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DIRECTED POLYMERS IN A RANDOM ENVIRONMENT *

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Abstract. In this lecture, I will define a system of walks or paths traversing an environment which is random in space and time. First, I will discuss some rigorous results on diffusive behavior above the upper critical dimension (joint work with T. Spencer). Then I will discuss what is believed to happen in lower dimensions. In particular, I will show how in one spacial dimension one can explain the exact value of the critical exponent describing the long-time displacement of the polymers.

To begin with, let us consider a noninteracting system of walks (no random environment). The walks are paths in Z^d, parametrized by an integer-valued time coordinate. Thus each walk ω is a function from the integers in $[0,T]$ to Z^d, with

$$|\omega(t+1)-\omega(t)|=1\,. \tag{1}$$

The measure is the uniform weighting of such walks, denoted as follows:

$$\int \cdot\, dW_0^T = \sum_{\omega} \cdot \left(\frac{1}{2d}\right)^T . \tag{2}$$

The subscript 0 indicates that all walks begin at the origin. The mean-square displacement of the walk is given by

$$\int \omega(T)^2\, dW_0^T = T\,, \tag{3}$$

which is diffusive behavior, with diffusion constant (coefficient of T) equal to unity. We can obtain (3) easily by defining a "free propagator"

$$p_0(T,x) = \int \delta(\omega(T)-x)\, dW_0^T \tag{4}$$

and its Fourier transform

$$p_0(T,k) = \left(1+\frac{1}{2d}\sum_{i=1}^{d} 2(\cos k_i - 1)\right)^T . \tag{5}$$

Then we calculate

$$\int \omega(T)^2\, dW_0^T = -\sum_{i=1}^{d} \frac{\partial^2}{\partial k_i^2}\, p_0(T,k) = T\,. \tag{6}$$

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By perturbing this measure, we can obtain several interesting systems. For example, the weakly self-avoiding walk is defined by a measure proportional to

$$\prod_{0<s<t\leq T} (1-\lambda\delta(\omega(s)-\omega(t)))\ dW_0^T\ , \tag{7}$$

where $0<\lambda\leq 1$ describes the strength of self-repulsion. The system we will focus on here is defined using an environment which is random in space and time. Thus, we let $\{h(t,x)\}_{t\in[0,T],x\in Z^d}$ be a collection of independent, identically distributed random variables. We suppose the distribution is bounded and symmetric about $h=0$; for example $h(t,x)=\pm 1$ with equal probabilities.

The partition function is

$$Z(T)=\int\prod_{t=1}^{T}(1+\epsilon h(t,\omega(t)))\ dW_0^T\ , \tag{8}$$

where ϵ parametrizes the strength of the disorder. The mean-square displacement is

$$\langle\omega(T)^2\rangle=\frac{N(T)}{Z(T)}=\frac{1}{Z(T)}\int\omega(T)^2\prod_{t=1}^{T}(1+\epsilon h(t,\omega(t)))\ dW_0^T\ . \tag{9}$$

We are interested in statements about $\langle\omega(T)^2\rangle$ which can be made with probability one (with respect to h), and also in statements about the average value of $\langle\omega(t)^2\rangle$.

The effect of the random environment is to enhance or suppress the tendency of the walk to traverse various regions of space-time. It is perhaps better to think of time as an extra spacial dimension, singled out by the fact that walks must move at a constant rate in that direction. This explains the term "directed polymer."

Diffusive Behavior in High Dimension. If the spacial dimension d is 1 or 2, nondiffusive behavior is expected even for small ϵ. This will be discussed more fully below. For $d>2$, however, the following result holds [5]:

THEOREM 1. *Let $d>2$ and ϵ be small. Then there is a $\theta>0$ such that for almost every environment h the following estimate holds for all T:*

$$\langle\omega(T)^2\rangle=T\left(1+O(T^{-\theta})\right)\ . \tag{10}$$

Note that the diffusion constant is independent of ϵ — it is not renormalized by the random environment. The constants implicit in the $O(T^{-\theta})$ estimate may depend on h. However, they do so in a controlled manner so that

$$\overline{\langle\omega(T)^2\rangle}=T(1+O(T^{-\theta}))\ . \tag{11}$$

(Here the bar denotes averaging with respect to h.)

In order to show what lies behind this result and why $d=2$ is the borderline dimension, we consider a simpler problem. Let us show that diffusive behavior occurs with large probability. This may be demonstrated relatively simply by bounding the fluctuation of numerators and denominator separately. For the partition function we have

$$\overline{Z(T)}=1\ , \tag{12}$$

since all nonconstant terms in

$$Z(T) = \int \prod_t \prod_x (1 + \epsilon h(t,x)\delta(\omega(x) - t))\, dW_0^T \tag{13}$$

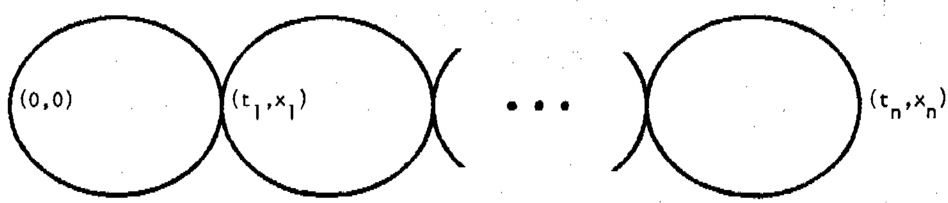

FIG. 1.

are linear in each $h(t,x)$, and $\overline{h(t,x)} = 0$. Next, we evaluate

$$Z(T)^2 = \int\int \textstyle\prod_t \prod_x [(1 + \epsilon h(t,x)\delta(\omega_1(t) - x))(1 + \epsilon h(t,x)\delta(\omega_2(t) - x))] \times dW_0^T(\omega_1)\, dW_0^T(\omega_2)\,. \tag{14}$$

Upon averaging, only terms constant or quadratic in each $h(t,x)$ survive, and using $\overline{h(t,x)^2} = 1$ we have

$$\overline{Z(T)^2} = \iint \prod_t \prod_x (1 + \epsilon^2 \delta(\omega_1(t) - x)\delta(\omega_2(t) - x))\, dW_0^T(\omega_1)\, dW_0^T(\omega_2)\,. \tag{15}$$

When this product is expanded out and integrated, we obtain the perturbation expansion for $\overline{Z(T)^2}$. The terms can be represented diagrammatically as a chain, where each line between two vertices represents a free propagator $p_0(t_{j+1} - t_j,\ x_{j+1} - x_j)$ (see Fig. 1). The reason for this structure is that the two walks are forced to coincide in space each time an interaction occurs. In these diagrams, the space-time locations of the vertices must be summed over, subject to the condition $t_1 < t_2 < \ldots < t_n$. There is a factor ϵ^{2n} for a graph with n interaction vertices, and we sum n from 0 to ∞.

The series is well behaved if a single bubble is convergent. We compute

$$\sum_x p_0(t,x)^2 = p_0(2t,0) \sim t^{-d/2}\,, \tag{16}$$

which is summable in t if $d > 2$. This implies that

$$\overline{(Z(T) - 1)^2} \le O(\epsilon^2)\,. \tag{17}$$

Chebyshev's inequality now yields small fluctuations of $Z(T)$ with high probability. Choose $0 < \eta < 1$, and we have

$$\mathrm{Prob}(|Z(T) - 1| > \epsilon^{1-\eta}) \le O(\epsilon^{2\eta})\,. \tag{18}$$

A similar argument can be used to control the numerator $N(T)$. We have

$$\overline{N(T)} = T \tag{19}$$

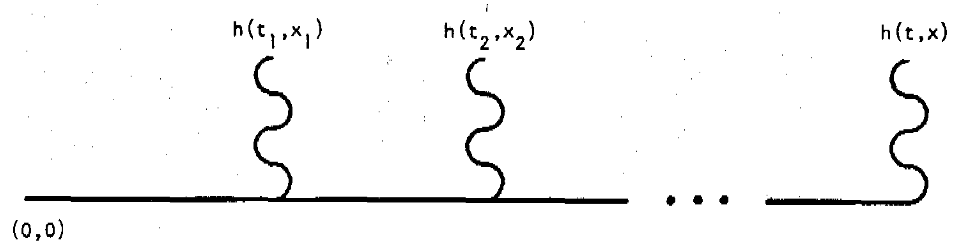

FIG. 2.

$$\overline{(N(T)-T)^2} \leq O(\epsilon^2)T \tag{20}$$

$$\text{Prob}(|N(T)-T| > O(\epsilon^{1-\eta})T) \leq O(\epsilon^{2\eta}) . \tag{21}$$

Combining (18) and (21) we obtain diffusion with large probability:

$$\text{Prob}(|\langle\omega(T)^2\rangle - T| > O(\epsilon^{1-\eta})T) < O(\epsilon^{2\eta}) . \tag{22}$$

These arguments show clearly that above two dimensions, fluctuations are typically too small to alter the long-time behavior. We would like to be able to obtain diffusion even in the presence of exceptional fluctuations. To do this we need a procedure for handling the set of h's where the above arguments fail. Specifically, we need to be able to identify the times at which the behavior of h leads to exceptional fluctuations.

Define an "irreducible" kernel

$$\begin{aligned} p_I(t,x) = \; & \textstyle\sum_{n=1}^{t} \prod_{i=1}^{n-1} \left[\sum_{t_i,x_i} \epsilon h(t_i,x_i) p_0(t_i - t_{i-1},\; x_i - x_{i-1})\right] \\ & \times \epsilon h(t,x) p_0(t - t_{n-1},\; x - x_{n-1}) . \end{aligned} \tag{23}$$

This kernel, which depends on h, is useful for the pointwise bounds we seek. Graphically, we draw diagrams with unintegrated h's indicated with wavy lines (see Fig. 2). These diagrams are simply the expansion of $Z(T)$ in powers of h. (The final line from t to T sums up to unity.) In fact, it is easy to see that

$$Z(T) = 1 + \sum_{x,t} p_I(t,x) . \tag{24}$$

Next, we partially resum (24) using the following quantities:

$$Z(T_1,T_2) = \sum_{t=T_1}^{T_2} \sum_{x} p_I(t,x) . \tag{25}$$

We can express $Z(T)$ as a sum of terms $Z(2^j, 2^j+\ell)$ with $0 \leq \ell < 2^j$. We expect that $Z(T_1,T_2)$ will usually decrease like $T_1^{-(d/2-1)/2}$, and we define "nonexceptional" events E_j accordingly:

$$E_j = \{h : |Z(2^j, 2^j+\ell)| \leq c\epsilon^{1-\eta}2^{-j\theta} \quad \text{for} \quad \ell = 0,1,\ldots,2^j-1\} . \tag{26}$$

Here $\theta = (d-2)/4 - \eta > 0$. On the set $\bigcap_j E_j$ we have for all T

$$(27) \qquad |Z(T) - 1| \le c\epsilon^{1-\eta} \sum_j 2^{-j\theta} = O(\epsilon^{1-\eta}) .$$

A similar analysis on $N(T)$ leads to diffusive behavior on the "good" set:

$$(28) \qquad |\langle \omega(T)^2 \rangle - T| \le O(T^{1-\theta}) .$$

We will, of course, need an estimate on $\mathrm{Prob}(E_j^c)$, the probability of a large fluctuation between time 2^j and 2^{j+1}. We use Chebyshev estimates again, but with larger powers of $Z(T_1, T_1)$ to achieve better control over probabilities. With m fixed at some moderately large integer, perturbative estimates lead to

$$(29) \qquad \overline{Z(T_1, T_2)^{2m}} \le T_1^{-2m(d-2)/4} \cdot (c\epsilon)^{2m} .$$

Hence, we have

$$(30) \qquad \mathrm{Prob}(|Z(T_1, T_2)| > \epsilon^{1-\eta} T_1^{-\theta}) < (c\epsilon)^{2m\eta} T_1^{-2m\eta} ,$$

and for large enough m we can sum the right-hand side over $\ell = 0, 1, \ldots, 2^j - 1$. Thus,

$$(31) \qquad P(E_j^c) \le (c\epsilon)^{2m\eta} T_1^{-(2m\eta - 1)}$$

and

$$(32) \qquad P(\cap_j E_j) \ge 1 - O(\epsilon^{2m\eta}) .$$

Now that we have control over the times of large fluctuations, let us suppose that E_j is the first event to fail. Thus, on the set

$$(33) \qquad F_j = E_0 \cap \ldots \cap E_{j-1} \cap E_j^c$$

we restart the procedure at time $s = 2^{j+1}$. We define new "good" sets

$$(34)\ E_{j,s} = \{h : Z_{s,y}(2^j, 2^j + \ell) < cs\epsilon^{1-\eta} 2^{-j\theta} \text{ for } \ell = 0, 1, \ldots, 2^j - 1 \text{ and for all } |y| \le s\}$$
$$\cap \{h : Z_{s,y_0}(2^j, 2^j + \ell) < c\epsilon^{1-\eta} 2^{-j\theta} \quad \text{for } \ell = 0, 1, \ldots, 2^j + 1, \text{ where } y_0 \text{ maximizes } p(s, y)\} .$$

Here $Z_{s,y}(T_1, T_2)$ is the same as $Z(T_1, T_2)$ except we have shifted the origin of space time to (s, y). Also, $p(s, y) = \langle \delta(\omega(s) - y) \rangle$ is the interacting diffusion on the time interval $[0, s]$. (We should also assume bounds on quantities based on $N(T)$ as well, but we omit any discussion of such terms for the sake of brevity.)

Again, we can show that $\mathrm{Prob}(\cap_j E_{j,s})$ is close to unity. On this set we have diffusive behavior at large times, no matter how badly behaved $p(s, y)$ is. For example, even if the behavior is ballistic through time s, we control the diffusion from all points accessible at time s, and at large times the initial behavior is washed out.

As s grows, the constants in our bounds deteriorate, as is evident in (34). To prevent the partition function from getting too close to zero, however, we need to control one term in an s-independent manner. This explains the second set in (34).

Of course, we need to analyze the sets $E_{j,s}^c$, and so on. The process continues until a complete string of successes, $\cap_j E_{j,s}$, occurs for some s. This will happen eventually for almost every h. In this manner the Chebyshev estimates can be pushed to obtain diffusion with probability one.

Directed Polymers in Low Dimension. If the spacial dimension is one or two, we expect nondiffusive behavior, even for weak coupling. Let us define an exponent $\varsigma = \varsigma(d)$ which measures the asymptotic behavior of the distance the path moves from the origin:

$$\langle \omega(T)^2 \rangle \sim T^{2\varsigma} . \tag{35}$$

Diffusive behavior corresponds to $\varsigma = 1/2$. There are several arguments for the value $\varsigma = 2/3$ in one spacial dimension. I will discuss one line of argument below. Numerical results [4,6] confirm the exponent $\varsigma = 2/3$ in one dimension. In higher dimensions, a value near 2/3 is obtained, but the data do not confirm or deny the possibility of a "super-universal" exponent which is independent of dimension. See [8] for further discussion of this point. (In high dimensions we expect nondiffusive behavior for strong disorder, and we use this regime in making comparisons with the low dimension cases.)

The key to our understanding of the value of ς in one dimension lies in the relation between directed polymers and a forced Burgers equation. Let $w(t,x)$ denote the unnormalized weight of paths ending at t, x. This weight obeys the following equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + hw . \tag{36}$$

Let us work in $d = 1$, and with continuous space and time. The random environment h acts as a potential perturbing a diffusion equation.

Although this is a linear equation, we perform a Hopf-Cole transformation to obtain a forced Burgers equation. We put $f = -\log w$, and a few simple computations transform the equation to

$$f_t = f_{xx} - (f_x)^2 + h , \tag{37}$$

where we use subscripts to denote partial differentiation. Now putting $u = f_x$ we obtain

$$u_t = u_{xx} - (u^2)_x + h_x , \tag{38}$$

which is Burgers equation with a conservative forcing term (derivative of white noise). This transformation is usually used in the other direction to convert Burgers equation into a linear equation. Here we exploit some special properties of Burgers equation to gain insights into the directed polymer system.

Several approaches have been used to obtain the scaling $x \sim t^{2/3}$ in the context of Burgers equation. Fisher, Huse, and Henley [2] observed that an invariant distribution exists for the equation. If the data are distributed like $\exp(-\int dx\, f_x^2)$ at one time, then they are so distributed for all subsequent times. Assuming that this distribution reflects the long-time behavior, this implies that $|f(t,0) - f(t,x)| \sim x^{1/2}$. Balancing this against the extra free energy x^2/t from stretching the polymer to a height x in time t, we obtain the desired scaling.

Other approaches are based on renormalization group analysis of the forced Burgers equation [3], and on a Bethe-ansatz solution of a related system (a commensurate-incommensurate interface model with impurities) [7].

We will discuss still another approach that goes back to Burgers [1], who understood the scaling in the context of the (unforced) Burgers equation with random initial data. He studied solutions to the equation

$$u_t = \nu u_{xx} - u u_x \tag{39}$$

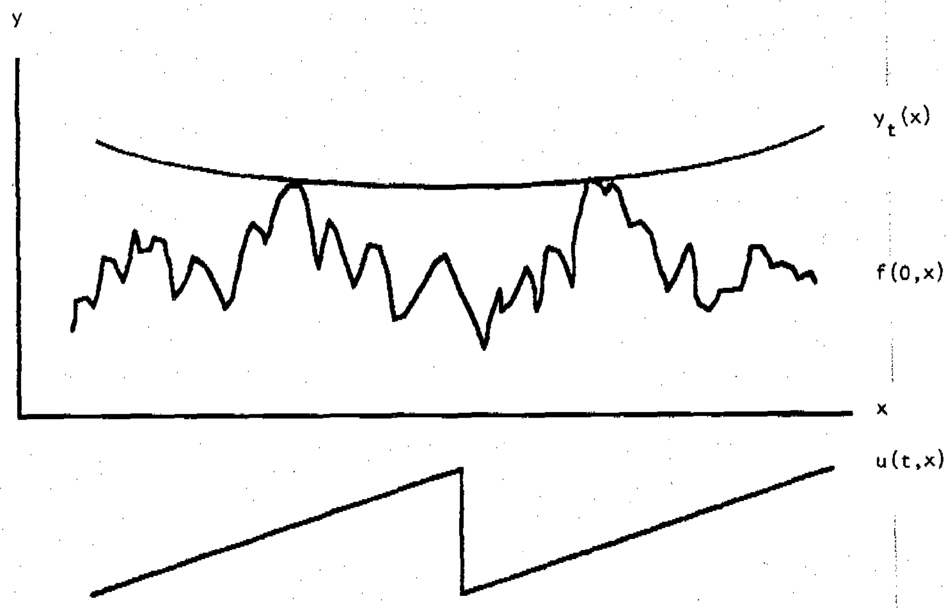

FIG. 3.

in the limit as the viscosity ν tends to zero. The Hopf-Cole transformation is

(40) $$u = -2\nu(\log w)_x \ ,$$

and gives us the diffusion equation

(41) $$w_t = \nu w_{xx} \ .$$

The randomness is put back in by taking initial data distributed like $\exp(-\int dx\, f_x^2)$, as above, where $f_x = -u$. The problem is admittedly rather far from the one we started with, but one can argue that the long-time behavior should be the same.

The solution to the equation can be written down exactly for arbitrary initial data. It involves a series of shock fronts, separated by regions where u varies almost linearly with x. The scaling between x and t shows up in the typical distance between shock fronts. The solution can be constructed as follows. To find $u(t,x)$, first plot the initial data $f(0,x)$ against x. Plot also the family of parabolas

(42) $$y_t(x) = (x - x_0)^2/2t + c \ ,$$

with c varying. As c decreases, at some point the parabola will first contact the initial data curve. Say the contact occurs at a single point ξ. Then the limit of the solution as $\nu \to 0$ is

(43) $$u(t,x) = \frac{x - \xi}{t} \ .$$

If t is large, ξ will vary slowly with x, and we have an almost linear behavior. At some values of x, however, the contact occurs at two points, and in that case there is a discontinuity or shock as we shift from one value of ξ to the other in (43) (see Fig. 3). As t increases, the parabola flattens out and the distance between double contacts (or shocks) increases.

Our assumption is that the initial data curve is a sample path from Brownian motion $(\exp - \int dx\ f_x^2)$. The problem, then, is to determine the typical distance between double contacts when the parabolas (42) are dropped onto the plot of a Brownian path. While this seems at first sight like a difficult problem, the t-dependence can be determined simply by a rescaling of coordinates on the plot. We put $x' = x/t^{2/3}$, $y' = y/t^{1/3}$, where y is the vertical coordinate. Such a scaling leaves Brownian motion invariant, while putting the parabola into a t-independent form

$$y'(x') = (x' - x_0')^2/2 + c' \tag{44}$$

Whatever the distance between contacts is in the primed variables, it must be that the distance scales as $t^{2/3}$ in the original variables. Thus, we obtain the claimed dependence of the intershock distance on t.

Despite all the indications that $\varsigma = 2/3$ in one dimension, no proofs have yet been devised. It is doubtless a challenging, but worthwhile problem.

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