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Debye Screening for Jellium and Other Coulomb Systems

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Abstract. Debye screening is proven for a large class of classical Coulomb gases at low densities. Among the models treated are jellium systems (where particles interact with a fixed background charge), systems with arbitrarily dilute fractional charges, and systems where the charges are not integrally related. The interaction potentials of the corresponding sine-Gordon models may have no symmetry and can have infinitely many stationary points which are degenerate or nearly degenerate in energy.

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Introduction

The classical Coulomb gas has been the subject of several rigorous investigations in the last few years. Brydges [1] established Debye screening for a lattice Coulomb gas. His work was greatly generalized by Brydges and Federbush [3] who considered the continuous statistical mechanics situation with a large class of

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allowable short range forces; see also [4]. These results were obtained in three dimensions in a region of parameters corresponding to a dilute gas. Analogous results hold in two dimensions at high temperatures and low activities. Fröhlich and Spencer [10] have shown that the two-dimensional lattice gas does not screen in a regime of low temperatures and moderate activities. In their study of the three dimensional U(1) lattice gauge theory [15], Göpfert and Mack expanded significantly the domain of activities for screening in a lattice Coulomb gas.

The Brydges-Federbush analysis allowed for non-charge-symmetric systems, but failed to deal with several interesting situations. As the charge of a species tends to zero, or as its activity becomes large, the convergence estimates deteriorate. Since a particle should decouple as its charge tends to zero (for appropriate short-range forces), one ought to be able to handle these situations – at least for slow enough growth of an activity as the corresponding charge tends to zero. If the activity grows as the inverse of the charge, then the jellium limit of a fixed background charge is approached. (Jellium is used as a model for ions moving in a sea of conduction electrons in a metal.) Another situation which falls outside the domain of [3] is where a species with a charge that is fractional with respect to the other species has an activity much smaller than the other species' activities. One would hope that such a species would not affect the system much. Finally, if not all charges present are integral multiples of an elementary charge, then the analysis of [3] fails. Integrally related charges are needed even in the basic thermodynamic estimates of [18]. They are also important in Fröhlich and Spencer's analysis of the two dimensional Coulomb gas [10].

In this paper we extend the class of models known to exhibit screening to the cases described above. We require a dilute system. Our restrictions on the size of activities are considerably weaker than the ones in [3]. This is achieved with an iterated Mayer expansion along the lines of [14].

In the sine-Gordon (or ϕ-) representation, the gas becomes a field theory with interaction of the form $\sum_i z_i(1-e^{i\beta^{1/2}e_i\phi})$, where e_i is the charge of the i^{th} species and z_i is its activity. The situations described above correspond to local minima of the interaction becoming nearly degenerate with the global minimum at $\phi=0$. The situation of nonintegrally related charges corresponds to the interaction being a nonperiodic function of ϕ. The background charge gives rise to an additional interaction $-i\beta^{1/2}z_s e_s\phi$ for some constant $z_s e_s$. It can be obtained from $z_s(1-e^{i\beta^{1/2}e_i\phi})$ by taking the limit $z_s\to\infty$, $e_s\to 0$ with $z_s e_s$ fixed.

As was pointed out in [3], the failure to deal with nearly degenerate minima of the interaction can be traced to difficulties with bounds on ratios of partition functions. This is symptomatic of models close to first order phase transitions, see for example [17]. The development of techniques to handle systematically ratios of partition functions [19, 16, 17] clarified the issues involved here and led to the current investigation. As we shall see, the "phase transition" in the models we consider can only occur with negative activities. Thus the basic physical input we need is an estimate that guarantees that the $\phi=0$ minimum dominates the others.

The fact that the $\phi=0$ minimum always dominates may be somewhat surprising. In $P(\phi)_2$ models, for example, it quite often happens that local minima dominate global minima [17]. It was our initial hope that similar phenomena in the Coulomb gas would shed light on some behavior of ionic solutions: behavior strongly dependent on concentration before the limiting law is reached in some charge-asymmetric solutions [9], and phase separation [8]. In fact, no amount of meddling with short range forces, charges, or activities within the domain of convergence of our expansion can coax the model away from the $\phi=0$ minimum.

We set up the model in a finite volume as in [3]. There are $s-1$ species of particles, with species i having charge e_i and bare activity $z_i \geqq 0$. There is a uniform background charge (the "jelly"). We can think of it as the s^{th} particle, where the limit $e_s \to 0$, $z_s \to \infty$ has been taken with $z_s e_s$ fixed. We can handle charges and activities arbitrarily close to this limit as well. A careful analysis of the Mayer series shows that if the radius of a particle tends to zero appropriately as $z_i \to \infty$ and $e_i \to 0$, then the system converges to a jellium system. We will absorb the unit of electric charge into the inverse temperature β, so e_i is dimensionless.

We put $l_D = \left(\sum_i z_i e_i^2 \beta\right)^{-1/2}$. Let $\Lambda \subseteqq \Lambda'$ be rectangular boxes in $\mathbb{R}^3$, with Λ built from $\tilde{l}_D$-lattice cubes ($\tilde{l}_D \cong l_D$ will be defined below). Let $\Delta_{\partial\Lambda}$ be the Laplacian with Dirichlet boundary conditions at $\partial\Lambda$. We split the Coulomb interaction into a long-range part and a short-range part, the former acting mainly at length scales from λl_D to ∞. (The parameter λ is at our disposal, and is taken to be small.) The short-range part must be cut off appropriately to avoid collapse, but the cutoff length scale can be taken of the order of β, which is much smaller than λl_D. The long-range part is given by

$$u_{\partial\Lambda}(x,y) = ((-\Delta_{\partial\Lambda})^{-1} - (-\Delta_{\partial\Lambda} + \lambda^{-2} l_D^{-2})^{-1})(x,y). \tag{1.1}$$

Let σ_i be the density of species i. It is a sum of δ-functions at the positions of particles of species i for $i \neq s$, and it is a constant $z_s|e_s|$ for $i=s$. Let $J = \sum_{i \neq s} e_i \sigma_i + (\operatorname{sgn} e_s)\sigma_s$ be the charge density. The interaction of the particles in the system is a sum of three terms:

$$V = U + W - d_0, \tag{1.2}$$

$$U = \frac{1}{2} \int_{\Lambda \times \Lambda} J(x)\, u_{\partial\Lambda}(x,y)\, J(y)\, dx\, dy, \tag{1.3}$$

$$d_0 = \sum_{\alpha:\, i(\alpha) \neq s} \left[\frac{1}{2} u_0(x_\alpha, x_\alpha) e_{i(\alpha)}^2 - \int_{\Lambda'} v_{i(\alpha)s}(x_\alpha - x)\, \sigma_s(x)\, dx \right], \tag{1.4}$$

$$W = \frac{1}{2} \sum_{i,j \neq s} \int_{\Lambda' \times \Lambda'} :\sigma_i(x)\, v_{ij}(x-y)\, \sigma_j(y):\, dx\, dy. \tag{1.5}$$

The kernel u_0 is constructed as in (1.1) but with the infinite volume Laplacian Δ replacing $\Delta_{\partial\Lambda}$. Subtraction of d_0 thus corresponds approximately to removing self-interaction terms. The colons in (1.5) indicate that such terms are not included in W. The kernels $v_{ij} = v_{ji}$ are the short-range part of the interaction; we shall limit their size below by requiring certain estimates on their Mayer series. If we include the force arising from the second term in (1.1), the actual non-Coulomb part of the interparticle force is (in infinite volume)

$$v_{ij}(x-y) - e_i \frac{e^{-|x-y|/\lambda l_D}}{4\pi|x-y|} e_j, \qquad i,j \neq s.$$

The second term in d_0 supplies a short-range force between the particles and the jelly. Both terms in d_0 will be absorbed into the activities.

To keep a specific example in mind, take a simple Coulomb system with hard cores. Then we take for $i, j \neq s$

$$v_{ij}(x-y)=\begin{cases} e_i \dfrac{e^{-|x-y|/\lambda l_D}}{4\pi|x-y|} e_j, & |x-y| \geqq R_{ij} \\ \infty, & |x-y| < R_{ij} \end{cases}, \tag{1.6}$$

$$v_{is}(x-y)=e_i \frac{e^{-|x-y|/\lambda l_D}}{4\pi|x-y|} (\operatorname{sgn} e_s), \tag{1.7}$$

with

$$\sup_i \sum_j \frac{|e_i e_j|}{R_{ij}} \leqq \frac{1}{R}, \tag{1.8}$$

$$R_{ij}^3 \leqq c|e_i e_j| R^3 . \tag{1.9}$$

The length R parametrizes the stability of the system; it is a short distance cutoff. The condition (1.9) is needed only because we are interested in the limit $e_i \to 0$, $z_i e_i$ fixed. We could send e_i to zero with z_i fixed; then (1.9) would not be necessary. Conditions (1.8) and (1.9) force R_{ij} to go to zero with e_i, but not too quickly.

There is a case of particular interest where all particles have charges of the same sign, which must then be opposite to the sign of the background charge. In this case stability is not a problem and we can study the pure Coulomb interaction (1.6), (1.7) with some or all $R_{ij}=0$. The constraint (1.8) is omitted and we put $R=\beta$.

The infinite volume limit is taken in two stages. With A a functional of the σ_i inside Λ, put

$$I(A)=\sum_N \frac{z^N}{N!} \int_{(\Lambda')^N} e^{-\beta V} A . \tag{1.10}$$

The multiindex $N=(N_1, \ldots, N_{s-1})$ specifies the number of particles of each species present, and $z^N/N! \equiv \prod_i z_i^{N_i}/N_i!$. The integral is over the positions of the $\sum_{i \neq s} N_i$ particles in Λ'. Taking Λ' to infinity, we obtain the expectation

$$\langle A \rangle_\Lambda = \lim_{\Lambda' \to \mathbb{R}^3} \frac{I(A)/Z_0}{I(1)/Z_0} = Z^{-1} \lim_{\Lambda' \to \mathbb{R}^3} I(A)/Z_0 , \tag{1.11}$$

where

$$Z_0=\sum_N \frac{z^N}{N!} \int_{(\Lambda')^N} e^{-\beta(W-d_0)}, \tag{1.12}$$

$$Z=\lim_{\Lambda' \to \mathbb{R}^3} I(1)/Z_0 . \tag{1.13}$$

Before discussing the $\Lambda \to \infty$ limit and stating our main theorems, we perform a sine-Gordon transformation and a Mayer expansion.

2. The Mayer Series and the Main Results

This section is in five parts. We begin with the sine-Gordon transformation. We then digress on the Mayer series. Estimates needed for the cluster expansion are stated, and conditions on the short-range forces sufficient to prove the estimates are given. The conditions are verified for the standard hard core system. In the third part, the neutrality condition is discussed, and in the fourth we state our main theorems. We conclude with an outline of the proofs which form the body of this paper.

2.1. The Sine-Gordon Transformation

Let $d\mu_{0,\partial\Lambda}(\phi)$ be the Gaussian measure with covariance $u_{\partial\Lambda}$. We have the identity

$$e^{-\beta U}=\int \exp\Big(i\beta^{1/2}\sum_{\alpha} e_{i(\alpha)}\phi(x_\alpha)+\int_{\Lambda} i\beta^{1/2}z_s e_s\phi(x)\,dx\Big)\,d\mu_{0,\partial\Lambda}(\phi), \tag{2.1}$$

and as a consequence

$$Z=\int Z(\phi)\,d\mu_{0,\partial\Lambda}(\phi), \tag{2.2}$$

where

$$\begin{aligned}Z(\phi)&=\exp\Big(\int_{\Lambda} i\beta^{1/2}z_s e_s\phi(x)\,dx\Big)\lim_{\Lambda'\to\mathbb{R}^3} Z_{\bullet}^{-1}\sum_N \frac{\tilde{z}^N}{N!}\int_{(\Lambda')^N} e^{-\beta W}\exp\Big(i\beta^{1/2}\sum_{\alpha} e_{i(\alpha)}\phi(x_\alpha)\Big)\\ &\equiv e^M.\end{aligned} \tag{2.3}$$

Here we have put

$$\tilde{z}_i=z_i\exp\Big(\beta e_i^2 u_0(x,x)/2-\beta\int_{\mathbb{R}^3} v_{i(\alpha)s}z_s|e_s|\,dx\Big),\qquad i\neq s. \tag{2.4}$$

2.2. The Mayer Series

The limit $\Lambda'\to\mathbb{R}^3$ in (1.11), (1.13), (2.3) is governed by a Mayer expansion. We suppose that M can be written as

$$M=\sum_{i=1}^{s}\int \varrho_i\varepsilon_i(x)+\frac{1}{2!}\sum_{i_1,i_2=1}^{s-1}\int \varrho_{i_1,i_2}(x_1,x_2)\varepsilon_{i_1}(x_1)\varepsilon_{i_2}(x_2)+\dots, \tag{2.5}$$

where

$$\begin{gathered}\varepsilon_i(x)=e^{i\beta^{1/2}e_i\phi(x)}-1,\\ \varrho_s\varepsilon_s(x)=i\varrho_s\beta^{1/2}e_s\phi(x),\qquad \varrho_s=z_s=\tilde{z}_s,\end{gathered} \tag{2.6}$$

and where each $\varrho_{i_1,\dots,i_t}(x_1,\dots,x_t)$ is independent of ϕ.

We define the basic length for exponential decay,

$$\tilde{l}_D=\Big(\sum_i \varrho_i e_i^2\beta\Big)^{-1/2}. \tag{2.7}$$

We shall use units where $\tilde{l}_D=1$ in Sects. 3–12. If $a_1,\dots,a_t$ are a set of unit lattice cubes, a length $L(\{a_u\})$ is defined in (A.13) (see also [3]). It satisfies

$$e^{-\alpha L(\{a_u\})}=\sum_{\eta^A} b_{\eta^A}e^{-\alpha L_{\eta^A}(\{a_u\})}, \tag{2.8}$$

with $b_{\eta^A} \geqq 0$, $\sum_{\eta^A} b_{\eta^A} = 1$ and

$$\sum_{(a_1, \ldots, a_t): a_{u_*} = a} e^{-\gamma L_{\eta^A}(\{a_u\})/\tilde{l}_D} \leqq c_\gamma^{t-1} \tag{2.9}$$

for any $\gamma > 0$. The sum in (2.9) is over ordered sets of t unit lattice cubes, one of which is fixed. Furthermore $L_{\eta^A}(\{a_u\})$ is the length of some tree on $\{a_1, \ldots, a_t\}$ and possibly other points.

We require the following estimates on $\varrho_{i_1, \ldots, i_t}$. It should be symmetric, translation invariant, and it should satisfy

$$\frac{1}{t!} \|\varrho_{i_1, \ldots, i_t}\|_{L^1(a_2 \times \ldots \times a_t)} \leqq (C_1 \beta \lambda^2 \tilde{l}_D^2)^{t-1} \prod_{u=1}^{t} |\varrho_{i_u} e_{i_u}| e^{-\alpha L(\{a_u\})} \tag{2.10}$$

(the norm taken with one variable fixed, and $t \geqq 2$),

$$\varrho_i = \tilde{z}_i(1 + O(C_2 \lambda^2)) = z_i(1 + O(C_2 \lambda^2)) e^{O(C_3 \beta/(\lambda l_D))}, \tag{2.11}$$

$$\sum_{i=1}^{s} \varrho_i |e_i| \leqq C_4 \sum_i \varrho_i e_i^2 . \tag{2.12}$$

The parameter λ is the same as the one in (1.1); we need to take it small at a number of points in this paper. Since z_i is positive, (2.11) implies that ϱ_i is also positive for λ small. Note that (2.12) is compatible with the jellium limit $e_i \to 0$, $z_i e_i$ fixed. We also assume a bound

$$\sup_i |e_i| \equiv e_m \leqq C_5 . \tag{2.13}$$

We obtain these estimates for suitable z_i, e_i, β, v_{ij}, λ using the Mayer expansion in the appendix. Convergence depends on stability estimates and estimates on two-body forces. The following measure of the size of two-body forces will be used:

$$\|v\|_\alpha = \int d^3x e^{\alpha|x|} |v(x)| . \tag{2.14}$$

Our most general result is contained in

Proposition 2.1. *Suppose there are splittings* $v_{ij} = v_{ij}^0 + v_{ij}^1$, $v_{ij}^1 = v_{ij}^n + v_{ij}^R$ *satisfying conditions* (i), (ii):

(i) *Let* W_N^0, W_N^1, W_N^n *be the N-body interactions constructed from* v_{ij}^0, v_{ij}^1, v_{ij}^n *as in* (1.5). *Then*

$$\beta W_N^0 \geqq -C_6(\beta/R)N \quad \text{whenever} \quad W_N^1 < \infty , \tag{2.15}$$

$$\beta W_N^n \geqq -C_7 N \quad \text{independently of} \quad W_N^1 , \tag{2.16}$$

$$v_{ij}^R \geqq 0 . \tag{2.17}$$

(ii) *The following two-body estimates hold:*

$$\|v_{ij}^0\|_\alpha \leqq C_8 |e_i e_j| \beta^2 , \tag{2.18}$$

$$\|v_{ij}^n\|_\alpha \leqq C_9 |e_i e_j| \lambda^2 l_D^2 , \tag{2.19}$$

$$\|\beta^{-1}(e^{-\beta v_{ij}^R} - 1)\|_\alpha \leqq C_9 |e_i e_j| \lambda^2 l_D^2 . \tag{2.20}$$

Suppose in addition that $e_m \leqq C_5$ *and that*

$$\|v_{is}\|_{L^1} \leqq C_9 \lambda^2 l_D^2, \tag{2.21}$$

$$\sum_{i=1}^{s} z_i |e_i| \leqq \tfrac{1}{2} C_4 \sum_i z_i e_i^2 . \tag{2.22}$$

Then $\log Z(\phi)$, *defined in* (2.3), *admits an expansion* (2.5) *satisfying* (2.10)–(2.12), *provided* λ *is small and*

$$\sum_i z_i e_i^2 \beta^3 = (\beta/l_D)^2 \leqq c_1 \lambda^2 e^{-c_2 \beta/R} . \tag{2.23}$$

Here c_1, c_2, C_1, C_2, C_3 *depend only on* $C_4, \ldots, C_9$.

We prove this proposition in the appendix, using the iterated Mayer expansion formalism of [14]. We expand in the least stable part of the interaction first, and afterwards expand in v_{ij}^1. Conditions (2.18), (2.19) are best understood if one thinks of Yukawa potentials with ranges β, λl_D. The basic expansion parameter for the v_{ij}^0 expansion is

$$\sup_{i \neq s} \sum_{j \neq s} \|v_{ij}^0\|_\alpha z_j \beta e^{C_6 \beta/R} \leqq c \sum_j z_j e_j^2 \beta^3 e^{C_6 \beta/R} ;$$

for the v_{ij}^1 expansion it is

$$\sup_{i \neq s} \sum_{j \neq s} (\|v_{ij}^n\|_\alpha + \|\beta^{-1}(e^{-\beta v_{ij}^R} - 1)\|_\alpha) z_j \beta e^{C_7} \leqq c\lambda^2 .$$

Since v_{ij}^0 has a relatively short range β [see (2.18)] the large factors $e^{C_6 \beta/R}$ coming from (2.15) can be handled using (2.23). The longer-ranged interaction v_{ij}^1 has improved stability (2.16), (2.17) and so causes no problems. Thus we do not require $\lambda^2 \ll e^{-c\beta/R}$ as in [3]. This leads to much improved conditions on activities in Theorem 2.3 below. For the same reasons, a similar splitting was used in [15] to treat a lattice Coulomb gas. There the self-energies of the particles (not included in our model) were sufficient to ensure the analog of (2.23) for β large and z of order unity.

We now apply Proposition 2.1 to our standard hard core system.

Proposition 2.2. *Suppose that* (2.23) *holds,* $e_m \leqq C_5$, $\alpha < (2\lambda l_D)^{-1}$, *and*

$$\sum_i z_i e_i^2 R^3 = R^3/(\beta l_D^2) \leqq C_{10} \lambda^2 . \tag{2.24}$$

Then the hard core system (1.6)–(1.9) *can be split into* v_{ij}^0, v_{ij}^n, v_{ij}^R, v_{is} *satisfying* (2.15)–(2.21). *Thus if* λ *is small and* (2.22) *holds, then* $\varrho_{i_1, \ldots, i_t}$ *satisfies* (2.10)–(2.12).

If all charges have the same sign and $R_{ij} = 0$ *for all i, j, then condition* (2.24) *can be omitted and R can be set equal to* β *in* (2.15), (2.23).

Proof. We put

$$v_{ij}^0(x-y) = e_i(-\Delta + \tilde{\beta}^{-2})^{-1}(x,y) e_j ,$$

$$v_{ij}^n(x-y) = e_i[(-\Delta + \lambda^{-2} l_D^{-2})(x,y) - (-\Delta + \tilde{\beta}^{-2})^{-1}(x,y)] e_j ,$$

$$v_{ij}^R = \begin{cases} \infty & \text{if } |x-y| \leqq R_{ij} \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\beta}=\min\{\beta, \lambda l_D\}$. Since v_{ij}^n is a positive operator, βW_N^n is bounded below by the sum of the self-energies, which is $\sum_{\alpha=1}^{N} \beta e_{i(\alpha)}^2 O(\beta^{-1}) \leqq C_7 N$. We have

$$\|v_{ij}^0\|_\alpha = |e_i e_j| \int d^3x (4\pi|x|)^{-1} \exp[(\alpha - \tilde{\beta}^{-1})|x|] \leqq C_8 |e_i e_j| \beta^2 ,$$

since $\alpha < (2\lambda l_D)^{-1} < (2\tilde{\beta})^{-1}$. We obtain (2.19) and (2.21) similarly after noting that $|v_{ij}^n(x-y)| \leqq |e_i e_j| (-\Delta + \lambda^{-2} l_D^{-2})^{-1}(x, y)$. Note that (2.23) and (2.24) imply that $R/l_D < c\lambda$. Thus $\alpha R < c$, and so

$$\|\beta^{-1}(e^{-\beta v_{ij}^R} - 1)\|_\alpha \leqq c R_{ij}^3 \beta^{-1} \leqq c |e_i e_j| R^3 \beta^{-1} \leqq C_9 |e_i e_j| \lambda^2 l_D^2 ,$$

where we have used (1.9) and (2.24). Of course this estimate is not needed if all R_{ij} vanish.

We now prove (2.15). This is trivial when all charges have the same sign. Stability is proven in the general case by considering first a comparison system where each charge e_i is smeared over the surface of a sphere of radius $R_{ii}/2$ and multiplied by a constant so that it interacts with charges outside the sphere with the original Yukawa potential. The constant is bounded by 1 for all values of $R_{ii}/\tilde{\beta}$. With self-energies included, the interaction is positive [7] and thus the N-body interaction $\hat{W}_N^0$ is bounded below by $-c \sum_{\alpha=1}^{N} e_{i(\alpha)}^2 / R_{ii}$, the sum of the self-energies. It remains for us to bound the difference $W_N^0 - \hat{W}_N^0$ from below, assuming the hard core conditions $|x_\alpha - x_\beta| \geqq R_{i(\alpha)\,i(\beta)}$ are satisfied. We need only be concerned with pairs $\alpha \neq \beta$ such that

$$R_{i(\alpha)\,i(\beta)} \leqq |x_\alpha - x_\beta| < (R_{i(\alpha)\,i(\alpha)} + R_{i(\beta)\,i(\beta)})/2 , \tag{2.25}$$

since terms violating the second inequality cancel between W_N^0 and $\hat{W}_N^0$. We can assume $R_{ii} < R_{jj}$ for $i < j$ by a relabeling of species. Then we need only consider pairs α, β with $i(\alpha) < i(\beta)$. For each α and each $i > i(\alpha)$ there are no more than some fixed number of β with $i(\beta) = i$ and satisfying (2.25). (All such β are spaced apart by at least $R_{i(\beta)\,i(\beta)}$ but are within $R_{i(\beta)\,i(\beta)}$ of x_α.) The $\alpha\beta$ term in W_N^0 or in $\hat{W}_N^0$ can be estimated by the corresponding Coulomb interaction. For W_N^0 the $\alpha\beta$ term is therefore bounded below by $-|e_{i(\alpha)} e_{i(\beta)}| / R_{i(\alpha)\,i(\beta)}$ by (2.25). For $\hat{W}_N^0$ we know that the potential created by the larger particle at x_β is nowhere greater in magnitude than $c e_{i(\beta)} / R_{i(\beta)\,i(\beta)}$. Thus the $\alpha\beta$ term in $-\hat{W}_N^0$ is bounded below by $-c|e_{i(\alpha)} e_{i(\beta)}| / R_{i(\beta)\,i(\beta)} \geqq -c|e_{i(\alpha)} e_{i(\beta)}| / R_{i(\alpha)\,i(\beta)}$. Combining the above bounds yields

$$W_N^0 \geqq \sum_{\alpha=1}^{N} \sum_{i \neq s} -c|e_{i(\alpha)} e_i| / R_{i(\alpha) i} ,$$

and (2.15) follows by using (1.8). This completes the proof. □

2.3. Neutrality

We require a neutrality condition as in [3]:

$$\sum_{i=1}^{s} \varrho_i e_i = 0 . \tag{2.26}$$

This condition puts implicit constraints on the activities z_i. The origin of the condition is the need for $\phi=0$ to be a stationary point of M, whose most important term is $\sum_{i=1}^{s} \int \varrho_i(e^{i\beta^{1/2}e_i\phi(x)}-1)$. Condition (2.26) may seem more natural when one considers the behavior of $M_1(\phi)=\sum_{i=1}^{s} \varrho_i e^{i\beta^{1/2}e_i\phi}$ in the complex ϕ-plane. On the imaginary axis, $\phi=ia$, $a\in\mathbb{R}$ we find that $M_1(ia)$ is a convex function of a since

$$\frac{d^2}{da^2}M_1(ia)=\sum_{i=1}^{s} \varrho_i e_i^2\beta e^{-\beta^{1/2}e_i a}>0.$$

Thus $M_1(ia)$ has a unique minimum for real a, and it would be advisable to pass the ϕ-integration contours through this saddle point. [In fact $\mathrm{Re}M_1(ia+b) \leqq \mathrm{Re}M_1(ia)$ for $a, b\in\mathbb{R}$.] A complex translation $\phi\to\phi+ia$ is equivalent to sending $\varrho_i\to\varrho_i'=\varrho_i e^{-\beta^{1/2}e_i a}$ in $M_1(ia)$. After the translation we find

$$0=\frac{d}{da}M_1(ia)|_{\alpha=0}=\sum_{i=1}^{s} -\varrho_i' e_i\beta^{1/2}, \tag{2.27}$$

and so (2.26) is satisfied with ϱ_i' replacing ϱ_i.

Physically we expect that a system with a set of activities not satisfying (2.26) would expel charges to $\partial\Lambda$, thereby placing most of the system in a background potential a. This would "renormalize" the activities as above, and neutrality would be recovered. Lacking the ability to prove that this occurs, we settle for condition (2.26) above. Of course one can always adjust the background charge density $\varrho_s e_s$ to obtain neutrality.

2.4. The Main Theorems

In the sine-Gordon language the main objects of study are the $\Lambda\to\mathbb{R}^3$ limit of

$$\langle\mathscr{A}\rangle_\Lambda^\phi=\frac{\int \mathscr{A}e^M d\mu_{0,\partial\Lambda}(\phi)}{\int e^M d\mu_{0,\partial\Lambda}(\phi)}, \tag{2.28}$$

where $\mathscr{A}$ is a functional of $\phi(x)$. The observables we will consider in (1.11) are of the form

$$A=\int f_A(x_1,\dots,x_{w_A})\prod_{\alpha=1}^{w_A}\sigma_{i_\alpha}(x_\alpha), \tag{2.29}$$

with f_A a continuous function of compact support, or

$$A=\prod_{\alpha=1}^{w_A}\sigma_{i_\alpha}(x_\alpha) \tag{2.30}$$

with $x_\alpha\neq x_\beta$ if $i_\alpha=i_\beta$. In (2.28) we take

$$\mathscr{A}=\prod_{\alpha=1}^{w_{\mathscr{A}}} e^{i\beta^{1/2}a_\alpha\phi(x_\alpha)} \tag{2.31}$$

with $|a_\alpha|\leqq A_0$. Given the results on the convergence of the Mayer series, it is an exercise to show that $\langle A\rangle_\Lambda$ can be expressed as convergent sums and integrals of expectations $\langle\mathscr{A}\rangle_\Lambda^\phi$. We now state our main results about these expectations.

Theorem 2.3. *Consider the purely Coulombic systems with hard cores given in* (1.1)–(1.13), (2.28)–(2.31). *Let* C_4, C_5, C_{10}, δ *be given, and suppose that*

$$\sum_{i=1}^{s} z_i|e_i| \leqq \tfrac{1}{2} C_4 \sum_i z_i e_i^2\,, \qquad e_m \leqq C_5\,,$$

and

$$\sum_i z_i e_i^2 R^3 = R^3/(\beta l_D^2) \leqq C_{10}\lambda^2\,. \tag{2.32}$$

Then there are constants c_2, c_3, c_4 *(depending on* C_4, C_5, C_{10}, δ*) such that for* $\lambda \leqq c_4$ *and*

$$\sum_i z_i e_i^2 \beta^3 = (\beta/l_D)^2 \leqq e^{-c_3/\lambda} e^{-c_2\beta/R} \tag{2.33}$$

the following result holds. Consider the neutral models satisfying $\sum_{i=1}^{s} \varrho_i e_i = 0$ *(or else adjust the background charge density* $\varrho_s e_s$ *to achieve neutrality); for these the infinite volume limits of* $\langle \mathscr{A} \rangle_\Lambda^\phi$ *and* $\langle A \rangle_\Lambda$ *exist and satisfy*

$$|\langle \mathscr{A} \rangle^\phi| \leqq c^{w_A}\,, \tag{2.34}$$

$$|\langle \mathscr{A}\mathscr{B} \rangle^\phi - \langle \mathscr{A} \rangle^\phi \langle \mathscr{B} \rangle^\phi| \leqq c^{w_{\mathscr{A}}} c^{w_{\mathscr{B}}} e^{-(1-\delta)\mathrm{dist}(\mathrm{suppt}\,\mathscr{A},\,\mathrm{suppt}\,\mathscr{B})/l_D}\,, \tag{2.35}$$

and

$$|\langle A \rangle| \leqq c_A\,, \tag{2.36}$$

$$|\langle AB \rangle - \langle A \rangle \langle B \rangle| \leqq c_A c_B e^{-(1-\delta)\mathrm{dist}(\mathrm{suppt}\,A,\,\mathrm{suppt}\,B)/l_D}\,. \tag{2.37}$$

The constant c_A *depends only on* f_A, w_A *and the activities of the species in* A; *likewise for* c_B.

If the charges of the particles all have the same sign, and if $R_{ij} = 0$ *for all i,j, then* (2.32) *can be omitted and* R *can be replaced with* β *in* (2.33).

Theorem 2.4. *For systems with more general short-range forces, let* $C_4, \ldots, C_9, \delta$ *be given, and put* $\alpha = (1-\delta/2)/l_D$. *Suppose* $e_m \leqq C_5$, $\sum_{i=1}^{s} z_i|e_i| \leqq \frac{1}{2} C_4 \sum_i z_i e_i^2$, *and let* v_{ij} *split into* v_{ij}^0, v_{ij}^n, v_{ij}^R, v_{is} *satisfying* (2.15)–(2.21). *Then there are constants* c_2, c_3, c_4 *(depending on* $C_4, \ldots, C_9, \delta$*) such that for* $\lambda \leqq c_4$, $\beta/l_D \leqq e^{-c_3/\lambda} e^{-c_2\beta/R}$, *and* $\sum_{i=1}^{s} \varrho_i e_i = 0$, *the infinite volume limits of* $\langle \mathscr{A} \rangle_\Lambda^\phi$ *and* $\langle A \rangle_\Lambda$ *exist and satisfy* (2.34)–(2.37).

Theorem 2.5. *Consider the sine-Gordon theories defined by the measure* $e^M d\mu_{0,\partial\Lambda}(\phi)$ *with* M *given by* (2.5), (2.6) *and with the covariance of* $d\mu_{0,\partial\Lambda}(\phi)$ *given in* (1.1). *Suppose that* $e_m \leqq C_5$, $\sum_{i=1}^{s} \varrho_i|e_i| \leqq C_4 \sum_i \varrho_i e_i^2$, $\sum_{i=1}^{s} \varrho_i e_i = 0$, $\varrho_i \geqq 0$, *and* $l_D = \tilde{l}_D(1 + O(C_2\lambda^2))e^{O(C_3\beta/(\lambda l_D))}$. *Suppose further that* $\varrho_{i_1,\ldots,i_t}$ *is symmetric, translation invariant, and satisfies* (2.10) *with* $\lambda \leqq c_4$ *and* $\alpha = (1-\delta/2)/l_D$. *Then for* $\beta/\tilde{l}_D \leqq e^{-c_3/\lambda}$, *the infinite volume limit of* $\langle \mathscr{A} \rangle_\Lambda^\phi$ *exists and satisfies* (2.34), (2.35). *Here* c_3, c_4 *depend only on* $C_1, \ldots, C_5, \delta$.

Using Proposition 2.2 we reduce Theorem 2.3 to Theorem 2.5. With $\alpha=(1-\delta/2)/l_D$ the condition $(2\lambda l_D)^{-1}>\alpha$ is satisfied for λ small. The estimate (2.23) is implied by (2.33), so we obtain (2.10)–(2.12). The hypotheses of Theorem 2.5 are immediate consequences. By expressing particle density expectations in terms of ϕ-expectations and using (2.10)–(2.12), (2.36), (2.37), we obtain (2.34), (2.35). Similarly, Proposition 2.1 reduces Theorem 2.4 to Theorem 2.5. We shall henceforth concentrate on proving clustering in the ϕ-representation for systems as in Theorem 2.5.

The expansion involves two lengths besides $\tilde{l}_D$: $L'\gg\tilde{l}_D\gg L$. We take $L'/\tilde{l}_D$ and $\tilde{l}_D/L$ to be large integers which may depend on the constants C_1, C_2, etc. appearing in Theorem 2.5, but which are chosen before λ and $\beta/\tilde{l}_D$.

2.5. *Outline of the Proof*

In Sect. 3 we define a partition of unity for the set of field configurations, basing the construction on the shape of the leading term in the interaction potential. Stationary points can come arbitrarily close in energy to the $\phi=0$ stationary point, and they can move off the real axis. Thus it is important to set up a precise tradeoff between the energy of a stationary point (extracted in Sects. 7 and 8) and the size of the interaction coefficients (estimated in Sect. 9). A space-dependent field translation is made for each term in the partition of unity.

In Sect. 4 we prove the basic estimate on ratios of partition functions with different boundary conditions. We show that partition functions in which $\phi\to 0$ at the boundary dominate corresponding ones where ϕ takes some other value at the boundary. This is the essence of the thermodynamic stability of the $\phi=0$ stationary point. The proof is quite short, however the proof of the corresponding result for the constrained partition functions generated by our expansion is much more difficult. In Sect. 11 we use our expansion to reduce the problem to the unconstrained case of Sect. 4.

Section 5 presents the expansion. Dirichlet decoupling is needed in order to produce the right partition functions for Sects. 4 and 11, and this necessitates a number of new features. In Sect. 6 we control the combinatorics of the expansion and state a sequence of convergence estimates proven there and in Sects. 7–10. The final section uses the expansion to prove exponential clustering and existence of the infinite volume limit.

3. The Peierls Expansion

In this section we consider the leading term

$$S(\phi)=\sum_{i=1}^{s}\varrho_i(1-e^{i\beta^{1/2}e_i\phi}) \tag{3.1}$$

in the action $-M$ of Sect. 2 and choose the values of ϕ that will make important contributions to the partition function. The point $\phi=0$ is a global minimum for S, and in units where $\tilde{l}_D=1$,

$$S''(0)=\sum_i \varrho_i e_i^2\beta=1\,. \tag{3.2}$$

We need to consider values of ϕ for which $S(\phi)$ comes close to this minimum. The set of such values will be denoted $\mathscr{H}$. The set $\mathscr{H}$ is constructed in the following lemma. Let γ be a fixed constant close to, but less than, 1/2. We use

$$\eta = \sqrt{\frac{1-2\gamma}{4e_m^2}} \tag{3.3}$$

as a measure of how close minima need to come to the $\phi=0$ minimum to be relevant.

Lemma 3.1. *There exists a set of real numbers* $\mathscr{H}$ *containing* 0 *and a set of intervals* $\mathscr{I}=\{[h-\delta_h, h+\delta_h]: h\in\mathscr{H}\}$ *with the following properties:*

(i) $$\operatorname{Re} S(\phi) \geqq \gamma(\phi-h)^2 + \eta_h^2\beta^{-1} \quad \textit{for} \quad \phi\in I_h \equiv [h-\delta_h, h+\delta_h], \tag{3.4}$$

where

$$\eta_h \equiv \min\{(\beta \operatorname{Re} S(h))^{1/2}, \eta\}, \tag{3.5}$$

(ii) $$\bigcup_{h\in\mathscr{H}} I_h = \mathbb{R},$$

(iii) $$I_h \cap I_{h'} \textit{ contains at most one point if } h \neq h',$$

(iv) $$\frac{1}{2\sqrt{2}}\eta\beta^{-1/2} \leqq \delta_h \leqq 4\eta\beta^{-1/2} \quad \textit{for all} \quad h\in\mathscr{H}. \tag{3.6}$$

Proof. Suppose ϕ is such that $\operatorname{Re} S(\phi) \leqq 4\eta^2\beta^{-1}$. Then by (3.2), (3.3),

$$\begin{aligned} 1 \geqq (\operatorname{Re} S)''(\phi) &= \sum_i \varrho_i e_i^2 \beta \cos \beta^{1/2} e_i \phi \\ &= 1 - \sum_i \varrho_i e_i^2 \beta (1-\cos\beta^{1/2} e_i\phi) \\ &\geqq 1 - e_m^2 \beta \operatorname{Re} S(\phi) \geqq 2\gamma. \end{aligned} \tag{3.7}$$

Since $\operatorname{Re} S>0$, the first inequality in (3.7) implies that

$$|(\operatorname{Re} S)'| \leqq 2\sqrt{2}\eta\beta^{-1/2} \tag{3.8}$$

for $\operatorname{Re} S \leqq 4\eta^2\beta^{-1}$.

Now suppose $\operatorname{Re} S(\phi_0) \leqq 2\eta^2\beta^{-1}$. The remarks above imply that there exist $\phi_1 < \phi_2 \leqq \phi_0 \leqq \phi_3 < \phi_4$ and $h\in[\phi_2, \phi_3]$ a local minimum of $\operatorname{Re} S$, such that

$$\operatorname{Re} S(\phi_1) = \operatorname{Re} S(\phi_4) = 4\eta^2\beta^{-1}, \tag{3.9}$$

$$\operatorname{Re} S(\phi_2) = \operatorname{Re} S(\phi_3) = 2\eta^2\beta^{-1}, \tag{3.10}$$

$$\operatorname{Re} S(\phi) \in [2\eta^2\beta^{-1}, 4\eta^2\beta^{-1}] \quad \text{for} \quad \phi\in[\phi_1,\phi_2]\cup[\phi_3,\phi_4], \tag{3.11}$$

$$\operatorname{Re} S(\phi) \in [0, 2\eta^2\beta^{-1}] \quad \text{for} \quad \phi\in[\phi_2,\phi_3], \tag{3.12}$$

$$\operatorname{Re} S(\phi) \geqq \gamma(\phi-h)^2 + \eta_h^2\beta^{-1} \quad \text{for} \quad \phi\in[\phi_1,\phi_4], \tag{3.13}$$

$$|\phi_1-\phi_2| \geqq \frac{1}{\sqrt{2}}\eta\beta^{-1/2}, \ |\phi_3-\phi_4| \geqq \frac{1}{\sqrt{2}}\eta\beta^{-1/2}. \tag{3.14}$$

Each excursion of $\mathrm{Re}\,S$ below $2\eta^2\beta^{-1}$ will be contained in a neighborhood $[\phi_1(h), \phi_4(h)]$ satisfying (3.9)–(3.14) above. We define $\mathscr{H}_1$ to be the set composed of all h's arising from this construction. Since $S(0)=0$, $\mathscr{H}_1$ contains 0 and $\eta_0=0$. For $h\in\mathscr{H}_1$, define

$$\delta_h=\min\{|h-\phi_4(h)|, |h-\phi_1(h)|\}. \tag{3.15}$$

By (3.14), $\delta_h \geqq \frac{1}{\sqrt{2}}\eta\beta^{-1/2}$, so the lower bound in (iv) is satisfied. By (3.13), $|h-\phi_4(h)|$ and $|h-\phi_0(h)|$ are no larger than $4\eta\beta^{-1/2}$, so the upper bound in (iv) is also satisfied.

It is clear that no two I_h's constructed so far intersect. In fact, (3.7) and (3.14) imply that

$$\begin{aligned}&(\mathrm{Re}\,S)'(\phi)\leqq 0 \quad \text{for} \quad \phi\in[h-\delta_h, h], (\mathrm{Re}\,S)'(h-\delta_h)\leqq -\frac{\gamma}{\sqrt{2}}\eta\beta^{-1/2},\\ &(\mathrm{Re}\,S)'(\phi)\geqq 0 \quad \text{for} \quad \phi\in[h, h+\delta_h], (\mathrm{Re}\,S)'(h+\delta_h)\geqq \frac{\gamma}{\sqrt{2}}\eta\beta^{-1/2},\end{aligned} \tag{3.16}$$

and so $(\mathrm{Re}\,S)''\leqq 1$ implies that $\mathrm{dist}(I_h, I_{h'})\geqq \sqrt{2}\gamma\eta\beta^{-1/2}$. Notice that (3.7) shows

$$\begin{aligned}\delta_h &\geqq \sqrt{2}\sqrt{4\eta^2\beta^{-1}-\eta_h^2\beta^{-1}}\geqq 2\sqrt{2\eta^2\beta^{-1}-\eta_h^2\beta^{-1}}\\ &\geqq \max\{|h-\phi_2(h)|, |h-\phi_3(h)|\},\end{aligned} \tag{3.17}$$

so that $\mathrm{Re}\,S(h\pm\delta_h)\geqq 2\eta^2\beta^{-1}$. Thus all regions not yet covered by intervals I_h have $\mathrm{Re}\,S\geqq 2\eta^2\beta^{-1}$.

The set $\mathscr{H}_1$ will contain arbitrarily large positive and negative h's. Let J be an interval $[h_1+\delta_{h_1}, h_2-\delta_{h_2}]$ with h_2 the smallest element of $\mathscr{H}_1$ larger than h_1. Since $\mathrm{Re}\,S(\phi)\geqq 2\eta^2\beta^{-1}$ for $\phi\in J$, any $h\in J$ has the property that

$$\mathrm{Re}\,S(\phi)\geqq \gamma(\phi-h)^2+\eta_h^2\beta^{-1} \quad \text{for} \quad \phi\in J\cap[h-L_0/2, h+L_0/2], \tag{3.18}$$

where $L_0=2\sqrt{2}\eta\beta^{-1/2}$. Cover J with $n=[|J|/L_0+1]$ equal intervals of length $|J|/n\leqq L_0$. (We are writing $|J|$ for the length of J and $[\cdot]$ denotes integer part.) These intervals are centered at

$$h_{Jk}=h_1+\delta_{h_1}+(k+\tfrac{1}{2})|J|/n, k=0, 1, \ldots, n-1, \tag{3.19}$$

and they cover J precisely, with no overlapping. We put

$$\delta_{h_{Jk}}=|J|/(2n)\leqq \sqrt{2}\eta\beta^{-1/2}, \tag{3.20}$$

and define $\mathscr{H}_2$ to be the set of all h_{Jk} arising from this construction, letting both J and k vary. It is now easy to check that the lemma holds with $\mathscr{H}=\mathscr{H}_1\cup\mathscr{H}_2$. Equations (3.13) and (3.18) yield (i) for the two types of h's; (ii) and (iii) have also been satisfied. Condition (iv) has been proven for $h\in\mathscr{H}_1$. For $h\in\mathscr{H}_2$, (3.20) supplies the upper bound. We showed above that $|J|\geqq \sqrt{2}\gamma\eta\beta^{-1/2}$ so that $\delta_{h_{Jk}}\geqq \frac{1}{2\sqrt{2}}\eta\beta^{-1/2}$. □

The Peierls expansion can now be generated by inserting an appropriate partition of unity into the measure. We divide Λ into L-lattice cubes Ω_α. In each such cube we define an average field and a fluctuation field:

$$A_\alpha = L^{-3} \int_{\Omega_\alpha} \phi(x)\,dx\,, \tag{3.21}$$

$$\delta(x) = \phi(x) - A_\alpha(x) \quad \text{for} \quad x \in \Omega_\alpha\,. \tag{3.22}$$

Let h denote a function on some region (such as Λ) taking values in $\mathscr{H}$ and constant on L-lattice cubes. The Peierls expansion is the identity

$$Z = \sum_h \int e^M \frac{\exp\left(-\frac{1}{2}\int_\Lambda (\phi(x) - h(x))^2\,dx\right)}{\prod_\alpha \left(\sum_{h(\Omega_\alpha)} \exp\left(-\frac{1}{2}\int_{\Omega_\alpha} (\phi(x) - h(\Omega_\alpha))^2\,dx\right)\right)}\,d\mu_{0,\partial\Lambda}(\phi)\,. \tag{3.23}$$

Define E by writing

$$M = \sum_{i=1}^{s} \int \varrho_i \varepsilon_i(x)\,dx + E\,. \tag{3.24}$$

Using the identities

$$\sum_{i=1}^{s} \varrho_i e_i = 0, \int \delta(x) f(A(x)) = 0, \sum_i \varrho_i e_i^2 \beta = 1\,,$$

we can rewrite (3.23) as

$$Z = \sum_h \int e^G e^E \exp(-\tfrac{1}{2}\int (\phi - h)^2)\,d\mu_{0,\partial\Lambda}(\phi)\,, \tag{3.25}$$

where

$$G = G_1 + G_2\,, \tag{3.26}$$

$$e^{G_1} = \prod_\alpha r(A_\alpha)\,, \tag{3.27}$$

$$r(A) = \frac{\exp\left(\sum_{i=1}^{s} \varrho_i (e^{i\beta^{1/2} e_i A} - 1) L^3\right)}{\sum_{h \in \mathscr{H}} \exp(-\frac{1}{2}(A - h)^2 L^3)}\,, \tag{3.28}$$

$$\begin{aligned} G_2 &= \int \sum_{i=1}^{s} \varrho_i (e^{i\beta^{1/2} e_i \phi} - e^{i\beta^{1/2} e_i A}) + \tfrac{1}{2}\int \delta^2 \\ &= \sum_{i=1}^{s} \varrho_i \int [(e^{i\beta^{1/2} e_i A} - 1)(e^{i\beta^{1/2} e_i \delta} - i\beta^{1/2} e_i \delta - 1) \\ &\quad + (e^{i\beta^{1/2} e_i \delta} + \tfrac{1}{2}\beta e_i^2 \delta^2 - i\beta^{1/2} e_i \delta - 1)]\,. \end{aligned} \tag{3.29}$$

We next translate the Gaussian measure from ϕ to $\psi = \phi - g$ using

$$\exp(-\tfrac{1}{2}\int (\phi - h)^2)\,d\mu_{0,\partial\Lambda}(\phi) = \exp(-\tfrac{1}{2}\int (\psi + g - h)^2 - \tfrac{1}{2}\int g u_{\partial\Lambda}^{-1} g - \int \psi u_{\partial\Lambda}^{-1} g)\,d\mu_{0,\partial\Lambda}(\psi)\,. \tag{3.30}$$

We also absorb the mass term $-\frac{1}{2}\psi^2$ into the measure, which changes the inverse covariance to

$$C_{\partial\Lambda}^{-1}=u_{\partial\Lambda}^{-1}+1=\lambda^2 l_D^2(-\Delta_{\partial\Lambda})^2-\Delta_{\partial\Lambda}+1\,. \tag{3.31}$$

Denote the new measure by $d\mu_{\partial\Lambda}(\psi)$ and put

$$N=\int\exp(-\tfrac{1}{2}\int\psi^2)d\mu_{0,\partial\Lambda}(\psi)\,. \tag{3.32}$$

The result of translation and mass shift is

$$Z=N\sum_h\int e^G e^E e^R d\mu_{\partial\Lambda}(\psi)\,, \tag{3.33}$$

where

$$R=-\tfrac{1}{2}\int(\psi+g-h)^2-\tfrac{1}{2}\int g u_{\partial\Lambda}^{-1}g-\int\psi u_{\partial\Lambda}^{-1}g+\tfrac{1}{2}\int\psi^2\,. \tag{3.34}$$

Before defining the translation $g(x)$ we need to generalize the set of partition functions we will need to consider.

Definition. Let V be a connected region composed of unit lattice cubes. Let $\{\sigma_\alpha(V)\}$ denote the components of ∂V, and let $\sigma_0(V)$ denote the external boundary component of V. Let $\mathbf{h}_0(V)$ be a function on the components of ∂V with values in $\mathscr{H}$. It specifies the boundary condition $\phi\to\mathbf{h}_0(\sigma)$ at σ.

We need a restriction on what functions h can occur in V with boundary conditions $\mathbf{h}_0$.

Condition A. If there is a σ with $\mathbf{h}_0(\sigma)\neq\mathbf{h}_0(\sigma_0)$, then $h(x)=\mathbf{h}_0(\sigma')$ for $\mathrm{dist}(x,\sigma')<L'$ whenever $\mathbf{h}_0(\sigma')\neq\mathbf{h}_0(\sigma)$.

Definition. If there is a σ with $\mathbf{h}_0(\sigma)\neq\mathbf{h}_0(\sigma_0)$, then we call $\mathbf{h}_0(\sigma)\in\mathscr{H}$ the leading boundary condition of $(\mathbf{h}_0, V)$ and denote it $h_0^l(\mathbf{h}_0, V)$. If all σ have $\mathbf{h}_0(\sigma)=\mathbf{h}_0(\sigma_0)$, then $h_0^l(\mathbf{h}_0, V)$ is defined to be $\mathbf{h}_0(\sigma_0)$. The union of all σ with $\mathbf{h}_0(\sigma)=h_0^l(\mathbf{h}_0, V)$ is called the leading boundary of V and is denoted $L\partial V$.

We will need to exercise some care not to violate this condition, especially in Sect. 11. The leading boundary condition of V is the only boundary condition that discontinuities in h can approach.

Let $\mathscr{A}$ be a bounded function of ϕ that factors across unit lattice cubes. Write $\mathscr{A}(V)$ for the part of $\mathscr{A}$ localized in V. We define some fixed h partition functions by analogy with (3.34):

$$\begin{aligned}Z_h(\mathbf{h}_0, V, \mathscr{A})&=\int\mathscr{A}(V)e^{G(V)}e^{E(V)}e^{R(V)}d\mu_{\partial V}(\psi)\,,\\ Z_h(\mathbf{h}_0, V)&=Z_h(\mathbf{h}_0, V, 1)\,.\end{aligned} \tag{3.35}$$

The measure $d\mu_{\partial V}(\psi)=d\mu_{\partial V}(\phi-g)$ has covariance

$$C_{\partial V}=(\lambda^2 l_D^2(-\Delta_{\partial V})^2-\Delta_{\partial V}+1)^{-1}\,, \tag{3.36}$$

and $G(V)$, $E(V)$ are defined as in (3.24) and (3.26)–(3.29) but with all integrals restricted to V. The following formula for $R(V)$ generalizes (3.34):

$$\begin{aligned}R(V)=&-\tfrac{1}{2}\int_V(\psi+g-h)^2-\tfrac{1}{2}\int_V(g-h_0^l(\mathbf{h}_0, V))u_{L\partial V}^{-1}(g-h_0^l(\mathbf{h}_0, V))\\ &-\int_V\psi u_{L\partial V}^{-1}(g-h_0^l(\mathbf{h}_0,V))+\tfrac{1}{2}\int_V\psi^2\,.\end{aligned} \tag{3.37}$$

The full partition function can now be written as

$$Z = N \sum_h Z_h(0, \Lambda). \tag{3.38}$$

We now proceed to define $g(x)$ for the partition function $Z_h(\mathbf{h}_0, V, \mathscr{A})$. We modify the definition in [3] only slightly. Pave $\mathbb{R}^3$ with cubes of edge length $L'/4$. Let $\mathscr{S}$ denote the intersection of V with the union of all $L'/4$-lattice cubes that are at least $L'/4$ from discontinuities in h. If the L-lattice cube Ω_α has a face in common with σ, we consider $\Omega_\alpha \cap \sigma$ to be a discontinuity in h if $h(\Omega_\alpha) \neq \mathbf{h}_0(\sigma)$. On $\mathscr{S}$ we put $g = h$. Let $\{\mathscr{J}_\beta\}$ be the connected components of $V \backslash \mathscr{S}$.

Define h^e_β by extending $h \upharpoonright \mathscr{J}_\beta$ smoothly to $\mathbb{R}^3$ with no discontinuities at $\partial \mathscr{J}_\beta \backslash \partial V$ or in $\sim \mathscr{J}_\beta$. On components of $\sim V$ touching $\mathscr{J}_\beta$ we have $h^e_\beta = h^l_0(\mathbf{h}_0, V)$. Let $\Gamma_\beta = \partial V \cap \mathscr{J}_\beta$ and put

$$\tilde{g}_\beta - h^l_0(\mathbf{h}_0, V) = (\lambda^2 l_D^2 (-\Delta_{\Gamma_\beta})^2 - \Delta_{\Gamma_\beta} + 1)^{-1} (h^e_\beta - h^l_0(\mathbf{h}_0, V)). \tag{3.39}$$

Note that $\tilde{g}_\beta \to h^l_0(\mathbf{h}_0, V)$ at $\partial \mathscr{J}_\beta \cap \partial V$. We define g by smoothing near $\partial \mathscr{J}_\beta$, as in [3]. Define $B\mathscr{J}_\beta$ to be the union of the unit lattice cubes of $\mathscr{J}_\beta$ touching $\partial \mathscr{J}_\beta \backslash \partial V$. Let χ_β be a C^∞ function equal to zero outside $\mathscr{J}_\beta$, equal to one in $\mathscr{J}_\beta \backslash B\mathscr{J}_\beta$, with $0 \leqq \chi_\beta \leqq 1$, and with $\nabla \chi_\beta$ normal to ∂V at ∂V. In addition we assume that $\chi_\beta(x) = 0$ or 1 if $\operatorname{dist}(x, \partial f) \leqq \frac{1}{4}$ for any face of the unit lattice making up ∂V. We now define

$$g = \chi_\beta \tilde{g}_\beta + (1 - \chi_\beta) h \quad \text{in} \quad \mathscr{J}_\beta. \tag{3.40}$$

We use Γ_β in (3.38) instead of ∂V or $\partial \Lambda$ so that the definitions will be invariant if V is reduced by inserting Dirichlet data outside $\mathscr{J}_\beta$. The restriction on where χ_β changes from 0 to 1 is to avoid possible singularities at edges or corners of ∂V. Note that we do not absorb quadratic expressions from $\int \varrho_{i,j}(x, y) \varepsilon_i(x) \varepsilon_j(y)$ into the measure as in [3]. Doing so would not help with convergence when expanding about minima which are not copies of the $\phi = 0$ minimum.

4. Stability of the $\phi = 0$ Stationary Point

In generating the cluster expansion of the next section, Dirichlet data is inserted on surfaces in the unit lattice that are at least a distance L' from discontinuities in h. When we have full Dirichlet data on a surface, nonlocal terms in E connecting the inside to the outside will have been interpolated away. Thus the measure factorizes across the Dirichlet surface.

Suppose we have a connected region V whose boundary is a Dirichlet surface. If V is free from terms differentiated down from the exponent $G + E + R$, and if each component of ∂V is in a region where $h(x) = h_0 \in \mathscr{H}$, then we resum all the terms of the Peierls expansion in V to yield a (slightly modified) partition function in V. Modifications arise from the constraint that discontinuities in h lie at least a distance L' from ∂V, and from other constraints arising in the expansion in the next section. The expansion depends on being able to replace this partition function by one where h_0 is replaced by 0. The error is a ratio of partition functions which must be bounded by a surface effect.

In Sect. 11 we control ratios of partition functions by an inductive procedure to remove the constraints. The induction terminates when an unmodified partition function

$$Z(h_0, V) = \sum_h Z_h(\mathbf{h}_0, V) \tag{4.1}$$

is obtained. Here $\mathbf{h}_0(\sigma)=h_0$ for all σ and there are no restrictions on the sum over h. Thus the whole procedure depends on an *a priori* bound on ratios of unmodified partition functions. We supply the bound here since it is the physics behind the stability of the $\phi=0$ stationary point.

A more convenient form for $Z(h_0, V)$ can be obtained by translating from ψ to $\tilde{\psi}=\psi+g-h_0$. The construction of g through a χ_β with $\nabla\chi_\beta$ normal to ∂V at ∂V insures that $g-h_0=\Delta_{\partial V}(g-h_0)=0$ at ∂V. Thus the translation is compatible with the Dirichlet data (see [3]). We have

$$\begin{aligned} e^{R(V)}d\mu_{\partial V}(\psi) &= N_V^{-1}e^{R(V)}\exp\Big(-\tfrac{1}{2}\int_V \psi^2\Big)\, d\mu_{0,\partial V}(\psi) \\ &= N_V^{-1}\exp\Big(-\tfrac{1}{2}\int_V(\psi+g-h)^2-\tfrac{1}{2}\int_V(g-h_0)u_{\partial V}^{-1}(g-h_0)-\int_V \psi u_{\partial V}^{-1}(g-h_0) \\ &\quad -\int_V \tilde{\psi}u_{\partial V}^{-1}(h_0-g)-\tfrac{1}{2}\int_V(h_0-g)u_{\partial V}^{-1}(h_0-g)\Big)\, d\mu_{0,\partial V}(\tilde{\psi}) \\ &= N_V^{-1}\exp\Big(-\tfrac{1}{2}\int_V(\phi-h)^2\Big)\, d\mu_{0,\partial V}(\tilde{\psi}). \end{aligned} \tag{4.2}$$

Here $d\mu_{0,\partial V}(\psi)$ has covariance $u_{\partial V}$ and

$$N_V=\int \exp\Big(\int_V -\tfrac{1}{2}\psi^2\Big)\, d\mu_{0,\partial V}(\psi). \tag{4.3}$$

As in the derivation of (3.25) from (3.23) we have

$$\begin{aligned} Z(h_0, V) &= N_V^{-1}\sum_h \int e^{M(V)} \frac{\exp\Big(-\tfrac{1}{2}\int_V(\phi-h)^2\Big)}{\prod_{\Omega_\alpha\subseteq V}\Big(\sum_{h(\Omega_\alpha)}\exp\Big(-\tfrac{1}{2}\int_{\Omega_\alpha}(\phi-h(\Omega_\alpha))^2\Big)\Big)}\, d\mu_{0,\partial V}(\tilde{\psi}) \\ &= N_V^{-1}\int e^{M(V)}d\mu_{0,\partial V}(\tilde{\psi}). \end{aligned} \tag{4.4}$$

The second step resummed the Peierls expansion in V.

The next proposition contains the main result on ratios of partition functions. It is a kind of correlation inequality and is not amenable to proof with expansion techniques.

Proposition 4.1. *The unmodified partition functions with Dirichlet boundary conditions defined in* (4.1) *satisfy the following inequality:*

$$|Z(h_0, V)|\leqq Z(0, V). \tag{4.5}$$

Proof. We write $e^{M(V)}$ in (4.4) in grand canonical form again. We can eliminate the restriction on integrations in $M(V)$ by setting $\phi(x)=0$ for $x\notin V$ because then $\varepsilon_i(x)=e^{i\beta^{1/2}e_i\phi(x)}-1=0$. Using (2.3), (4.4), we have

$$\begin{aligned} Z(h_0, V) &= N_V^{-1}\int \exp\Big(\int i\beta^{1/2}z_s e_s\phi\Big) \\ &\quad \cdot \lim_{\Lambda'\to\mathbb{R}^3} Z_0^{-1}\sum_N \frac{\tilde{z}^N}{N!}\int_{(\Lambda')^N} e^{-\beta W}\exp\Big(i\beta^{1/2}\sum_\alpha e_{i(\alpha)}\phi(x_\alpha)\Big)\, d\mu_{0,\partial V}(\tilde{\psi}) \\ &= N_V^{-1}\lim_{\Lambda'\to\mathbb{R}^3} Z_0^{-1}\sum_N \frac{\tilde{z}^N}{N!}\int_{(\Lambda')^N} e^{-\beta W}\exp\Big(i\beta^{1/2}\sum_{\alpha:\, x_\alpha\in V} e_{i(\alpha)}h_0+\int_V i\beta^{1/2}z_s e_s h_0\Big) \\ &\quad \cdot \int \exp\Big(i\beta^{1/2}\sum_{\alpha:\, x_\alpha\in V} e_{i(\alpha)}\tilde{\psi}(x_\alpha)+\int i\beta^{1/2}z_s e_s\tilde{\psi}\Big)\, d\mu_{0,\partial V}(\tilde{\psi}). \end{aligned} \tag{4.6}$$

We have replaced ϕ with $\tilde{\psi}+h_0$ everywhere in V. The reason for using this representation is now clear: The integral over $\tilde{\psi}$ is manifestly positive and so it can be left alone when we take absolute values. The phase factors involving h_0 go away, and we can reverse the manipulations in (4.6) to obtain

$$
\begin{aligned}
|Z(h_0, V)| &\leqq N_V^{-1} \lim_{\Lambda' \to \mathbb{R}^3} Z_0^{-1} \sum_N \frac{\tilde{z}^N}{N!} \int_{(\Lambda')^N} e^{-\beta W} \int \exp\Big(i\beta^{1/2} \sum_{\alpha: x_\alpha \in V} e_{i(\alpha)} \tilde{\psi}(x_\alpha) \\
&\quad + \int_V i\beta^{1/2} z_s e_s \tilde{\psi} \Big) d\mu_{0,\partial\Lambda}(\tilde{\psi}) \\
&= N_V^{-1} \int e^{M(V)} d\mu_{0,\partial V}(\phi) \\
&= Z(0, V)\,. \qquad (4.7)
\end{aligned}
$$

We have substituted ϕ for $\tilde{\psi}$ in V to make the last equality more transparent. □

Remark. If we had not eliminated the constraint that $h=h_0$ near ∂V, then resumming the Peierls expansion in V would have left remnants of the partition of unity near ∂V. Pushed back into grand canonical form, these functions are Fourier transformed. After taking absolute values and returning to sine-Gordon language, the functions are drastically modified. Any reasonable approximate characteristic function $\chi(A)$ has the property that $\||\tilde{\chi}|^\urcorner\|_{L^\infty} \sim c|\log\beta|$. These divergent factors would cause problems with our expansion because they would have to be beaten by factors of λ while $\beta<\beta_0(\lambda)=ce^{-c/\lambda}$.

5. The Cluster Expansion

We develop the cluster expansion along the lines of [3], but a number of new devices are needed to handle the special requirements of our situation. An inductively defined expansion is needed to allow sufficient flexibility. Dirichlet decoupling is needed so that partition functions inside clusters will have Dirichlet boundary conditions to enforce the condition $\phi=h_0$ at ∂V. With the Dirichlet decoupling procedure we use, many contractions to a cube could occur without long contraction distances to compensate. As in [11], the region to be isolated is expanded appropriately to avoid this problem. We also expand the region to be isolated to be as connected as possible, though this is probably not essential.

We derive the expansion first for a fixed $h(x)$ in a connected region $V \subseteqq \Lambda$, with Dirichlet boundary conditions enforcing $\phi=\mathbf{h}_0(\sigma)$ at each component σ of ∂V. Suppose that the support of $\mathscr{A}(V)$ is contained in S, a union of "special" lattice cubes. We think of $\mathscr{A}(V)$ as determining S, even though $\mathscr{A}(a)=1$ is possible for a cube $a \subseteqq S$. We order the unit lattice cubes lexicographically. Write $a_i=[i_1,i_1+1]\times[i_2,i_2+1]\times[i_3,i_3+1]$, where $i=(i_1, i_2, i_3)\in\mathbb{Z}^3$.

Then

$$
\begin{aligned}
a_i<a_j \quad &\text{if} \quad i_1<j_1 \\
&\text{or} \quad i_1=j_1 \qquad \text{and} \quad i_2<j_2 \\
&\text{or} \quad i_1=j_1, i_2=j_2, \quad \text{and} \quad i_3<j_3\,. \qquad (5.1)
\end{aligned}
$$

If X_1,X_2 are two unions of unit lattice cubes, then we say X_1 is before X_2 if the first cube in X_1 is before the first cube in X_2. The basic quantities to be expanded are $Z_h(\mathbf{h}_0, V, \mathscr{A})$ and $Z_h(\mathbf{h}_0, V)$, defined in (3.35), where h satisfies Condition A of Sect. 3.

The expansion generates an increasing sequence of regions $X_1, X_2, \ldots, X_k$. Their boundaries $\gamma_i = \partial X_i$ are contours for the insertion of Dirichlet data. Given a sequence of interpolation parameters $s = \{s_1, \ldots, s_{l-1}\}$ we define interpolating covariances $C(s)$ with full Dirichlet data on ∂V and partial data on $\gamma_1, \ldots, \gamma_{l-1}$. Put

$$C_\gamma = (\lambda^2 l_D^2(-\Delta_\gamma)^2 - \Delta_\gamma + 1)^{-1}, \tag{5.2}$$

where Δ_γ has Dirichlet boundary conditions on γ. Then we define inductively

$$C_\gamma(s_1, \ldots, s_l) = s_l C_\gamma(s_1, \ldots, s_{l-1}) + (1 - s_l) C_{\gamma \cup \gamma_l}(s_1, \ldots, s_{l-1}). \tag{5.3}$$

Finally for $s = \{s_1, \ldots, s_{l-1}\}$ we write

$$C(s) = C_{\partial V}(s_1, \ldots, s_{l-1}). \tag{5.4}$$

We shall need a formula for $\frac{d}{ds_{l-1}} C(s_1, \ldots, s_{l-1})$. In terms of operators e_γ, δ_γ,

$$e_\gamma C_\Gamma(s) = C_{\Gamma \cup \gamma}(s), \tag{5.5}$$

$$\delta_\gamma C_\Gamma(s) = C_\Gamma(s) - C_{\Gamma \cup \gamma}(s), \tag{5.6}$$

we have

$$\begin{aligned} C_\gamma(s_1, \ldots, s_{l-1}) &= e_{\gamma_{l-1}} C_\gamma(s_1, \ldots, s_{l-2}) + s_{l-1} \delta_{\gamma_{l-1}} C_\gamma(s_1, \ldots, s_{l-2}) \\ &= \ldots = \prod_{\alpha=1}^{l-1} (e_{\gamma_\alpha} + s_\alpha \delta_{\gamma_\alpha}) C_\gamma. \end{aligned} \tag{5.7}$$

Since each γ_α separates γ_β from γ_δ if $\beta < \alpha < \delta$ we have $\delta_{\gamma_\beta} e_{\gamma_\alpha} \delta_{\gamma_\delta} C_\gamma = 0$. Thus

$$C_\gamma(s_1, \ldots, s_{l-1}) = \sum_{\alpha=1}^{l-1} \sum_{\beta=\alpha}^{l-1} e_{\gamma_1} \ldots e_{\gamma_{\alpha-1}} (s_\alpha \delta_{\gamma_\alpha}) \ldots (s_\beta \delta_{\gamma_\beta}) e_{\gamma_{\beta+1}} \ldots e_{\gamma_{l-1}} C_\gamma + e_{\gamma_1} \ldots e_{\gamma_{l-1}} C_\gamma. \tag{5.8}$$

This yields the formula

$$\frac{d}{ds_{l-1}} C(s_1, \ldots, s_{l-1}) = \sum_{\eta(l)=1}^{l-1} s_{\eta(l)} \ldots s_{l-2} \delta_{\gamma_{\eta(l)}} \ldots \delta_{\gamma_{l-1}} C_{\gamma_1 \cup \ldots \cup \gamma_{\eta(l)-1} \cup \partial V}. \tag{5.9}$$

Nonlocal terms in $E(V)$ coupling across contours γ_l are interpolated with a factor s_l. For unit lattice cubes $a_1, \ldots, a_t$ we define

$$\mathscr{E}_t(a_1, \ldots, a_t) = \frac{1}{t!} \sum_{i_1, \ldots, i_t} \int_{a_1 \times \ldots \times a_t} \varrho_{i_1, \ldots, i_t}(x_1, \ldots, x_t) \varepsilon_{i_1}(x_1) \ldots \varepsilon_{i_t}(x_t). \tag{5.10}$$

Then we have

$$E(V) = \sum_{t=2}^{\infty} \sum_{(a_1, \ldots, a_t): a_i \subseteq V} \mathscr{E}_t(a_1, \ldots, a_t). \tag{5.11}$$

When X_l is being decoupled from $V \setminus X_l$, $E(V, s_1, \ldots, s_{l-1})$ is replaced by $E(V, s_1, \ldots, s_l)$. The new version is the same as $E(V, s_1, \ldots, s_{l-1})$ except that terms with some pair of cubes (a_i, a_j) separated by γ_l are multiplied by s_l. We say $(a_1, \ldots, a_t)$ are partitioned by γ_l. Thus all the terms in $\frac{d}{ds_{l-1}} E(V, s_1, \ldots, s_{l-1})$ are partitioned by γ_{l-1}. For each such term define $\eta(l)$ to be the smallest integer such that the cubes are partitioned by $\gamma_{\eta(l)}$. Since the X_l are increasing, the term acquires factors $s_{\eta(l)} \dots s_{l-1}$ in the interpolation process, and we have

$$\frac{d}{ds_{l-1}} E(V, s_1, \dots, s_{l-1}) = \sum_{\eta(l)=1}^{l-1} s_{\eta(l)} \dots s_{l-2} \sum_{t=2}^{\infty} \sum_{\substack{(a_1, \dots, a_t) \text{ partitioned by } \gamma_{\eta(l)} \text{ and} \\ \text{by } \gamma_{l-1} \text{ but not by } \gamma_{\eta(l)-1}}} \mathscr{E}_t(a_1, \dots, a_t). \tag{5.12}$$

Definition. Let $\mathscr{B}(h, \mathbf{h}_0, V)$ be the set of all faces of unit cubes in V that are at least a distance L' from discontinuities in h. [For this purpose we define $h(x) = \mathbf{h}_0(\sigma)$ for $x \notin V$ near σ.] The union of all faces in $\mathscr{B}(h, \mathbf{h}_0, V)$ breaks V into a number of connected regions. We call the resulting regions the elementary regions associated to h. Each X_l will be a union of elementary regions.

Let a_1 be the first cube in S if $S \neq \emptyset$ or the first cube in V if $S = \emptyset$. In the latter case a_1 will be at the boundary of V so we can define Y_1 to be the elementary region containing a_1, and then Y_1 will have $g = \mathbf{h}_0(\sigma_0)$ at its outer boundary. In general we arrange for this to be the case by including in Y_1 all the elementary regions that surround or contain a_1.

Definition. A region R is said to surround a cube a if $a \nsubseteq R$ and if every curve from a to infinity intersects R in a curve of finite length. A curve γ surrounds a set γ' if $\gamma' \nsubseteq \gamma$ and if every curve from γ' to infinity intersects γ.

Draw the shortest path from a_1 to the elementary region surrounding a_1 that has the largest diameter. Let T_1 be the set of cubes in $\mathbb{R}^3$ that touch this path. Then let Y_1 be the set of all elementary regions that have a cube in common with T_1. If no elementary regions surround a_1, then Y_1 is the elementary region containing a_1. Finally we put $X_1 = Y_1$ and $\gamma_1 = \partial X_1$.

The first interpolation attempts to remove interactions across γ_1. There is a decoupled term ($s_1 = 0$) which factors across γ_1, and there is an interaction term $\int_0^1 \frac{d}{ds_1} Z_h(\mathbf{h}_0, V, \mathscr{A}, s_1) ds_1$.

The interaction term is expanded further and several new regions are defined, depending on what term in the expansion is being considered. We have

$$\frac{d}{ds_1} Z_h(\mathbf{h}_0, V, \mathscr{A}, s_1) = \int e^{E(V, s_1)} \bar{\kappa}_1 \mathscr{A}(V) e^{G(V)} e^{R(V)} d\mu_s(\psi), \tag{5.13}$$

where $d\mu_s(\psi)$ has covariance $C(s)$ and where

$$\begin{aligned} \bar{\kappa}_l = {} & \frac{d}{ds_l} E(V, s_1, \dots, s_l) + \int_{V \times V} dx dy \frac{d}{ds_l} C(x, y, s_1, \dots, s_l) \\ & \cdot \left(\frac{\delta}{\delta \psi(x)} + \frac{\delta}{\delta \psi(x)} E(V, s_1, \dots, s_l) \right) \left(\frac{\delta}{\delta \psi(y)} + \frac{\delta}{\delta \psi(y)} E(V, s_1, \dots, s_l) \right). \end{aligned} \tag{5.14}$$

There is only one term $\eta(2) = 1$ in the sums (5.9), (5.12). Let $\tau_2 \in \{1, 2, 3, 4, 5\}$ signify which of the five terms in (5.14) is being considered. There are sums over t for each appearance of E in (5.14). The values are given by t_2, t'_2 (one or both may be superfluous, depending on τ_2). The integrals over x, y in (5.14) are expanded into unit lattice localizations, and dE/ds_1, $\delta E/\delta\psi$ are written as a sum of terms where each vertex is localized in a unit cube. The cubes in which $\mathscr{E}$ or $\delta/\delta\psi$ vertices are localized will be denoted $a_{2\alpha}$. Some of these cubes are distinguished and denoted $a'_{2\alpha}$. Any cube containing a $\delta/\delta\psi$ is distinguished. Cubes in which $g=h$ are distinguished. For each factor $\mathscr{E}_{t_2}$ or $\mathscr{E}_{t'_2}$ we distinguish the first three cubes (two cubes if t_2 or t'_2 equals 2). Distinguished cubes are ones where number divergences can occur.

Having expanded out the interaction term, we construct the region to be isolated at the next interpolation. For each factor $\mathscr{E}_{t_2}$ or $\mathscr{E}_{t'_2}$, draw the shortest tree graph that connects all the cubes $a_{2\alpha}$ of the factor. Tree graphs connecting a set of cubes must have a vertex in each cube but can have additional vertices. If there is more than one shortest tree, choose one but let the choice depend only on the set of cubes to be connected. When τ_2 is such that there are functional derivatives, draw in addition a shortest path that travels from one $\delta/\delta\psi$ cube to ∂Y_1 and then to the other $\delta/\delta\psi$ cube. It should depend only on the cubes and γ_1. Define T_2 to be the set of cubes in $\mathbb{R}^3$ that touch one of the trees connecting the $a_{2\alpha}$ or that touch the path from $\delta/\delta\psi(x)$ to ∂Y_1 to $\delta/\delta\psi(y)$. The region T_2 is a connected set intersecting X_1. Like T_1, it may run outside of V. Define Y_2 to be the union of all elementary regions not in X_1 that have a cube in common with T_2. Finally put $X_2=X_1\cup Y_2$ and $\gamma_2=\partial X_2$. The region $X_2\cup T_1\cup T_2$ is connected. Using an interpolation parameter s_2, Dirichlet data is inserted at γ_2 and terms in E coupling across γ_2 are interpolated away. Again there is a decoupled term and an interaction term which we expand further.

The general step is similar to the step just described. When X_{l-1} is being isolated, the operator $\bar{\kappa}_{l-1}$ is inserted as in (5.13). Using (5.8) and (5.10) we find sums over $\eta(l)=1,\dots,l-1$, over $\tau_l=1,\dots,5$, over t_l, t'_l, and over localization cubes $a_{l\alpha}$. The distinguished cubes $a'_{l\alpha}$ are defined as follows. As before, any cube containing a $\delta/\delta\psi$ is distinguished. We distinguish cubes in which $g=h$ and which have not been distinguished at earlier steps. The first three (or two) cubes in $\mathscr{E}_{t_l}$ or $\mathscr{E}_{t'_l}$ are distinguished. We draw shortest trees connecting the cubes of $\mathscr{E}_{t_l}$, $\mathscr{E}_{t'_l}$ and a shortest path from $\delta/\delta\psi$ to $\partial Y_{\eta(l)}$ to $\delta/\delta\psi$. Cubes touching the trees or the path are included in T_l. In addition any cube a satisfying

$$\operatorname{dist}(a, a'_{l\alpha}) < (N(a'_{l\alpha})-1)^{1/4} \tag{5.15}$$

is included in T_l. Here $N(a'_{l\alpha})$ is the number of interpolation steps in which $a'_{l\alpha}$ has been distinguished. We define Y_l to be the union of all elementary regions not in X_{l-1} that overlap T_l, and we put $X_l=X_{l-1}\cup Y_l$, $\gamma_l=\partial X_l$. Each region $X_l\cup\bigcup_{i=1}^{l} T_i$ is connected.

The expansion proceeds until each term has its largest region X_k decoupled from $V\backslash X_k$ or until $X_k=V$. At this point we have

$$Z_h(\mathbf{h}_0, V, \mathscr{A}) = \sum_{\mathbb{Z}} \varrho(\mathbb{Z}) \prod_{V_i \subseteq V\backslash X_k(\mathbb{Z})} Z_h(\mathbf{h}_0(V_i), V_i, \mathscr{A}). \tag{5.16}$$

Here $\mathbb{Z}$ specifies all the information needed to construct a term of the expansion. We call it a cluster. It specifies:

A region V with boundary conditions $\mathbf{h}_0(V)$;
An integer $k \geqq 1$ for which $s_k = 0$;
A cube a_1;
Regions T_1 and $Y_1 = X_1$, and a compatible $h \upharpoonright X_1$;
A contour $\gamma_1 = \partial X_1$;
For each $l = 2, \ldots, k$:
 An interpolation parameter s_{l-1},
 A value of $\eta(l) \in \{1, \ldots, l-1\}$,
 A type of term τ_l,
 Indices t_l, t'_l for $\mathscr{E}$-factors,
 Localization cubes $a_{l\alpha}$,
 Distinguished localization cubes $a'_{l\alpha}$,
 A region T_l based on the trees and paths connecting the $a_{l\alpha}$,
 A region Y_l and a compatible $h \upharpoonright Y_l$,
 A region $X_l = X_{l-1} \cup Y_l$, and
 A contour $\gamma_l = \partial X_l$;
An observable $\mathscr{A}(X_k)$.

These data are compatible in the sense that $\mathbb{Z}$ must arise from the expansion of some $Z_h(\mathbf{h}_0, V, \mathscr{A})$ as described above. For example V affects the covariances $C(s)$ and determines what part of T_l is not contained in X_l. The sum over $\mathbb{Z}$ in (5.16) is restricted to clusters arising from the expansion of $Z_h(\mathbf{h}_0, V, \mathscr{A})$. When $\mathbb{Z}$ is written in parenthesis after one of the above symbols, as in $X_k(\mathbb{Z})$, we mean the value specified by $\mathbb{Z}$.

We have the following formula for $\varrho(\mathbb{Z})$:

$$\varrho(\mathbb{Z}) = \int e^{E(X_k, s_1, \ldots, s_{k-1})} \prod_{l=2}^{k} [s_{\eta(l)} \cdots s_{l-2} \kappa_{l-1}(\tau_l, t_l, t'_l, a_{l\alpha})] \mathscr{A}(X_k) e^{G(X_k)} e^{R(X_k)} d\mu_s(\psi), \tag{5.17}$$

where the operators κ_l are written in the order $\kappa_{k-1} \ldots \kappa_1$. The form κ_{l-1} takes depends on τ_l; for example we let $\tau_l = 3$ specify the term

$$\kappa_{l-1}(3, t_l, t'_l, a_{l\alpha}) = \int_{a_{l\bar{\alpha}_1} \times a_{l\bar{\alpha}_2}} dx dy \delta_{\gamma_{\eta(l)}} \ldots \delta_{\gamma_{l-1}} C_{\gamma_1 \cup \ldots \cup \gamma_{\eta(l)-1} \cup \partial V}(x, y)$$
$$\cdot \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)} \mathscr{E}_{t_l}(a_{l\alpha_3}, \ldots, a_{l\alpha_{t_l+2}}). \tag{5.18}$$

In $V \backslash X_k$ we have a product of decoupled partition functions, one for each component V_i of $V \backslash X_k$ [see (5.16)]. Components do not interact since $\mathscr{E}$-factors coupling across γ_k have been interpolated away. Each component has boundary conditions $\mathbf{h}_0(V_i)$. For σ a component of ∂V_i the boundary condition $\mathbf{h}_0(\sigma)$ is defined to be the constant value of g at σ. The function h in $Z_h(\mathbf{h}_0(V_i), V_i, \mathscr{A})$ is understood to be restricted to V_i. It must equal $\mathbf{h}_0(\sigma)$ within a distance L' of $\sigma \backslash \partial V$. It cannot specify an elementary region that surrounds $X_k(\mathbb{Z})$, by our choice of T_1 and Y_1. There are no other constraints on h in V_i arising from $\mathbb{Z}$.

The partition functions $Z_h(\mathbf{h}_0(V_i), V_i, \mathscr{A})$ will not necessarily satisfy Condition A unless some further conditions are placed on h, $\mathbf{h}_0(V)$, and V. These will be imposed and Condition A will be checked each time (5.16) is applied. Assuming Condition A for the moment, we need to check two things before we can say that the partition functions $Z_h(\mathbf{h}_0(V_i), V_i, \mathscr{A})$ produced in (5.16) agree with our earlier definition (3.35). In order that $R(V)$, when restricted to V_i, should agree with $R(V_i)$, defined in (3.37), we need

$$g_V = g_{V_i} \quad \text{on} \quad V$$

and

$$u_{L\partial V}^{-1}(g - h_0^l(\mathbf{h}_0(V), V)) = u_{L\partial V_i}^{-1}(g - h_0^l(\mathbf{h}_0(V_i), V_i)) \quad \text{on} \quad V_i .$$

The subscript on g indicates which region was used in the construction of g as in Sect. 3. In fact it does not matter whether V or V_i is used to construct g on V_i because only the part of ∂V touching $\mathscr{J}_\beta$ affects g in $\mathscr{J}_\beta$. New Dirichlet surfaces are placed at least L' from discontinuities in h. Hence they are well outside of $\mathscr{J}_\beta$ and do not affect g there. The second condition is easily checked when $h_0^l(\mathbf{h}_0(V), V) = h_0^l(\mathbf{h}_0(V_i), V_i)$, because new Dirichlet surfaces in $L\partial V_i$ are at least L' from places where $h \neq h_0^l(\mathbf{h}_0(V), V)$, and hence $g = h_0^l(\mathbf{h}_0(V), V)$ near the surfaces. When $h_0^l(\mathbf{h}_0(V), V) \neq h_0^l(\mathbf{h}_0(V_i), V_i)$ we verify the sequence

$$\begin{aligned} u_{L\partial V}^{-1}(g - h_0^l(\mathbf{h}_0(V), V)) &= u_0^{-1}(g - h_0^l(\mathbf{h}_0(V), V)) \\ &= u_0^{-1}(g - h_0^l(\mathbf{h}_0(V_i), V_i)) \\ &= u_{L\partial V_i}^{-1}(g - h_0^l(\mathbf{h}_0(V_i), V_i)) , \end{aligned}$$

where u_0 has free boundary conditions. The first step follows because Condition A for V_i implies that $L\partial V$ is far from places in V_i where h (and hence g) differs from $h_0^l(\mathbf{h}_0(V), V)$. The next step is just the fact that $u_0^{-1} = \lambda^2 l_D^2(-\Delta)^2 - \Delta$ annihilates constants. The third step follows because $L\partial V_i \cap \partial V$ is far from places where $g \neq h_0(\mathbf{h}_0(V_i), V_i)$ by Condition A for V; the same is true for $L\partial V_i \setminus \partial V$ because it is all new.

We wish to use (5.16) to derive an expansion for some full partition functions (with h no longer fixed) which we define now.

Definition. Suppose we are given V, $\mathbf{h}_0(V)$, and a set of cubes $\widehat{\partial V}$, where h is required to equal $\mathbf{h}_0(\sigma_0(V))$, the external boundary condition of V. In addition, for each interior component σ of ∂V we specify $i(\sigma) = y$ or n, depending on whether h's specifying elementary regions surrounding σ are allowed or not. We may as well assume $i(\sigma) = y$ for $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$ – otherwise no h's would be allowed. All this information will be denoted $\mathbb{V}$. Then we say that h is compatible with $(\mathbf{h}_0, \mathbb{V})$ if it satisfies the above requirements.

We can now define

$$\begin{aligned} Z(\mathbf{h}_0, \mathbb{V}, \mathscr{A}) &= \sum_{h \text{ compatible with } (\mathbf{h}_0, \mathbb{V})} Z_h(\mathbf{h}_0, V, \mathscr{A}) , \\ Z(\mathbf{h}_0, \mathbb{V}) &= Z(\mathbf{h}_0, \mathbb{V}, 1) . \end{aligned} \tag{5.19}$$

We will always deal with $(\mathbf{h}_0, \mathbb{V})$ that satisfy the following two conditions.

Condition B. Every cube in $\widehat{\partial V}$ is less than L' from σ_0 or from a σ with $i(\sigma) = n$.

Condition A'. If there is a σ with $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$, then all cubes less than L' from a σ' with $\mathbf{h}_0(\sigma') \neq \mathbf{h}_0(\sigma)$ are in $\widehat{\partial V}$.

Condition A' is just Condition A applied at the level of full partition functions.

Some comments on Conditions A' and B should help us to understand what sorts of $(\mathbf{h}_0, \mathbb{V})$ can occur. If $\mathbf{h}_0(V)$ specifies more than one boundary condition, then an interior component σ has $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$. The partition function would vanish unless all such σ have $i(\sigma) = y$. Condition B keeps cubes of $\widehat{\partial V}$ away from such σ. If a different boundary condition occurs on an interior σ', then Condition A' includes nearby cubes in $\widehat{\partial V}$. Thus we must have $\mathbf{h}_0(\sigma') = \mathbf{h}_0(\sigma_0)$. To summarize, at most two boundary conditions are present, and if two are present then components with $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$ have no constraints [$i(\sigma) = y$ and no nearby ∂V cubes], and components with $\mathbf{h}_0(\sigma) = \mathbf{h}_0(\sigma_0)$ have full constraints [$i(\sigma) = n$ and all nearby cubes in $\widehat{\partial V}$] (see Fig. 1).

Note that $Z(h_0, V)$, defined in (4.1), is equal to $Z(\mathbf{h}_0, \mathbb{V})$, where $\mathbf{h}_0(\sigma) = h_0$ for all σ and where $\mathbb{V}$ specifies $\widehat{\partial V} = \emptyset$ and $i(\sigma) = y$ for all σ.

Equation (5.16) now yields the expansion

$$Z(\mathbf{h}_0, \mathbb{V}, \mathscr{A}) = \sum_{\mathbb{Z}} \varrho(\mathbb{Z}) \prod_{V_i \subseteqq V \setminus X_k(\mathbb{Z})} Z(\mathbf{h}_0(V_i), \mathbb{V}_i, \mathscr{A}). \tag{5.20}$$

Here $\mathbb{Z}$ runs over clusters that arise in the expansion of $Z_h(\mathbf{h}_0, V, \mathscr{A})$ for some h compatible with $(\mathbf{h}_0, \mathbb{V})$. We have summed over all h in $\mathbb{V}$ compatible with a fixed $\mathbb{Z}$ to obtain the factors $Z(\mathbf{h}_0(V_i), \mathbb{V}_i, \mathscr{A})$. All cubes of V_i closer than L' to $\partial V_i \setminus \partial V$ are in $\widehat{\partial V_i}$. In addition, any cube in $V_i \cap \widehat{\partial V}$ is in $\widehat{\partial V_i}$. A component of ∂V_i has $i(\sigma) = n$ if it surrounds or equals some σ' of ∂V with $i(\sigma') = n$. A component of $\widehat{\partial V_i}$ that surrounds $X_k(\mathbb{Z})$ has $i(\sigma) = n$. Otherwise $i(\sigma') = y$.

We now apply (5.20) to $Z(0, \Lambda, \mathscr{A})$, putting $\widehat{\partial \Lambda} = \emptyset$. Equation (5.20) is to be iterated by applying it to the partition function $Z(\mathbf{h}_0(V_i), \mathbb{V}_i, \mathscr{A})$ that contains the first cube of S that is not already part of some $X_k(\mathbb{Z})$. The process continues until all cubes of S are used up. The clusters produced so far will be denoted with letters $\mathbb{X}_r$. The expansion takes the form

$$Z(0, \Lambda, \mathscr{A}) = \sum_{\{\mathbb{X}_r\}} \prod_r \varrho(\mathbb{X}_r) \prod_{V_i \subseteqq \Lambda \setminus \bigcup_r X_k(\mathbb{X}_r)} Z(\mathbf{h}_0(V_i), \mathbb{V}_i). \tag{5.21}$$

Note that all partition functions produced by this expansion have constant boundary conditions and all interior boundary components σ have $i(\sigma) = n$. Thus Conditions A' and B are satisfied.

For each V_i with $\mathbf{h}_0(V_i) \neq 0$ we multiply and divide by $Z(0, \mathbb{V}_i)$. Note that for these V_i, $\widehat{\partial V_i}$ is always the set of cubes at a distance less than L' from ∂V_i. Also, any interior components σ of ∂V_i have $i(\sigma) = n$ and $\mathbf{h}_0(\sigma) = \mathbf{h}_0(\sigma_0)$. We also divide (5.21) by $Z(0, \mathbf{a})^{|\Lambda|}$, where a is some unit cube, $\mathbf{a}$ specifies $\widehat{\partial a} = a$, and $|\Lambda|$ is the volume of Λ (the number of unit cubes in Λ). We put

$$\begin{aligned} \tilde{Z}(\mathbf{h}_0, \mathbb{V}, \mathscr{A}) &= Z(\mathbf{h}_0, \mathbb{V}, \mathscr{A}) Z(0, \mathbf{a})^{-|V|}, \\ \tilde{Z}(\mathbf{h}_0, \mathbb{V}) &= Z(\mathbf{h}_0, \mathbb{V}) Z(0, \mathbf{a})^{-|V|}, \\ \tilde{\varrho}(\mathbb{Z}) &= \varrho(\mathbb{Z}) Z(0, \mathbf{a})^{-|X_k(\mathbb{Z})|}, \end{aligned} \tag{5.22}$$

so that (5.21) becomes

$$\tilde{Z}(0, \Lambda, \mathscr{A}) = \sum_{\{\mathbb{X}_r\}} \Xi(\{\mathbb{X}_r\}) \prod_{V_i \subseteqq \Lambda \setminus \bigcup_r X_k(\mathbb{X}_r)} \tilde{Z}(0, \mathbb{V}_i), \tag{5.23}$$

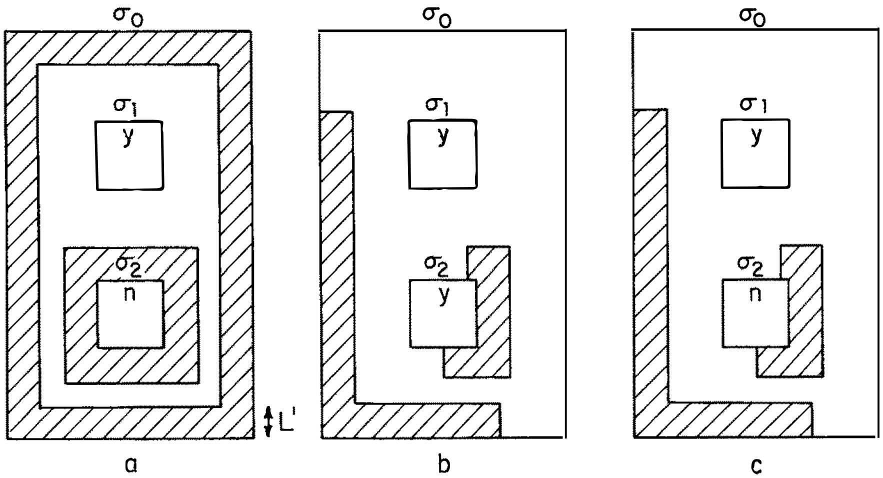

Fig. 1a–c. Situations violating or satisfying Conditions A′ and B. In each of **a–c** let V be the large rectangle minus the squares enclosing a y or an n. The shaded region is $\widehat{\partial V}$, and the σ_i label the components of ∂V. The values of $i(\sigma_1)$ and $i(\sigma_2)$ are indicated by y or n. Suppose that $\mathbf{h}_0(\sigma_0) = \mathbf{h}_0(\sigma_2) \neq \mathbf{h}_0(\sigma_1)$, so that $\mathbf{h}_0(\sigma_1)$ is the leading boundary condition of V. Then **a** satisfies Conditions A′ and B, **b** satisfies neither, and **c** satisfies B but not A′. If we changed $i(\sigma_2)$ to y in **a**, then A′ would be satisfied but not B. Now suppose that $\mathbf{h}_0(\sigma_0) = \mathbf{h}_0(\sigma_1) = \mathbf{h}_0(\sigma_2)$. Then Condition A′ is irrelevant, and Condition B is satisfied only in **a** and **c**

where

$$\Xi(\{\mathbb{X}_r\}) = \prod_r \tilde{\varrho}(\mathbb{X}_r) \prod_{V_i : \mathbf{h}_0(V_i) \neq 0} \frac{Z(\mathbf{h}_0(V_i), \mathbb{V}_i)}{Z(0, \mathbb{V}_i)}. \tag{5.24}$$

There are some compatibility conditions on all these sums over clusters. They arise from the fact that at each application of (5.20) the set of clusters generated depends on the available volume. Each $\mathbb{X}_r$ must arise in the expansion of some component $(\mathbf{h}_0(\mathbb{V}_i), \mathbb{V}_i)$ of $\Lambda \setminus \bigcup_{\mathbb{X}_{r'}, \text{before } \mathbb{X}_r} X_k(\mathbb{X}_{r'})$.

The next several sections will be concerned with proving good bounds on $\varrho(\mathbb{Z})$. To state the estimates we need some measures for how many convergent factors are produced in the expansion.

Definition. Let $d(\mathbb{Z})$ be the sum of the lengths of the trees and paths connecting the $a_{l\alpha}$, $l = 2, \ldots, k$. Let $\delta(\mathbb{Z})$ be the number of functional derivatives $\delta/\delta\psi$ specified by $\mathbb{Z}$. Put t_l or $t'_l = 0$ when τ_l is such that the corresponding $\mathscr{E}$-factor is absent. Then define

$$t(\mathbb{Z}) = \max\left\{\delta(\mathbb{Z}), \sum_{l=2}^{k} (t_l + t'_l)\right\}. \tag{5.25}$$

For f a face of a unit cube in $X_k(\mathbb{Z})$, let $\delta h(f)$ be the discontinuity in h across f. Define

$$\|\mathscr{A}\| = \sup_{\phi, \{x_i, n_i\}} \left| \prod_i \left(\frac{1}{c\lambda} \frac{\delta}{\delta\phi(x_i)} \right)^{n_i} \mathscr{A}(\phi) \right|.$$

Proposition 5.1. *Consider the sum of clusters produced in the expansion* (5.20) *for some fixed* $\mathbf{h}_0$, $\mathbb{V}$, *and* $\mathscr{A}$. *Under the conditions of Theorem 2.4,*

$$\sum_{\mathbb{Z}} |\varrho(\mathbb{Z})|(c(L)\lambda)^{-t(\mathbb{Z})/2} e^{(1-3\delta/4)d(\mathbb{Z})} e^{\varepsilon|X_k(\mathbb{Z})|} \prod_f e^{\varepsilon L^3(\delta h(f))^2} \|\mathscr{A}(X_k(\mathbb{Z}))\|^{-1} \leqq c\,. \quad (5.26)$$

We obtain in Sect. 8 a bound

$$|Z(0,\mathbf{a})|^{-1} \leqq 1 + c\lambda\,, \quad (5.27)$$

and so estimates like (5.26) will hold for $\tilde{\varrho}(\mathbb{Z})$. The ratios of partition functions in (5.24) will be estimated by a very mild surface term $\exp\left(e^{-cL^3\eta^2\beta^{-1}}\left|\bigcup_r X_k(\mathbb{X}_r)\right|\right)$. The surface effect is absent in the bound of Sect. 4 and arises from the constraints $\widehat{\partial V_i} \neq \emptyset$ and $i(\sigma)=n$ in each $\mathbb{V}_i$. It is controlled by the expansion (5.20) applied appropriately in an inductive argument. In the process we obtain bounds on the "external" ratio of partition functions in

$$\langle \mathscr{A} \rangle_\Lambda^\phi = \frac{\tilde{Z}(0,\Lambda,\mathscr{A})}{\tilde{Z}(0,\Lambda)} = \sum_{\{\mathbb{X}_r\}} \Xi(\{\mathbb{X}_r\}) \frac{\prod\limits_{V_i \subseteq \Lambda \setminus \bigcup_r X_k(\mathbb{X}_r)} \tilde{Z}(0,\mathbb{V}_i)}{\tilde{Z}(0,\Lambda)}\,, \quad (5.28)$$

bounding the ratio by $\exp\left(c(L)\lambda\left|\bigcup_r X_k(\mathbb{X}_r)\right|\right)$.

6. Estimation of the Expansion

We wish to prove Proposition 5.1, assuming some bounds on parts of $\varrho(\mathbb{Z})$. Our first task is to bound the sum over $\mathbb{Z}$ in (5.26) by a supremum with appropriate combinatoric coefficients included. Define

$$\varrho_1(\mathbb{Z}) = |\varrho(\mathbb{Z})|(c(L)\lambda)^{-t(\mathbb{Z})/2} e^{(1-3\delta/4)d(\mathbb{Z})} e^{\varepsilon|X_k(\mathbb{Z})|} \prod_f e^{\varepsilon L^3(\delta h(f))^2} \|\mathscr{A}(X_k(\mathbb{Z}))\|^{-1}\,. \quad (6.1)$$

We adopt the convention that ε and c denote small and large constants, respectively, and different appearances can denote different constants. The size of c depends only on the constants δ, C_1, C_2, c_1, c_2, etc. appearing in Theorems 2.2–2.4, and not on λ, β/l_D, L, or L'. Dependence on one of these parameters will be written explicitly, as in $c(L)$. One can take ε small, but not depending on λ, β/l_D, L, or L', so that $\varepsilon^{-1} \leqq c$. Recall the discussion of $L(\{a_\alpha\})$ in (2.8)–(2.10). Let $\{a_{l\alpha_1}\}$ and $\{a_{l\alpha_2}\}$ be the subsets of $\{a_{l\alpha}\}$ involved in the trees $\mathscr{E}_{t_l}$, $\mathscr{E}_{t_l'}$, respectively.

Proposition 6.1.

$$\sum_{\mathbb{Z}} \varrho_1(\mathbb{Z}) \leqq \sup_{\mathbb{Z}} \varrho_1(\mathbb{Z}) e^{\varepsilon|X_k(\mathbb{Z})|} c^{t(\mathbb{Z})+1} \prod_f e^{\varepsilon L^3(\delta h(f))^2} e^{\varepsilon d(\mathbb{Z})}$$
$$\cdot \prod_{l=2}^{k} \left[s_{\eta(l)} \cdots s_{l-2} \left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_1}\})} \right) \left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_2}\})} \right) \right]^{-1}. \quad (6.2)$$

Proof. Write the sum over $\mathbb{Z}$ as

$$\sum_k \sum_{Y_1, h \restriction Y_1} \int ds_1 \sum_{\tau_2} \sum_{t_2, t_2'} \sum_{\{a_{2\alpha}\}} \sum_{Y_2, h \restriction Y_2} \int ds_2 \sum_{\eta(3)=1}^{2} \sum_{\tau_3} \cdots \sum_{Y_k, h \restriction Y_k} \quad (6.3)$$

with each summation variable compatible with the ones on its left. We convert sums into supremums from left to right using the identity $\sum_T f(T) \leqq \sup_T C_T f(T)$, valid for $f(T)$, $C_T \geqq 0$ when

$$\sum_T C_T^{-1} \leqq 1 . \tag{6.4}$$

For the sum over k we put

$$C_k = c^k , \tag{6.5}$$

and (6.4) is satisfied.

For the sum over $H_l = (Y_l, h \upharpoonright Y_l)$ we put

$$C_{H_l} = \prod_{f \subseteqq Y_l} e^{\varepsilon L^3 (\delta h(f))^2} (1 + c e^{-\varepsilon L^3 \eta^2 \beta^{-1}})^{|Y_l|/L^3} . \tag{6.6}$$

Consider the case $l=1$. We first want to sum over T_1, or equivalently the cube a_c closest to a_1 in the elementary region of largest diameter surrounding a_1. By (3.6) the smallest value of $\delta h(f)$ is $2^{-3/2} \eta \beta^{-1/2}$. Therefore

$$\begin{aligned} \sum_{a_c} \prod_{f \subseteqq Y_l} e^{-\varepsilon L^3 (\delta h(f))^2} &\leqq 1 + \sum_{a_c \neq a_1} e^{-\varepsilon L^3 \eta^2 \beta^{-1} (\mathrm{dist}(a_1, a_c) + 1)} \\ &\leqq 1 + c e^{-\varepsilon L^3 \eta^2 \beta^{-1}} . \end{aligned} \tag{6.7}$$

A similar argument bounds the sum over the L-lattice cube Ω_0 farthest from a_1 whose boundary contains a discontinuity in h. The cube Ω_0 contains or borders on regions where $h(x) = h_0(\sigma_0(V))$ because this holds at the outer boundary of Y_1. Therefore we can sum over $h(\Omega_0)$ using

$$\sum_{h(\Omega_0)} e^{-\varepsilon L^3 (\delta h(f))^2} \leqq 1 + 2 \sum_{n=1}^{\infty} e^{-\varepsilon L^3 \eta^2 \beta^{-1} n/8} \leqq 1 + c e^{-\varepsilon L^3 \eta^2 \beta^{-1}} . \tag{6.8}$$

The term 1 in (6.8) corresponds to no discontinuity (in which case Ω_0 is the first L-cube in a_1). We continue applying (6.8) to L-cubes adjacent to regions where h has been fixed. When h has been fixed in the cube containing Ω_0, it determines a minimal region that must be contained in Y. The region consists of all cubes less than L' from discontinuities in h. We continue summing over h in new regions using (6.8) and expanding the minimal region to accomodate discontinuities. Eventually the region will contain $T_1 \cap V$. The process stops when h has been fixed in a region such that no cubes in V are less than L' from discontinuities. At this point Y_1 and $h \upharpoonright Y_1$ are determined. Gathering the estimates yields

$$\sum_{Y_1, h \upharpoonright Y_1} \prod_{f \subseteqq Y_1} e^{-\varepsilon L^3 (\delta h(f))^2} (1 + c e^{-\varepsilon L^3 \eta^2 \beta^{-1}})^{-(|Y_1|/L^3 + 2)} \leqq 1 . \tag{6.9}$$

The corresponding estimate for $l > 1$ is easier, because T_l is determined by the $\{a_{l\alpha}\}$ and we can take Ω_0 to be any cube in T_l bordering on X_{l-1} or on ∂V. The estimate (6.4) follows from (6.6) and (6.9).

We next consider sums over $T_l = (s_{l-1}, \eta(l))$. Define

$$A_{l-1} = \sum_{\alpha=1}^{l-1} \varepsilon s_\alpha \dots s_{l-2} |Y_\alpha| , \tag{6.10}$$

$$C_{T_l}^{-1} = e^{(s_{l-1} - 1) A_{l-1}} \varepsilon s_{\eta(l)} \dots s_{l-2} |Y_{\eta(l)}| . \tag{6.11}$$

Then we verify (6.4):

$$\sum_{T_l} C_{T_l}^{-1} = \int ds_{l-1} \sum_{\eta(l)=1}^{l-1} e^{(s_{l-1}-1)A_{l-1}} \varepsilon s_{\eta(l)} \dots s_{l-2} |Y_{\eta(l)}|$$
$$= \int ds_{l-1} e^{(s_{l-1}-1)A_{l-1}} A_{l-1}$$
$$= 1 - e^{-A_{l-1}} \leqq 1 . \tag{6.12}$$

We can take $C_{\tau_l}=5$, $C_{t_l'}=c^{t_l'}$, $C_{t_l}=c^{t_l}$, and (6.4) will hold for the τ_l, t_l, and t_l' sums.

Let $d_l(\mathbb{Z})$ denote the part of $d(\mathbb{Z})$ arising from differentiation with respect to s_{l-1}. We wish to bound the sum over $\{a_{l\alpha}\}$ using

$$C_l^{-1} = \frac{e^{-\varepsilon d_l(\mathbb{Z})}}{c^{t_l+t_l'}|Y_{\eta(l)}|}\left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_1}\})}\right)\left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_2}\})}\right) \tag{6.13}$$

as combinatoric coefficient. (Some of these factors may be omitted, depending on τ_l.) The sums over the $\delta/\delta\psi$ cubes $a_{l\bar\alpha_1}$, $a_{l\bar\alpha_2}$ are handled by the estimate

$$\sum_{a_{l\bar\alpha_1}, a_{l\bar\alpha_2}} e^{-\varepsilon d_l(\mathbb{Z})} \leqq c|Y_{\eta(l)}| , \tag{6.14}$$

which holds because $d_l(\mathbb{Z})$ has a contribution from a path from $a_{l\bar\alpha_1}$ to $\partial Y_{\eta(l)}$ to $a_{l\bar\alpha_2}$. If we are dealing with the dE/ds_{l-1} term in (5.14), then we use the fact that $\{a_{l\alpha}\}$ are partitioned by $\gamma_{\eta(l)}$ but not by $\gamma_{\eta(l)-1}$ to show that the tree connecting the $\{a_{l\alpha}\}$ must intersect $\partial Y_{\eta(l)}$. Thus we can sum over one cube on each side of $\partial Y_{\eta(l)}$ using (6.14). The remaining sums over the $a_{l\alpha}$ can be estimated by using (2.9). Putting these estimates together yields (6.4).

We need only gather the coefficients to complete the estimation. In (6.5) we note that $k \leqq t(\mathbb{Z})+1$; in (6.6) we note that

$$\prod_{l=1}^{k} (1+ce^{-\varepsilon L^3 \eta^2 \beta^{-1}})^{|Y_l|/L^3} \leqq e^{\varepsilon |X_k(\mathbb{Z})|} . \tag{6.15}$$

The factors $|Y_{\eta(l)}|$ cancel between (6.11) and (6.13). Finally, we observe some cancellations between coefficients in (6.11),

$$\prod_{l=2}^{k} e^{(1-s_{l-1})A_{l-1}} = \exp\left(\sum_{l=2}^{k}\sum_{\alpha=1}^{l-1} \varepsilon(s_\alpha \dots s_{l-2} - s_\alpha \dots s_{l-1})|Y_\alpha|\right)$$
$$= \exp\left(\sum_{\alpha=1}^{k-1} \varepsilon(1-s_\alpha \dots s_{k-1})|Y_\alpha|\right)$$
$$\leqq e^{\varepsilon|X_k(\mathbb{Z})|} , \tag{6.16}$$

completing the proof of the proposition. □

Proposition 6.2.

$$\sup_{\mathbb{Z}} |\varrho(\mathbb{Z})| (c(L)\lambda)^{-t(\mathbb{Z})/2} e^{(1-2\delta/3)d(\mathbb{Z})} e^{\varepsilon|X_k(\mathbb{Z})|} \prod_f e^{\varepsilon L^3(\delta h(f))^2}$$
$$\cdot \|\mathscr{A}(X_k(\mathbb{Z}))\|^{-1} \prod_{l=2}^{k}\left[s_{\eta(l)} \dots s_{l-2}\left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_1}\})}\right)\left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_{l\alpha_2}\})}\right)\right]^{-1} \leqq c . \tag{6.17}$$

Comparison of (6.17) with (6.2), (6.1) yields Proposition 5.1.

Proof. Recall the formulae (5.17), (5.18) for $\varrho(\mathbb{Z})$. Let us rewrite $R(X_k)$ using g_c, defined by

$$g_c - h_0^l = C_{L\partial V}(h - h_0^l), \tag{6.18}$$

where $C_{L\partial V}^{-1} = u_{L\partial V}^{-1} + 1$ and $h_0^l = h_0^l(\mathbf{h}_0(V), V)$. By (3.37), we have

$$\begin{aligned} R(X_k) &= -\tfrac{1}{2} \int_{X_k} (g-h)^2 - \tfrac{1}{2} \int_{X_k} (g-h_0^l) u_{L\partial V}^{-1}(g-h_0^l) - \int_{X_k} \psi(g-h+u_{L\partial V}^{-1}(g-h_0^l)) \\ &= -\tfrac{1}{2} \int_{X_k} [(g-h)^2 + g u_0^{-1} g] - \int_{X_k} \psi C_{L\partial V}^{-1}(g-g_c) \\ &= -F_1 - F_2 \,. \end{aligned} \tag{6.19}$$

Our construction of g insures that $g \to h_0^l$ appropriately at $L\partial V$, so that $g - h_0^l \in D(u_{L\partial V}^{-1})$ and the replacement of $u_{L\partial V}$ with u_0 is justified.

We now proceed with a series of estimates, some of which will be proven in later sections. Combining them will yield the proposition. We have

$$F_1 \geqq cL^3 \sum_f (\delta h(f))^2 \,, \tag{6.20}$$

because $\int g u_0^{-1} g \geqq \int |\nabla g|^2$ (see the proof of Lemma 5.2 in [1]).

The other term in $R(X_k)$ satisfies the estimate

$$\|e^{-F_2}\|_{L^{p_1}(d\mu_s)} \leqq e^{\varepsilon F_1} \,, \tag{6.21}$$

where p_1 can be large. The proof of (6.21) is similar to the one in [3]. The left-hand side is calculated as

$$\exp(\tfrac{1}{2} p_1 \int (g-g_c) C_{L\partial V}^{-1} C(s) C_{L\partial V}^{-1}(g-g_c)) \leqq \exp(c \int (C_{L\partial V}^{-1}(g-g_c))^2) \,,$$

using $\int |C(s,x,y)| \, dy \leqq c$ (see Sect. 10). Where $\chi_\beta = 0$, we have $g = h$ and $C_{L\partial V}^{-1}(g-g_c) = 0$. In $\mathscr{I}_\beta$ we are away from $L\partial V \backslash \Gamma_\beta$ and $h_\beta^e = h$ (recall that $\Gamma_\beta = L\partial V \cap \mathscr{I}_\beta$). Thus where $\chi_\beta = 1$, we have $g - h_0^l = \tilde{g}_\beta - h_0^l = C_{\Gamma_\beta}(h_\beta^e - h_0^l)$, and again $C_{L\partial V}^{-1}(g-g_c) = 0$. Thus the integral on the right is supported where $\chi_\beta \notin \{0,1\}$ for some β. Call this set $\tilde{B}$. At $y \in \tilde{B} \cap \mathscr{I}_\beta$ we note that $\Gamma_\beta = \emptyset$ if $h(y) \neq h_0^l$ so that

$$C_{L\partial V}^{-1}(g-g_c) = C_{L\partial V}^{-1} \chi_\beta C_{\Gamma_\beta}(h_\beta^e - h(y)) \,.$$

Note that $h_\beta^e(x) = h(y)$ unless $\mathrm{dist}(x,y) \geqq L'/8$. In Sect. 10 we bound $|\partial_y^\alpha C_{\Gamma_\beta}(y,x)|$ by $ce^{-|x-y|/2}$ for $|x-y| \geqq L'/8$, $y \in \tilde{B}$, $|\alpha| \leqq 4$. Thus we can apply the derivatives in $C_{L\partial V}^{-1}$ to obtain

$$\int_{\mathscr{I}_\beta} (C_{L\partial V}^{-1}(g-g_c))^2 \leqq c \int_{\mathscr{I}_\beta} dy (\int dx e^{-|x-y|/2} |h_\beta^e(x) - h(y)|)^2 \,.$$

The right-hand side can be bounded by $e^{-cL'} \sum_{f \subseteq \mathscr{I}_\beta} (\delta h(f))^2$ as in [3]. By summing over β, taking L' large, and using (6.20), we obtain (6.21).

We consider next the functional derivatives $\delta/\delta\psi$ in κ_l in (5.17). Each $\delta/\delta\psi$ in a cube a distinguishes a and counts in $N(a)$. Let $t(a)$ be the number of factors ε_i that are localized in a. The number of terms resulting from the application of all functional derivatives is bounded by

$$\prod_a (t(a) + N(a) + 4)^{N(a)} \leqq c^{t(\mathbb{Z})} \prod_a (cN(a))^{cN(a)} \,. \tag{6.22}$$

To control derivatives of F_2 we use the estimate

$$\prod_{i=1}^{n}\left(\int_{a_i}\left|\frac{\delta F_2}{\delta\psi}\right|\right) \leqq e^{\varepsilon F_1}\prod_a N(a)^{cN(a)}\lambda^n. \qquad (6.23)$$

This follows from the proof of (6.21), where we bounded $\int\left|\frac{\delta F_2}{\delta\psi}\right|^2$ by εF_1. The factor λ^n can be extracted because a_i must be within L' of a discontinuity in h. Thus a piece of F_1 is available for the bound $(\lambda N(a))^{-N(a)} \leqq e^{c/\lambda} \leqq e^{\varepsilon L^3\eta^2\beta^{-1}/L'^3}$. By the definition of $\|\mathscr{A}\|$ we have

$$\left|\prod_{i=1}^{n}\frac{\delta}{\delta\psi(x_i)}\mathscr{A}(X_k)\right| \leqq (c\lambda)^n\|\mathscr{A}(X_k)\|. \qquad (6.24)$$

We will prove in Sect. 9 the following estimates on functional derivatives of G_2 and E:

$$\begin{aligned}\left|\frac{\delta^n G_2}{\delta\psi^n}(x)\right| \leqq (c\beta)^{\max\{1/2,(n-2)/2\}}(1+\eta_{h(x)}\beta^{-1/2}+|\psi(x)|&\\ +|\delta(x)|+|g(x)-h(x)|)(1+|\delta(x)|)^2, \quad n\geqq 0,&\end{aligned} \qquad (6.25)$$

$$\begin{aligned}\left|\prod_{\alpha=1}^{t}\left(\frac{\delta}{\delta\psi(x_\alpha)}\right)_\alpha^{n_\alpha}\mathscr{E}_t(a_1,\ldots,a_t)\right| \leqq (c\lambda^2)^{t-1}e^{-(1-\delta/2)L(\{a_\alpha\})}\beta^{-1}\prod_{\alpha:n_\alpha>0}(c\beta)^{n_\alpha/2}&\\ \cdot\int\prod_{\substack{\text{distinguished}\,\alpha\\ n_\alpha=0}}[\beta^{1/2}(|\psi(y_\alpha)|+|g(y_\alpha)-h(y_\alpha)|&\\ +\eta_{h(y_\alpha)}\beta^{-1/2})dy_\alpha]f(\{y_\alpha\}).&\end{aligned} \qquad (6.26)$$

Here f is symmetric and $\|f\|_{L^1}\leqq 1$ (the norm taken with one variable fixed). The subscript α on $(\delta/\delta\psi(x_\alpha))^{n_\alpha}$ indicates that the operator acts only on ε_{i_α} in (5.10).

We bound the factors produced in (6.25) and (6.26) by taking a supremum over $\{y_\alpha\}$. For the factors $|g-h|$ we use

$$\prod_i|g(x_i)-h(x_i)| \leqq e^{\varepsilon F_1}\prod_a(cN(a)L^{-3})^{cN(a)}, \qquad (6.27)$$

which follows from the estimate $L^3(g(x)-h(x))^2\leqq cF_1(\Omega_x)$, where Ω_x is the L-lattice cube containing x. The factors $|\psi|$ and $|\delta|$ are controlled using

$$\left\|\prod_{i=1}^{b}(|\psi(x_i)|+|\delta(x_i)|)\right\|_{L^{p_2}(d\mu_s)} \leqq e^{\varepsilon F_1}\left(\frac{c}{\lambda}\right)^{b/2}\prod_a(cN(a)L^{-1})^{cN(a)}, \qquad (6.28)$$

with p_2 large and even. This is proven using

$$\begin{aligned}(|\psi(x)|+|\delta(x)|)^2 &\leqq c\left(\psi(x)^2+\left(L^{-3}\int_{\Omega_x}\psi\right)^2+\left(g(x)-L^{-3}\int_{\Omega_x}g\right)^2\right)\\ &\leqq c\left(\psi(x)^2+L^{-3}\int_{\Omega_x}\psi^2+L^{-1}\int_{\Omega_x}(\nabla g)^2\right).\end{aligned}$$

The last factor is less than $cL^{-1}F_1(\Omega_x)$; the other factors are handled by the usual Gaussian integration estimates. We need $|C(s,x,y)|\leqq c\lambda^{-1}e^{-|x-y|/2}$, which is implied by (6.34) below. The factors $\eta_{h(x_i)}\beta^{-1/2}$ are produced because we are not quite at a stationary point of the action when $h(x) \neq 0$. Hence large coefficients are possible, but they are compensated by a vacuum energy term as follows:

$$\prod_i (\eta_{h(x_i)}\beta^{-1/2}) \exp\left(\sum_{\Omega_\alpha \subsetneq X_k} -\varepsilon L^3 \eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \leq \prod_a (cN(a)L^{-3})^{cN(a)}. \qquad (6.29)$$

We now state the main vacuum energy estimates to be proven in Sects. 7 and 8. The main estimate produces the energy factors used in (6.29) and controls the remaining functional derivatives. With p_3 near 1 we have

$$\left\| e^{G_2(X_k)} \prod_{i=1}^{n} \frac{\delta}{\delta\psi(x_i)} e^{G_1(X_k)} \right\|_{L^{p_3}(d\mu_s)}$$
$$\leq (\beta^{1/6}e^{c/\lambda})^n e^{3F_1/4} \prod_a N(a)^{cN(a)} \exp\left(\sum_{\Omega_\alpha \subsetneq X_k} -cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \exp(\beta^{1/6}e^{c/\lambda}|X_k|). \qquad (6.30)$$

The nonlocal interaction terms are estimated using

$$\|e^{E(X_k, s_1, \ldots, s_{k-1})}\|_{L^{p_4}(d\mu_s)} \leq e^{c\lambda|X_k|} e^{\varepsilon F_1} \exp\left(\sum_{\Omega_\alpha \subsetneq X_k} \varepsilon L^3 \eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \qquad (6.31)$$

with p_4 large.

The various number divergences produced above are handled with exponential pinning:

$$\prod_a (cN(a)L^{-3})^{cN(a)} \leq c(L)^{t(\mathbb{Z})} e^{\varepsilon d(\mathbb{Z})}. \qquad (6.32)$$

This is proven by noting that there must be $N(a)/2$ lines in $d(\mathbb{Z})$ starting at a and going a distance at least $[(N(a)-3)/2]^{1/4}$. Hence a factor $\exp(-\varepsilon N(a)^{5/4})$ is available in $e^{-\varepsilon d(\mathbb{Z})}$ for each a, and (6.32) follows by noting that the total number of distinguished cubes is bounded by $2t(\mathbb{Z})$.

The factor $\exp(-\varepsilon N(a)^{5/4})$ also beats the volume divergence associated with the region around a of size $cN(a)^{3/4}$ included in X_k. There is also a contribution to X_k of size $cd(\mathbb{Z})$ from the cubes touching the trees and paths in $d(\mathbb{Z})$. Finally, we have contributions from cubes within L' of discontinuities in h. Altogether we can estimate the volume divergences by

$$\exp[(\varepsilon + c(L)\lambda + \beta^{1/6}e^{c/\lambda})|X_k|]e^{-\varepsilon d(\mathbb{Z})}e^{-\varepsilon F_1} \leq c^{t(\mathbb{Z})+1}. \qquad (6.33)$$

We need the following estimate on differences of covariances, proven in Sect. 10:

$$|\delta_{\gamma_{\eta(l)}} \ldots \delta_{\gamma_{l-1}} C_{\gamma_1 \cup \ldots \cup \gamma_{\eta(l)-1} \cup \partial V}(x, y)| \leq c\lambda^{-1} e^{-(1-\delta/2)\mathrm{dist}(x, \partial Y_{\eta(l)}, y)}. \qquad (6.34)$$

Here $\mathrm{dist}(x, \partial Y_{\eta(l)}, y)$ is the length of the shortest path from the cube containing x to $\partial Y_{\eta(l)}$ to the cube containing y. The remaining convergence in $d(\mathbb{Z})$ comes from the $L(\{a_\alpha\})$ in (6.26), using the estimate

$$e^{-(1-\delta/2)L(\{a_\alpha\})} e^{(1-2\delta/3+2\varepsilon)d_T(\{a_\alpha\})} \left(\sum_{\eta^A} b_{\eta^A} e^{-\varepsilon L_{\eta^A}(\{a_\alpha\})}\right)^{-1} \leq 1. \qquad (6.35)$$

Here $d_T(\{a_\alpha\})$ is the length of the shortest tree connecting $\{a_\alpha\}$. The bound (6.35) is easily proven using $d_T(\{a_\alpha\}) \leq L_{\eta^A}(\{a_\alpha\})$ and (2.8) with $\alpha = 1 - \delta/2$.

The proposition now follows from (6.20)–(6.35). We split up the $d\mu_s$ integral using Hölder's inequality, and each part is estimated above. The factors $s_{\eta(l)} \ldots s_{l-2}$

cancel between (6.17) and (5.17). By (6.26), we have factors λ^{2t-2} for each $\mathscr{E}_t$. Each distinguished ε_i in $\mathscr{E}_t$ can come with a factor $|\psi|$ or it can be functionally differentiated; in each case we must allow a factor $\lambda^{-1/2}$ from (6.28) or (6.34). Altogether we have a factor $\lambda^{(3t-4)/2} \leqq \lambda^{t/2}$ for each $\mathscr{E}_t$. There must be at least $\delta(\mathbb{Z}) - \sum_{l=2}^{k} (t_l + t_l')$ other functional derivatives which hit F_2, $\mathscr{A}$, G_2, G_1, or some ε_i a second time. These get factors $\lambda^{-1/2}$ from (6.34) and factors λ, $\beta^{1/2}$, or $\beta^{1/6}$ from (6.23)–(6.26), (6.30). [The first derivative of ε_i produces a $\beta^{1/2}$, but we use it to cancel the β^{-1} in (6.26).] Extra powers of $|\psi|$ or $|\delta|$ (and hence $\lambda^{-1/2}$) are generated in (6.25) but these are associated with a $\beta^{1/2}$. Overall an extra $\delta(\mathbb{Z}) - \sum_{l=2}^{k} (t_l + t_l')$ factors of $\lambda^{1/2}$ are produced and thus a $\lambda^{t(\mathbb{Z})/2}$ is available, as required by (6.17). All other factors are accounted for in (6.20)–(6.35), and Proposition 6.2 is proven. □

7. Derivatives of $r(A)$

In this section we use Cauchy's integral formula to estimate derivatives of $r(A)$, defined in (3.28).

Lemma 7.1. *For any $h \in \mathscr{H}$,*

$$\left|\frac{d^N}{dA^N} r(A)\right| \leqq c N^{cN} \beta^{N/6} e^{-cL^3\eta_h^2\beta^{-1}} e^{L^3(A-h)^2/8}. \tag{7.1}$$

Proof. As in [3] we write

$$r(x+iy) = \frac{\exp(L^3 \sum \varrho_i(e^{i\beta^{1/2}e_i x} - 1)) \exp(L^3 \sum \varrho_i e^{i\beta^{1/2} e_i x}(e^{-\beta^{1/2}e_i y} - 1))}{\exp(L^3 y^2/2) \sum_{h' \in \mathscr{H}} \exp(-L^3[(x-h')^2 + 2iy(x-h')]/2)}$$
$$= \frac{\mathrm{I} \cdot \mathrm{II}}{\mathrm{III} \cdot \mathrm{IV}}. \tag{7.2}$$

Consider first the region

$$|x-h| \leqq 2\beta^{-1/6}, \quad |y| \leqq \min\{L^{-3}\eta_h^{-2}\beta^{1/2}, \beta^{-1/6}\}. \tag{7.3}$$

By Lemma 3.1 we have

$$|\mathrm{I}| \leqq e^{-L^3\eta_h^2\beta^{-1}} e^{-\gamma L^3(x-h)^2}. \tag{7.4}$$

We expand in x and y, use $\sum \varrho_i e_i = 0$ and $(\mathrm{Re}\, S)'(h) = 0$ to obtain

$$\begin{aligned}
|\mathrm{II}| &\leqq \exp(L^3 \sum \varrho_i(e^{-\beta^{1/2}e_i y} - 1) + L^3 \sum \varrho_i(\cos\beta^{1/2}e_i x - 1)(e^{-\beta^{1/2}e_i y} - 1)) \\
&\leqq \exp(L^3 \sum \varrho_i \beta e_i^2 y^2/2 - L^3 \sum \varrho_i \beta^{3/2} e_i^3 \hat{y}^3 e^{-\beta^{1/2}e_i\hat{y}}/6 \\
&\quad + L^3 \sum \varrho_i(1 - \cos\beta^{1/2}e_i x) c\beta^{1/2}|y|) \\
&\leqq \exp(L^3 y^2/2 + c + cL^3\eta_h^2\beta^{-1/2}|y| + cL^3 \sum \varrho_i \beta^{3/2} e_i^2 |y| (\hat{x} - h)^2) \\
&\leqq c e^{L^3y^2/2} \leqq c|\mathrm{III}|,
\end{aligned} \tag{7.5}$$

for some $\hat{y}$ between 0 and y and some $\hat{x}$ between x and h. Only the $h'=h$ term contributes significantly in IV, so we have

$$|\mathrm{IV}|^{-1} \leqq c e^{L^3(x-h)^2/2}. \tag{7.6}$$

Thus

$$|r(x+iy)| \leqq c e^{-L^3\eta_h^2\beta^{-1}} e^{(1-2\gamma)L^3(x-h)^2/2}, \tag{7.7}$$

and (7.1) follows for $|A-h| \leqq \beta^{-1/6}$ if we note that $1-2\gamma$ is small and

$$(L^3\eta_h^2\beta^{-1/2})^N e^{-L^3\eta_h^2\beta^{-1/2}} \leqq cN^{cN}\beta^{N/2}. \tag{7.8}$$

Next we consider the region

$$x \in I_{h''}, \quad |x-h| \geqq \beta^{-1/6}/2, \quad |y| \leqq L^{-3}\eta^{-1}\beta^{1/2}/16, \tag{7.9}$$

where the following bounds hold:

$$\begin{aligned} |\mathrm{I}| &\leqq e^{-L^3\eta_{h''}^2\beta^{-1}} e^{-\gamma(x-h'')^2}, \\ |\mathrm{II}| &\leqq \exp(cL^3 \textstyle\sum \varrho_i \beta |e_i| \leqq c, \\ |\mathrm{III}|^{-1} &\leqq c, \\ |\mathrm{IV}|^{-1} &\leqq c e^{L^3(x-h'')^2/2}. \end{aligned} \tag{7.10}$$

For the last bound we used the bound on $|y|$ to show that all terms in the sum comparable to the h'' term are correlated in phase. Thus all but the h'' term can be neglected for an upper bound. Altogether we have

$$\begin{aligned} |r(x+iy)| &\leqq c e^{-L^3\eta_{h''}^2\beta^{-1}} e^{(1-2\gamma)L^3(x-h'')^2/2} \\ &\leqq c e^{-cL^3\eta_h^2\beta^{-1}} e^{L^3(x-h)^2/16} \\ &\leqq c e^{-cL^3\eta_h^2\beta^{-1}} e^{L^3(x-h)^2/8} e^{-cL^3\beta^{-1/3}}. \end{aligned} \tag{7.11}$$

The transition from h'' to h in the second inequality was made using (3.5), (3.6), assuming γ is close to $\frac{1}{2}$ and the coefficient of $L^3\eta_h^2\beta^{-1}$ is fairly small. The resulting Cauchy estimate for $|A-h| \geqq \beta^{-1/6}$ is

$$\left|\frac{d^N}{dA^N} r(A)\right| \leqq c e^{-cL^3\eta_h^2\beta^{-1}} e^{L^3(A-h)^2/8} N! \beta^{N/6} (cL^3\eta\beta^{-2/3})^N e^{-cL^3\beta^{-1/3}}, \tag{7.12}$$

and (7.1) follows immediately. □

We use Lemma 7.1 to obtain more precise estimates on $r(A)$ with no derivatives.

Lemma 7.2. *For any* $h \in \mathscr{H}$,

$$|r(A)| \leqq e^{-cL^3\eta_h^2\beta^{-1}}(1 + cL^{-3/2}\beta^{1/6} e^{L^3(A-h)^2/6}). \tag{7.13}$$

In addition,

$$|r(A) - 1| \leqq cL^{-3/2}\beta^{1/6} e^{L^3A^2/6}. \tag{7.14}$$

Proof. The first bound is just the fundamental theorem of calculus:

$$|r(A)| \leqq \frac{\exp(L^3 \sum \varrho_i(e^{i\beta^{1/2}e_i h} - 1))}{\sum_{h' \in \mathscr{H}} \exp(-L^3(h-h')^2/2)} + \int_h^A c\beta^{1/6} e^{-cL^3\eta_h^2\beta^{-1}} e^{L^3(A'-h)^2/8} dA'$$

$$\leqq e^{-cL^3\eta_h^2\beta^{-1}}(1 + c\beta^{1/6}|A-h| e^{L^3(A-h)^2/8}). \quad (7.15)$$

For the second bound we take $h=0$, use the estimate

$$\left|1 - \sum_{h' \in \mathscr{H}} \exp(-L^3 h'^2/2)\right| \leqq e^{-cL^3\eta^2\beta^{-1}}, \quad (7.16)$$

and handle the interpolation term as above. □

8. Vacuum Energy Estimates

This section is devoted to proving (6.30), (6.31), and (5.27). We need to extract some of the small β and small λ behavior of the vacuum energy to avoid swamping the convergence factors of Sect. 6. Other new features arise from the Dirichlet boundary conditions, which prevent us from following [3] exactly.

Lemma 8.1. *Let p_3 be close to* 1. *Then*

$$\left\| e^{G_2(X_k)} \prod_{i=1}^n \frac{\delta}{\delta\psi(x_i)} e^{G_1(X_k)} \right\|_{L^{p_3}(d\mu_s)}$$
$$\leqq (\beta^{1/6} e^{c/\lambda})^n e^{3F_1/4} \prod_a N(a)^{cN(a)} \exp\left(\sum_{\Omega_\alpha \subseteq X_k} - cL^3 \eta^2_{h(\Omega_\alpha)} \beta^{-1}\right) \exp(\beta^{1/6} e^{c/\lambda} |X_k|). \quad (8.1)$$

Proof. Let D be the union of the L-lattice cubes containing x_i's. We apply Lemmas 7.1 and 7.2 to the factors $r(A)$ in e^{G_1} to obtain

$$\int \left| e^{G_2(X_k)} \prod_{i=1}^n \frac{\delta}{\delta\psi(x_i)} e^{G_1(X_k)} \right|^{p_3} d\mu_s(\psi)$$
$$\leqq \int |e^{p_3 G_2}| \prod_{\Omega_\alpha \subseteq D} (1 + cL^{-3/2}\beta^{1/6} e^{L^3(A(\Omega_\alpha) - h(\Omega_\alpha))^2/6})^{p_3}$$
$$\cdot \prod_{\Omega_\alpha \subseteq D} (c e^{p_3 L^3(A(\Omega_\alpha) - h(\Omega_\alpha))^2/6}) \exp\left(\sum_{\Omega_\alpha} - cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \prod_a N(a)^{cN(a)} \beta^{p_3 n/6}$$
$$\leqq \sum_{S \subseteq X_k \setminus D} \int |e^{p_3 G_2}| (cL^{-3/2}\beta^{1/6})^{|S|} \prod_{\Omega_\alpha \subseteq \bar{S}} e^{p_3 L^3 (A(\Omega_\alpha) - h(\Omega_\alpha))^2/6}$$
$$\cdot \exp\left(\sum_{\Omega_\alpha} - cL^3 \eta^2_{h(\Omega_\alpha)} \beta^{-1}\right) \prod_a N(a)^{cN(a)} (c\beta)^{p_3 n/6}. \quad (8.2)$$

We have used $(1+x)^p \leqq 1 + px(1+x)^{p-1}$ and expanded the product over $\Omega_\alpha \nsubseteq D$ into subsets S of $X_k \setminus D$ composed of L-lattice cubes. We take $\bar{S}$ to be the union of the unit lattice cubes intersecting D or S nontrivially.

The next step is to use the bound

$$|e^{p_3 G_2(a)}| \leqq 1 + \int p_3 |G_2(a) e^{t p_3 G_2(a)}| dt$$
$$\leqq 1 + c\beta^{1/2} \left[\int_a (1 + \eta_{h(x)} \beta^{-1/2} + |\psi(x)| + |\delta(x)| + |g(x) - h(x)|)(1 + |\delta(x)|)^2 dx\right]$$
$$\cdot \exp\left(\int_a 2p_3 \delta(x)^2 dx\right), \quad (8.3)$$

which follows from (6.25) and $|G_2(a)| \leqq \int_a 2\delta^2$. We expand the product over $a \subsetneqq X_k \setminus \bar{S}$ into a sum over subsets T of unit lattice cubes. Denote by $V(a)$ the quantity in square brackets. Noting that $(A-h)^2 \leqq (\psi+g-h)^2$, we have the following bound on (8.2):

$$\sum_{S \subsetneqq X_k \setminus D} \sum_{T \subsetneqq X_k \setminus \bar{S}} \int \exp\left[\int_{T \cup \bar{S}} (2p_3\delta^2 + p_3(\psi+g-h)^2/6)\right] (cL^{-3/2}\beta^{1/6})^{|S|}(c\beta)^{|T|/2}$$
$$\cdot \prod_{a \subsetneqq T} V(a) \exp\left(\sum_{\Omega_\alpha} -cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \prod_a N(a)^{cN(a)}(c\beta)^{p_3 n/6}. \tag{8.4}$$

The factors $V(a)$ can be handled separately with Hölder's inequality, still keeping the coefficient of $(\psi+g-h)^2/6$ close to unity. In each $a \subsetneqq T$, we pick up a factor $c\lambda^{-3/2}$ from the Gaussian integration (6.28). We prove below that

$$\int \exp\left[\int_{T \cup \bar{S}} (2q\delta^2 + q(\psi+g-h)^2/6)\right] d\mu_s(\psi) \leqq e^{3F_1/4} e^{c|T \cup \bar{S}|/\lambda}, \tag{8.5}$$

for $q > p_3$ near 1. Thus we can sum over S and T in (8.4) to obtain the estimate

$$\sum_{S \subsetneqq X_k \setminus D} \prod_{a \subsetneqq X_k \setminus \bar{S}} \left[1 + c\beta^{1/2}\lambda^{-3/2}\left(1 + \int_a |g-h| + \int_a \eta_h^2 \beta^{-1}\right) e^{c/\lambda}\right] e^{c|\bar{S}|/\lambda}$$
$$\cdot e^{3p_3F_1/(4q)} (cL^{-3/2}\beta^{1/6})^{|S|} \exp\left(\sum_{\Omega_\alpha} -cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \prod_a N(a)^{cN(a)}(c\beta)^{p_3 n/6}$$
$$\leqq \sum_{S \subsetneqq X_k \setminus D} \exp(\beta^{1/2} e^{c/\lambda} |X_k \setminus \bar{S}|) e^{c|\bar{S}|/\lambda} e^{3p_3F_1/4} (cL^{-3/2}\beta^{1/6})^{|S|} \tag{8.6}$$
$$\cdot \exp\left(\sum_{\Omega_\alpha} -cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \prod_a N(a)^{cN(a)}(c\beta)^{p_3 n/6}$$
$$\leqq \exp(\beta^{1/6} e^{c/\lambda} |X_k|) e^{3p_3F_1/4} \exp\left(\sum_{\Omega_\alpha} -cL^3\eta^2_{h(\Omega_\alpha)}\beta^{-1}\right) \prod_a N(a)^{cN(a)}(\beta^{1/6} e^{c/\lambda})^{p_3 n},$$

which proves the lemma.

To prove (8.5) note that $C(s)$ is a convex combination of covariances of the form

$$C_\Gamma = (\lambda^2 l_D^2(-\Delta_\Gamma)^2 - \Delta_\Gamma + 1)^{-1},$$

so by [3, Eq. (9.67)] it suffices to consider that case only. We use $(\psi+g-h)^2 \leqq 2\psi^2 + 2(g-h)^2$ to obtain

$$\int \exp\left(\int_{T \cup \bar{S}} (2q\delta^2 + q(\psi+g-h)^2/6)\right) d\mu(\psi) e^{-2qF_1(T \cup \bar{S})/3}$$
$$\leqq \int \exp\left(\int_{T \cup \bar{S}} 2q\delta^2 + q\psi^2/3)\right) d\mu(\psi) \exp\left(-\int_{T \cup \bar{S}} qgu_0^{-1}g/3\right). \tag{8.7}$$

Let P be the operator which projects out the constant component of a function in each L-lattice cube. Then

$$\delta^2 \equiv (P\phi)^2 = (P\psi + Pg)^2 \leqq 2(P\psi)^2 + 2(Pg)^2. \tag{8.8}$$

We have $u_0^{-1} \geqq -\Delta_N$ (Neumann boundary conditions on all L-lattice cubes) and $-\Delta_N \geqq cL^{-2}P$, so that

$$4q(Pg)^2 \leqq cL^2 g u_0^{-1} g. \tag{8.9}$$

Thus for small L we can estimate (8.7) by

$$\int \exp\left(\int_{T \cup \bar{S}} (4q(P\psi)^2 + q\psi^2/3\right) d\mu(\psi). \tag{8.10}$$

We now bound (8.10) by $e^{c|T\cup\bar{S}|/\lambda}$ as in [3, Appendix 3]. We have

$$\|C_\Gamma^{1/2}(8qP\chi_{T\cup\bar{S}}P+2q\chi_{T\cup\bar{S}}/3)C_\Gamma^{1/2}\| \leqq \|C_\Gamma^{1/2}(8qP+2q/3)C_\Gamma^{1/2}\|$$
$$\leqq \|C_\Gamma^{1/2}(cL^2(-\Delta_N)+2q/3)C_\Gamma^{1/2}\| \leqq 2q/3<1\,, \tag{8.11}$$

again with L small. In addition,

$$\operatorname{tr} C_\Gamma(8qP\chi_{T\cup\bar{S}}P+2q\chi_{T\cup\bar{S}}/3) \leqq c \operatorname{tr}(C_\Gamma\chi_{T\cup\bar{S}}) \leqq c|T\cup\bar{S}|/\lambda\,, \tag{8.12}$$

where we have used (6.34) for the estimate $C_\Gamma(x,x)\leqq c/\lambda$. This completes the proof of (8.5) and the lemma. □

Lemma 8.2. *Take* $p_4<\infty$. *Then*

$$\|e^{E(X_k,s_1,\ldots,s_{k-1})}\|_{L^{p_4}(d\mu_s)} \leqq e^{c\lambda|X_k|}e^{\varepsilon F_1}\exp\Big(\sum_{\Omega_\alpha\subseteqq X_k}\varepsilon L^3\eta_{h(\Omega_\alpha)}^2\beta^{-1}\Big)\,. \tag{8.13}$$

Proof. We use (6.26) with $n_i=0$ and two ε_i's distinguished for each $\mathscr{E}_t$ to bound

$$|E(X_k)| \leqq \sum_{t=2}^{\infty}\sum_{\{a_\alpha\}}(c\lambda^2)^{t-1}e^{-(1-\delta/2)L(\{a_\alpha\})}\int dy_1dy_2 f(y_1,y_2)$$
$$\cdot(|\psi(y_1)|+|g(y_1)-h(y_1)|+\eta_{h(y_1)}\beta^{-1/2})(|\psi(y_2)|+|g(y_2)-h(y_2)|+\eta_{h(y_2)}\beta^{-1/2})$$
$$\leqq c\lambda^2\int_{X_k}(\psi^2+(g-h)^2+\eta_h^2\beta^{-1})\,. \tag{8.14}$$

We have used [3, Lemma 9.6] and the fact that the norm of the integral operator associated to f is bounded by $\|f(y_1,\cdot)\|_{L^1}\leqq 1$. The second and third terms are bounded by $\varepsilon F_1+\sum\varepsilon L^3\eta_{h(\Omega_\alpha)}^2\beta^{-1}$. For the first term we use

$$\int\exp\Big(\int cp_4\lambda^2\psi^2\Big)d\mu_s(\psi)\leqq e^{c\lambda|X_k|}\,, \tag{8.15}$$

which follows from

$$\begin{gathered}\|C(s)^{1/2}2cp_4\lambda^2\chi_{X_k}C(s)^{1/2}\|<1\,,\\ \operatorname{tr}(C(s)2cp_4\lambda^2\chi_{X_k})\leqq c\lambda\,,\end{gathered} \tag{8.16}$$

using again $|C(s,x,y)|\leqq c/\lambda$. This completes the proof. □

Proposition 8.3.

$$|Z(0,\mathbf{a})|^{-1}\leqq 1+c\lambda\,. \tag{8.17}$$

Proof. We have

$$Z(0,\mathbf{a})=Z_0(0,a)=\int e^{G_2(a)}e^{G_1(a)}e^{E(a)}d\mu_{\partial a}(\phi)\,. \tag{8.18}$$

Use the bound of Lemma 7.2 on $|r(A)-1|$ to show that

$$|Z(0,\mathbf{a})-\int e^{G_2(a)}e^{E(a)}d\mu_{\partial a}(\phi)|$$
$$\leqq \sum_{S\subseteqq a,\,S\neq\emptyset}\int|e^{G_2(a)}e^{E(a)}|(cL^{-3/2}\beta^{1/6})^{|S|}\prod_{\Omega_\alpha\subseteqq S}e^{L^3A(\Omega_\alpha)/6}d\mu_{\partial a}(\phi)$$
$$\leqq cL^{-9/2}\beta^{1/6}\int|e^{E(a)}|\exp\Big(\int_a 2\delta^2+\phi^2/6\Big)d\mu_{\partial a}(\phi)$$
$$\leqq cL^{-9/2}\beta^{1/6}e^{c\lambda}e^{c/\lambda}\leqq c\lambda\,. \tag{8.19}$$

We have used Hölder's inequality, Lemma 8.2, and (8.5). Next we use (6.25) to show that

$$|G_2(a)| \leqq c\beta^{1/2} \int_a (1+|\phi|)^3 , \tag{8.20}$$

and hence (6.28), Lemma 8.2, and (8.5) imply that

$$\begin{aligned} &\left|\int e^{G_2(a)} e^{E(a)} d\mu_{\partial a}(\phi) - \int e^{E(a)} d\mu_{\partial a}(\phi)\right| \\ &\quad \leqq c\beta^{1/2} \sup_{x\in a} \int (1+|\phi(x)|)^3 |e^{E(a)}| \exp\left(\int_a 2\delta^2\right) d\mu_{\partial a}(\phi) \\ &\quad \leqq c\beta^{1/2}\lambda^{-3/2} e^{c\lambda} e^{c/\lambda} \leqq c\lambda . \end{aligned} \tag{8.21}$$

Finally we use (8.14), (8.15) to estimate

$$\begin{aligned} \left|\int e^{E(a)} d\mu_{\partial a}(\phi) - 1\right| &\leqq \int c\lambda^2 \left(\int_a \psi^2\right) \exp\left(c\lambda^2 \int_a \psi^2\right) d\mu_{\partial a}(\phi) \\ &\leqq c\lambda e^{c\lambda} \leqq c\lambda . \end{aligned} \tag{8.22}$$

The proposition follows from (8.19), (8.21), and (8.22). □

9. Functional Derivatives

We prove (6.25) and (6.26). Functional derivatives of G_2 and E need not be small when $h \neq 0$. However, we can bound them by small factors times functions involving $\eta_h \beta^{-1/2}$, which can be controlled by the corresponding vacuum energy term.

Lemma 9.1. *For any* $n \geqq 0$,

$$\begin{aligned} \left|\frac{\delta^n G_2}{\delta \psi^n}(x)\right| &\leqq (c\beta)^{\max\{1/2,(n-2)/2\}} (1+\eta_{h(x)}\beta^{-1/2} + |\psi(x)| + |\delta(x)| \\ &\quad + |g(x)-h(x)|)(1+|\delta(x)|)^2 . \end{aligned} \tag{9.1}$$

Proof. We have by (3.29)

$$G_2 = \int \sum_{i=1}^{s} \varrho_i (G_a^i(x) G_b^i(x) + G_c^i(x)) dx , \tag{9.2}$$

where

$$\begin{aligned} G_a^i &= e^{i\beta^{1/2} e_i A} - 1 , \\ G_b^i &= e^{i\beta^{1/2} e_i \delta} - i\beta^{1/2} e_i \delta - 1 , \\ G_c^i &= e^{i\beta^{1/2} e_i \delta} + \tfrac{1}{2}\beta e_i^2 \delta^2 - i\beta^{1/2} e_i \delta - 1 . \end{aligned} \tag{9.3}$$

Since $A = \psi + g - \delta$, we have

$$\begin{aligned} |G_a^i| &= |(e^{i\beta^{1/2} e_i(\psi+g-h-\delta)} - 1) e^{i\beta^{1/2} e_i h} + e^{i\beta^{1/2} e_i h} - 1| \\ &\leqq \beta^{1/2} |e_i| (|\psi| + |g-h| + |\delta|) + \sqrt{2}\sqrt{1-\cos \beta^{1/2} e_i h} . \end{aligned} \tag{9.4}$$

Other possible derivatives are easily estimated as follows:

$$\left|\frac{\delta^n}{\delta\psi^n} G_a^i\right| \leqq \beta^{n/2}|e_i|^n, \quad n \geqq 1,$$

$$|G_b^i| \leqq \beta e_i^2 \delta^2/2, \quad \left|\frac{\delta}{\delta\psi} G_b^i\right| \leqq \beta e_i^2 \delta,$$

$$\left|\frac{\delta^n}{\delta\psi^n} G_b^i\right| \leqq \beta^{n/2}|e_i|^n, \quad n \geqq 2, \tag{9.5}$$

$$|G_c^i| \leqq \beta^{3/2}|e_i|^3\delta^3/6, \quad \left|\frac{\delta}{\delta\psi} G_c^i\right| \leqq \beta^{3/2}|e_i|^3\delta^2/2, \quad \left|\frac{\delta^2}{\delta\psi^2} G_c^i\right| \leqq \beta^{3/2}|e_i|^3\delta,$$

$$\left|\frac{\delta^n}{\delta\psi^n} G_c^i\right| \leqq \beta^{n/2}|e_i|^n, \quad n \geqq 3.$$

If at least one derivative hits G_a^i, or if we consider the G_c^i term, then we have the bound

$$\begin{aligned}\sum_{i=1}^{s} \varrho_i(\beta^{1/2}|e_i|)^{\max\{3,n\}}(1+|\delta|)^3 &\leqq (c\beta)^{\max\{1/2,(n-2)/2\}} \sum_{i=1}^{s} \varrho_i e_i^2 \beta(1+|\delta|)^3 \\ &= (c\beta)^{\max\{1/2,(n-2)/2\}}(1+|\delta|)^3 .\end{aligned} \tag{9.6}$$

We have used $|e_i| \leqq c$, $\sum \varrho_i e_i^2 \beta = \tilde{l}_D = 1$. If G_a^i is undifferentiated, the first term in (9.4) is handled similarly, leading to the bound

$$(c\beta)^{\max\{1/2,(n-2)/2\}}(|\psi| + |g-h| + |\delta|)(1+|\delta|)^2 . \tag{9.7}$$

The second term leads to the estimate

$$\begin{aligned}&\sum_{i=1}^{s} \varrho_i \sqrt{2}\sqrt{1-\cos\beta^{1/2}e_i h}\,(\beta^{1/2}|e_i|)^{\max\{2,n\}}(1+|\delta|)^2 \\ &\leqq \left(\sum_{i=1}^{s} \varrho_i e_i^2 \beta\right)^{1/2} \left(\sum_{i=1}^{s} \varrho_i(1-\cos\beta^{1/2}e_i h)\right)^{1/2} (c\beta)^{\max\{1/2,(n-1)/2\}}(1+|\delta|)^2 \\ &= (\mathrm{Re}\,S(h))^{1/2}(c\beta)^{\max\{1/2,(n-1)/2\}}(1+|\delta|)^2 ,\end{aligned} \tag{9.8}$$

when $\eta_h = (\beta\,\mathrm{Re}\,S(h))^{1/2}$ [see (3.1) and (3.5)]. When $\eta_h = \eta$, we use $\eta^{-1} = 2e_m(1-2\gamma)^{-1/2} \leqq c$ to bound the first line by

$$\sum_{i=1}^{s} 2\varrho_i e_i^2 \beta (c\beta)^{\max\{0,(n-2)/2\}}(1+|\delta|)^2 \leqq \eta_h \beta^{-1/2}(c\beta)^{\max\{1/2,(n-2)/2\}}(1+|\delta|)^2 , \tag{9.9}$$

to prove (9.8). This completes the proof. □

Lemma 9.2. *The estimate*

$$\begin{aligned}&\left|\prod_{\alpha=1}^{t}\left(\frac{\delta}{\delta\psi(x_\alpha)}\right)_\alpha^{n_\alpha} \mathscr{E}_t(a_1,\dots,a_t)\right| \\ &\leqq (c\lambda^2)^{t-1} e^{-(1-\delta/2)L(\{a_\alpha\})}\beta^{-1} \prod_{\alpha: n_\alpha > 0} (c\beta)^{n_\alpha/2} \\ &\quad \cdot \int \prod_{\substack{\text{distinguished } \alpha \\ n_\alpha = 0}} [\beta^{1/2}(|\psi(y_\alpha)| + |g(y_\alpha) - h(y_\alpha)| + \eta_{h(y_\alpha)}\beta^{-1/2}) dy_\alpha] f(\{y_\alpha\})\end{aligned} \tag{9.10}$$

holds for some symmetric f with $\|f\|_{L^1} \leqq 1$ (the norm taken with one variable fixed).

Proof. We bound $\varepsilon_i(y_\alpha)$ as in (9.4):

$$|\varepsilon_i| = |(e^{i\beta^{1/2}e_i(\psi+g-h)} - 1)e^{i\beta^{1/2}e_ih} + e^{i\beta^{1/2}e_ih} - 1|$$
$$\leqq \beta^{1/2}|e_i|(|\psi| + |g-h|) + \sqrt{2}\sqrt{1-\cos\beta^{1/2}e_ih}\,. \tag{9.11}$$

For the undistinguished α's we use $|\varepsilon_i| \leqq 2$. For the α's with $n_\alpha > 0$ we use

$$\left|\frac{\delta^{n_\alpha}}{\delta\psi^{n_\alpha}}\varepsilon_i(x_\alpha)\right| \leqq \beta^{n_\alpha/2}|e_i|^{n_\alpha}\,. \tag{9.12}$$

Applying (2.10) with $\alpha = 1 - \delta/2$, we obtain

$$\left|\prod_{\alpha=1}^{t}\left(\frac{\delta}{\delta\psi(x_\alpha)}\right)_\alpha^{n_\alpha} \int_{a_1\times\ldots\times a_t} \frac{1}{t!}\varrho_{i_1,\ldots,i_t}(y_1,\ldots,y_t)\varepsilon_{i_1}(y_1)\ldots\varepsilon_{i_t}(y_t)\right|$$
$$\leqq \int_{a_1\times\ldots\times a_t} f_{i_1,\ldots,i_t}(y_1,\ldots,y_t)(c\lambda^2)^{t-1}\beta^{-1}e^{-(1-\delta/2)L(\{a_\alpha\})}\prod_{\alpha=1}^{t}|\varrho_{i_\alpha}e_{i_\alpha}\beta|$$
$$\cdot \prod_{\alpha:n_\alpha>0}(\beta^{n_\alpha/2}|e_i|^{n_\alpha})$$
$$\cdot \prod_{\substack{\text{distinguished }\alpha\\ n_\alpha=0}}(\beta^{1/2}|e_i|(|\psi(y_\alpha)| + |g(y_\alpha) - h(y_\alpha)|) + \sqrt{2}\sqrt{1-\cos\beta^{1/2}e_ih(y_\alpha)}), \tag{9.13}$$

where $\|f_{i_1,\ldots,i_t}\|_{L^1} \leqq 1$. We sum over species indices using

$$\sum_{i=1}^{s}\varrho_ie_i^2\beta = 1\,, \qquad \sum_{i=1}^{s}|\varrho_ie_i\beta| \leqq c\,,$$
$$\sum_{i=1}^{s}|\varrho_ie_i\beta^{1/2}|\sqrt{1-\cos\beta^{1/2}e_ih} \leqq c\eta_h\beta^{-1/2}\,, \tag{9.14}$$

the third bound proven as in (9.8) and (9.9). Estimate (9.10) is immediate. □

10. Derivatives of Covariances

We establish here all the covariance estimates used in this paper. The following formula will prove useful:

$$C_\Gamma = (\lambda^2 l_D^2(-\Delta_\Gamma)^2 - \Delta_\Gamma + 1)^{-1}$$
$$= (1 - 4\lambda^2 l_D^2)^{-1/2}[(-\Delta_\Gamma + r_-)^{-1} - (-\Delta_\Gamma + r_+)^{-1}]\,, \tag{10.1}$$

where

$$r_\pm = (2l_D^2\lambda^2)^{-1}[1 \pm \sqrt{1-4\lambda^2 l_D^2}]\,. \tag{10.2}$$

Recall that $l_D = 1 + O(\lambda^2) + O(\beta/\lambda)$, so for small λ and β we have

$$|r_- - 1| \leqq \delta/4\,, \qquad |r_+ - \lambda^{-2}| \leqq c\,. \tag{10.3}$$

The following lemma contains the main result (6.34) on differences of covariances.

Lemma 10.1. *Let* $\mathrm{dist}(x, \partial Y_{\eta(l)}, y)$ *be the length of the shortest path from the unit cube containing* x *to* $\partial Y_{\eta(l)}$ *to the unit cube containing* y. *Then*

$$|\delta_{\gamma_{\eta(l)}}\ldots\delta_{\gamma_{l-1}}C_{\gamma_1\cup\ldots\cup\gamma_{\eta(l)-1}\cup\partial V}(x,y)| \leqq c\lambda^{-1}e^{-(1-\delta/2)\mathrm{dist}(x,\partial Y_{\eta(l)},y)}\,. \tag{10.4}$$

Proof. We use the path space representation [12, 6] for the two covariances $(-\Delta+r_\pm)^{-1}$:

$$(-\Delta_\Gamma+r_\pm)^{-1}(x,y)=\int_0^\infty dt e^{-r_\pm t}\int P^t_{x,y}(d\omega)\chi_\Gamma(\omega). \tag{10.5}$$

Here $P^t_{xy}(d\omega)$ is the conditional Wiener measure on the set of all paths starting at x at time zero and ending at y at time t, and

$$\chi_\Gamma(\omega)=\begin{cases}0 & \text{if } \omega(t')\in\Gamma \text{ for some } t'\in[0,t]\\ 1 & \text{otherwise.}\end{cases} \tag{10.6}$$

We see that δ_γ acts on C_Γ by inserting a factor $(1-\chi_\gamma(\omega))$ in the Wiener integral:

$$\begin{aligned}\delta_{\gamma_{\eta(l)}}\dots\delta_{\gamma_{l-1}}C_\Gamma(x,y)&=(1-4\lambda^2 l_D^2)^{-1/2}\int_0^\infty dt(e^{-r_-t}-e^{-r_+t})\\ &\quad\cdot\int P^t_{x,y}(d\omega)\prod_{\alpha=\eta(l)}^{l-1}(1-\chi_{\gamma_\alpha}(\omega))\chi_\Gamma(\omega)\geqq 0.\end{aligned} \tag{10.7}$$

Consider the case where $\operatorname{dist}(x,\partial Y_{\eta(l)},y)\geqq 1$. Drop the e^{-r_+t} term and the factors $(1-\chi_{\gamma_\alpha}(\omega))$ for $\eta(l)+1\leqq\alpha\leqq l+1$ for an upper bound. Note also that $(1-\chi_{\gamma_{\eta(l)}}(\omega))\chi_\Gamma(\omega)\leqq(1-\chi_{\partial Y_{\eta(l)}}(\omega))$, since

$$\gamma_{\eta(l)}\backslash\partial Y_{\eta(l)}\subseteqq\Gamma=\gamma_1\cup\dots\cup\gamma_{\eta(l)-1}\cup\partial V.$$

Let $\{F_\alpha\}$ be the unit cube faces comprising $\partial Y_{\eta(l)}$. Then $(1-\chi_{\partial Y_{\eta(l)}}(\omega))\leqq\sum_\alpha(1-\chi_{F_\alpha}(\omega))$, and we have the following estimate on the left-hand side of (10.4):

$$\begin{aligned}\sum_\alpha c\int_0^\infty dt e^{-r_-t}\int P^t_{x,y}(d\omega)(1-\chi_{F_\alpha}(\omega))&\leqq\sum_\alpha ce^{-(1-\delta/4)\operatorname{dist}(x,F_\alpha,y)}\\ &\leqq ce^{-(1-\delta/2)\operatorname{dist}(x,\partial Y_{\eta(l)},y)}.\end{aligned} \tag{10.8}$$

The first step made use of standard estimates on conditional Wiener integrals [20, 12, 6].

When $\operatorname{dist}(x,\partial Y_{\eta(l)},y)<1$ we drop all characteristic functions $\chi_\Gamma(\omega)$ and $(1-\chi_{\gamma_\alpha}(\omega))$. This yields the free covariance which is exactly calculable as

$$\begin{aligned}C_0(x,y)&=(1-4\lambda^2 l_D^2)^{-1/2}\frac{(e^{-r_-^{1/2}|x-y|}-e^{-r_+^{1/2}|x-y|})}{4\pi|x-y|}\\ &\leqq c(r_-^{1/2}+r_+^{1/2})\leqq c\lambda^{-1}.\end{aligned} \tag{10.9}$$

This completes the proof. □

In Sect. 6 we used the estimate

$$|\partial_y^\alpha C_{\Gamma_\beta}(y,x)|\leqq ce^{-|x-y|/2}, \tag{10.10}$$

for $|\alpha|\leqq 4$, $|x-y|>L'/8$, and $\chi_\beta(y)\notin\{0,1\}$. We use some Poisson kernel techniques from [5]. It suffices to prove the estimate for each term in (10.1). Let B be a unit cube containing y such that $\operatorname{dist}(y,\partial B\backslash\Gamma_\beta)\geqq\frac{1}{4}$ and $B\cap\Gamma_\beta\subseteqq\partial B$. Such a cube exists because of the condition $\chi_\beta(y)\in\{0,1\}$ for $\operatorname{dist}(y,\partial f)\leqq\frac{1}{4}$, f a face of Γ_β. Let $f_0(z)=(-\Delta_{\Gamma_\beta}+r)^{-1}(z,x)\equiv C_{r,\Gamma_\beta}(z,x)$ be defined for $z\in\partial B$, with $r=r_+$ or r_-. We have

$|f_0| \leqq c e^{-|x-y|/2}$, with $f_0 = 0$ at $\partial B \cap \Gamma_\beta$. Considered as a function of y, $C_{r,\Gamma_\beta}(y,x)$ solves the Dirichlet problem for $(-\Delta + r)$ with boundary values f_0 on ∂B. Therefore we have the Poisson kernel formula

$$C_{r,\Gamma_\beta}(y,x) = \int_{\partial B} \frac{\partial}{\partial n_z} C_{r,\partial B}(y,z) f_0(z) dz. \tag{10.11}$$

Note that $C_{r,\partial B}$ has an explicit series representation in terms of $(-\Delta + r)^{-1}$, using the method of images. Thus derivatives of $C_{r,\partial B}(y,z)$ are bounded for $|y-z| \geqq \frac{1}{4}$, where the integral in (10.11) is supported. Thus $|\partial_y^\alpha C_{r,\Gamma_\beta}(y,x)| \leqq c e^{-|x-y|/2}$, and (10.10) follows.

The estimate $\int |C(s,x,y)| dy \leqq c$ was needed in Sect. 6. It follows from the fact that $C(s)$ is a convex combination of C_Γ's, each of which is bounded by $c(-\Delta + r_-)^{-1}$ by (10.7). The bound is obvious for this kernel.

11. Ratios of Partition Functions

Our task is to control the ratios of partition functions produced in the expansion of Sect. 5. We reduce inductively the constraints on h implied by $\mathbb{V}$. When all constraints are gone, Proposition 4.1 can be applied.

Definition. Let $v(\mathbf{h}_0, \mathbb{V}) = 1$ if a component of ∂V has $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$, and let $v(\mathbf{h}_0, \mathbb{V}) = 0$ otherwise. Let $n(\mathbb{V})$ denote the number of components of ∂V for which $i(\sigma) = n$. The function $c(\mathbb{V}) = |\partial \hat{V}| + n(\mathbb{V})$ measures the effect of constraints on our estimates for ratios of partition functions.

Proposition 11.1. *Suppose $\mathbf{h}_0, \mathbb{V}$ are such that Conditions A' and B of Sect. 5 hold. Under the conditions of Theorem 2.4,*

$$\left| \frac{Z(\mathbf{h}_0, \mathbb{V})}{Z(0, \mathbb{V})} \right| \leqq \exp\left(e^{-cL^3\eta^2\beta^{-1}} c(\mathbb{V}) - cL^3\eta^2\beta^{-1} v(\mathbf{h}_0, \mathbb{V})\right). \tag{11.1}$$

Proof. We use a double induction. We assume the proposition for all $\bar{V}$ that are strictly contained in V and for all $\bar{\mathbb{V}}$ with $c(\bar{\mathbb{V}}) < c(\mathbb{V})$. When $c(\mathbb{V}) = v(\mathbf{h}_0, \mathbb{V}) = 0$ the proposition reduces to Proposition 4.1, so we can take $c(\mathbb{V}) > 0$ when $v(\mathbf{h}_0, \mathbb{V}) = 0$. We may as well assume $\mathbf{h}_0 \neq 0$.

We define $S(\mathbf{h}_0, \mathbb{V})$ to be the first cube adjacent to a component of ∂V with $\mathbf{h}_0(\sigma) \neq \mathbf{h}_0(\sigma_0)$ [Case 1]. If no such component exists, $S(\mathbf{h}_0, \mathbb{V})$ is the first cube in $\widehat{\partial V}$ [Case 2] and if $\widehat{\partial V} = \emptyset$ it is the first cube adjacent to a component of ∂V with $i(\sigma) = n$ [Case 3]. In Case 2 we define $\mathbb{V}'$ by deleting $S(\mathbf{h}_0, \mathbb{V})$ from $\widehat{\partial V}$. In Case 3 $\mathbb{V}'$ is defined by switching $i(\sigma)$ from n to y for σ adjacent to $S(\mathbf{h}_0, \mathbb{V})$.

In Cases 2, 3 we write

$$\begin{aligned} Z(\mathbf{h}_0, \mathbb{V}) &= Z(\mathbf{h}_0, \mathbb{V}') - \sum_{h \text{ compatible with } (\mathbf{h}_0, \mathbb{V}') \text{ but not } (\mathbf{h}_0, \mathbb{V})} Z_h(\mathbf{h}_0, V) \\ &= \frac{Z(\mathbf{h}_0, \mathbb{V}')}{Z(0, \mathbb{V}')} \left[Z(0, \mathbb{V}) + \sum_{h \text{ compatible with } (0, \mathbb{V}') \text{ but not } (0, \mathbb{V})} Z_h(0, V) \right] \\ &\quad - \sum_{h \text{ compatible with } (\mathbf{h}_0, \mathbb{V}') \text{ but not } (\mathbf{h}_0, \mathbb{V})} Z_h(\mathbf{h}_0, V), \end{aligned} \tag{11.2}$$

and apply the expansion (5.16) to each $Z_h(\cdot, V)$, using $S=S(\mathbf{h}_0, \mathbb{V})$ as the special cube. This yields

$$Z(\mathbf{h}_0, \mathbb{V}) = \frac{Z(\mathbf{h}_0, \mathbb{V}')}{Z(0, \mathbb{V}')} \left[Z(0, \mathbb{V}) + \sum_h \sum_{\mathbb{X}} \varrho(\mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} Z_h(\mathbf{h}_0(V_i), V_i) \right] - \sum_h \sum_{\mathbb{X}} \varrho(\mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} Z_h(\mathbf{h}_0(V_i), V_i). \tag{11.3}$$

Fix $\mathbb{X}$ and sum over h compatible with $\mathbb{X}$. We obtain a product of partition functions $Z(\mathbf{h}_0(V_i), \mathbb{V}_i)$. A cube is in $\widehat{\partial V_i}$ if it is in $\widehat{\partial V} \cap V_i$ or if it is less than L' from $X_k(\mathbb{X})$. Note that h must generate an elementary region surrounding a boundary component adjacent to $S(\mathbf{h}_0, \mathbb{V})$ in Case 3. However, all such regions are included in $X_k(\mathbb{X})$. Hence a boundary component σ of V_i has $i(\sigma)=n$ if it surrounds $X_k(\mathbb{X})$ or if it surrounds or equals a boundary component σ' of V with $i(\sigma')=n$. Otherwise $i(\sigma)=y$. The result of resummation is

$$Z(\mathbf{h}_0, \mathbb{V}) = \frac{Z(\mathbf{h}_0, \mathbb{V}')}{Z(0, \mathbb{V}')} \left[Z(0, \mathbb{V}) + \sum_{\mathbb{X}} \varrho(\mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} Z(\mathbf{h}_0(V_i), \mathbb{V}_i) \right] - \sum_{\mathbb{X}} \varrho(\mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} Z(\mathbf{h}_0(V_i), \mathbb{V}_i). \tag{11.4}$$

The first sum over $\mathbb{X}$ is over all clusters generated in the expansion of $Z_h(0, V)$ for some h compatible with $(0, \mathbb{V}')$ but not with $(0, \mathbb{V})$ [Class 1]. The second sum over $\mathbb{X}$ is similar but with $\mathbf{h}_0$ replacing 0 [Class 2]. Class 3 contains $\mathbb{X}=\emptyset$ only, representing the first term in (11.4).

As in Sect. 5 we multiply and divide by $Z(0, \mathbb{V}_i)$ and incorporate the ratios $Z(\mathbf{h}_0(V_i), \mathbb{V}_i)/Z(0, \mathbb{V}_i)$ into a connected object

$$\Xi(\mathbf{h}_0, \mathbb{V}, \mathbb{X}) = \begin{cases} \dfrac{Z(\mathbf{h}_0, \mathbb{V}')}{Z(0, \mathbb{V}')} \tilde{\varrho}(\mathbb{X}) \displaystyle\prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} \dfrac{Z(\mathbf{h}_0(V_i), \mathbb{V}_i)}{Z(0, \mathbb{V}_i)}, & \mathbb{X} \text{ in Class 1} \\ -\tilde{\varrho}(\mathbb{X}) \displaystyle\prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} \dfrac{Z(\mathbf{h}_0(V_i), \mathbb{V}_i)}{Z(0, \mathbb{V}_i)}, & \mathbb{X} \text{ in Class 2} \\ \dfrac{Z(\mathbf{h}_0, \mathbb{V}')}{Z(0, \mathbb{V}')}, & \mathbb{X} \text{ in Class 3}. \end{cases} \tag{11.5}$$

If we recall that

$$\begin{aligned} \tilde{Z}(\mathbf{h}_0, \mathbb{V}) &= Z(\mathbf{h}_0, \mathbb{V}) Z(0, \mathbf{a})^{-|V|}, \\ \tilde{\varrho}(\mathbb{X}) &= \varrho(\mathbb{X}) Z(0, \mathbf{a})^{-|X_k(\mathbb{X})|}, \end{aligned} \tag{11.6}$$

then (11.4) becomes

$$\tilde{Z}(\mathbf{h}_0, \mathbb{V}) = \sum_{\mathbb{X}} \Xi(\mathbf{h}_0, \mathbb{V}, \mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} \tilde{Z}(0, \mathbb{V}_i). \tag{11.7}$$

We can derive an equation similar to (11.7) in Case 1 by applying the expansion (5.20) directly to $Z(\mathbf{h}_0, \mathbb{V})$ with $\mathscr{A}=1$ and $S=S(\mathbf{h}_0, \mathbb{V})$. We multiply the expansion by ratios of partition functions, and (11.7) follows with

$$\Xi(\mathbf{h}_0, \mathbb{V}, \mathbb{X}) = \tilde{\varrho}(\mathbb{X}) \prod_{V_i \subsetneq V \setminus X_k(\mathbb{X})} \frac{Z(\mathbf{h}_0(V_i), \mathbb{V}_i)}{Z(0, \mathbb{V}_i)}. \tag{11.8}$$

We would like to apply the proposition to the ratios of partition functions in $\Xi(\mathbf{h}_0, \mathbb{V}, \mathbb{X})$. It will be useful to recall the comments on Conditions A′ and B below Eq. (5.19).

Condition B is satisfied for the ratios of partition functions occurring in (11.5) and (11.8). This is because we only attempt to change $i(\sigma)$ from n to y when $\widehat{\partial V} = \emptyset$. New σ's are generated with $i(\sigma) = n$.

To understand why Condition A′ is satisfied, consider first Cases 2 and 3. The function $\mathbf{h}_0(V)$ is constant over the boundary components of V. New boundary conditions can arise in V_i, but always with full constraints. Thus Condition A′ in the proposition is not violated. In Case 1, consider new exterior boundary components $\sigma_0(V_i)$. These components can only surround other $\sigma(V_i)$ with $i(\sigma(V_i)) = y$ [otherwise the $\mathbb{X}$ generating V_i would have been incompatible with $(\mathbf{h}_0(V), \mathbb{V}')$]. Hence all interior $\sigma(V_i)$ have no constraints and $\sigma_0(V_i)$ has full constraints, which verifies the condition. Now consider new interior boundary components of V_i. Again these arise with full constraints and so do not violate the condition.

Finally, we note that the induction hypotheses apply to the ratios of partition functions in (11.5) and (11.8). They occur either in regions strictly contained in V or else in $\mathbb{V}'$ and $c(\mathbb{V}') < c(\mathbb{V})$.

Before estimating the expansion (11.7), we need a lemma to control ratios of partition functions with zero boundary conditions. Suppose we have $(0, \mathbb{V})$ satisfying Conditions A′ and B. If X is a collection of cubes in V, we want to control the effect of deleting X from $\mathbb{V}$. Let $\{V_i\}$ be the components of $V \backslash X$. Define $\mathbb{V}_i$ as in (11.4). Put a cube in $\widehat{\partial V_i}$ if it is in $\widehat{\partial V}$ or if it is less than L' from X. Put $i(\sigma) = n$ if it surrounds any part of X or if it surrounds or equals a boundary component σ' of V with $i(\sigma') = n$. Define

$$\tilde{Z}(0, \mathbb{V} \backslash X) = \prod_{V_i \subsetneq V \backslash X} \tilde{Z}(0, \mathbb{V}_i). \tag{11.9}$$

Lemma 11.2. *Under the conditions of Proposition 11.1,*

$$\left| \frac{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup X_2))}{\tilde{Z}(0, \mathbb{V} \backslash X_1)} \right| \leqq (1 + c(L)\lambda)^{|X_2|}. \tag{11.10}$$

Proof. The induction for this lemma proceeds in parallel with the one for the proposition. We assume the proposition for $\bar{V}$ strictly contained in V. This is allowed since the proof of the proposition for $\bar{V}$ uses this lemma only for regions contained in or equal to $\bar{V}$. We also assume the lemma for larger X_1. Then for Δ a cube of X_2,

$$\begin{aligned} \left| \frac{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup X_2))}{\tilde{Z}(0, \mathbb{V} \backslash X_1)} \right| &= \left| \frac{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup X_2))}{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup \Delta))} \right| \left| \frac{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup \Delta))}{\tilde{Z}(0, \mathbb{V} \backslash X_1)} \right| \\ &\leqq (1 + c(L)\lambda)^{|X_2| - 1} \left| \frac{\tilde{Z}(0, \mathbb{V} \backslash (X_1 \cup \Delta))}{\tilde{Z}(0, \mathbb{V} \backslash X_1)} \right|, \end{aligned} \tag{11.11}$$

and we are reduced to the case where $X_2 = \Delta$, $|\Delta| = 1$. Only the component of $\mathbb{V} \backslash X_1$ containing Δ is modified from numerator to denominator in (11.10). Calling this component $\mathbb{V}_0$, we need only prove that

$$\left| \frac{\tilde{Z}(0, \mathbb{V}_0 \backslash \Delta)}{\tilde{Z}(0, \mathbb{V}_0)} \right| \leqq 1 + c(L)\lambda. \tag{11.12}$$

Expand the denominator using (5.20) with $\mathscr{A}=1$ and $S=\Delta$. After multiplying by ratios of partition functions we obtain

$$\begin{aligned}\tilde{Z}(0,\mathbb{V}_0)&=Z(0,\mathbf{a})^{-|V_0|}\sum_{\mathbb{Y}}\varrho(\mathbb{Y})\prod_{V_i\subsetneqq V_0\setminus X_k(\mathbb{Y})}Z(\mathbf{h}_0(V_i),\mathbb{V}_i)\\&=\sum_{\mathbb{Y}}\tilde{\varrho}(\mathbb{Y})\prod_{V_i:\mathbf{h}_0(V_i)\neq 0}\frac{Z(\mathbf{h}_0(V_i),\mathbb{V}_i)}{Z(0,\mathbb{V}_i)}\prod_{V_i\subsetneqq V_0\setminus X_k(\mathbb{Y})}\tilde{Z}(0,\mathbb{V}_i)\\&=\tilde{Z}(0,\mathbb{V}_0\setminus\Delta)+\sum_{\mathbb{Y}\text{ nontrivial}}\tilde{\varrho}(y)\prod_{V_i:\mathbf{h}_0(V_i)\neq 0}\frac{Z(\mathbf{h}_0(V_i),\mathbb{V}_i)}{Z(0,\mathbb{V}_i)}\tilde{Z}(0,\mathbb{V}_0\setminus X_k(\mathbb{Y}))\,.\end{aligned}\tag{11.13}$$

The last step used the fact that $\tilde{\varrho}(\mathbb{Y})=1$ for trivial $\mathbb{Y}$'s, that is, for $\mathbb{Y}$'s with $|X_k(\mathbb{Y})|=1$ and h constant. We now have

$$\frac{\tilde{Z}(0,\mathbb{V}_0)-\tilde{Z}(0,\mathbb{V}_0\setminus\Delta)}{\tilde{Z}(0,\mathbb{V}_0\setminus\Delta)}=\sum_{\mathbb{Y}\text{ nontrivial}}\tilde{\varrho}(\mathbb{Y})\prod_{V_i:\mathbf{h}_0(V_i)\neq 0}\frac{Z(\mathbf{h}_0(V_i),\mathbb{V}_i)}{Z(0,\mathbb{V}_i)}\frac{\tilde{Z}(0,\mathbb{V}_0\setminus X_k(\mathbb{Y}))}{\tilde{Z}(0,\mathbb{V}_0\setminus\Delta)},\tag{11.14}$$

and we can apply the induction hypotheses to each ratio of partition functions. The last ratio is bounded by $(1+c(L)\lambda)^{|X_k(\mathbb{Y})|-1}$ because more is deleted from $\mathbb{V}_0$ in the denominator than in (11.12). As mentioned above, we can apply the proposition to the ratios $Z(\mathbf{h}_0(V_i),\mathbb{V}_i)/Z(0,\mathbb{V}_i)$ because each $\mathbb{V}_i$ is strictly contained in $\mathbb{V}$. Also, Condition B for V implies Condition B for $\mathbb{V}_0\setminus\Delta$ and the $\mathbb{V}_i$'s. Condition A′ is satisfied because boundary components of V_i have $\mathbf{h}_0(\sigma)=0$ except for $\sigma_0(V_i)$, which arises with full constraints.

Note that V_i can surround boundary components of $\mathbb{V}_0$, but any components it surrounds must have $i(\sigma)=y$. [Otherwise the $\mathbb{Y}$ specifying V_i with $\mathbf{h}_0(V_i)\neq 0$ would never have arisen in the expansion.] Condition B implies that there will be no contribution to $c(\mathbb{V}_i)$ from interior boundary components of $\mathbb{V}_i$. Thus we have

$$\sum_{V_i:\mathbf{h}_0(V_i)\neq 0}c(\mathbb{V}_i)\leqq cL'^3|X_k(\mathbb{Y})|\,,\tag{11.15}$$

so that

$$\prod_{V_i:\mathbf{h}_0(V_i)\neq 0}\frac{Z(\mathbf{h}_0(V_i),\mathbb{V}_i)}{Z(0,\mathbb{V}_i)}\leqq\exp\left(ce^{-cL^3\eta^2\beta^{-1}}L'^3|X_k(\mathbb{Y})|\right).\tag{11.16}$$

This and $|Z(0,\mathbf{a})|^{-1}\leqq 1+c\lambda$ yield the following bound on (11.14):

$$\left|\frac{\tilde{Z}(0,\mathbb{V}_0)}{\tilde{Z}(0,\mathbb{V}_0\setminus\Delta)}-1\right|\leqq\sum_{\mathbb{Y}\text{ nontrivial}}|\varrho(\mathbb{Y})|e^{c(L)\lambda|X_k(\mathbb{Y})|}\leqq c(L)\lambda\,.\tag{11.17}$$

We have used Proposition 5.1 to bound the sum over $\mathbb{Y}$, noting that $t(\mathbb{Y})\geqq 2$ or $\sum_f(\delta h(f))^2\geqq c\eta^2\beta^{-1}$ for nontrivial $\mathbb{Y}$'s, so that an extra factor $c(L)\lambda$ can be extracted. The bound (11.12) and the lemma follow from (11.17). □

We return to the proof of Proposition 11.1 by writing (11.7) as

$$\frac{Z(\mathbf{h}_0,\mathbb{V})}{Z(0,\mathbb{V})}=\sum_{\mathbb{X}}\Xi(\mathbf{h}_0,\mathbb{V},\mathbb{X})\frac{\tilde{Z}(0,\mathbb{V}\setminus X_k(\mathbb{X}))}{\tilde{Z}(0,\mathbb{V})}.\tag{11.18}$$

Consider the total of all the constraints $c(\mathbb{V}_i)$ or $c(\mathbb{V}')$ in the ratios of partition functions in (11.5) and (11.8). We can ignore V_i with $\mathbf{h}_0(V_i)=0$ since the ratio vanishes in that case. We have $c(\mathbb{V}')\leqq c(\mathbb{V})-1$. For $\mathbb{X}$ in Class 1 all the remaining constraints are bounded by $cL'^3|X_k(\mathbb{X})|$. This is because the regions V_i not surrounded by $\mathbb{X}$ have $\mathbf{h}_0(V_i)=0$ and contribute no constraints. Also, internal boundary components of a V_i with $\mathbf{h}_0(V_i)\neq 0$ must have $i(\sigma)=y$, and there are no constraints associated with such components by Condition B. For $\mathbb{X}$ in Class 2 there are constraints coming from regions not surrounded by $\mathbb{X}$ since they will have $\mathbf{h}_0(V_i)\neq 0$. However, by arguments as above we can bound the total constraint by $c(\mathbb{V})-1+cL'^3|X_k(\mathbb{X})|$. The same bound holds in Case 1 for the ratios in (11.8).

We now sum over $\mathbb{X}$ in Classes 1, 2 (Cases 2, 3) and over $\mathbb{X}$ in Case 1 using Proposition 5.1. If $\mathbb{X}$ is compatible with $\mathbb{V}'$ but not $\mathbb{V}$ then there is at least one face f in $X_k(\mathbb{X})$ with $\delta h(f)\neq 0$. The same holds for $\mathbb{X}$ in Case 1, since there must be a discontinuity in h surrounding a σ with $\mathbf{h}_0(\sigma)\neq\mathbf{h}_0(\sigma_0)$, and this discontinuity is automatically incorporated into $\mathbb{X}$. We obtain

$$\begin{aligned}
&\sum_{\mathbb{X}} \Xi(\mathbf{h}_0, \mathbb{V}, \mathbb{X})\frac{\tilde{Z}(0, \mathbb{V}\backslash X_k(\mathbb{X}))}{\tilde{Z}(0,\mathbb{V})}\\
&\leqq \sum_{\mathbb{X}} |\tilde{\varrho}(\mathbb{X})| \exp(e^{-cL^3\eta^2\beta^{-1}}(c(\mathbb{V})-1+cL'^3|X_k(\mathbb{X})|))\frac{\tilde{Z}(0, \mathbb{V}\backslash X_k(\mathbb{X}))}{\tilde{Z}(0,\mathbb{V})}\\
&\leqq \sum_{\mathbb{X}} |\varrho(\mathbb{X})| \exp(e^{-cL^3\eta^2\beta^{-1}}(c(\mathbb{V})-1))e^{c\lambda|X_k(\mathbb{X})|}\\
&\leqq e^{-cL^3\eta^2\beta^{-1}}\exp(e^{-cL^3\eta^2\beta^{-1}}(c(\mathbb{V})-1)).
\end{aligned} \tag{11.19}$$

We have used the lemma on $\tilde{Z}(0,\mathbb{V}\backslash X_k(\mathbb{X}))/\tilde{Z}(0,\mathbb{V})$ and the factors $Z(0,\mathbf{a})^{-1}$ have been bounded by $1+c\lambda$. This proves the theorem in Case 1 (when $v(\mathbf{h}_0,\mathbb{V})=1$). If we combine (11.19) with the contribution from Class 3, we obtain

$$\left|\frac{Z(\mathbf{h}_0,\mathbb{V})}{Z(0,\mathbb{V})}\right| \leqq (1+e^{-cL^3\eta^2\beta^{-1}})\exp(e^{-cL^3\eta^2\beta^{-1}}(c(\mathbb{V})-1)), \tag{11.20}$$

which completes the proof when $v(\mathbf{h}_0,\mathbb{V})=0$. □

12. Proof Completed

We complete the proof of Theorem 2.5 by establishing convergence of the expansion, exponential clustering, and the infinite volume limit. Proposition 11.1 and Lemma 11.2 bound the ratios of partition functions in (5.28) and (5.24) by mild surface effects; the factors $Z(0,\mathbf{a})^{-1}$ in $\tilde{\varrho}$ are bounded by (5.27). Thus we can estimate (5.28) by

$$\begin{aligned}
|\langle\mathscr{A}\rangle_\Lambda^\phi| &\leqq \sum_{\mathbb{X}_1} |\varrho(\mathbb{X}_1)|\, e^{c(L)\lambda|X_k(\mathbb{X}_1)|}\dots\sum_{\mathbb{X}_n}|\varrho(\mathbb{X}_n)|e^{c(L)\lambda|X_k(\mathbb{X}_n)|}\\
&\leqq \sum_{n\leqq w_{\mathscr{A}}} c^n \leqq c^{w_{\mathscr{A}}},
\end{aligned} \tag{12.1}$$

where we have used Proposition 5.1 and $\|\mathscr{A}\|\leqq c$ for each sum over $\mathbb{X}_r$.

Clustering is first proven in a finite volume using the doubled measure trick [12]. We write

$$\langle \mathscr{A}\mathscr{B}\rangle_\Lambda^\phi - \langle \mathscr{A}\rangle_\Lambda^\phi \langle \mathscr{B}\rangle_\Lambda^\phi = \tfrac{1}{2}\langle\langle(\mathscr{A}_1 - \mathscr{A}_2)(\mathscr{B}_1 - \mathscr{B}_2)\rangle_\Lambda^{\phi_1}\rangle_\Lambda^{\phi_2}, \tag{12.2}$$

where ϕ_1 and ϕ_2 are two independent fields. Expand the ϕ_1 partition function $Z(0, \Lambda, \mathscr{A}_1 - \mathscr{A}_2)$ using (5.21) with S taken as the set of cubes intersecting the support of $\mathscr{A}$. In this and subsequent expansions, consider clusters that intersect or surround the support of $\mathscr{B}$ later. Promote the rest of the expansion to the form (5.23) by multiplying by ratios of partition functions. Next expand the ϕ_2 partition function with S taken as $\bigcup_r X_k(\mathbb{Z}_r)$ from the first expansion. Proceed as above, alternating ϕ_1 and ϕ_2 expansions, until at some point the union of the ϕ_1 clusters equals the union of the ϕ_2 clusters. Outside this region we have the difference $\tilde{Z}(0, V, \mathscr{B}_1) - \tilde{Z}(0, V, \mathscr{B}_2) = 0$ multiplying the term. Thus we need only consider the terms which extend as far as suppt$\mathscr{B}$.

For these terms we expand once more each measure, taking S as the set of cubes intersecting suppt$\mathscr{B}$. We now bound all the ratios of partition functions to yield an estimate like (12.1):

$$2|\langle \mathscr{A}\mathscr{B}\rangle_\Lambda^\phi - \langle \mathscr{A}\rangle_\Lambda^\phi \langle \mathscr{B}\rangle_\Lambda^\phi| \leqq \sum_{\mathbb{Z}_1} |\tilde{\varrho}(\mathbb{Z}_1)| e^{c(L)\lambda|X_k(\mathbb{Z}_1)|} \dots \sum_{\mathbb{Z}_n} |\tilde{\varrho}(\mathbb{Z}_n)| e^{c(L)\lambda|X_k(\mathbb{Z}_n)|}. \tag{12.3}$$

The $\mathbb{Z}_r$ can be clusters associated with either measure, and the sum of their diameters must be at least dist(suppt$\mathscr{A}$, suppt$\mathscr{B}$). Note that

$$d(\mathbb{Z}) + \varepsilon L^3 \sum_f (\delta h(f))^2 + ct(\mathbb{Z}) \geqq \operatorname{diam}(X_k(\mathbb{Z})), \tag{12.4}$$

so that the required exponential decay can be extracted from (5.26). When $\mathbb{Z}_r$ does not intersect suppt$\mathscr{A}$ or suppt$\mathscr{B}$, we either have $\tilde{\varrho}(\mathbb{Z}_r) = 1$ or else we can extract at least a factor $c(L)\lambda$ from (5.26). Thus if we allow a factor $c^{W_\mathscr{A}} c^{W_\mathscr{B}}$, each sum is bounded by $1 + c(L)\lambda$. Each such factor can be absorbed into some $e^{c(L)\lambda|X_k(\mathbb{Z}_r)|}$, so that (12.3) is bounded by

$$c^{W_\mathscr{A}} c^{W_\mathscr{B}} \exp(-(1-\delta)\operatorname{dist}(\text{suppt}\mathscr{A}, \text{suppt}\mathscr{B})).$$

It remains for us to consider the infinite volume limit. The cluster functions $\varrho(\mathbb{Z})$ depend on Λ through the differences of covariances

$$\delta_{\gamma_{\eta(l)}} \dots \delta_{\gamma_{l-1}} C_{\gamma_1 \cup \dots \cup \gamma_{\eta(l)-1} \cup \partial V} = (1 - \delta_{\partial \Lambda}) \delta_{\gamma_{\eta(l)}} \dots \delta_{\gamma_{l-1}} C_{\gamma_1 \cup \dots \cup \gamma_{\eta(l)-1} \cup (\partial V \setminus \partial \Lambda)}. \tag{12.5}$$

The second term, representing the deviation from the infinite volume limit, is estimated by the methods of Sect. 10, yielding an extra factor $e^{-\varepsilon \operatorname{dist}(X_k, \partial\Lambda)}$. Thus $\varrho(\mathbb{Z})$ converges as $\Lambda \to \mathbb{R}^3$, and $\delta\varrho(\mathbb{Z}) \equiv \varrho(\mathbb{Z}) - \lim_{\Lambda \to \mathbb{R}^3} \varrho(\mathbb{Z})$ satisfies (5.26) with a factor $e^{\varepsilon \operatorname{dist}(X_k, \partial\Lambda)}$ included on the left-hand side. We also need convergence of $Z(0, \Lambda \setminus X)/Z(0, \Lambda)$. As in [1] we write

$$\frac{Z(0, \Lambda\setminus X)}{Z(0, \Lambda)} - \frac{Z(0, \Lambda'\setminus X)}{Z(0, \Lambda')} = \frac{Z(0, \Lambda\setminus X) Z(0, \Lambda') - Z(0, \Lambda'\setminus X) Z(0, \Lambda)}{Z(0, \Lambda) Z(0, \Lambda')}. \tag{12.6}$$

Expand each term in the numerator as in the clustering proof, going back and forth between the Λ and Λ' partition functions until the same region is deleted from both or until the clusters reach the boundary. If the clusters have not reached the boundary, then the terms cancel in (12.6) up to errors involving $\delta\varrho(\mathbb{Z})$. Altogether we can extract a factor $e^{-\varepsilon \operatorname{dist}(X, \partial\Lambda \cup \partial\Lambda')}$ from the $\varrho(\mathbb{Z})$'s and from the $\delta\varrho(\mathbb{Z})$'s, and the expansion can be estimated as before by $e^{c(L)\lambda|X|}e^{-\varepsilon \operatorname{dist}(X, \partial\Lambda \cup \partial\Lambda')}$. Thus $Z(0, \Lambda \backslash X)/Z(0, \Lambda)$ converges as $\Lambda \to \mathbb{R}^3$. This yields convergence of $\langle \mathscr{A} \rangle_{\Lambda}^{\phi}$.

Appendix. An Iterated Mayer Expansion

This appendix is devoted to the proof of Proposition 2.1. The iterated Mayer expansion formalism of [14] is well adapted to this problem, and we follow the notation of that paper closely. The least stable interaction is v_{ij}^0, and we expand in it first. We have the stability estimate (2.15) only when the hard core conditions in v_{ij}^1 are satisfied. Thus we must preserve the hard core conditions when expanding in v_{ij}^0.

A 0-vertex is a single particle b with coordinates $\xi_b = (e_{i(b)}, x_b)$ and vertex function

$$\sigma^0(\xi_b) = \tilde{z}_{i(b)}. \tag{A.1}$$

In general an l-vertex α' is a finite non-empty collection $\{\alpha\}$ of $(l-1)$-vertices, no two sharing any constituents. [A 0-vertex is its own constituent, and the constituents of α are the constituents of the $(l-1)$-vertices in α.] Let $C(\alpha)$ be the set of constituents of α, and write $b \subseteq \alpha$ when $b \in C(\alpha)$. The type $[\alpha']$ of an l-vertex is an equivalence class of l-vertices which contain the same number of $(l-1)$-vertices of each type $[\alpha]$. We denote by T_l the set of types of l-vertices. An l-vertex α has coordinates $\{\xi_b\}_{b \subseteq \alpha}$.

We can define 1-vertex functions through the formula

$$\begin{aligned} \sigma_\alpha^1(\xi_\alpha) = \frac{(-\beta)^{t-1}}{t} \mathbb{S}_b \int_0^1 ds_1 \dots ds_{t-1} \sum_\eta \prod_{l=2}^t [s_{\eta(l)} \dots s_{l-2} v^0(\xi_{b_l}, \xi_{b_{\eta(l)}})] \\ \cdot e^{-\beta W^0(s,\alpha)} \prod_{m=1}^t \sigma^0(\xi_{b_m}), \end{aligned} \tag{A.2}$$

where

$$v^*(\xi_b, \xi_c) = v_{i(b)i(c)}^*(x_b - x_c), \qquad * = 0, 1, n, R. \tag{A.3}$$

Here α consists of t particles $b_1, \dots, b_t$. The symbol $\mathbb{S}_b$ stands for symmetrization of the expression following in $b_1, \dots, b_t$. Interpolation parameters s and the tree η are as in Sect. 5; we have $1 \leqq \eta(l) < l$. The interpolated interaction is

$$W^0(s, \alpha) = \sum_{1 \leqq m < n \leqq t} s_m s_{m+1} \dots s_{n-1} v^0(\xi_{b_m}, \xi_{b_n}). \tag{A.4}$$

This is a convex combination of interactions satisfying (2.15), so we have $\beta W^1(s, \alpha) \geqq -C_6(\beta/R)t$. [We define $\sigma_\alpha^1(\xi_\alpha)$ only for ξ_α satisfying the hard core conditions $|x_b - x_c| \geqq R_{i(b)i(c)}$ for $b, c \in \alpha$.]

The 2-vertex functions are defined using interpolations as above for v^n, and like $e^{-\beta v^R} = 1 + s(e^{-\beta v^R} - 1)$ for the repulsive interactions v^R. Let α' be a 2-vertex that contains t 1-vertices $\alpha_1, \ldots, \alpha_t$. The type of a 1-vertex is just the number of constituents. Let $N^{\alpha'}_{[\beta]}$ be the number of 1-vertices of type $[\beta]$ in α'. Then

$$\sigma^2_{\alpha'} = \frac{t!}{\prod\limits_{[\beta]\in T_1} N^{\alpha'}_{[\beta]}!}\hat{\sigma}^2_{\alpha'},$$

$$\hat{\sigma}^2_{\alpha'}(\xi_{\alpha'}) = \frac{(-\beta)^{t-1}}{t}\mathbb{S}_\alpha \int_0^1 ds_1 \ldots ds_{t-1} \sum_\eta \prod_{l=2}^t \{s_{\eta(l)} \ldots s_{l-2} \tag{A.5}$$
$$\cdot [v^n(\xi_{\alpha_l}, \xi_{\alpha_{\eta(l)}}) + u^R(\xi_{\alpha_l}, \xi_{\alpha_{\eta(l)}})]\} H^1(s,\alpha')e^{-\beta W^n(s,\alpha')} \prod_{m=1}^t \sigma^1_{\alpha_m}(\xi_{\alpha_m}).$$

Here we have defined

$$v^*(\xi_{\alpha_l}, \xi_{\alpha_m}) = \sum_{\substack{b\in\alpha_l, c\in\alpha_m\\ b\neq c}} v^*(\xi_b, \xi_c), \qquad * = n, R,$$

$$u^R(\xi_{\alpha_l}, \xi_{\alpha_{\eta(l)}}) = \sum_{b\in\alpha_l, c\in\alpha_m} \frac{\beta^{-1}(e^{-\beta v^R(\xi_b,\xi_c)} - 1)}{1 + s_{\eta(l)} \ldots s_{l-1}(e^{-\beta v^R(\xi_b,\xi_c)} - 1)},$$

$$H^1(s,\alpha') = \prod_{m=1}^t \exp[-\beta v^R(\xi_{\alpha_m}, \xi_{\alpha_m})] \prod_{1\leqq m<n\leqq t} \prod_{b\in\alpha_l, c\in\alpha_m} \tag{A.6}$$
$$\cdot [1 + s_m \ldots s_{n-1}(e^{-\beta v^R(\xi_b,\xi_c)} - 1)],$$

$$W^n(s,\alpha') = \sum_{1\leqq m<n\leqq t} s_m \ldots s_{n-1} v^n(\xi_{\alpha_m}, \xi_{\alpha_n}) + \tfrac{1}{2}\sum_{m=1}^t v^n(\xi_{\alpha_m}, \xi_{\alpha_m}),$$

and $\mathbb{S}_\alpha$ denotes symmetrization in $\alpha_1, \ldots, \alpha_t$. The factors $\exp[-\beta v^R(\xi_{\alpha_m}, \xi_{\alpha_m})]$ in $H^1(s,\alpha')$ enforce the hard core conditions assumed in defining $\sigma^1_{\alpha_m}(\xi_{\alpha_m})$.

We now use these vertex functions to write a formula for the potentials in the Mayer series (2.5). Writing $\int d\xi_b$ for $\sum\limits_{i(b)=1}^{s-1} \int\limits_{\Lambda'} dx_b$, $d\xi_\alpha$ for $\int \prod\limits_{b\in\alpha} d\xi_b$, $\varepsilon(\xi_b) = \varepsilon_{i(b)}(x_b)$, we have an expression for (2.3):

$$Z(\phi) = \exp\left(\int_\Lambda i\beta^{1/2} z_s e_s \phi(x)dx\right) \lim_{\Lambda'\to\mathbb{R}^3} \exp\left(\sum_{[\alpha]\in T_2} \int d\xi_\alpha \sigma^2_\alpha(\xi_\alpha)\left[-1 + \prod_{b\in\alpha}[1 + \varepsilon(\xi_b)]\right]\right). \tag{A.7}$$

There follows the formula ($i_u \neq s$)

$$\varrho_{i_1,\ldots,i_r}(x_1,\ldots,x_r) = r! \sum_{t\geqq r}\binom{t}{r} \sum_{\substack{[\alpha]\in T_2\\ C(\alpha)=\{1,\ldots,t\}}} \int d\xi_{r+1} \ldots d\xi_t \sigma^2_\alpha(\xi_\alpha), \tag{A.8}$$

where the x-integrals now extend over $\mathbb{R}^3$. The derivation of (A.8) follows the corresponding derivation in [15] using the formalism of [14] and estimates on $\sigma^l_\alpha(\xi_\alpha)$ given below. The formulas for σ^2 are modified to allow for the different interpolation procedure used (see the appendix in [2]).

We state bounds on σ^1, σ^2, and ϱ using augmented tree graphs η^A, defined in [3]. An augmented tree graph on $\{1,\ldots,s\}$ is a tree η on s' vertices, $s' \geqq s$, together with an injective map A from $\{1,\ldots,s\}$ to $\{1,\ldots,s'\}$. Let α be a 1-vertex with $t \geqq 2$ and constituents $b_1,\ldots,b_t$. We claim the 1-vertex functions satisfy the following estimates:

$$|\sigma^1_\alpha(\xi_\alpha)| \leqq \sum_{\eta^A} b'_{\eta^A}\sigma^1_\alpha(\xi_\alpha, \eta^A),$$
$$\int \prod_{\substack{b\in\alpha\\ b\neq b_0}} dx_b |\sigma^1_\alpha(\xi_\alpha, \eta^A)| \exp[\alpha L_{\eta^A}(x_{b_1},\ldots,x_{b_t})] \leqq \prod_{u=1}^t |\tilde{z}_{i(b_u)} e_{i(b_u)}| (c\beta\lambda^2 l_D^2)^{t-1}. \tag{A.9}$$

Here η^A runs over augmented tree graphs with $s=s'=t$. The coefficients b'_{η^A} are positive and sum to 1. We have let $L_{\eta^A}(x_{b_1}, \ldots, x_{b_t})$ denote the length of the tree on $x_{b_1}, \ldots, x_{b_t}$ defined by η^A.

In (A.2), a tree η and a term specified by $\mathbb{S}_b$ determine η^A and $\sigma^1_\alpha(\xi_\alpha, \eta^A)$. We integrate out each x_{b_u} using (2.15), (2.18). This absorbs the factor $\exp[\alpha L_{\eta^A}(x_{b_1}, \ldots, x_{b_2})]$ and produces $t-1$ factors of β^2 and at least a factor $e_{i(b_u)}$ at each vertex. We bound $e^{-\beta W^0(s,\alpha)} \leqq e^{c\beta(t-1)R}$, and normalize the η^A-measure defined by the s-integrals by including a factor bounded by e^{t-1}. This yields the coefficients b'_{η^A}. We then use $\beta^3 e^{c\beta/R} \leqq c\beta\lambda^2 l_D^2$ to obtain (A.9). When $t=1$ we have simply $\sigma^1_\alpha(\xi_\alpha) = \tilde{z}_{i(b_1)}$.

A similar analysis applies to $\hat{\sigma}^2$. If α has $t \geqq 2$ constituents, then

$$|\hat{\sigma}^2_\alpha(\xi_\alpha)| \leqq \sum_{\eta^A} b''_{\eta^A} \hat{\sigma}^2_\alpha(\xi_\alpha, \eta^A),$$
$$\int \prod_{\substack{b \in \alpha \\ b \neq b_0}} dx_b |\hat{\sigma}^2_\alpha(\xi_\alpha, \eta^A)| \exp[\alpha L_{\eta^A}(x_\alpha)] \leqq \prod_{b \in \alpha} |\tilde{z}_{i(b)} e_{i(b)}| (c\beta\lambda^2 l_D^2)^{t-1}. \tag{A.10}$$

Each η^A is an augmented tree with $s=s'=t$; $L_{\eta^A}(x_\alpha)$ is the length of the corresponding graph on $x_\alpha = \{x_b\}_{b \in \alpha}$. We obtain (A.10) by substituting (A.9) in (A.5). We cancel denominators in factors u^R against corresponding factors in $H^1(s, \alpha')$. The rest of $H^1(s, \alpha')$ can be bounded by 1 by (2.17). We have in addition $e^{-\beta W^n(s,\alpha')} \leqq c^{t-1}$ by (2.16). Next we apply (2.19), (2.20) to the x_b-integrals. This yields factors $t(\alpha_l) t(\alpha_{\eta(l)})$, where each element α_l of α has $t(\alpha_l)$ constituents. Thus a factor $e^{2(t-1)}$ should be included to allow for normalizing the η^A-measure. Again we have $\sigma^2_\alpha(\xi_\alpha) = \tilde{z}_{i(b)}$ if $t=1$.

We can now prove (2.10). Write the sums over t and $[\alpha]$ in (A.8) as a sum over $n_1, n_2, \ldots$, where n_j is the number of elements of α containing j constituents. Then we have $\sum_j n_j j = t \geqq r$, and $t_2 = \sum_j n_j$ is the number of elements of α. We have for $r \geqq 2$

$$\begin{aligned} &\|\hat{\varrho}_{i_1, \ldots, i_r}\|_{L^1(a_2 \times \ldots \times a_r)} \\ &\leqq r! \sum_{n_1, n_2, \ldots} \binom{t}{r} \frac{t_2!}{n_1! n_2! \ldots} \int_{a_2} dx_2 \ldots \int_{a_r} dx_r \int d\xi_{r+1} \ldots \int d\xi_t \sum_{\eta^A} b''_{\eta^A} |\hat{\sigma}^2_\alpha(\xi_\alpha, \eta^A)|. \end{aligned} \tag{A.11}$$

Note that by (2.4), (2.21), and $u_0(x,x) = (4\pi\lambda l_D)^{-1}$ we have $\tilde{z}_i = z_i(1 + O(\lambda^2)) e^{O(\beta/(\lambda l_D))}$. Thus (2.22), (2.23) imply that $\sum_i \tilde{z}_i |e_i| \beta l_D^2 \leqq c$, and using (A.10) we can bound (A.11) by

$$r! \prod_{u=1}^{r} |\tilde{z}_{i_u} e_{i_u}| (c\beta\lambda^2 l_D^2)^{r-1} \sum_{n_1, n_2, \ldots} \binom{t}{r} \frac{t_2!}{n_1! n_2! \ldots} (c\lambda^2)^{t-r} \sum_{\eta^A} b''_{\eta^A} e^{-\alpha L_{\eta^A}(\{a_u\})}. \tag{A.12}$$

Here $L_{\eta^A}(\{a_u\})$ is defined by taking the minimum of $L_{\eta^A}(x_\alpha)$ over $x_1, x_2, \ldots, x_t$ with $x_u \in a_u$, $u = 1, \ldots, r$. For this minimal tree we can remove unnecessary vertices to produce an augmented tree $\bar{\eta}^A$ with $s=r$, $s' \leqq 2r-1$. A new convex combination

$$\sum_{\bar{\eta}^A} b_{\bar{\eta}^A} \exp^{[-\alpha L_{\bar{\eta}^A}(\{a_u\})]} \equiv e^{-\alpha L(\{a_u\})} \tag{A.13}$$

results. Putting $y = c\lambda^2$, we have

$$\sum_{n_1, n_2, \ldots} \binom{t}{r} \frac{t_2!}{n_1! n_2! \ldots} y^{t-r} = \frac{1}{r!} \frac{d^r}{dy^r} \sum_{t_2=1}^{\infty} \left(\sum_{n=1}^{\infty} y^n \right)^{t_2}$$

$$= \frac{1}{r!} \frac{d^r}{dy^r} \frac{y}{1-2y} \leqq c^r \tag{A.14}$$

for λ small. Thus we have the bound

$$\|\varrho_{i_1, \ldots, i_r}\|_{L^1(a_2 \times \ldots \times a_r)} \leqq r! \prod_{u=1}^{r} |\tilde{z}_{i_u} e_{i_u}| (c\beta\lambda^2 l_D^2)^{r-1} e^{-\alpha L(\{a_u\})} . \tag{A.15}$$

When $r=1$ we must include the contribution from an α with only one constituent. We obtain

$$|\varrho_i - \tilde{z}_i| \leqq |\tilde{z}_i e_i| \left(\frac{d}{dy} \frac{y}{1-2y} - 1 \right) \leqq |\tilde{z}_i e_i| c\lambda^2 , \tag{A.16}$$

completing the proof of (2.11). Estimates (A.15) and (2.22) now yield (2.10) and (2.12).

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