Hermitian indices and factorization of selfadjoint operators on a Kreĭn space

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In memory of Heinz Langer, for his many contributions and leadership in the study of indefinite inner product spaces.

Abstract. The hermitian indices of a selfadjoint operator C on a Kreĭn space $\mathcal{H}$ are defined as geometric measures of positivity and negativity of the operator. A different pair of indices arises in the Bognár-Krámli factorization of C, which writes C as a product AA^* where A acts on a Kreĭn space $\mathcal{A}$ into $\mathcal{H}$ and has zero kernel; the new indices are the positive and negative indices of $\mathcal{A}$. Such factorizations are far from unique. When $\mathcal{H}$ is separable, it is known that the two notions of indices always coincide, and this has applications to index formulas in the theory of Julia operators and completion problems for operator matrices. A new proof of the equality of indices that does not require separability is given in this work.

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1. Introduction

The notion of hermitian indices $h_\pm(C)$ for a selfadjoint operator C on a Kreĭn space $\mathcal{H}$, defined below, is closely related to representations of C as the product of an operator A and its adjoint. The existence of such representations with $A \in \mathcal{L}(\mathcal{H})$ was characterized by János Bognár and András Krámli [4, Th. 1] in terms of ranks of positivity and negativity of C (see also Bognár [3, Th. VII.2.1]). A related factorization $C = AA^*$ with A acting from an external Kreĭn space $\mathcal{A}$ into $\mathcal{H}$ and $\ker A = \{0\}$ was encountered in applications by two of the present authors [11, 12]. Of primary interest in the applications is the fact that the indices $\operatorname{ind}_\pm \mathcal{A}$ of the external Kreĭn space $\mathcal{A}$ do not depend on the choice of factorization. For separable Kreĭn spaces, it is shown in [11] that this is always the case, and in fact $\operatorname{ind}_\pm \mathcal{A}$ coincide with an ad hoc definition of hermitian indices $h_\pm(C)$ for separable Kreĭn spaces [11, §1.2]. In this paper the ad hoc definition of hermitian indices is extended to arbitrary Kreĭn spaces and it is shown that $\operatorname{ind}_\pm \mathcal{A} = h_\pm(C)$ for nonseparable as well as separable Kreĭn spaces. This effectively removes the assumption of separability in index formulas that appear in [11, 12].

Notation and terminology generally follow [11, 12]. A Kreĭn space is a complex inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ which admits a **fundamental decomposition** $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\mathcal{H}_+$ is a Hilbert space and $\mathcal{H}_-$ is the antispace of a Hilbert space. The indices of $\mathcal{H}$, $\operatorname{ind}_\pm \mathcal{H} = \dim \mathcal{H}_\pm$, do not depend on the choice of fundamental decomposition. Every fundamental decomposition induces an **associated Hilbert space** $|\mathcal{H}|$ which coincides with $\mathcal{H}$ as a vector space and replaces $\mathcal{H}_-$ by the original Hilbert space. A corresponding **norm** for the Kreĭn space $\mathcal{H}$ is defined by $\|f\|^2 = \langle f_+, f_+\rangle_{\mathcal{H}} + |\langle f_-, f_-\rangle_{\mathcal{H}}|$, where $f_\pm$ are the components of f in $\mathcal{H}_\pm$. Any two norms are equivalent and determine a unique **strong topology** on $\mathcal{H}$. A **subspace** is simply a linear manifold and need not be closed. The **dimension** of a closed subspace is its dimension in any associated Hilbert space (the choice does not matter). Operators are assumed everywhere defined and continuous. If $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces, $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H},\mathcal{K})$ are the spaces of continuous operators on $\mathcal{H}$ into itself and on $\mathcal{H}$ into $\mathcal{K}$. The Kreĭn space **adjoint** of an operator A is denoted A^*. If C is a selfadjoint operator on a Kreĭn space $\mathcal{H}$, the formula

$$\langle f, g\rangle_C = \langle Cf, g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H},$$

defines a linear and symmetric C**-inner product** on $\mathcal{H}$. The induced quadratic form $\langle Cf, f\rangle_{\mathcal{H}}$, $f \in \mathcal{H}$, thus assumes only real values. A subspace $\mathcal{M}$ of $\mathcal{H}$ is C**-strictly positive** if $\langle f, f\rangle_C > 0$ for every $f \neq 0$ in $\mathcal{M}$, and C**-strictly negative** if $\langle f, f\rangle_C < 0$ for every $f \neq 0$ in $\mathcal{M}$; C**-orthogonality** for vectors in $\mathcal{H}$, $\perp_C$, refers to orthogonality in the C-inner product.

Section 2 presents a self-contained account of hermitian indices and includes a technical result, Theorem 2.10, on signature operators on a Hilbert space. The theorem is used in Section 3 to prove our main result, Theorem 3.1, which is also stated in an equivalent form for continuously contained Kreĭn spaces.

2. Hermitian indices

The concepts of ranks of positivity and negativity for operators and quadratic forms are classical and appear already in elementary matrix theory. For operators on Kreĭn spaces, they are formalized in Bognár [3] in terms of decomposable inner product spaces. Hermitian indices provide an equivalent viewpoint that reduces to the approach of Potapov [18, Ch. 2] for finite-dimensional Kreĭn spaces.

Some preliminaries on the notion of congruence are needed. Kreĭn space operators $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are said to be **congruent** if $A = X^*BX$ for some invertible operator $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Congruence is an equivalence relation.

Lemma 2.1. *Every selfadjoint operator on a Kreĭn space is congruent to a selfadjoint operator on a Hilbert space.*

Proof. Let $\mathcal{H}$ be a Kreĭn space, $C \in \mathcal{L}(\mathcal{H})$ a selfadjoint operator. Choose any invertible operator X from $\mathcal{H}$ onto a Hilbert space $\mathcal{K}$. Then since $X^{*\,-1} = X^{-1\,*}$, the operator $D = X^{*\,-1}CX^{-1}$ is selfadjoint on $\mathcal{K}$ and $C = X^*DX$. □

Lemma 2.2. *Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ be congruent selfadjoint operators on Kreĭn spaces $\mathcal{H}$ and $\mathcal{K}$, and suppose $A = X^*BX$ where $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is invertible. Then the mapping*

$$\mathcal{X} \colon \mathcal{M} \to X\mathcal{M}$$

is a one-to-one and onto correspondence between the set $\mathcal{C}_\pm(\mathcal{H}, A)$ of all closed A-strictly positive/negative subspaces of $\mathcal{H}$ and the set $\mathcal{C}_\pm(\mathcal{K}, B)$ of all closed B-strictly positive/negative subspaces of $\mathcal{K}$. Moreover:

(1) *The correspondence $\mathcal{X}$ preserves dimension:* $\dim \mathcal{M} = \dim X\mathcal{M}$.
(2) *If vectors f, g in $\mathcal{H}$ are A-orthogonal, their images $u = Xf$, $v = Xg$ in $\mathcal{K}$ are B-orthogonal.*

Proof. Since $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $X \in \mathcal{L}(|\mathcal{H}|, |\mathcal{K}|)$ for any associated Hilbert spaces. Thus by standard Hilbert space methods, if $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, $X\mathcal{M}$ is a closed subspace of $\mathcal{K}$ of the same dimension. For any $f, g \in \mathcal{H}$ and $u = Xf$, $v = Xg$ in $\mathcal{K}$,

$$\langle f, g \rangle_A = \langle Af, g \rangle_{\mathcal{H}} = \langle BXf, Xg \rangle_{\mathcal{K}} = \langle u, v \rangle_B. \tag{2.1}$$

It follows that $\mathcal{X}$ maps $\mathcal{C}_\pm(\mathcal{H}, A)$ into $\mathcal{C}_\pm(\mathcal{K}, B)$. Similarly, $\mathcal{X}$ is onto. A straightforward verification shows that $\mathcal{X}$ is one-to-one, which proves (1). Then (2) follows from (2.1). □

Definition 2.3. *Let C be a selfadjoint operator on a Kreĭn space $\mathcal{H}$. The* **positive/negative hermitian indices** *of C, denoted $h_\pm(C)$, are defined as the maximum dimensions of a closed C-strictly positive/negative subspace of $\mathcal{H}$.*

Theorem 2.4 shows that hermitian indices are well defined, that is, the required subspaces of maximum dimension always exist. The hermitian indices of the identity operator 1 on a Kreĭn space $\mathcal{H}$ are $h_\pm(1) = \operatorname{ind}_\pm \mathcal{H}$, where $\operatorname{ind}_\pm \mathcal{H}$ are the positive and negative indices of $\mathcal{H}$. The zero operator has hermitian indices $h_\pm(0) = 0$.

Theorem 2.4. *The hermitian indices $h_\pm(C)$ are well defined for every selfadjoint operator C on a Kreĭn space $\mathcal{H}$ and are invariant under congruence. If $\mathcal{H}$ is a Hilbert space, then $h_\pm(C) = \dim \mathcal{H}_\pm$, where $\mathcal{H}_\pm$ are the spectral subspaces of C for the sets $(0, \infty)$ and $(-\infty, 0)$.*

Corollary 2.5. *Every selfadjoint operator on a Kreĭn space is congruent to a selfadjoint operator on a Hilbert space having the same hermitian indices.*

Proof of Theorem 2.4. Assume first that $\mathcal{H}$ is a Hilbert space, and let $E(\cdot)$ be the spectral measure for a selfadjoint operator C on $\mathcal{H}$. Set

$$\mathcal{H}_+ = E((0, \infty)), \quad \mathcal{H}_- = E((-\infty, 0)), \quad \mathcal{H}_0 = E(\{0\}).$$

By the spectral theorem, $\mathcal{H}_\pm$ are C-strictly positive/negative subspaces of $\mathcal{H}$. Consider any closed C-strictly positive subspace $\mathcal{N}$ of $\mathcal{H}$, and let $T\colon \mathcal{N} \to \mathcal{H}_+$ be orthogonal projection onto $\mathcal{H}_+$. If $f \in \mathcal{N}$ and $Tf = 0$, then $f \in \mathcal{H}_- \oplus \mathcal{H}_0$ and hence $\langle f, f\rangle_C \leq 0$. Then $f = 0$ because $\mathcal{N}$ is C-strictly positive. Since T is one-to-one, $\dim \mathcal{N} \leq \dim \mathcal{H}_+$. Therefore $\mathcal{H}_+$ has maximum dimension among all closed C-strictly positive subspaces of $\mathcal{H}$. The negative case is handled similarly. Hermitian indices are thus well defined for selfadjoint operators on a Hilbert space.

By Lemma 2.1, every selfadjoint operator C on a Kreĭn space is congruent to a selfadjoint operator D on a Hilbert space. The congruence maps D-strictly positive/negative subspaces onto C-strictly positive/negative subspaces and preserves dimension by Lemma 2.2. Thus hermitian indices are well defined on Kreĭn spaces. Another application of Lemma 2.2 shows that the indices are invariant under congruence. □

The well-known theorem of J. J. Sylvester that characterizes the congruence of matrices by their numbers of positive, negative, and zero eigenvalues has a straightforward generalization to finite-dimensional Kreĭn spaces.

Theorem 2.6 (Sylvester theorem). *Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ be selfadjoint operators on Kreĭn spaces $\mathcal{H}$ and $\mathcal{K}$ that have the same finite dimension. Then A is congruent to B if and only if $h_\pm(A) = h_\pm(B)$.*

Proof. Assume $h_\pm(A) = h_\pm(B)$. By Corollary 2.5, A and B are congruent to selfadjoint operators A_1 and B_1 on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{K}_1$ having the same hermitian indices as A and B. Then $h_\pm(A_1) = h_\pm(B_1)$, and the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{K}_1$ have the same finite dimension as $\mathcal{H}$ and $\mathcal{K}$. By Theorem 2.4, $h_\pm(A_1)$ and $h_\pm(B_1)$ are the numbers of positive/negative eigenvalues of A_1 and B_1. Therefore A_1 and B_1 are congruent by the classical theorem of Sylvester [15, p. 223], and hence A and B are congruent. The other direction follows from Theorem 2.4. □

Theorem 2.7. *For every selfadjoint operator C on a Kreĭn space $\mathcal{H}$, there exist subspaces $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ of $\mathcal{H}$ such that*

$$\mathcal{H} = \mathcal{M}_+ + \mathcal{M}_- + \mathcal{M}_0, \tag{2.2}$$

where (i) $\mathcal{M}_\pm$ are closed and C-strictly positive/negative and $\mathcal{M}_0 = \ker C$; (ii) the sum of any two of the three subspaces is closed; (iii) the subspaces are pairwise C-orthogonal; and (iv) $\dim \mathcal{M}_\pm = h_\pm(C)$.

It follows that the hermitian indices $h_\pm(C)$ coincide with the ranks of positivity and negativity of C defined in Bognár [3, pp. 95, 149]. For by their definition, the latter indices are determined by any fundamental decomposition of $\mathcal{H}$ in the C-inner product [3, pp. 24, 95], and (2.2) is one such fundamental decomposition. In this context, the term "fundamental decomposition" is in the sense of Bognár [3, p. 24]. See also Azizov and Iokhvidov [2] and Gheondea [13] for analogous notions of fundamental decompositions and ranks of positivity and negativity.

Proof of Theorem 2.7. By Lemma 2.1, one can choose a selfadjoint operator D on a Hilbert space $\mathcal{K}$ and an invertible operator $Y \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $D = Y^*CY$. Let $\mathcal{K}_+, \mathcal{K}_-, \mathcal{K}_0$ be the spectral subspaces for D for $(0, \infty)$, $(-\infty, 0)$, $\{0\}$, so $\mathcal{K}_\pm$ are D-strictly positive/negative and $\mathcal{K}_0 = \ker D$. Then (2.2) holds with

$$\mathcal{M}_+ = Y\mathcal{K}_+, \quad \mathcal{M}_- = Y\mathcal{K}_-, \quad \mathcal{M}_0 = Y\mathcal{K}_0.$$

Since $\mathcal{M}_0 = Y \ker D = \ker C$, the subspaces $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ satisfy (i), (ii), (iv) by Lemma 2.2(1) because $\mathcal{K}_+, \mathcal{K}_-, \mathcal{K}_0$ have these properties in $\mathcal{K}$. The subspaces $\mathcal{K}_+, \mathcal{K}_-, \mathcal{K}_0$ are pairwise orthogonal in the inner product of $\mathcal{K}$. They are also pairwise D-orthogonal by the special fact that they are invariant under D. Hence $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ satisfy (iii) by Lemma 2.2(2). □

Remark 2.8. Decompositions of the type (2.2) in Theorem 2.7 are not unique in general. The proof of Theorem 2.7 shows that when $\mathcal{H}$ is a Hilbert space, a decomposition (2.2) can be chosen such that the subspaces $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ are invariant under C. This fails in general for Kreĭn spaces.

In Theorem 2.7, the conditions $(i) - (iii)$ alone imply (iv). This is shown in the next result, which gives additional information.

Theorem 2.9. *Let C be a given selfadjoint operator on a Kreĭn space $\mathcal{H}$, and let $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ be any subspaces of $\mathcal{H}$ that satisfy* (2.2) *and conditions* $(i), (ii), (iii)$ *in Theorem* 2.7. *Then the subspaces also satisfy* (iv). *Moreover:*

1. *The sum* (2.2) *is direct; that is, if $f_+ + f_- + f_0 = 0$ with $f_\pm \in \mathcal{M}_\pm$ and $f_0 \in \mathcal{M}_0$, then $f_+ = f_- = f_0 = 0$.*
2. *There exist projections $Q_\pm, Q_0$ in $\mathcal{L}(\mathcal{H})$ such that Q_+ has range $\mathcal{M}_+$ and kernel $\mathcal{M}_- + \mathcal{M}_0$; Q_- has range $\mathcal{M}_-$ and kernel $\mathcal{M}_+ + \mathcal{M}_0$; Q_0 has range $\mathcal{M}_0$ and kernel $\mathcal{M}_+ + \mathcal{M}_-$.*

Proof. Let $\mathcal{M}_+, \mathcal{M}_-, \mathcal{M}_0$ satisfy (2.2) and conditions $(i), (ii), (iii)$ in Theorem 2.7. Suppose $f_\pm \in \mathcal{M}_\pm$, $f_0 \in \mathcal{M}_0$ and $f_+ + f_- + f_0 = 0$. By (i), $\langle f_+, f_+ \rangle_C \geq 0$ with equality only when $f_+ = 0$; $\langle f_-, f_- \rangle_C \leq 0$ with equality only when $f_- = 0$; and $\langle f_0, f_0 \rangle_C = \langle Cf_0, f_0 \rangle_{\mathcal{H}} = 0$ because $Cf_0 = 0$. Since by (iii), $\mathcal{M}_- \perp_C \mathcal{M}_0$, it follows that

$$\langle f_+, f_+ \rangle_C = \langle -f_- - f_0, -f_- - f_0 \rangle_C = \langle f_-, f_- \rangle_C + \langle f_0, f_0 \rangle_C \leq 0.$$

Therefore $\langle f_+, f_+ \rangle_C = 0$, and hence $f_+ = 0$. Then also $f_- + f_0 = 0$, so

$$\langle f_-, f_- \rangle_C = \langle -f_0, -f_0 \rangle_C = 0,$$

and hence $f_- = 0$. So $f_+ = f_- = f_0 = 0$, which proves (1).

By (ii) and a closed graph argument, there are projection operators $Q_\pm, Q_0$ in $\mathcal{L}(\mathcal{H})$ such that

$$Q_+ f = f_+, \qquad Q_- f = f_-, \qquad Q_0 f = f_0,$$

whenever $f = f_+ + f_- + f_0$ with $f_\pm \in \mathcal{M}_\pm$ and $f_0 \in \mathcal{M}_0$. They have the properties listed in (2) by construction.

It remains to show that $\mathcal{M}_+, \mathcal{M}_-$ satisfy (iv), that is, $\dim \mathcal{M}_\pm = h_\pm(C)$, or, equivalently, $\mathcal{M}_\pm$ have maximum dimensions among all closed

C-strictly positive/negative subspaces. Consider any closed C-strictly positive subspace $\mathcal{N}$ of $\mathcal{H}$, and define $T\colon \mathcal{N} \to \mathcal{M}_+$ by $Tf = Q_+f$, $f \in \mathcal{N}$. Then T is continuous in the strong topology of $\mathcal{H}$. It is also one-to-one. For suppose $f \in \mathcal{N}$ and $Tf = 0$. Then $Q_+f = 0$, and hence $f = f_- + f_0$, where $f_- \in \mathcal{M}_-$ and $f_0 \in \mathcal{M}_0$. Since $\mathcal{M}_- \perp_C \mathcal{M}_0$,

$$\langle f, f\rangle_C = \langle f_-, f_-\rangle_C + \langle f_0, f_0\rangle_C \le 0.$$

Therefore $\langle f, f\rangle_C = 0$ and hence $f = 0$. Thus $\dim \mathcal{N} \le \dim \mathcal{M}_+$, which proves (iv) for the positive case. The negative case is proved similarly. □

A **signature operator** on a Hilbert space $\mathcal{H}$ is an operator $J \in \mathcal{L}(\mathcal{H})$ which is both selfadjoint and unitary. If J is a signature operator on $\mathcal{H}$, it coincides with ± 1 on orthogonal closed subspaces $\mathcal{M}_\pm$ that span $\mathcal{H}$, and $h_\pm(J) = \dim \mathcal{M}_\pm$.

Theorem 2.10. *Let $\mathcal{H}$ be a Hilbert space, $C \in \mathcal{L}(\mathcal{H})$ a selfadjoint operator such that* $\ker C = \{0\}$*. Suppose $C = T^*J_AT$, where J_A is a signature operator on a Hilbert space $\mathcal{K}$, and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has zero kernel and dense range in $\mathcal{K}$. Then $h_\pm(C) = h_\pm(J_A)$.*

The assumption that T has zero kernel is redundant since $\ker C = \{0\}$, but is retained for convenience in the proof. What follows is a Kreĭn space proof based on a theorem of Phillips [17] as given in Arsene and Gheondea [1].

Proof. Let $\mathcal{A}$ be the Kreĭn space which is $\mathcal{K}$ as a vector space and has J_A as fundamental symmetry. Then $\langle f, g\rangle_{\mathcal{A}} = \langle J_Af, g\rangle_{\mathcal{K}}$, $f, g \in \mathcal{A}$. Suppose $J_A = \pm 1$ on the subspaces $\mathcal{A}_\pm$ of $\mathcal{K}$. Then $h_\pm(J_A) = \dim \mathcal{A}_\pm$ and $\mathcal{A}$ has fundamental decomposition $\mathcal{A} = \mathcal{A}_+ \oplus_{\mathcal{A}} \mathcal{A}_-$. The corresponding associated Hilbert space $|\mathcal{A}|$ coincides with the Hilbert space $\mathcal{K}$. In particular, $\mathcal{A}$ and $\mathcal{K}$ have the same strong topology.

Write $\mathcal{H} = \mathcal{H}_+ \oplus_{\mathcal{H}} \mathcal{H}_-$, where $\mathcal{H}_\pm$ are the spectral subspaces of C for the positive and negative real axes. Since $\ker C = \{0\}$, $h_\pm(C) = \dim \mathcal{H}_\pm$ by Theorem 2.4. Let J_C be the signature operator on $\mathcal{H}$ with $J_C = \pm 1$ on $\mathcal{H}_\pm$, and let $|C| = (C^2)^{\frac{1}{2}}$ be the operator modulus of C. Then

$$|C|^{\frac{1}{2}} J_C |C|^{\frac{1}{2}} = C = T^*J_AT \tag{2.3}$$

and

$$|C|^{\frac{1}{2}} \mathcal{H}_\pm \subseteq \mathcal{H}_\pm. \tag{2.4}$$

The problem is to show that $\dim \mathcal{H}_\pm = \dim \mathcal{A}_\pm$.

Define subspaces $\mathcal{G}_\pm$ of $\mathcal{A}$ by $\mathcal{G}_\pm = T\mathcal{H}_\pm$. Then $\mathcal{G} := T\mathcal{H} = \mathcal{G}_+ + \mathcal{G}_-$. As $\mathcal{A}$ and $\mathcal{K}$ have the same strong topology and T has dense range in $\mathcal{K}$, $\mathcal{G}$ is dense in $\mathcal{A}$. The subspaces $\mathcal{G}_\pm$ are nonnegative/nonpositive and orthogonal in $\mathcal{A}$. For suppose $g_\pm \in \mathcal{G}_\pm$ with $g_\pm = Tf_\pm$, $f_\pm \in \mathcal{H}_\pm$, then by (2.3) and (2.4),

$$\langle g_\pm, g_\pm\rangle_{\mathcal{A}} = \langle J_ATf_\pm, Tf_\pm\rangle_{\mathcal{K}} = \left\langle J_C|C|^{\frac{1}{2}} f_\pm, |C|^{\frac{1}{2}} f_\pm \right\rangle_{\mathcal{H}},$$

where the last term on the right side is nonnegative/nonpositive. Similarly,

$$\langle g_+, g_- \rangle_{\mathcal{A}} = \langle J_A T f_+, T f_- \rangle_{\mathcal{K}} = \left\langle J_C |C|^{\frac{1}{2}} f_+, |C|^{\frac{1}{2}} f_- \right\rangle_{\mathcal{H}} = 0,$$

which shows orthogonality.

Let

$$\mathcal{G}_+ = \{x + G_+ x\}_{x \in \mathcal{M}_+}, \qquad \mathcal{G}_- = \{G_- y + y\}_{y \in \mathcal{M}_-}, \tag{2.5}$$

be the graph representations of $\mathcal{G}_\pm$ relative to the fundamental decomposition $\mathcal{A} = \mathcal{A}_+ \oplus_{\mathcal{A}} \mathcal{A}_-$ of $\mathcal{A}$. Thus $G_+ \colon \mathcal{M}_+ \to |\mathcal{A}_-|$ is a contraction from a subspace $\mathcal{M}_+$ of $\mathcal{A}_+$ into $|\mathcal{A}_-|$, and $G_- \colon \mathcal{M}_- \to \mathcal{A}_+$ is a contraction from a subspace $\mathcal{M}_-$ of $|\mathcal{A}_-|$ into $\mathcal{A}_+$. Let $P_\pm \in \mathcal{L}(\mathcal{A})$ be the selfadjoint projections onto $\mathcal{A}_\pm$. The subspace $\mathcal{M}_+$ in (2.5) is the one-to-one image of $\mathcal{H}_+$ under $P_+ T$ (T is one-to-one by assumption, and the restriction of P_+ to $\mathcal{G}_+$ is one-to-one by the graph representation of $\mathcal{G}_+$), and hence

$$\dim \mathcal{H}_+ = \dim \overline{\mathcal{M}}_+ \leq \dim \mathcal{A}_+. \tag{2.6}$$

Similarly, $\mathcal{M}_-$ is the one-to-one image of $\mathcal{H}_-$ under $P_- T$, and hence

$$\dim \mathcal{H}_- = \dim \overline{\mathcal{M}}_- \leq \dim \mathcal{A}_-. \tag{2.7}$$

Since $\mathcal{G}_+$ is nonnegative, $\mathcal{G}_-$ nonpositive, and $\mathcal{G}_+ \perp_{\mathcal{A}} \mathcal{G}_-$, by Phillips' Theorem [17, 1], there is a maximal nonnegative subspace $\widetilde{\mathcal{G}}_+$ and a maximal nonpositive subspace $\widetilde{\mathcal{G}}_-$ such that $\mathcal{G}_\pm \subseteq \widetilde{\mathcal{G}}_\pm$ and $\widetilde{\mathcal{G}}_+ \perp_{\mathcal{A}} \widetilde{\mathcal{G}}_-$. Then

$$\widetilde{\mathcal{G}}_+ = \{x + Gx\}_{x \in \mathcal{A}_+}, \qquad \widetilde{\mathcal{G}}_- = \{G^* y + y\}_{y \in |\mathcal{A}_-|},$$

where $G \in \mathcal{L}(\mathcal{A}_+, |\mathcal{A}_-|)$ is a contraction operator. The inclusions $\mathcal{G}_\pm \subseteq \widetilde{\mathcal{G}}_\pm$ mean that $G_+ = G\big|\mathcal{M}_+$ and $G_- = G^*\big|\mathcal{M}_-$, and hence (2.5) takes the form

$$\mathcal{G}_+ = \{x + Gx\}_{x \in \mathcal{M}_+}, \qquad \mathcal{G}_- = \{G^* y + y\}_{y \in \mathcal{M}_-}, \tag{2.8}$$

To complete the proof, consider any $h \in \mathcal{A}$. Since $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ is dense in $\mathcal{A}$, by (2.8) there exist sequences $\{f_n^\pm\}_{n=1}^\infty \subseteq \mathcal{M}_\pm$ such that

$$\lim_{n\to\infty} \left[f_n^+ + G f_n^+ + G^* f_n^- + f_n^- \right] = h. \tag{2.9}$$

If $h \in \mathcal{A}_+$, one can apply $P_\pm$ to both sides of (2.9) to get $f_n^+ + G^* f_n^- \to h$ and $G f_n^+ + f_n^- \to 0$, and hence

$$(1 - G^* G) f_n^+ = f_n^+ + G^* f_n^- - G^*(G f_n^+ + f_n^-) \to h.$$

It follows that $1 - G^* G$ maps $\overline{\mathcal{M}}_+$ onto a dense subspace of $\mathcal{A}_+$. Therefore

$$\dim \mathcal{A}_+ \leq \dim \overline{\mathcal{M}}_+ = \dim \mathcal{H}_+.$$

The reverse inequality was proved above, so $\dim \mathcal{A}_+ = \dim \mathcal{H}_+$. If $h \in \mathcal{A}_-$, then $f_n^+ + G^* f_n^- \to 0$ follows from (2.9) and so $G f_n^+ + f_n^- \to h$. Thus

$$(1 - GG^*) f_n^- = G f_n^+ + f_n^- - G(f_n^+ + G^* f_n^-) \to h.$$

Therefore $1 - GG^*$ maps $\overline{\mathcal{M}}_-$ onto a dense subspace of $\mathcal{A}_-$, and

$$\dim \mathcal{A}_- \leq \dim \overline{\mathcal{M}}_- = \dim \mathcal{H}_-.$$

Hence $\dim \mathcal{A}_- = \dim \mathcal{H}_-$, and so $\dim \mathcal{A}_\pm = \dim \mathcal{H}_\pm$. The result follows. □

3. Bognár-Krámli factorization

Following terminology introduced in [12], a **Bognár-Krámli factorization** of a selfadjoint operator C on a Kreĭn space $\mathcal{H}$ is any representation $C = AA^*$ such that $A \in \mathcal{L}(\mathcal{A}, \mathcal{H})$ for some Kreĭn space $\mathcal{A}$ and $\ker A = \{0\}$. From the viewpoint of Constantinescu and Gheondea [6] and Gheondea [13, Ch. 6], $\mathcal{A}$ is an **induced Kreĭn space**. Another equivalent notion is that of **continuously contained Kreĭn spaces**, which is discussed below.

Bognár-Krámli factorizations of a selfadjoint operator C are not unique. In a special case, any two are identical modulo an isomorphism between the external domain spaces, and then the factorization is said to be **essentially unique**. This occurs if and only if C is congruent to a selfadjoint operator on a Hilbert space whose spectrum omits an interval $(0, \varepsilon)$ or $(-\varepsilon, 0)$ for some $\varepsilon > 0$. For details and other conditions, see Hara [14], Dritschel [10], Constantinescu and Gheondea [5, 6], and Gheondea [13, Ch. 6]. The indices of the Kreĭn space $\mathcal{A}$ in any Bognár-Krámli factorization are nevertheless unique and determined by the selfadjoint operator C.

Theorem 3.1. *A selfadjoint operator C on a Kreĭn space $\mathcal{H}$ admits a Bognár-Krámli factorization*

$$C = AA^*, \quad A \in \mathcal{L}(\mathcal{A}, \mathcal{H}), \quad \ker A = \{0\} \tag{3.1}$$

for some Kreĭn space $\mathcal{A}$ if and only if $\operatorname{ind}_\pm \mathcal{A} = h_\pm(C)$.

By Theorem 3.1, the standing assumption of separability in [12, p. 144] can be removed, assuring that [12, Th. 2.1] holds for all Kreĭn spaces.

The sufficiency part in Theorem 3.1 is a known variant of [4, Th. 1]. Proofs can be found in [5, 11, 12, 13], and a proof is included here for completeness. A proof of necessity is given in [11] for separable Kreĭn spaces. The proof presented here is new and based on Theorem 2.10. Its key feature is that it does not require separability.

Proof of Theorem 3.1. Necessity: Assume one is given a Kreĭn space $\mathcal{A}$ and factorization $C = AA^*$ with $A \in \mathcal{L}(\mathcal{A}, \mathcal{H})$ and $\ker A = \{0\}$. It must be shown that $\operatorname{ind}_\pm \mathcal{A} = h_\pm(C)$. Without loss of generality, assume that $\mathcal{H}$ is a Hilbert space. For suppose the conclusion holds in this case. In the general case when $\mathcal{H}$ is a Kreĭn space, by Corollary 2.5, C is congruent to a selfadjoint operator D on a Hilbert space $\mathcal{K}$ such that $h_\pm(D) = h_\pm(C)$. Suppose $C = X^*DX$, where $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is invertible. Since $X^{*\,-1} = X^{-1\,*}$,

$$D = X^{*\;-1} C X^{-1} = X^{-1\,*} AA^* X^{-1} = BB^*,$$

where $B = X^{-1\,*}A \in \mathcal{L}(\mathcal{A}, \mathcal{K})$ and $\ker B = \{0\}$. Since $\mathcal{K}$ is a Hilbert space, $h_\pm(D) = \operatorname{ind}_\pm \mathcal{A}$ by the special case, and hence $h_\pm(C) = \operatorname{ind}_\pm \mathcal{A}$.

Thus it can be assumed that $\mathcal{H}$ is a Hilbert space. It can be further assumed that $\ker C = \{0\}$. For suppose the conclusion is known in this case. Consider a selfadjoint operator C on a Hilbert space $\mathcal{H}$ with possibly $\ker C \neq \{0\}$. Let $\mathcal{B}$ be a Kreĭn space, $B \in \mathcal{L}(\mathcal{B}, \mathcal{H})$, $\ker B = \{0\}$, and $C = BB^*$. Using the spectral theorem, write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$ and $\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm$

and $\mathcal{H}_0 = \ker C$ are the spectral subspaces for C for $(0,\infty)$, $(-\infty,0)$, and $\{0\}$. Notice that

$$B(B^*\mathcal{H}) = C\mathcal{H} \subseteq \mathcal{H}_1,$$

that is, B maps the dense subspace $B^*\mathcal{H}$ of $\mathcal{B}$ into $\mathcal{H}_1$. Therefore $\operatorname{ran} B \subseteq \mathcal{H}_1$. Set $C_1 = E^*CE = B_1B_1^*$, where $E \in \mathcal{L}(\mathcal{H}_1,\mathcal{H})$ is the natural embedding of $\mathcal{H}_1$ into $\mathcal{H}$, and $B_1 = E^*B \in \mathcal{L}(\mathcal{B},\mathcal{H}_1)$. Then E^* is orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$, and $\ker B_1 = \{0\}$ because $\operatorname{ran} B \subseteq \mathcal{H}_1$ and E^* is the identity on $\mathcal{H}_1$. Moreover, $C_1 = C\big|\mathcal{H}_1$ and hence $\ker C_1 = \{0\}$. Since the result is granted in the zero kernel case, $\operatorname{ind}_\pm \mathcal{B} = h_\pm(C_1)$. Then noting that $h_\pm(C_1) = \dim \mathcal{H}_\pm = h_\pm(C)$, one obtains $\operatorname{ind}_\pm \mathcal{B} = h_\pm(C)$, as was to be shown.

For the remaining case, let $\mathcal{H}$ be a Hilbert space, and assume $\ker C = \{0\}$. Let $\mathcal{A}$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{A},\mathcal{H})$ an operator such that $\ker \mathcal{A} = \{0\}$ and $C = AA^*$. Choose a fundamental symmetry $J_\mathcal{A}$ for $\mathcal{A}$ and corresponding associated Hilbert space $|\mathcal{A}|$. Then $C = AJ_\mathcal{A}A^\times$, where $A^\times \in \mathcal{L}(\mathcal{H},|\mathcal{A}|)$ is the adjoint of A viewed as an operator in $\mathcal{L}(|\mathcal{A}|,\mathcal{H})$. Since $\ker A = \{0\}$, $A^\times$ has dense range. Since $\ker C = \{0\}$, $A^\times$ has zero kernel. Hence by Theorem 2.10 applied with $T = A^\times \in \mathcal{L}(\mathcal{H},|\mathcal{A}|)$ gives $h_\pm(C) = h_\pm(J_\mathcal{A}) = \operatorname{ind}_\pm \mathcal{A}$, as was to be shown.

Sufficiency: Suppose first that $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is a Kreĭn space such that $\operatorname{ind}_\pm \mathcal{A} = h_\pm(C)$. By Theorem 2.4, $h_\pm(C) = \dim \mathcal{H}_\pm$, where $\mathcal{H}_\pm$ are the spectral subspaces of C for the intervals $(0,\infty)$ and $(-\infty,0)$. Since $\mathcal{A}$ can be replaced by an isomorphic copy and $h_\pm(C) = \dim \mathcal{H}_\pm$, one is free to replace $\mathcal{A}$ by $\mathcal{A} = \mathcal{H}_+ + \mathcal{H}_-$ with fundamental symmetry $J_\mathcal{A}(x_+ + x_-) = x_+ - x_-$, $x_\pm \in \mathcal{H}_\pm$, and inner product

$$\langle x,y\rangle_\mathcal{A} = \langle J_\mathcal{A}x,y\rangle_\mathcal{H}, \qquad x,y \in \mathcal{A}.$$

Define $A \in \mathcal{L}(\mathcal{A},\mathcal{H})$ by $Ax = |C|^{\frac{1}{2}}x$ for all $x \in \mathcal{A}$. Then $\ker A = \{0\}$ and $AA^*h = |C|^{\frac{1}{2}}J_\mathcal{A}|C|^{\frac{1}{2}}h = Ch$ for all $h \in \mathcal{H}$. Thus $C = AA^*$, where $A \in \mathcal{L}(\mathcal{A},\mathcal{H})$ and $\ker A = \{0\}$, as required. The general case of sufficiency, in which $\mathcal{H}$ is a Kreĭn space, can be reduced to the Hilbert space case using Corollary 2.5, in a similar manner to the argument given above. □

A Kreĭn space $\mathcal{A}$ is **continuously contained** in a Kreĭn space $\mathcal{H}$ if $\mathcal{A}$ is a linear subspace of $\mathcal{H}$ and the inclusion mapping A from $\mathcal{A}$ into $\mathcal{H}$ is continuous.

Theorem 3.2. *If $\mathcal{A}$ is a Kreĭn space which is continuously contained in a Kreĭn space $\mathcal{H}$ with inclusion mapping A, then* $\operatorname{ind}_\pm \mathcal{A} = h_\pm(C)$*, where $C = AA^*$. Every selfadjoint operator C on $\mathcal{H}$ arises in this way.*

Proof. The first statement is a special case of Theorem 3.1. For the second, first factor the given selfadjoint operator C in the form $C = AA^*$, where $A \in \mathcal{L}(\mathcal{A},\mathcal{H})$ for some Kreĭn space $\mathcal{A}$ and $\ker A = \{0\}$, Then replace $\mathcal{A}$ by the range of A in the inner product that makes A an isomorphism. □

Heinz Langer took an early interest in continuously contained Kreĭn spaces and de Branges' theory [9] of complementation of contractively contained spaces (1988 private communication to Dritschel and Rovnyak). He pursued this topic to great depth in joint work with Branko Ćurgus [7, 8], showing novel interactions between de Branges' theory, definitizable selfadjoint operators [16], and the problem of Kreĭn space completions of nondegenerate inner product spaces. Details are beyond the present scope, but a re-examination of these topics may well shed further light on this area.

Data Availability No data were used or generated in this work.

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