Journal of Statistical Physics, Vol. 52, Nos. 3/4, 1988

Diffusion of Directed Polymers in a Random Environment

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Received April 26, 1988

We consider a system of random walks or directed polymers interacting weakly with an environment which is random in space and time. In spatial dimensions $d>2$, we establish that the behavior is diffusive with probability one. The diffusion constant is not renormalized by the interaction.

KEY WORDS: Random walks; diffusion; directed polymers; random environment.

1. INTRODUCTION

A directed polymer system is a statistical ensemble of walks or paths in $\mathbb{Z}^d$ parametrized by time. The graph of the walk in $\mathbb{Z}^{d+1}$ is the "polymer" which moves at a constant rate in the time direction and so is called "directed." Directed polymers can also be defined in continuous space and time, but we consider here only the lattice version. We consider walks interacting with a weak random space-time environment, and show that they behave diffusively for $d>2$. Directed polymers in a random environment have appeared in recent physics literature[1] as a model for the interface in two-dimensional Ising models with random exchange interactions. In this case $d=1$, and nondiffusive behavior is conjectured. We discuss the background more fully after defining the model and stating our main results.

We consider walks ω: $[0, T] \cap \mathbb{Z} \to \mathbb{Z}^d$ such that $\omega(0)=0$, $|\omega(t+1)-$

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0022-4715/88/0800-0609$06.00/0

$\omega(t)| = 1$. Giving each walk a weight $(2d)^{-T}$, we obtain a probability measure dW_0^T for noninteracting walks. The function

$$p_0(T, x) = \int \delta(\omega(T) - x)\, dW_0^T \tag{1.1}$$

gives the probability that $\omega(T) = x$. (Here δ is a Kronecker δ-function for $\mathbb{Z}^d$.) The mean-square displacement from the origin for the free diffusion is

$$\langle \omega(T)^2 \rangle_{T,0} \equiv \sum_x x^2 p_0(T, x) = T \tag{1.2}$$

as can be easily shown using Fourier transforms. (Here $x^2 = x_1^2 + \cdots + x_d^2$.)

The random environment is a real-valued function $h(t, x)$ for $t > 0$. For simplicity we take $h(t, x)$ to be independent for each x and t, with $h(t, x) = \pm 1$ with equal probabilities. This environment is weakly coupled to the diffusion, producing an interacting density

$$p(T, x) = \int \delta(\omega(T) - x) \prod_{0 < t < T} [1 + \varepsilon h(t, \omega(t))]\, dW_0^T \tag{1.3}$$

Paths traversing space-time regions with $h = 1$ have enhanced weight, while those traversing regions with $h = -1$ are suppressed. This density is unnormalized, and so to obtain a probability for paths to reach x at time T we define

$$p_N(T, x) = p(T, x)/Z(T) \tag{1.4}$$

Here $Z(T)$ is the partition function

$$Z(T) = \sum_x p(T, x) = \int \prod_{0 < t \leqslant T} [1 + \varepsilon h(t, \omega(t))]\, dW_0^T \tag{1.5}$$

Our results concern the mean square displacement for the interacting system:

$$\langle \omega(T)^2 \rangle_{T,h} = \sum_x x^2 p_N(T, x)$$

Theorem 1. For any $d > 2$, let ε be small. There is a $\theta > 0$ such that the following bound holds with probability one:

$$|\langle \omega(T)^2 \rangle_{T,h} - T| \leqslant c(h) T^{1-\theta} \qquad \text{for all} \quad T > 0 \tag{1.6}$$

We see that diffusive behavior holds for almost every realization of the environment. Furthermore, the diffusion constant is equal to one, the same as for the noninteracting system: it is not renormalized by the interaction as one might at first expect.

The random constant $c(h)$ in (1.6) is unbounded; however, we can estimate its distribution. This leads to the following bound:

$$|\overline{\langle \omega(T)^2 \rangle_{T,h}} - T| \leqslant \varepsilon^{1-\eta} T^{1-\theta} \qquad \text{for some} \quad 1-\eta > 0 \text{ and all } T \tag{1.7}$$

Here the bar denotes averaging over the random environment h. We obtain these results with θ slightly smaller than $\min\{(d-2)/4, 3/4\}$.

1.1. Background

Directed polymers have received considerable attention in the recent physics literature. The situation in spatial dimension $d=1$ is especially interesting, with a conjectured superdiffusive exponent $\zeta = 2/3$ in

$$\langle \omega(T)^2 \rangle_{T,h} \sim T^{2\zeta}$$

This behavior was observed numerically by Huse and Henley[1] and soon thereafter explained heuristically by Fisher *et al.*[2] and Kardar and Nelson.[3] The approach of ref. 2 was to exploit some special properties of a forced Burgers equation which the density $p(t, x)$ obeys in the continuum limit. In ref. 3 a completely different argument uses replicas and a Bethe-ansatz solution. The exponents for the forced Burgers equation were actually explained some time ago in ref. 4, using renormalization group ideas.

The renormalization group picture indicates that $d=1$ and $d=2$ are intrinsically strong coupling problems, with nontrivial exponents (e.g., $\zeta = 2/3$ in $d=1$). Even a weak random environment becomes effectively strong at large distances and long times. In higher dimensions, it is likely that nondiffusive behavior still occurs at strong coupling, while (as we show in this paper) weak coupling entails diffusive behavior. Numerical studies support this expectation[5]; see also refs. 6, 8.

1.2. The Upper Critical Dimension

It is instructive to examine fluctuations in $Z(T)$ to see why $d=2$ is borderline for our analysis. It is simple to rewrite (1.5) in the following form:

$$Z(T) = \int \prod_{0 < s \leqslant T} \prod_{x} [1 + \varepsilon h(s, x)\, \delta(\omega(s) - x)]\, dW_0^T \tag{1.8}$$

Since each $h(s, x)$ is an independent random variable with $\bar{h}=0$, we easily see that

$$\overline{Z(T)}=1 \tag{1.9}$$

Fluctuations in $Z(T)$ can be estimated as follows. We have

$$Z(T)^2=\iint \prod_{0<s\leqslant T} \prod_{x} \{[1+\varepsilon h(s, x)\,\delta(\omega_1(s)-x)] \\ \times [1+\varepsilon h(s, x)\,\delta(\omega_2(s)-x)]\}\, dW_0^T(\omega_1)\, dW_0^T(\omega_2) \tag{1.10}$$

and after averaging over h, only terms quadratic in each $h(s, x)$ survive. Using $\overline{h^2}=1$, we obtain

$$\overline{Z(T)^2}=\iint \prod_{0<s\leqslant T} \prod_{x} [1+\varepsilon^2\,\delta(\omega_1(s)-x)\,\delta(\omega_2(s)-x)] \\ \times dW_0^T(\omega_1)\, dW_0^T(\omega_2) \tag{1.11}$$

Expanding the products over s, x, we obtain a sum over subsets $\{s_i\}$ of $[0, T]$. At each time s_i, both walks must visit the same site x_i, which is also summed over. Between the s_i's, the two walks are independent, so integrating over ω_1 and ω_2 produces two free diffusions $p_0(s_i-s_{i-1}, x_i-x_{i-1})$. After s_n, the last time the walks are forced to meet, the walks are completely unconstrained and we use the fact that

$$\sum_x p_0(T-s_n, x-x_n)=1 \tag{1.12}$$

The result is the following expansion:

$$\overline{Z(T)^2}=\sum_{n=0}^{T} \sum_{0=s_0<s_1<\cdots<s_n\leqslant T} \varepsilon^{2n} \prod_{i=1}^{n} \left[\sum_{x_i} p_0(s_i-s_{i-1}, x_i-x_{i-1})^2\right] \tag{1.13}$$

We have for the free diffusion

$$p_0(s, x)\leqslant ce^{-cx^2/s}/s^{d/2} \tag{1.14}$$

so that

$$\sum_x p_0(s, x)^2\leqslant cs^{-d/2} \tag{1.15}$$

Since $d>2$, each sum over s_i converges nicely. Taking out the $n=0$ term (which equals unity), we obtain

$$\overline{[Z(T)-1]^2}\leqslant \sum_{n=1}^{T} (c\varepsilon)^{2n}\leqslant O(\varepsilon^2) \tag{1.16}$$

It is evident that for $d>2$, fluctuations in $Z(T)$ are quite small. This is an important simplifying feature of our system; it means that a good approximation the normalization of the measure can be ignored. In dimension $d \leqslant 2$, much larger fluctuations should occur, presumably behaving as a power of T.

1.3. Diffusion with Large Probability

A similar analysis can be performed on the numerator $N(T)$ in the expectation

$$\sum_x x^2 p_N(T, x) = \langle \omega(T)^2 \rangle_{T,h} = N(T)/Z(T)$$

$$= Z(T)^{-1} \int \prod_{0<t\leqslant T} [1+\varepsilon h(t, \omega(t))]\, \omega(T)^2\, dW_0^T \quad (1.17)$$

We have $\overline{N(T)} = T$, and the fluctuations can be estimated perturbatively as above. Skipping this analysis (which will be done in greater generality later), we obtain

$$\overline{[N(T)-T]^2} \leqslant c\varepsilon^2 T^2 \quad (1.18)$$

Using Chebyshev's inequality, (1.16) and (1.18) imply that for any $1>\eta>0$,

$$|Z(T)-1| \leqslant c\varepsilon^{1-\eta} \text{ with probability at least } 1-\varepsilon^{2\eta} \quad (1.19)$$

$$|N(T)-T| \leqslant c\varepsilon^{1-\eta} T \text{ with probability at least } 1-\varepsilon^{2\eta} \quad (1.20)$$

Hence, we have diffusion with high probability:

$$\langle \omega(T)^2 \rangle_{T,h} = T[1+O(\varepsilon^{1-\eta})] \text{ with probability at least } 1-2\varepsilon^{2\eta}$$

1.4. Diffusion with Probability One

The method we use to prove Theorem 1 is a repeated application of the above idea. We apply perturbative and Chebyshev estimates in such a way that the residual set on which atypical fluctuations occur, as in (1.19)–(1.20), has measure tending to zero.

The first step is to break up the partition function as

$$Z(T) = \sum_j Z(2^j, 2^{j+1}-1) \quad (1.21)$$

with each $Z(2^j, 2^{j+1}-1)$ defined as a partial sum of the perturbation expansion for $Z(T)$. Each $Z(2^j, 2^{j+1})$ depends only on $h(s, x)$ for $s \leqslant 2^{j+1}-1$. See (2.4) for a precise definition. Furthermore, since it involves graphs extending from 0 to t with $t \in [2^j, 2^{j+1})$, perturbation theory yields an estimate decreasing as a power of 2^j:

$$\overline{Z(2^j, 2^{j+1}-1)^{2m}} \leqslant (c\varepsilon)^{2m}\,(2^j)^{-(d-2)m/2} \tag{1.22}$$

By Chebyshev's inequality, (1.22) implies that

$$\text{Prob}(|Z(2^j, 2^{j+1}-1)| > c\varepsilon^{1-\eta}2^{-j\theta}) \leqslant \varepsilon^{2m\eta}2^{-2jm\eta} \tag{1.23}$$

where $\theta = (d-2)/4 + \eta$, and m is a large integer, chosen after η. If each term $Z(2^j, 2^{j+1}-1)$ is bounded by $c\varepsilon^{1-\eta}2^{-j\theta}$, and analogous bounds hold for "numerator" quantities, then (1.19), (1.20) hold and we have diffusion with high probability as before.

Next let us suppose there is a large fluctuation as in (1.23). Let j be the first instance of such a fluctuation. We consider afresh all possible interacting diffusions starting at time 2^{j+1} and position x with $|x| \leqslant 2^{j+1}$, and repeat the analysis leading to (1.22), (1.23). Note that $|x|$ is necessarily $\leqslant 2^{j+1}$, since the path takes nearest neighbor steps and $t = 2^{j+1}$. These new diffusions involve only values of $h(t, x)$ which were not used in defining the event that j is the first large fluctuation. If we can establish diffusive behavior no matter what the starting point, then the original system starting at (0, 0) has diffusive behavior as well. The right-hand side of (1.23) decreases as a large power of 2^{-j} for m large, and this allows us to ask for small fluctuations for all possible starting points. In this way we can prove diffusion for most of the cases not covered by the initial estimate.

Iterating the process, we map the measure space of the h's onto a stochastic process $\{X_j\}$, where $X_j \in \{S, F\}$. Each S (success) corresponds to situations where bounds like (1.21) hold between times 2^j and $2^{j+1}-1$, while each F (failure) corresponds to the complementary situation. Diffusion holds whenever an unbroken string of successes S occurs out to $j = \infty$. If we condition on any set of h's leading up to a failure F, then the probability that another failure occurs after a waiting time k is less than $\varepsilon 2^{-\kappa' k}$, with $\kappa' > 0$. Hence, an infinite string of successes will occur with probability one.

2. SUFFICIENT CONDITIONS FOR DIFFUSION

In this section we construct events which, in appropriate combinations, lead to diffusion. In the next section, we show that these conditions are satisfied with probability one, and so obtain the results stated in the introduction.

We work with a sort of irreducible kernel from which $Z(T)$, $N(T)$ can be derived. It is defined by its perturbation expansion:

$$p_I(t, x) = \sum_{n=1}^{t} \prod_{i=1}^{n-1} \left[\sum_{s_i, x_i} \varepsilon h(s_i, x_i)\, p_0(s_i - s_{i-1}, x_i - x_{i-1}) \right] \times \varepsilon h(t, x)\, p_0(t - s_{n-1}, x - x_{n-1}) \tag{2.1}$$

There is no problem of convergence because all sums are finite. Diagrammatically, $p_I(t, x)$ corresponds to graphs with unintegrated h's which end at $h(t, x)$ (Fig. 1). The solid lines between two vertices denote a free propagator $p_0(s_n - s_{n-1}, x_n - x_{n-1})$. Notice that

$$Z(T) = 1 + \sum_{t=1}^{T} \sum_{x} p_I(t, x) \tag{2.2}$$

since by expanding the products in (1.8) and integrating over dW_0^T we obtain the expansion (2.1) with t, x summed over. Similarly, we have

$$N(T) = T + \sum_{t=1}^{T} \sum_{x} p_I(t, x) \sum_{y} y^2 p_0(T - t, y - x) \tag{2.3}$$

where we use $\overline{N(T)} = T$ as the value of $N(T)$ when $h = 0$.

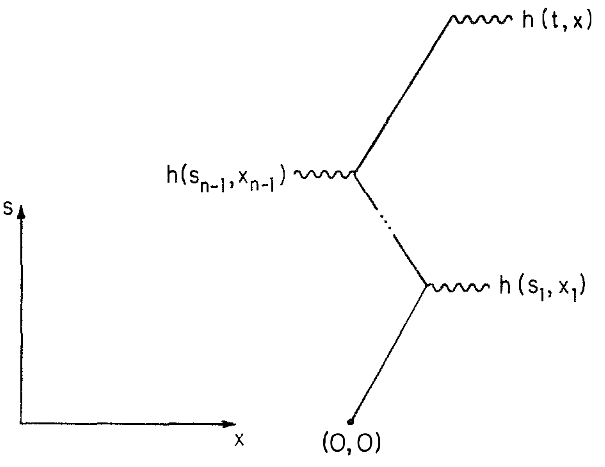

Fig. 1. The graphical expansion for $p_I(t, x)$. Wavy lines indicate unintegrated fields h. Solid lines are propagator $p_0(s_n - s_{n-1}, x_n - x_{n-1})$.

Next we define partial sums of $p_I(t, x)$, which will be the basic quantities we will have to estimate. We define

$$Z(T_1, T_2) = \sum_{t=T_1}^{T_2} \sum_x p_I(t, x) \tag{2.4}$$

$$N^{(x^a)}(T_1, T_2) = \sum_{t=T_1}^{T_2} \sum_x p_I(t, x) x^a, \qquad a = 1, 2 \tag{2.5}$$

$$N^{(t)}(T_1, T_2) = \sum_{t=T_1}^{T_2} \sum_x p_I(t, x) t \tag{2.6}$$

In the Appendix we show how to estimate the average of powers of these quantities. These perturbative bounds will form the basis for the probabilistic bounds of the next section.

First we consider events $E_j^{(0)}$ which will guarantee that diffusion from $t = 0$, $x = 0$ is well behaved. (Here j indexes scales of time, 2^j.) Later we will consider more general events $E_j^{(k)}$ relating to diffusions beginning at time 2^k. Let us put

$$\theta = \min\{(d-2)/4, 3/4\} - \eta > 0 \tag{2.7}$$

and define for $j = 0, 1, 2,...$

$$E_j^{(0)} = \{h\colon |Z(2^j, 2^j + l)| \leqslant c\varepsilon^{1-\eta} s^{-j\theta} \text{ and } |N^{(*)}(2^j, 2^j + l)| \leqslant c\varepsilon^{1-\eta} 2^{(1-\theta)j},$$
$$\text{for} \quad l = 0, 1, 2,..., 2^j - 1, \quad \text{and for} \quad * = x^2 \text{ or } t\} \tag{2.8}$$

Proposition 2.1. On the set $\bigcap_j E_j^{(0)}$ we have diffusion, in the sense that for all T

$$|\langle \omega(T)^2 \rangle_{T,h} - T| \leqslant c\varepsilon^{1-\eta} T^{1-\theta} \tag{2.9}$$

Proof. Any interval $[1, T]$ can be represented as a disjoint union of intervals $[T_1, T_2] = [2^j, 2^j + l_j]$, with one such interval for each $j \leqslant \log_2 T$. By (2.2), (2.4), we have

$$|Z(T) - 1| \leqslant \sum_{j=0}^{\log_2 T} |Z(2^j, 2^j + l_j)| \leqslant c\varepsilon^{1-\eta} \sum_j 2^{-j\theta} \leqslant c\varepsilon^{1-\eta} \tag{2.10}$$

For the numerator, we use (2.3), (2.5), and (2.6) to obtain an expression in terms of $N^{(*)}(T_1, T_2)$ and $Z(T)$. Since

$$\sum_y y^2 p_0(T-t, y-x) = x^2 + T - t \tag{2.11}$$

we have

$$N(T)-T=T[Z(T)-1]+\sum_{j=0}^{\log_2 T}[N^{(x^2)}(2^j, 2^j+l_j)-N^{(t)}(2^j, 2^j+l_j)] \tag{2.12}$$

Inserting the bounds valid on $\bigcap E_j^{(0)}$, we obtain

$$|N(T)-TZ(T)|\leqslant\sum_j c\varepsilon^{1-\eta}2^{(1-\theta)j}\leqslant c\varepsilon^{1-\eta}T^{1-\theta} \tag{2.13}$$

Combining this with (2.10), we obtain the proposition. ∎

Let us consider diffusions starting at general space-time points (s, y). We define $N_{s,y}(T)$, $Z_{s,y}(T)$, $N_{s,y}^{(x^2)}(T_1, T_2)$, $Z_{s,y}(T_1, T_2)$, etc., as before, only walks start at (s, y) instead of $(0, 0)$. More precisely, we can define $\tilde{h}(t, x)=h(t+s, x+y)$ and write, for example,

$$Z_{s,y}^{(h)}(T)=Z^{(\tilde{h})}(T-s) \tag{2.14}$$

$$Z_{s,y}^{(h)}(T_1, T_2)=Z^{(\tilde{h})}(T_1-s, T_2-s) \tag{2.15}$$

$$p_{l,s,y}^{(h)}(T, x)=p_l^{(\tilde{h})}(T-s, x-y) \tag{2.16}$$

with the superscripts indicating the noise field at which quantities are being evaluated. Notice that $Z_{s,y}(T)$ depends only on $h(t, x)$ for $s<t\leqslant T$. Similar statements hold for the irreducible quantities; for example, $Z_{s,y}(T_1, T_2)$ depends only on $h(t, x)$ for $s\leqslant t\leqslant T_2$. Putting $s=2^k-1$, we can now define for $j\geqslant k$ a more general event

$$\begin{aligned}E_j^{(k)}=\{h\colon &|Z_{s,y}(2^j, 2^j+l)|\leqslant cs\varepsilon^{1-\eta}2^{-(j-k)\theta} \text{ and}\\ &|N_{s,y}^{(*)}(2^j, 2^j+l)|\leqslant cs\varepsilon^{1-\eta}2^{j(1-\theta)}, \text{ for } *=x^2, x, \text{ or } t,\\ &\text{for } l=0, 1, 2,..., 2^j-1, \text{ and for all } |y|\leqslant s\}\\ &\cap\{h\colon |Z_{s,y_0}(2^j, 2^j+l)|\leqslant c\varepsilon^{1-\eta}2^{-(j-k)\theta} \text{ for } l=0, 1,..., 2^j-1\\ &\text{and for } y_0 \text{ equal to the largest } y \text{ that maximizes } p(s, y)\}\end{aligned} \tag{2.17}$$

The first event is similar to $E_j^{(0)}$, only the estimate is less restrictive, with an extra factor of s. The probability for violation of the bound is much smaller and this compensates for the fact that we consider all $|y|\leqslant s$. Note that $p(s, y)=0$ for $|y|>s$. We also need bounds for $N^{(x)}$ because we have diffusions starting "off-center" at $y\neq 0$. The second event is used solely to prevent partition functions $Z(T)$ from vanishing, and it is sufficient to consider only one site.

To relate the diffusion starting at (0, 0) to the ones starting at (s, y), we need a "semigroup property"

$$p(T, x) = \sum_y p(s, y)\, p_{s,y}(T, x), \qquad \text{for any} \quad s \in [0, T] \tag{2.18}$$

This is easily obtained by inserting $1 = \sum_y \delta(\omega(s) - y)$ into our defining expression for $p(T, x)$. The measure dW_0^T factors, and the second part of the walk gives an independent diffusion $p_{s,y}(T, x)$ from (s, y) to (T, x). [We use subscripts to denote shifted diffusions, as in (2.16).] Note that (2.18) fails for $p_N(T, x)$; the normalized propagator is not the transition function of a Markov process.

Proposition 2.2. For any $k > 0$ we have diffusion on the set $\bigcap_{j \geq k} E_j^{(k)}$. With $s = 2^k - 1$ and θ given by (2.7), we have for all T

$$|\langle \omega(T)^2 \rangle_{T,h} - T| \leq s^2 + cs^{d+2} \varepsilon^{1-\eta} T^{1-\theta} \tag{2.19}$$

Proof. On the set $\bigcap_{j \geq k} E_j^{(k)}$ we know little about $p(s, y)$ but we have good control over $p_{s,y}(T, x)$. We do know that $p(s, y) \geq 0$, that $p(s, y) = 0$ for $|y| > s$, and that $p(s, y)$ is maximized at $y = y_0$. Let us normalize $p(s, y)$ to a probability measure on y. We obtain for the partition function

$$\begin{aligned} Z(T) &= Z(s) \sum_y p_N(s, y)\, Z_{s,y}(T) \\ &= Z(s) \sum_y p_N(s, y) \left[1 + \sum_{j=k}^{[\log_2 T]} Z_{s,y}(2^j, 2^j + l_j) \right] \end{aligned}$$

and we can estimate this using the $Z_{s,y}$ bounds in (2.17). The result is

$$\left| \frac{Z(T)}{Z(s)} - 1 \right| \leq \sum_{j \leq k} cs\varepsilon^{1-\eta} 2^{-(j-k)\theta} \leq cs\varepsilon^{1-\eta} \tag{2.20}$$

$$\frac{Z(T)}{Z(s)} \geq cs^{-d} Z_{s,y_0}(T - s) \geq cs^{-d}(1 - c\varepsilon^{1-\eta}) \tag{2.21}$$

The first bound will not prevent $Z(T)$ from vanishing, but by noticing that $p_N(s, y_0) \geq cs^{-d}$, we obtain a lower bound from the single term in the sum over y. This is where the second condition in $E_j^{(k)}$ comes into play.

For the numerator we have

$$N(T) = Z(s) \sum_y p_N(s, y) \sum_s (y + z)^2\, p_{s,y}(T, y + z) \tag{2.22}$$

We write $(y+z)^2 = z^2 + y^2 + 2yz$ and consider the three terms in order. We can prove as in (2.13) that the z^2 terms is

$$\begin{aligned} Z(s) \sum_y p_N(s, y)[(T-s)\, Z_{s,y}(T) + O(s\varepsilon^{1-\eta} T^{1-\theta})] \\ = (T-s)\, Z(T) + Z(s)\, O(s\varepsilon^{1-\eta} T^{1-\theta}) \\ = [(T-s) + O(s^{1+d} \varepsilon^{1-\eta} T^{1-\theta})]\, Z(T) \end{aligned} \tag{2.23}$$

In the last equality we used $Z(s) < O(s^d)\, Z(T)$, which follows from (2.21). The y^2 term is equal to

$$\langle \omega(s)^2 \rangle_{T,h}\, Z(T) \leqslant s^2 Z(T) \tag{2.24}$$

The $2yz$ term is equal to

$$\begin{aligned} Z(s) \sum_y p_N(s, y) \sum_{r=s+1}^{T} \sum_w p_{I,s,y}(r, y+w) \sum_z 2yz p_0(T-r, z-w) \\ = Z(s) \sum_y p_N(s, y) \sum_{r,w} p_{I,s,y}(r, y+w) 2yw \end{aligned} \tag{2.25}$$

Here we have used the expansion $p = p_0 + p_I p_0$ [the unintegrated form of (2.2)], and also the symmetry of p_0. Now the bounds on $N_{s,y}^{(x)}$ in (2.17) can be used to show that

$$\left| \sum_{r,w} p_{I,s,y}(r, y+w) w \right| \leqslant cs\varepsilon^{1-\eta} T^{1-\theta} \tag{2.26}$$

and using again $|y| \leqslant s$, we bound the $2yz$ term by

$$Z(s) cs^2 \varepsilon^{1-\eta} T^{1-\theta} \leqslant cs^{2+d} \varepsilon^{1-\eta} T^{1-\theta} Z(T) \tag{2.27}$$

Altogether we have shown that

$$\frac{N(T)}{Z(T)} = T - s + O(s^2) + O(s^{1+d} \varepsilon^{1-\eta} T^{1-\theta}) + O(s^{2+d} \varepsilon^{1-\eta} T^{1-\theta}) \tag{2.28}$$

This can easily be replaced by the statement of the proposition. ∎

In conclusion, we have shown that $\bigcap_{j \geqslant k} E_j^{(k)}$ is a sufficient condition for diffusion. On these sets, the asymptotic behavior of $\langle x^2 \rangle_{T,h}$ is always T, with h-dependent corrections showing up only in terms growing as smaller powers of T.

3. PROBABILITY ESTIMATES

In this section we show that with probability one, there is some k such that $\bigcap_{j \geq k} E_j^{(k)}$ holds. By Proposition 2.2, this implies diffusion. Our basic estimate is on $\text{Prob}_s(E_j^{(k)c})$, where Prob_s denotes the probability conditioned on the values of $h(t, x)$ for $t \leq s = 2^k - 1$.

Proposition 3.1. For any $\kappa > 0$, let ε be sufficiently small. Then

$$\text{Prob}_s(E_j^{(k)c}) \leq \varepsilon^\kappa 2^{-\kappa(j-k)} \tag{3.1}$$

Proof. We rely on the following "perturbation theory" bounds, which are proven in the Appendix. Putting $d' = \min\{d, 5\}$, we have

$$\overline{Z(T_1, T_2)^{2m}} \leq (c\varepsilon)^{2m} T_1^{-(d-2)m/2}, \tag{3.2}$$

$$\overline{N^{(*)}(T_1, T_2)^{2m}} \leq (c\varepsilon)^{2m} T_2^{2m[1-(d'-2)/4]}, \qquad * = x^2, x, \text{ or } t \tag{3.3}$$

(The constants here are not uniform in m, but we keep m bounded.) These statements can be transformed into probabilistic statements:

$$\text{Prob}(|Z(T_1, T_2)| > cA\varepsilon^{1-\eta} T_1^{-(d-2)/4+\eta}) \leq A^{-2m} \varepsilon^{2m\eta} T_1^{-2m\eta} \tag{3.4}$$

$$\text{Prob}(|N^{(*)}(T_1, T_2)| > cA\varepsilon^{1-\eta} T_2^{1-(d'-2)/4+\eta}) \leq A^{-2m} \varepsilon^{2m\eta} T_2^{-2m\eta} \tag{3.5}$$

We take $\eta < 1/4$, so that for $d > 2$ we have $\theta = (d'-2)/4 - \eta > 0$. This additional decrease in T_1 or T_2 [compared with $N(T)$ or $Z(T)$] reflects the "irreducible" character of $p_I(t, x)$, from which the quantities $Z(T_1, T_2)$, $N^{(*)}(T_1, T_2)$ were defined.

Taking into account the shift of origin, we estimate

$$\text{Prob}(|Z_{s,y}(2^j, 2^j + l)| > cs\varepsilon^{1-\eta} 2^{-(j-k)\theta}) \leq s^{-2m} \varepsilon^{2m\eta} (2^{j-k})^{-2m\eta} \tag{3.6}$$

Here we use twice the fact that $2^j - s \geq 2^{j-k}$. We apply this bound for each of 2^j possible values of l and each of $O(s^d)$ possible values of y. Recalling that $s = 2^k - 1$, we see that for large enough m, the probability that any one of the $Z_{s,y}(2^j, 2^j + l)$ is large is less than $\varepsilon^\kappa 2^{-\kappa(j-k)}$. Similarly, we have

$$\text{Prob}(|N_{s,y}^{(*)}(2^j, 2^j + l)| > cs\varepsilon^{1-\eta} 2^{j(1-\theta)}) \leq s^{-2m} \varepsilon^{2m\eta} (2^j - s)^{-2m\eta} \tag{3.7}$$

and again the probability of a fluctuation larger than permitted in (2.17) is less than $\varepsilon^\kappa 2^{-\kappa(j-k)}$. Finally, we consider fluctuations of $Z_{s,y_0}(2^j, 2^j + l)$. We now have dependence on $h(t, x)$ for $t \geq s$, because y_0 is variable. However, if we condition on $h(t, x)$ for $t \leq s$, we can regard y_0 as fixed. Then by (3.4) we have

$$\text{Prob}_s(|Z_{s,y_0}(2^j, 2^j + l)| > c\varepsilon^{1-\eta} 2^{-(j-k)\theta}) \leq \varepsilon^{2m\eta} (2^{j-k})^{-2m\eta} \tag{3.8}$$

If $j>k$, this controls the sum over l, and the overall estimate is as in the right-hand side of (3.1). If $j=k$, we need to break up the range from s to 2^j+l into intervals $[s+2^i, s+2^i+l_i]$. Then we have

$$\text{Prob}_s(|Z_{s,y_0}(s+2^i, s+2^i+l_i)| > c\varepsilon^{1-\eta}2^{-i\theta}) \leqslant \varepsilon^{2m\eta}(2^i)^{-2m\eta} \tag{3.9}$$

Now we can sum over the l_i's and over i to obtain for $j=k$

$$\text{Prob}_s(|Z_{s,y_0}(2^j, 2^j+l)| > c\varepsilon^{1-\eta} \text{ for any } 0 \leqslant l < 2^j) < \varepsilon^{2m\eta} \tag{3.10}$$

This completes the proof of (3.1). ∎

We now discuss a method for finding k such that $\bigcap_{j\geqslant k} E_j^{(k)}$ holds. To keep track of the procedure, define an indicator sequence I_n as follows:

$$I_0 = 0$$

$$I_{n+1} = \begin{cases} I_n & \text{if } E_n^{(I_n)} \\ n+1 & \text{if } E_n^{(I_n)c} \end{cases} \tag{3.11}$$

This corresponds to a procedure whereby we look for the first n such that $E_n^{(0)c}$ holds, then look for the first $n'>n$ such that $E_{n'}^{(n+1)c}$ holds, etc. The value of I_n is thus the current index k on which we are testing if $\bigcap_{j\geqslant k} E_j^{(k)}$ holds. Our success at finding such a k depends on whether $I_\infty = \lim_{n\to\infty} I_n$ is finite or not.

Proposition 3.2. For any $\kappa'>0$, let ε be sufficiently small. Then for any $n>0$,

$$\text{Prob}(I_\infty = n) \leqslant (e2^{-n})^{\kappa'} \tag{3.12}$$

Furthermore,

$$\text{Prob}(I_\infty = \infty) = 0 \tag{3.13}$$

Proof. There is an annoying dependence among the events $E_j^{(k)}$ due to the dependence on y_0. We take care of this by conditioning in such a way that only probabilities covered by Proposition 3.1 appear. It is worthwhile noticing that the indicator sequence is determined by those k such that $I_k = k$. After any such k, the sequence is constant at k until the next jump. Thus, it is automatically true that

$$\text{Prob}(I_\infty = n) \leqslant \text{Prob}(I_n = n) \tag{3.14}$$

Put $s_k = 2^k - 1$, and assume that the values $h(t, x)$ for $t \leqslant s_k$ are such that $I_k = k$. Then we prove that

$$\text{Prob}_{s_k}(I_n = n) \leqslant (\varepsilon 2^{-(n-k)})^{\kappa'} \tag{3.15}$$

We work inductively, assuming the validity of (3.15) for $k' > k$. Conditioning on the time of the first discontinuity in I after k, we define $F_{k'}$ to be the event that $I_{k'} = k'$ and $I_l = k$ for $k \leqslant l < k'$. Then

$$\mathrm{Prob}_{s_k}(I_n = n) \leqslant \sum_{k'=k+1}^{n} \mathrm{Prob}_{s_k}(F_{k'})\,\mathrm{Prob}_{s_k}(I_n = n \mid F_{k'}) \tag{3.16}$$

By Proposition 3.1, we have

$$\mathrm{Prob}_{s_k}(F_{k'}) \leqslant \mathrm{Prob}_{s_k}(E_{k'-1}^{(k)c}) \leqslant (\varepsilon 2^{-(k'-k-1)})^{\kappa} \tag{3.17}$$

The second factor in (3.16) is covered by the induction hypothesis, since

$$\mathrm{Prob}_{s_k}(I_n = n \mid F_{k'}) \leqslant \sup \mathrm{Prob}_{s_{k'}}(I_n = n) \tag{3.18}$$

with the supremum over values of $h(t, x)$, $t \in (s_k, s_{k'}]$ such that $F_{k'}$ occurs. Thus, (3.16) is bounded by

$$\sum_{k'=k+1}^{n-1} (\varepsilon 2^{-(k'-k-1)})^{\kappa} (\varepsilon 2^{-(n-k')})^{\kappa'} + (\varepsilon 2^{-(n-k-1)})^{\kappa} \leqslant (\varepsilon 2^{-(n-k)})^{\kappa'} \tag{3.19}$$

where we choose $\kappa = \kappa' + 1$. This proves (3.15).

We now obtain (3.12) by setting $k = s_k = 0$ in (3.15) and applying (3.14). It is immediate that $\mathrm{Prob}(I_\infty = \infty) = 0$, since arbitrarily large values of I imply that there are arbitrarily large n such that $I_n = n$. By (3.15) this has vanishing probability. ∎

Proof of Theorem 1. We have just shown that $I_\infty < \infty$ with probability 1. If $I_\infty = k$, then by construction the event $\bigcap_{j \geqslant k} E_j^{(k)}$ holds, and Proposition 2.2 implies the conclusion of Theorem 1. To estimate averaged quantities, we use (3.12), (2.9), and (2.19) to obtain

$$\begin{aligned} |\overline{\langle \omega(T)^2 \rangle_{T,h}} - T| &\leqslant c\varepsilon^{1-\eta} T^{1-\theta} + \sum_{k=1}^{\infty} (s_k^2 + c s_k^{d+2} \varepsilon^{1-\eta} T^{1-\theta})(\varepsilon 2^{-k})^{\kappa'} \\ &\leqslant c\varepsilon^{1-\eta} T^{1-\theta} \end{aligned} \tag{3.20}$$

which agrees with our earlier claim (1.7). ∎

APPENDIX: PERTURBATIVE ESTIMATES

We prove (3.2) and (3.3) by directly controlling the graphical expansions of $\overline{Z(T_1, T_2)^{2m}}$ and $\overline{N^{(*)}(T_1, T_2)^{2m}}$. The collection of graphs is slightly more complicated than the bubble chain we considered in estimating $Z(T)^2$. We consider $Z(T_1, T_2)^{2m}$ first, with m fixed; no attempt is made to derive estimates uniform in m. There are $2m$ walks, and a number of junctions $(s_1, x_1), \ldots, (s_n, x_n)$. At each junction an even number of the walks are specified to meet. All walks last at least until T_1, and they terminate pairwise at junctions at times between T_1 and T_2, inclusive. There may be more than one junction at any time s_i, but two such junctions must involve disjoint subsets of the walks. There is a factor of ε^2 for each pair of walks at a junction, or $\varepsilon^{2n'}$ in all, $n' \geqslant n$. Note that $\overline{h(s, x)^{2k}} = 1$ for any k. Of course, each line $\mathscr{L} = \{(s_i, x_i), (s_j, x_j)\}$, corresponding to a walk running freely between junctions at times $s_i < s_j$, gives rise to a propagator $p_0(s_j - s_i, x_j - x_i) = p_0(\mathscr{L})$. The resulting expansion looks as follows:

$$\overline{Z(T_1, T_2)^{2m}} = \sum_{n=m}^{mT_2} \sum_{\{(s_i, x_i)\}_{i=1}^{n} : s_n \in [T_1, T_2]} \sum_G \varepsilon^{2n'} \prod_{\mathscr{L} \in G} p_0(\mathscr{L})$$

Here G is the collection of lines $\mathscr{L}$ of the graph, with a consistent set of labelings describing which walk each line is part of. It is simple to estimate the number of graphs G with n junctions by c^n, with c dependent on m. See Fig. 2.

We now consider a fixed topological arrangement of lines, and sum over the junctions (s_i, x_i). We would like to obtain a behavior like $(c\varepsilon)^{2m} T_1^{-(d-2)m/2}$. First, whenever four or more walks join at one junction, we may pretend that there are several junctions to be summed over independently, with any fixed assignment of walks to junctions. This gives an upper bound and simplifies the analysis. Next, we sum over (s_i, x_i)

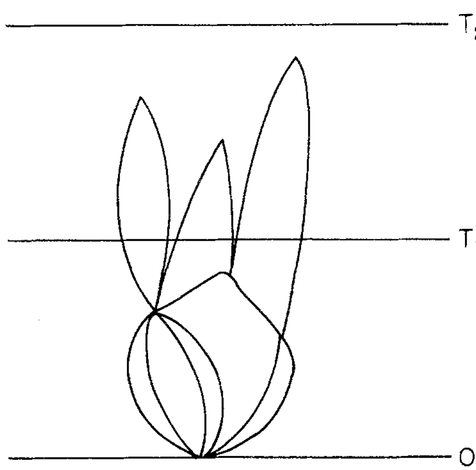

Fig. 2. A typical term in the graphical expansion for $Z(T_1, T_2)^{2m}$. In this case, $m = 3$.

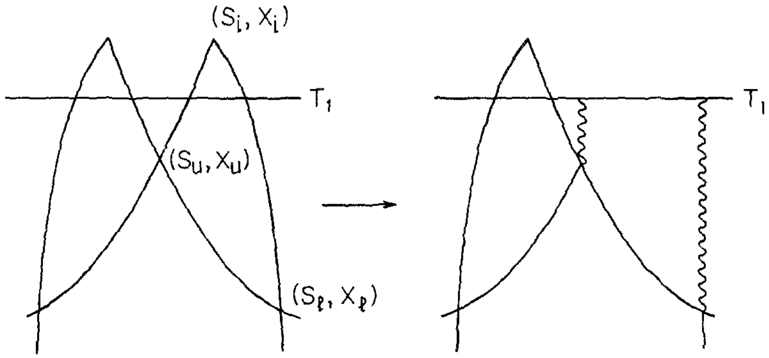

Fig. 3. After estimating the sum over (s_i, x_i), there remains a decrease $(T_1 - s_u)^{-(d'-2)/4}$ $(T_1 - s_l)^{-(d'-2)/4}$, which is indicated by the wavy lines.

where two walks terminate (necessarily between T_1 and T_2, though we can put $T_2 = \infty$ for an upper bound). The two lines extending down from (s_i, x_i) end at upper and lower junctions, (s_u, x_u) and (s_l, x_l), respectively, with $s_u \geqslant s_l$. We use $p_0(s_i - s_u,\ x_i - x_u)$ to control the sum over x_i, and replace $p_0(s_i - s_l,\ x_i - x_l)$ with its maximum value, which is $O(1)(s_i - s_l)^{-d/2}$. Summation over s_i yields an overall estimate of $c(\max\{s_u, T_1\} - s_l)^{-(d/2-1)}$, since $s_i \geqslant \max\{s_u, T_1\}$. At first, s_u and s_l may be greater than T_1, and this factor is not needed to sum over junctions at the top of the graph. As we proceed downward, however, s_l and then s_u will fall below T_1, and we may then assign factors $c(T_1 - s_u)^{-(d-2)/4}$ if $s_u < T_1$ and also $c(T_1 - s_l)^{-(d-2)/4}$ if $s_l < T_1$. (Diagrammatically, we draw wavy lines from T_1 down to s_u or s_l.) (See Fig. 3).

Notice that after removing the two lines to (s_i, x_i), the resulting graph still has the property that every junction has two lines emanating downward. Furthermore, junctions below T_1 have two lines emanating upward, either p_0 lines or wavy lines to T_1. These properties will be

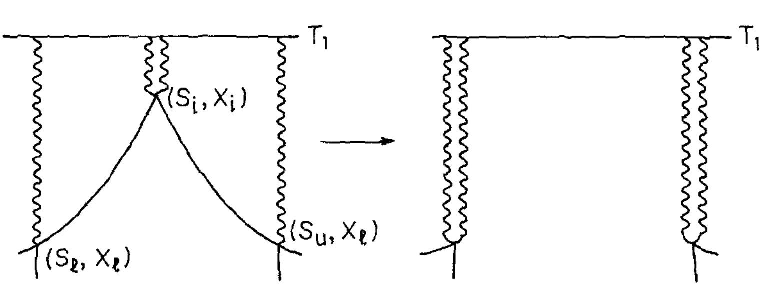

Fig. 4. The estimate for $s_l < T_1$ preserves the number of wavy lines dropping down from T_1.

preserved as we work down the graph. We continue as above until all junctions above T_1 are gone. At this point there are $2m$ wavy lines coming down from T_1, one for each line of the original graph that crosses T_1. (See Fig. 4).

We next sum over junctions with only wavy lines emanating upward. As before, (s_i, x_i) is joined down to (s_u, x_u) and (s_l, x_l). Summation over x_i yields a bound $c(T_1 - s_i)^{-(d-2)/2}(s_i - s_l)^{-d/2}$, the first factor coming from the wavy lines. Summation of s_i from s_u to T_1 then yields a bound

$$c(T_1 - s_l)^{-(d-2)/2} \leqslant c(T_1 - s_u)^{-(d-2)/4}(T_1 - s_l)^{-(d-2)/4}$$

as may be seen by considering separately the case $s_i > (T_1 + s_l)/2$, $s_i \leqslant (T_1 + s_l)/2$. Thus we have produced wavy lines emanating up from (s_u, x_u) and (s_l, x_l), and the process can continue.

There remain always $2m$ wavy lines, so after summing over all junctions we have the desired bound

$$\overline{Z(T_1, T_2)^{2m}} \leqslant \sum_{n=m}^{mT_2} (c\varepsilon)^{2n} (T_1^{-(d-2)/4})^{2m} \leqslant (c\varepsilon)^{2m} T_1^{-(d-2)m/2}$$

The corresponding estimate for $\overline{N^{(*)}(T_1, T_2)^{2m}}$ proceeds along the same lines ($* = x$, x^2, or t). After taking the $2m$th power and averaging, there are m factors of x_i^{2a} or s_i^2 associated with the final junctions of the walks.

The case $a = 1$ should yield better estimates, but for simplicity we use $x_i^2 \leqslant x_i^4$ and consider only $* = x^2$ or t. The first step is to write

$$x_i^4 \leqslant 4[(x_i - x_u)^2 + x_u^2][(x_i - x_l)^2 + x_l^2]$$

and consider each of the four terms that result. As before, we sum over x_i using $p_0(s_i - s_u, x_i - x_u)$, and the factor $(x_i - x_u)^2$ is traded for one of $s_i - s_u$. Likewise, in taking the maximum of $(x_i - x_l)^2\, p_0(s_i - s_l, x_i - x_l)$, we obtain an extra factor of $s_i - s_l$. In case $* = t$, we write $s_i^2 = [s_u + (s_i - s_u)]$ $[s_l + (s_i - s_l)]$. In this way the effect of x_i^4 or s_i^2 propagates down each walk, either as factors of x^2, s^2 lower down or as $(s_i - s_u)$ or $(s_i - s_l)$. Thus we have to estimate sums like

$$\sum_{s_i = s_u}^{T_2} (s_i - s_u)(s_i - s_l)^{1 - d/2} \leqslant c(T_2 - s_u)^{3 - d'/2}$$

$$\leqslant c(T_2 - s_u)^{1 - (d'-2)/4}(T_2 - s_l)^{1 - (d'-2)/4}$$

Here we use $d' = \min\{d, 5\}$ because for $d \geqslant 6$ the summation starts to become dominated by its lower limit, leading to different behavior, which we do not attempt to follow. The other cases can be treated similarly, yielding an overall factor

$$c[\hat{s}_u + (T_2 - s_u)^{1-(d'-2)/4}][\hat{s}_l + (T_2 - s_l)^{1-(d'-2)/4}]$$

where $\hat{s} = s$ $(* = t)$ or $\hat{s} = x^2$ $(* = x^2$ or $x)$.

As we proceed down the graph, each junction will have picked up factors

$$c[\hat{s}_i + (T_2 - s_i)^{1-(d'-2)/4}]^2 \leqslant c\hat{s}_i^2 + c(T_2 - s_i)^{2-(d'-2)/2}$$

We have analyzed the $\hat{s}_i^2$ term already; for the other term we obtain

$$\begin{aligned}\sum_{s_i, x_i} &(T_2 - s_i)^{2-(d'-2)/2}\, p_0(s_i - s_u, x_i - x_u)\, p_0(s_i - s_l, x_i - x_l) \\ &\leqslant \sum_{s_i = s_u}^{T_2} c(T_2 - s_i)^{2-(d'-2)/2} (s_i - s_l)^{-d/2} \\ &\leqslant c(T_2 - s_u)^{2-(d'-2)/2} \\ &\leqslant c(T_2 - s_u)^{1-(d'-2)/4} (T_2 - s_l)^{1-(d'-2)/4}\end{aligned}$$

which produces terms of the same type. In the end these are the only remaining terms, and we have the desired bound

$$\overline{N^{(*)}(T_1, T_2)^{2m}} \leqslant (c\varepsilon)^{2m}\, T_2^{2m[1-(d'-2)/4]}, \qquad * = x^2,\ x, \text{ or } t$$

ACKNOWLEDGMENTS

J. Z. I. is an Alfred P. Sloan Research Fellow. This research was supported in part by NSF grants PHY-87-06420 and PHY-86-45122 (at Harvard) and by NSF grant DMR-84-01225 (at the Courant Institute).

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