Self-Avoiding Walk in Four Dimensions

John Z. Imbrie[1,2]
Departments of Mathematics and Physics
Harvard University
Cambridge, MA 02138

ABSTRACT

Self-avoiding walk is equivalent to a certain supersymmetric field theory of the Φ^4-type. The equivalence is presented along with the prerequisite mathematical tools (Grassman Gaussian integrals). A Lévy process is defined on a hierarchical lattice and mimics the standard random walk. Self-avoiding walk on the hierarchical lattice is analyzed using the equivalence. The self-avoidance does not change the decay of the Green's function at the critical value of the killing rate (mass).

In this talk I will present an approach to self-avoiding walks which is based on an equivalence to a certain supersymmetric field theory. Through this equivalence, the local time τ maps to the square of a Gaussian field, Φ^2. Hence, the self-avoidance interaction τ^2 maps to Φ^4, and the result is a supersymmetric version of the frequently studied φ^4 field theory.

This equivalence has been exploited by Brydges, Evans, and myself [BEI] in a hierarchical model of self-avoiding walks. We show that the self-avoidance interaction reduces to no interaction at long distances, and that the Green's function decays like the simple random walk Green's function. Some extension of our methods should exhibit the logarithmic corrections expected for quantities such as the end-to-end distance for T step walks. Here, due to the slow approach to noninteracting walks as the length scale tends to infinity, one expects the end-to-end distance to behave as $T^{1/2}(\log T)^{1/8}$, rather than $T^{1/2}$ as for simple random walk. We hope these results may also be extended to the simple cubic lattice, a

[1] Research supported in part by National Science Foundation Grants DMS88–58073 and DMS90–08827.
[2] Alfred P. Sloan Foundation Fellow.

longstanding problem.

The most familiar form of the self-avoiding walk is obtained by enumerating all self-avoiding walks of length T. Such a walk is described by a function $\omega : \mathbf{Z}^+ \to \mathbf{Z}^d$ in d dimensions, with $\omega(0) = 0$, $|\omega(i) - \omega(i+1)| = 1$ and $\omega(i) \neq \omega(j)$ for $i \neq j$. One puts counting measure on the set of T step self-avoiding walks. Letting the associated expectation be denoted $\langle\cdot\rangle$, one can analyze the asymptotics of the end-to-end distance:

$$\langle\omega(T)^2\rangle^{1/2} \sim T^\nu \ .$$

Here $\nu = \nu(d)$ is one of the critical exponents of self-avoiding walks. One expects $\nu = \frac{1}{2}$ for $d > 4$, thought this is proven only for d larger than some d_0 [S]. A weakly self-avoiding walk is a bit easier to study, but should have the same exponents. In this case one does not assume $\omega(i) \neq \omega(j)$ for $i \neq j$, and instead weights each walk by a factor proportional to

$$\prod_{i<j}(1 - \lambda\delta(\omega(i) - \omega(j))) \ . \tag{1}$$

If $\lambda = 1$, we get the strictly self-avoiding walk, and if $0 < \lambda << 1$, we get the weakly self-avoiding walk. In the latter case it is proven that $\nu = \frac{1}{2}$ for $d > 4$ [BS].

Four dimensions can be seen as the borderline between "mean field" behavior ($\nu = \frac{1}{2}$) for $d > 4$, and nontrivial behavior ($\nu > \frac{1}{2}$) expected for $d > 4$. This should be signaled by the appearance of logarithms. Detailed predictions for the logarithms have been made by extending renormalization group calculations for n-vector ϕ^4 models to $n = 0$ [BLZ, De, Du]. The equivalence with the supersymmetric Φ^4 model can be thought of as a rigorous version of the $n = 0$ limit. At least it is the representation which appears most suited for rigorous analysis of self-avoiding walks.

We slightly change the underlying random walk process by working in continuous time. This means that the unperturbed measure is defined by a system of probabilities $p(t, x)$, $t \in \mathbf{R}^+$, $x \in \mathbf{Z}^d$, satisfying

$$\sum_x p(t, x) = 1 \ ,$$

and the diffusion equation

$$p_t = \Delta p \ ,$$

where Δ is the lattice Laplacian. We can define a Green's function for simple random walk in continuous time:

$$G_{0x} = \int_0^\infty dt \ p(t, x) = \int_0^\infty dt \ e^{t\Delta}(0, x)$$

$$= (-\Delta)^{-1}(0, x)$$

$$= \frac{1}{N} \int \phi_0 \phi_x e^{-\frac{1}{2} \sum_x (\nabla\phi)^2_x} \prod_y d\phi_y \; . \tag{2}$$

In the last line we have used the fact that the inverse of any operator can be obtained as the covariance of some Gaussian integral. In this case the operator is the Laplacian, leading to the $(\nabla\phi)^2$ term in the Gaussian measure.

Equation (2) expresses simple random walk as a path integral. Before attempting the same thing for self-avoiding walks, we need to replace the interaction (1) with something appropriate to the continuous time framework. We replace (1) with

$$\exp\left[-\lambda \int_0^T ds \int_0^T dt \; \delta(\omega(s) - \omega(t))\right]$$

$$= \exp\left[-\lambda \sum_x \int_0^T ds \; \delta(\omega(s) - x) \int_0^T dt \; \delta(\omega(t) - x)\right]$$

$$= \exp\left[-\lambda \sum_x \tau_x^2\right] \; .$$

Here

$$\tau_x = \int_0^T dt \; \delta(\omega(t) - x)$$

is the time in $[0, T]$ spent at x, the "local time." We can now define the Green's function for self-avoiding walk

$$G_{0x}^{\mathrm{SAW}}(s) = \int_0^\infty dT \; e^{sT} \left\langle \exp\left(-\lambda \sum_x \tau_x^2\right) \delta(\omega(T) - x) \right\rangle \; , \tag{3}$$

where $\langle \cdot \rangle$ is the expectation determined by simple random walk, i.e., by the probabilities $\{p(t, x)\}$.

Note the appearance of a Laplace transform variable s dual to T. This was suppressed in (2) because $s = 0$ was the critical value where G_{0x} exhibits power law decay. In (3) we need to adjust s to some nonzero value to compensate for the killing rate induced by the self-avoidance interaction. It is the critical value of s which is relevant for the long-time behavior of self-avoiding walk. If one wishes to evaluate quantities such as the end-to-end distance for T step walks, one would have to invert the Laplace transform. What is important then is the behavior of $G_{0x}(s)$

in the neighborhood of the critical value $s = s_c$. We believe that the analysis leading to the asymptotics of the end-to-end distance can be done, but we will consider here only the asymptotics of $G_{0x}(s_c)$.

Let us now proceed to the field theory representation for $G_{0x}^{\mathrm{SAW}}(s)$. It was originally derived by McKane [M] and Parisi-Sourlas [PS]. Let $F(\underline{\tau})$ be any function of local times $\underline{\tau} = \{\tau_x\}$ (for the process in $[0, T]$). We claim that

$$\int_0^\infty dT \, \langle F(\underline{\tau}) \delta(\omega(T) - x) \rangle = \int \phi_0 \phi_x F(\underline{\phi}^2) e^{-\frac{1}{2} \sum\limits_x (\nabla \phi)_x^2} \prod_y d\phi_y \, . \tag{4}$$

Note that taking $F = 1$, we recover (2). To obtain G_{0x}^{SAW} we need only take

$$F = \exp \left[s \sum_x \tau_x - \lambda \sum_x \tau_x^2 \right] \, ,$$

since $T = \sum_x \tau_x$. Then on the right-hand side we obtain

$$F(\underline{\phi}^2) = \exp \left[s \sum_x \phi_x^2 - \lambda \sum_x \phi_x^4 \right] \, ,$$

leading to a ϕ^4 field theory.

To prove the claim, note that it is linear in F so it is sufficient to verify it for functions of the form $\exp[\sum_x v_x \tau_x]$. In this case we have

$$\left\langle \exp \left(\sum_x v_x \tau_x \right) \delta(\omega(T) - x) \right\rangle = \left\langle \exp \left(\int_0^T v(\omega(t)) \, dt \right) \delta(\omega(T) - x) \right\rangle$$

$$= e^{TH}(0, x) \, ,$$

where $H = \Delta + v$. This is the Feynman-Kac formula. Therefore, applying steps as in (2), we obtain

$$\begin{aligned} \int_0^\infty dT \, \langle F(\underline{\tau}) \delta(\omega(T) - x) \rangle &= \int_0^\infty dT \, e^{TH}(0, x) \\ &= (-\Delta - v)^{-1}(0, x) \\ &= \frac{1}{N'} \int \phi_0 \phi_x e^{-\frac{1}{2} \sum\limits_x (\nabla \phi)_x^2 - \frac{1}{2} \sum\limits_x v_x \phi_x^2} \prod_y d\phi_y \\ &= \frac{1}{N'} \int \phi_0 \phi_x F(\underline{\phi}^2) e^{-\frac{1}{2} \sum\limits_x (\nabla \phi)_x^2} \prod_y d\phi_y \, . \end{aligned} \tag{5}$$

Here N' is the normalization for the v-dependent Gaussian measure exhibited on the third line. Note that except for this factor, the claim (4) is verified.

Grassman Integrals. Unfortunately, in the present framework, the factor N' will not go away. To obtain a true statement, we have to generalize our notion of Gaussian integral to allow for Grassman variables as well as ordinary variables. One can arrange for the two types of normalization factors to cancel, so that for an appropriate generalization of ϕ, $N' = 1$ and (4) holds. See [L], [BM] for some other applications.

It is fairly straightforward to define the needed Grassman algebra and its associated integration theory (the Bérezin integral [B]). Let $\mathbf{G}$ be the algebra over the ring of complex valued C^∞ functions on $\mathbf{R}^{2n}$ with a $2n$-tuple $\{\bar{\psi}_1, \psi_1, \ldots, \bar{\psi}_n, \psi_n\}$ of generators satisfying the anticommutation relations:

$$\psi_i^\# \psi_j^\# + \psi_j^\# \psi_i^\# = 0 \qquad i, j = 1, \ldots, n, \quad \psi_i^\# = \psi_i \text{ or } \bar{\psi}_i \ .$$

The elements of $\mathbf{G}$ can be uniquely represented in the form

$$g = \sum_\alpha g^{(\alpha)}(\phi)\psi^\alpha \ ,$$

where α is a multi-index with $2n$ components $(\bar{\alpha}_1, \alpha_1, \ldots, \bar{\alpha}_n, \alpha_n)$ which take values 0 or 1, and where

$$\psi^\alpha = \bar{\psi}_1^{\bar{\alpha}_1} \psi_1^{\alpha_1} \cdots \bar{\psi}_n^{\bar{\alpha}_n} \psi_n^{\alpha_n} \ .$$

For each α, $g^{(\alpha)}(\phi) \equiv g^{(\alpha)}(\phi, \bar{\phi})$ is a C^∞ function on $\mathbf{R}^{2n}$, but we use complex conjugate variables $(\bar{\phi}_1, \phi_1, \ldots, \bar{\phi}_n, \phi_n)$ to denote a point in $\mathbf{R}^{2n}$. The Bérezin integral is the map $\int d^\alpha \psi : \mathbf{G} \to \mathbf{G}$ which is linear over $C^\infty(\mathbf{R}^{2n})$ and satisfies

$$\int d^\alpha \psi (\psi^\alpha \psi^\beta) = \pi^{-|\alpha|/2} \psi^\beta$$

whenever $\psi^\alpha \psi^\beta \neq 0$. If $\alpha = (1, 1, \ldots, 1)$, we write $d^\alpha \psi = d\psi$. We may assemble all fields into a single vector $\Phi_i = (\bar{\phi}_i, \phi_i, \bar{\psi}_i \psi_i)$. Then it is natural to define combined Boson $(\bar{\phi}, \phi)$ and Fermion $(\bar{\psi}, \psi)$ integration by putting

$$\int d\Phi g = \int d\phi \int d\psi \ g$$

for any $g \in \mathbf{G}$. Here $d\phi = d^n(\text{Re } \phi) d^n(\text{Im } \phi)$.

With these definitions it is not hard to verify that

$$\int d\psi e^{-\psi A \bar{\psi}} = \pi^{-n} \det A$$

for any complex $n \times n$ matrix. In contrast,

$$\int d\phi e^{-\phi A\bar{\phi}} = \pi^{-n}(\det A)^{-1} ,$$

provided the real part $A + \bar{A}^t$ of A is positive definite. Thus, as desired, the normalizations cancel, and we define a combined Fermion-Boson Gaussian measure

$$d\mu_c(\Phi) = d\Phi e^{-\phi A\bar{\phi} - \psi A\bar{\psi}} ,$$

where $C = A^{-1}$. It may be surprising, given the completely different integration theories, that this Gaussian measure is calculationally very similar to the standard Gaussian integral. For example,

$$\int d\mu_c(\Phi)\bar{\phi}_i\phi_j = \int d\mu_c(\Phi)\bar{\psi}_i\psi_j = C_{ij} ,$$

and other moments can be evaluated using the usual Wick rules (but with minus signs coming from the anticommutation relations).

In an ordinary Gaussian integral where ϕ is an n-component vector, the normalization is $(\det A)^{-n/2}$. Since this cancels in the combined measure, one can think of the system as an $n = 0$ component vector, the two Fermionic components "cancel" the two Bosonic ones.

Using the combined Gaussian measure, we can now write down a corrected form of (5).

Theorem 1. *Let $F \in C^\infty(\mathbf{R}^n_+)$ satisfy $|F(\underline{\tau})| \le \text{const} \cdot \exp(-b \sum \tau_i)$ for some $b > 0$. Then*

$$\int_0^\infty dT \, \langle F(\underline{\tau})\delta(\omega(T) - x)\rangle = \int d\mu_c(\Phi) F(\underline{\Phi}^2) \begin{Bmatrix} \bar{\psi}_0\psi_x \\ \text{or} \\ \bar{\phi}_0\phi_x \end{Bmatrix} ,$$

where $C = (-\Delta)^{-1}$. More generally C is the inverse of the generator of any continuous time Markov chain ω.

Applying this theorem to self-avoiding walk we obtain

$$G^{\text{SAW}}_{0x}(s) = \int d\mu_c(\Phi)\bar{\phi}_0\phi_x \exp\left[s \sum_x \Phi_x^2 - \lambda \sum_x (\Phi_x^2)^2\right] . \tag{6}$$

The Hierarchical Model. In [BEI], a Lévy process (continuous time Markov chain) is constructed on a hierarchical lattice $\mathcal{G} = \bigoplus_{j=0}^\infty \mathbf{Z}_n$. The jumping rates are chosen in such a way that the Green's function mimics the inverse Laplacian. Define the length of any point $x \in \mathcal{G}$:

$$|x|_H = \begin{cases} 0, & x = 0, \\ L^p, & p = \inf\,\{k : x \in \mathcal{G}_k\}, \end{cases}$$

where $\mathcal{G}_k = \bigoplus_{j=0}^{k} \mathbf{Z}_n$. Taking $n = L^p$, we can think of $\mathcal{G}_k$ as a block of L^{dk} points such as would arise in a renormalization group scheme on the simple cubic lattice. Then, for most x, y, $|x-y|_H$ is comparable to the Euclidean distance $|x-y|$.

Endowing the hierarchical lattice with a group structure permits the use of the Fourier transform to compute the Green's function. Specifically, in Fourier transform variables, the Green's functions is the inverse of the infinitesimal generator of the process. The following result is obtained in [BEI].

Theorem 2. *Consider the Lévy process on $\mathcal{G}$ in which jumps from x to y have probability proportional to $|x-y|_H^{-(d+2)}$. The jumping rate may be chosen so that the Green's function is*

$$G_{xy} = \begin{cases} |x-y|_H^{-(d-2)}, & x \neq y \\ (1-L^{-d})/(1-L^{-2}), & x = y. \end{cases} \tag{7}$$

The Renormalization Group. Let us apply the field theory representation (6) to the hierarchical model. The covariance of the Gaussian measure is given by (7). It can be split as

$$G = G' + \Gamma$$

where

$$G'(x) = L^{-2} G(x/L)\ .$$

Here division by L corresponds to a shift in $\mathcal{G} = \bigoplus \mathbf{Z}_n$. The "fluctuation covariance" Γ vanishes if $|x-y|_H > L$. A renormalization group transformation can be defined by writing

$$\int d\mu_G(\Phi) F(\Phi) = \int d\mu_{G'}(\Phi') d\mu_\Gamma(\delta) F(\Phi' + \zeta)$$

$$= d\mu_G(\Phi)(TF)(\Phi)\ .$$

In the second step, we have rescaled Φ' and replaced blocks with points, using the constancy of G' on blocks.

In the case at hand, $F = \prod_x f(\Phi_x)$. Since Γ does not couple different L^d-blocks, TF is also a product. Hence, T can be viewed as a transformation on functions f. Except for the presence of Fermionic variables, this transformation is similar to ones studied by [BlS], [GK].

The following result on the action of the renormalization group on the hierarchical self-avoiding walk is proven in [BEI]. The formulation uses wick ordered powers of Φ, which are defined as

$$:P(\Phi): = \exp(-\Delta)P(\Phi) ,$$

$$\Delta \equiv \sum_{x,y} G(x-y) \left\{ \frac{\partial}{\partial\psi(x)} \frac{\partial}{\partial\bar{\psi}(y)} + \frac{\partial}{\partial\phi(x)} \frac{\partial}{\partial\bar{\phi}(y)} \right\} .$$

Theorem 3. *Let $d = 4$. Assume μ^2, η are $O(\lambda^2)$, with λ small. Suppose*

$$g(\Phi) = e^{-\lambda:(\Phi^2)^2:+\mu^2:\Phi^2:}(1 + \eta:(\Phi^2)^3:) + r(\Phi) ,$$

with remainder $r(\Phi)$ depending only on Φ^2, and satisfying

$$\sum_\alpha \sup_\phi \left| \frac{\partial^\alpha r(\Phi)}{\alpha!} e^{\lambda^{1/2}|\phi|^2} \right| \le O(\lambda) .$$

Then Tg satisfies the same properties, except that λ, μ^2, η are replaced with primed parameters obeying

$$\lambda' = \lambda - \beta\lambda^2 + O(\lambda^3)$$

$$\mu'^2 = L^2\mu^2 + \gamma\lambda^2 + O(\lambda^3) .$$

Theorem 3 shows that the model is asymptotically free. This means that for a certain choice of $\mu^2 = \mu_c^2(\lambda)$, the parameter λ is driven to zero under successive applications of the transformation T.

Further analysis of the "observables" $\phi_0\bar{\phi}_x$ in (6) leads to the following asymptotic statement for the self-avoiding walk Green's function (at the critical $\mu_c^2(\lambda)$):

$$G_{xy}^{\mathrm{SAW}}(\mu_c^2(\lambda)) = (1 + O(\lambda))G(x-y)(1 + \epsilon(x-y))$$

$$|\epsilon(x)| \le \lambda(1 + \lambda \log|x|)^{-1} .$$

Thus the decay of the Green's function for self-avoiding walk is the same as that of the Green's function of the hierarchical Lévy process.

References.

[B] Bérezin, F. A.: *The Method of Second Quantization.* Academic Press, New York (1966).

[BlS] Bleher, P. M., and Sinai, Ya. G.: Investigation of the Critical Point in Models of the Type of Dyson's Hierarchical Models. *Commun. Math. Phys.* **33**, 23–42 (1973).

[BLZ] Bérezin, E., Le Guillou, J. C., and Zinn-Justin, J.: *Phase Transitions and Critical Phenomena, Vol. VI.* Domb, C. and Green, M. S. (eds.): Academic Press, New York (1976).

[BEI] Brydges, D., Evans, S. N., and Imbrie, J. Z.: Self-Avoiding Walk on a Hierarchical Lattice in Four Dimensions. *Ann. Prob.*, to appear.

[BM] Brydges, D. C., and Munoz-Maya, I.: An Application of Berezin Integration to Large Deviations. University of Virginia preprint, 1989.

[BS] Brydges, D., and Spencer, T.: Self-Avoiding Walk in 5 or More Dimensions. *Commun. Math. Phys.* **97**, 125–148 (1985).

[De] De Gennes, P. G.: Exponents for the Excluded Volume Problem as Derived by the Wilson Method: *Physics Lett.* **38A**, 339–340 (1972).

[Du] Duplantier, B.: Polymer Chains in Four Dimensions. *Nucl. Phys.* **B275**, 319–355 (1986).

[GK] Gawedzki, K., and Kupiainen, A.: Asymptotic Freedom Beyond Perturbation Theory. In: *Critical Phenomena, Random Systems, Gauge Theories*, Les Houches 1984. Osterwalder, K., and Stora, R. (eds.): North Holland, Amsterdam (1986).

Gawedzki, K., and Kupiainen, A.: Triviality of φ_4^4 and All That in a Hierarchical Model Approximation. *J. Stat. Phys.* **29**, 683–698 (1982).

[L] Luttinger, J. M.: The Asymptotic Evaluation of a Class of Path Integrals II. *Journal of Mathematical Physics* **24**, 2070–2073 (1983).

[M] McKane, A. J.: Reformulation of $n \to 0$ Models Using Anticommuting Scalar Fields. *Phys. Lett.* **76A**, 22–24 (1980).

[PS] Parisi, G., and Sourlas, N.: Self-Avoiding Walk and Supersymmetry. *J. Phys. Lett.* **41**, L403–L406 (1980).

Parisi, G., and Sourlas, N.: Random Magnetic Fields, Supersymmetry and Negative Dimensions. *Phys. Lett.* **43**, 744–745 (1979).

[S] Slade, G.: The Diffusion of Self-Avoiding Random Walk in High Dimensions. *Commun. Math. Phys.* **110**, 661–683 (1987).