u_1} + Q(u_1)^*Q(u_1))\phi'\rangle \quad ,$$

where

$$f^{(1)}_{p'} = \frac{1}{ie(\varepsilon)} \log v(p') \quad ,$$

and where σ_1, $\Delta_1(u_1)$ are the resulting block field forms.

The block field terms $<f^{(1)},\sigma_1 f^{(1)}>$ and $<\psi,\Delta_1(u_1)\psi>$ are the quadratic terms of the effective action for v,ψ. If we rescale the L-lattice to the l-lattice, they have properties analogous to the original quadratic terms for u,ϕ: $<f,f>$ and $<\phi,D_u^* D_u\phi>$. The forms are massless, gauge-invariant, and obey lower bounds which have a local form:

$$<f^{(1)},\sigma_1 f^{(1)}> \geqslant c<f^{(1)},f^{(1)}>$$

$$<\psi,\Delta_1(u_1)\psi> \geqslant c\sum_{b'} |u_1(<b'_-,b'_+>)\psi(b'_+) - \psi(b'_-)|^2 \quad .$$

The forms are no longer local, however--they have exponential tails. The terms can be expressed simply as the original quadratic terms evaluated on the configurations u_1 and ϕ_1.

The scalar field self-interaction gives rise to some ϕ'-independent terms when we expand $\phi = \phi_1 + \phi'$. These are just

$$\sum_x (\lambda(\varepsilon)|\phi_1(x)|^4 - \frac{1}{4} m^2\varepsilon^2|\phi_1(x)|^2 - \frac{1}{2}\delta m^2\varepsilon^2|\phi_1(x)|^2) \ ,$$

and the other terms are small, having one power of $\lambda(\varepsilon)$ and three or less powers of ϕ_1, with $|\phi_1| \leqslant \lambda(\varepsilon)^{-1/4}|\log \varepsilon|^p$.

After the translations and expansions, the term $\Lambda = T_\varepsilon$ in our original density (1) becomes

$$c \exp\Big[-\frac{1}{2}<f^{(1)},\sigma_1 f^{(1)}> - \frac{1}{2}<\psi,\Delta_1(u_1)\psi> - \sum_x (\lambda(\varepsilon)|\phi_1(x)|^4 - \frac{1}{4} m^2\varepsilon^2|\phi_1(x)|^2 - \frac{1}{2}\delta m^2\varepsilon^2|\phi_1(x)|^2) - E\Big]$$

$$\cdot\int dA'd\phi' \ \delta(QA')\delta_{axial}(A')\chi_\Lambda \exp\Big[-\frac{1}{2}<\partial A',\partial A'> - \frac{1}{2}<\phi',(D^*_{u_1}D_{u_1} + Q(u_1)^*Q(u_1))\phi'> - V\Big] \quad .$$

Here V contains the ϕ',A'-dependent terms, all of which are bounded by some power of coupling constants $\lambda(\varepsilon)$, $e(\varepsilon)$ for all values of ϕ',A' permitted by χ_Λ. We introduce the Gaussian normalization

$$Z^{(0)}(u_1) = \int dA' d\phi' \ \delta(QA')\delta_{axial}(A')\exp\left[-\frac{1}{2}<\partial A',\partial A'> - \frac{1}{2}<\phi',(D^*_{u_1}D_{u_1} + Q(u_1)^* Q(u_1))\phi'>\right],$$

and the corresponding normalized measure $<\cdot>$. Thus we have the remaining terms in the effective action for v,ψ: $\log Z^{(0)}(u_1)$ and $\log<\chi_\Lambda e^{-V}>$.

We need to give a very precise expansion for $\log<\chi_\Lambda e^{-V}>$ because renormalization cancellations must be exhibited. This is especially important after many steps of the iteration have been performed and the divergences in perturbation theory start to manifest themselves. We have found it most convenient to exhibit cancellations on a purely perturbative level, as the cancellations or finiteness properties due to gauge invariance are fairly subtle. Thus we extract a few orders of perturbation theory using the cumulant expansion,

$$\log<\chi_\Lambda e^{-V}> = -<V> + \frac{1}{2!}<V;V> - \cdots$$

where

$$<V;V> = <V^2> - <V>^2$$

The restrictions in χ_Λ --A', ϕ' smaller than $|\log \varepsilon|^p$ --produce a change which is smaller than any power of ε, and hence is not seen in the perturbation expansion in ε. If we extract n orders of perturbation theory, for some sufficiently large n, then the remainder will be of the order of ε^κ, with $\kappa > d$. As in our large field estimate, this is small enough to be ignored henceforth. However, in order to address the question of whether there is a mass gap, we must exhibit the locality properties of the remainder. This is done with a cluster expansion. The result is

$$\log<\chi_\Lambda e^{-V}> = \sum_{j=1}^{n} \frac{1}{j!}<(-V;)^j> + \sum_X W(X) ,$$

where $<(\cdot;)^j>$ denotes the j-th-truncated correlation. Here $W(X)$ depends only on v,ψ in the connected set X, and $|W(X)| \leqslant \varepsilon^\kappa e^{-c|X|}$ with $|X|$ denoting the number of sites in X.

In the general case, with nonempty large field region, the small field calculations are performed with the values of the fields fixed in a neighborhood of the large field region (conditional integration).

After dropping or expanding out the $W(X)$ terms, we have the following form of the effective action:

$$S^{(1)}(v,\psi) = \frac{1}{2}\langle f^{(1)},\sigma_1 f^{(1)}\rangle + \frac{1}{2}\langle\psi,\Delta_1(u_1)\psi\rangle + \mathcal{P}^{(1)}(v,\psi) - \log Z^{(0)}(u_1) + E \quad .$$

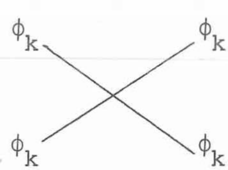
Here $\mathcal{P}^{(1)}(v,\psi)$ contains the terms $\lambda(\varepsilon)|\phi_1(x)|^4 + \cdots$ as well as the truncated correlations of V. A final step is to rescale all expressions and fields from the L-lattice to the 1-lattice. Relabeling v,ψ by u,ϕ, we are in a position to repeat the process.

2.9 The Effective Action After k Steps

Rather than describe the general step of the procedure, let us simply examine the k-th effective action to get some feel for the behavior of the model under iterated application of renormalization transformations. We stick to the small field region, and use block fields u,ϕ defined on the unit lattice, which is the k-times decimated version of the original lattice rescaled to spacing L^{-k}.

We have background fields $u_k(u)$ and $\phi_k = K_k(u_k)\phi$ defined on the L^{-k}-lattice. These fields represent the fields u,ϕ on the original lattice in a smooth way. (Note that as k increases, the L^{-k}-lattice becomes finer; eventually it is the ε-lattice and the procedure stops.) The background fields are the minimum energy L^{-k}-lattice configurations, given u,ϕ and taking into account only quadratic terms.

As before, the leading terms in the effective action are just the original action evaluated on the configurations u_k,ϕ_k. Thus we have, for example, $\Sigma_x L^{-dk}\lambda(L^k\varepsilon)|\phi_k(x)|^4$. The coupling constants are partially rescaled back to their original values--we have $\lambda(L^k\varepsilon) = \lambda\cdot(L^k\varepsilon)^{4-d}$ with $L^k\varepsilon$ increasing towards 1. The basic picture is that the renormalization transformation is driving the model away from a trivial fixed point, so the coupling constants are growing. It is convenient to introduce a diagrammatic representation for terms in the effective action. The above quadratic term is represented by



with a standard ϕ^4 vertex summed over the L^{-k}-lattice. The external lines in all diagrams are background fields ϕ_k or $f_k(p) = (ie(L^k\varepsilon)L^{-k})^{-1} \log u_k(p)$.

When ϕ_k and f_k are substituted into the quadratic terms of the original action, we obtain the main quadratic forms for the block fields $\phi(y)$ and $f^{(k)}(p') = (ie(L^k\varepsilon))^{-I} \log u(p')$. We denote these by $\Delta_k(u_k)$ and σ_k, and prove the basic stability bounds for them:

$$<\phi,\Delta_k(u_k)\phi> \geqslant \sum_{b'} c|u_k(<b'_-,b'_+>)\phi(b'_+) - \phi(b'_-)|^2$$

$$<f^{(k)},\sigma_k f^{(k)}> \geqslant c <f^{(k)},f^{(k)}> .$$

We also have a sequence of Gaussian normalization factors, $Z^{(0)}(u_k)\cdots Z^{(k-1)}(u_k)$, whose logarithms contribute to the effective action.

Finally we have the higher-order terms in $\lambda(L^k\varepsilon)$ and $e(L^k\varepsilon)$, which are given by diagrams with internal as well as external lines. The vertices in these diagrams are derived from the L^{-k}-lattice action, but there are some new vertices arising from the expansion of $Q(u)$ in the renormalization transformation. The propagators are derived from the original action also, but with the quadratic terms or δ-functions from the renormalization transformations included as well. Thus the scalar field propagator is

$$G_k(u_k) = (D^{L^{-k}*}_{u_k} D^{L^{-k}}_{u_k} + Q_k(u_k)^* Q_k(u_k))^{-1} ,$$

with $Q_k(u_k)$ being the k-th iterate of the averaging operation for scalar fields. The term $Q_k(u_k)^*Q_k(u_k)$ provides an effective mass for this L^{-k}-lattice propagator; thus we can prove its exponential decay. The short-distance behavior of $G_k(u_k)$ is also important for the analysis of ultraviolet divergences.

It is worth noting that our effective action is a purely perturbative one; it is given by the sum of diagrams of order less than some fixed n. The higher order and nonperturbative effects have been expanded out of the action and treated like large field regions. Thus the task of providing uniform bounds on the effective action is reduced to that of controlling the perturbation expansion. Renormalization cancellations are built into the perturbation expansion through the definition of counterterms.

We must consider effective observables as well as effective actions, since the original fields have been integrated out. The situation is quite analogous to that of the action--there are perturbative terms as well as nonperturbative or high-order terms. The latter can be estimated and essentially neglected, while the former generates diagrams like those considered for the action, except for new vertices coming from the observables on the L^{-k}-lattice.

2.10 Changes in Gauge

In the k-th renormalization transformation we perform the same basic steps that were outlined for the first step. However there is one new operation, the change of gauge, that is worth describing. The purpose is to improve the ultraviolet behavior of gauge field propagators in the effective action. This is accomplished by modifying the gauge fix for the fluctuation fields already integrated out.

Since we have been imposing axial gauge conditions each time we integrated out a fluctuation field, the gauge that would naturally occur in the vector field propagators in the k-th effective action is an axial-type gauge, with $A_b = 0$ for b in a maximal tree on each block of L^k sites. For large k these propagators are poorly behaved in the ultraviolet, and we are unable to prove the needed bounds on the perturbation expansion. To obtain better propagators, we need to change the background field u_k by a gauge transformation before expanding in the fluctuation field A'. Note that we have invariance of the effective action under the full group of L^{-k}-lattice gauge transformations of u_k even though that gauge freedom was broken when we integrated out the fluctuation fields. (When we transform u_k, the current fields u,ϕ must also be transformed by the restriction of the gauge function to the unit lattice.)

Unfortunately the required gauge transformation depends nonlocally on the current field u. This introduces nonlocal effects that have to be controlled with additional expansions. After the gauge transformation we have the background field written as

$$u_k = u_{k+1} \exp(ie(L^k\varepsilon)L^{-k}H_k A') \ ,$$

with H_k a regular, exponentially decaying kernel. Like u_k, the L^{-k}-lattice configuration $H_k A'$ is of minimal energy under certain constraints, but with a Feynman-like gauge fix used to measure energy instead of axial gauge restrictions. With the above form for u_k, we can expand the action with respect to A' as before. The next background field u_{k+1} remains in all expressions.

After integrating out A', we find that the regularity of H_k (and of earlier H_j, $j<k$) yields well-behaved gauge field propagators. In this way we see that it is possible to use one gauge for

integrating a field out, and another for representing the trace of that field in the effective action.

3. THE HIGGS MECHANISM

The renormalization transformations are continued until $L^k\varepsilon$ gets close to unity. The estimates begin to break down when $L^k\varepsilon$ approaches the smaller of m^{-1} (the inverse of the classical scalar field mass), and $(8\lambda/e^2)^{1/2}m^{-1}$ (the inverse of the classical vector field mass). At this point the payoff comes--we have a gauge invariant effective action for unit lattice fields with properties expected from perturbation theory. In particular, if we are interested in ultraviolet stability, the bounds on the effective action are independent of ε, and simple estimates for the last integration over u,ϕ suffices to prove ε-independent bounds on the original functional integrals. If we are interested in the mass generation and the infinite volume limit, we must extract mass terms from the effective action in order to integrate over u,ϕ--we can no longer rely on effective masses from renormalization transformations. We can extract mass terms by going to the unitary gauge for u,ϕ. The ultraviolet problem has already been treated; there are no difficulties associated with choosing $\theta = 0$ at each point of the unit lattice.

We conclude with a brief outline of the steps performed in integrating out the last fields and exhibiting the Higgs mechanism. The negative mass-squared term in the scalar potential has become significant, so we can use the "wine bottle" shape of the potential to introduce restrictions on ϕ. The small field region is defined to be where ϕ lies the annulus $\left||\phi| - (m^2/8\lambda)^{1/2}\right| \leqslant |\log e|^p$ and where $|D_{\bar{u}_k}\phi| \leqslant |\log e|^p$. [We define $\bar{u}_k(b') = u_k(\langle b'_-, b'_+\rangle)$.] The large field region has suppression factors coming from the corresponding terms in the action. For simplicity, we put $L^k\varepsilon = 1$.

It turns out that the analogous stability bound for the gauge field is best seen in axial gauge. Thus we again use our freedom to change u_k by a gauge transformation and put it in axial gauge ($u_k(b) = 1$ for b in maximal trees in L^k-blocks). The corresponding form of the action is almost like the standard unit lattice Wilson action because u_k takes the following simple form:

$u_k(b) = u(b')$ for b touching both neighboring blocks corresponding to the endpoints of b'

$u_k(b) = 1$ otherwise.

Actually, $u_k(b)$ differs from $u(b')$ or 1 by a small field; the difference is of the order of $e|\log e|^p$.

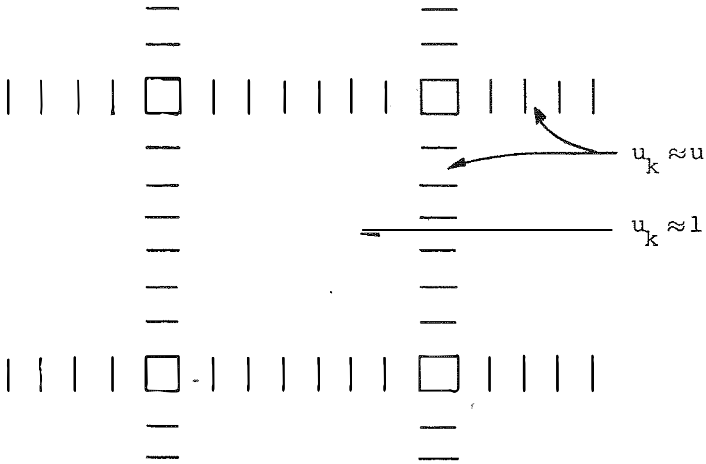


Figure 5.

For $d = 3$, these corridors become one higher dimensional. Of course this is a very singular gauge; when written in terms of Lie algebra elements, the concentrations at the corridors between blocks are almost δ-functions.

We recall the basic stability bound for the scalar field quadratic form:

$$\langle\phi, \Delta_k(u_k)\phi\rangle \geqslant c \sum_{b'} |u_k(\langle b'_-, b'_+\rangle)\phi(b'_+) - \phi(b'_-)|^2 \quad .$$

In axial gauge we have $u_k(\langle b'_-, b'_+\rangle) \approx u(b)$. Since ϕ is limited to a small neighborhood of $|\phi| = (m^2/8\lambda)^{1/2}$, this estimate allows us to restrict u to a small neighborhood of a pure gauge. Thus if we write

$$\phi = r\, e^{i\theta} \qquad\qquad r > 0$$

then we have

$$u = \exp[ie(A - \partial\theta)] \quad , \qquad |A_b| \leqslant c|\log e|^p$$

in the small field region. In the large field region we have convergence from the term

$$|u_k(\langle b'_-, b'_+\rangle)\phi(b'_-)|^2 \approx \frac{m^2}{8\lambda}\, |\exp[ieA_b] - 1|^2 \quad .$$

Now that we have the restrictions on A, we can change to a gauge in which the dependence of u_k on A is regular. We have

$$u_k = \exp[ie\varepsilon(H_k A + \partial^{\varepsilon}\theta')], \qquad \theta'(x) = -\theta(y(x)) + (D_k A)_x \quad ,$$

with $D_k(x,b)$ bounded; $y(x)$ is the corner of the block containing x. The term $\partial^{\varepsilon}\theta'$ is removed from u_k by a gauge transformation. The phase of ϕ is canceled--there is no longer any dependence on θ, and it can be integrated out trivially. This leaves us in unitary gauge $\phi = r > 0$. However, there is a small residual phase $\exp[ie(D_k A)_y]$ multiplying each $r(y)$. This cannot be gauged away and is a new interaction.

We expand the action with respect to $H_k A$ and $D_k A$. A shift $r = r_0 + r'$, $r_0 = (m^2/8\lambda)^{1/2}$ to the minimum of the scalar potential allows us to extract an explicit mass term for the scalar field. The gauge field mass term is also extracted at this point. The quadratic form resulting from perturbing $\langle r_0, \Delta_k(u_k) r_0\rangle$ with respect to $H_k A$ and $D_k A$ contains terms like Figure 6.

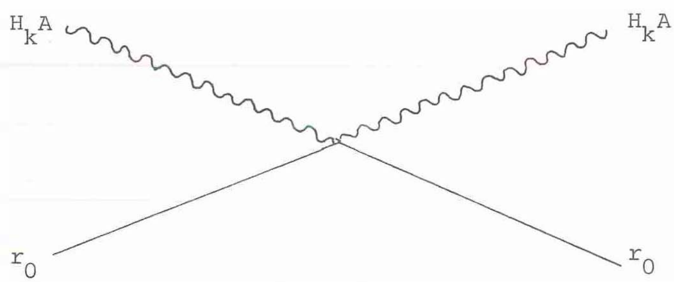


Figure 6.

We combine this form with the kinetic energy term $\langle f^{(k)}, \sigma_k f^{(k)}\rangle = \langle \partial A, \sigma_k \partial A\rangle$ to obtain a quadratic form with a strictly positive lower bound $\sim r_0^2 e^2 \langle A,A\rangle$. The coefficient $r_0^2 e^2 = m^2 e^2/8\lambda$ is the square of the semiclassical gauge field mass.

A cluster expansion can now be performed, since the mass terms lead to exponentially decaying propagators for r' and A, and since all interaction terms are small in the small field region. A convergent cluster expansion yields exponentially decaying correlations and the existence of the infinite volume limit.

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