AN INDEFINITE ANALOG OF SARASON'S GENERALIZED INTERPOLATION THEOREM

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Abstract. By Sarason's generalized interpolation theorem, a contraction operator R on a model space that commutes with the compression T of the shift operator has the form $R = f(T)$, where f is a Schur function on the unit disk. An analogous result is obtained when $1 - RR^*$ has κ negative squares. By a theorem of Alpay, Dijksma, and the author, such an operator R satisfies an operator equation that depends on a finite Blaschke product and a Schur function. An analysis of the operator equation depends on properties of the root subspaces for the operator T. Such a study is carried out in a broader class of spaces, namely, $\mathcal{H}(B)$ spaces that satisfy the identity for difference quotients. Specialized to the inner case, these results yield an indefinite analog of Sarason's theorem. A parallel result holds when R commutes with T^* and $1 - R^*R$ has κ negative squares.

1. Introduction and preliminaries

This paper is an excursion in the theory of $\mathcal{H}(B)$ spaces [6], motivated by a problem in indefinite interpolation. The main application is to spaces that are contained isometrically in the Hardy space H^2 and is based on a theorem by Alpay, Dijksma, and the author [1]. This result appears in Theorem 3.4 and may be viewed as an indefinite analog of Donald Sarason's generalized interpolation theorem [11].

For notational reasons, $\mathcal{H}(B)$ spaces are here denoted $\mathcal{H}(C)$. If C is an analytic function which is defined and bounded by one on the unit disk $\mathbb{D}$, $\mathcal{H}(C)$ is the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel

$$K_C(w,z) = \frac{1 - C(z)\overline{C(w)}}{1 - z\bar{w}}, \qquad w, z \in \mathbb{D}.$$

Equivalently, $\mathcal{H}(C)$ is the space of all functions h in the Hardy space H^2 such that

$$\|h\|_C^2 = \sup\left[\|h + Cg\|_2^2 - \|g\|_2^2\right] < \infty, \tag{1.1}$$

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where the supremum is over all g in H^2 and $\|\cdot\|_2$ is the norm of H^2. We assume basic familiarity with such spaces, as may be found, for example, in [2, 4, 5, 6, 8, 13, 17]. Every space $\mathcal{H}(C)$ contains all difference quotients

$$\frac{h(z)-h(w)}{z-w} \quad \text{and} \quad \frac{C(z)-C(w)}{z-w}$$

whenever $h(z)$ is in the space and $w \in \mathbb{D}$. The formula

$$T\colon h(z) \to zh(z) - C(z)\left\langle h(z), \frac{C(z)-C(0)}{z} \right\rangle_C \tag{1.2}$$

defines a contraction operator on $\mathcal{H}(C)$ with adjoint

$$T^*\colon h(z) \to \frac{h(z)-h(0)}{z}. \tag{1.3}$$

By the difference-quotient inequality (see (2.3) below), the only h in $\mathcal{H}(C)$ such that $\|T^{*n}h\|_C = \|h\|_C$ for all $n \geqslant 1$ is $h = 0$. Therefore the operators T and T^* are completely nonunitary by [16, Ch. I, Th. 3.2]. This allows us to use the functional calculus of Schreiber [14] and Sz.-Nagy and Foias [15, 16]: for any completely nonunitary operator A on a Hilbert space $\mathcal{H}$ and any H^∞ function $f(z) = \sum_{j=0}^{\infty} f_j z^j$ on the unit disk, a bounded operator $f(A)$ on $\mathcal{H}$ is defined by the formula

$$f(A) = \operatorname*{s-lim}_{r\uparrow 1} \sum_{j=0}^{\infty} f_j\, r^j A^j. \tag{1.4}$$

The operator $f(A)$ commutes with A and satisfies

$$\|f(A)\| \leqslant \|f\|_\infty. \tag{1.5}$$

The correspondence $f \to f(A)$ is an algebra homomorphism from H^∞ into $\mathcal{L}(\mathcal{H})$; see [16, Ch. III, Th. 2.1].

The inner case and Sarason's theorem. When C is an inner function, $\mathcal{H}(C)$ is a model space and contained isometrically in H^2 as $H^2 \ominus CH^2$. Let S be the operator multiplication by z on H^2, and let P_C be the selfadjoint projection on H^2 with range $\mathcal{H}(C)$. Then

$$T = P_C S|_{\mathcal{H}(C)}.$$

For any H^∞ function f on $\mathbb{D}$, $f(S)$ is multiplication by $f(z)$ on H^2. One can also show that

$$f(T) = P_C f(S)|_{\mathcal{H}(C)}.$$

The operator $f(T)$ commutes with T. Sarason's generalized interpolation theorem [11, Th. 1] states that, conversely, if R is a bounded operator on $\mathcal{H}(C)$ that commutes with T, there is an $f \in H^\infty$ such that $R = f(T)$, and f can be chosen such that $\|f\|_\infty = \|R\|$. See [7, Th. 14.38] and [9, Th. 10.9] for recent accounts.

Sarason's theorem can be restated in an equivalent form for contraction operators.

Theorem 1.1. *Let C be an inner function. If R is a contraction operator on $\mathcal{H}(C)$ that commutes with T, there is a Schur function f on $\mathbb{D}$ such that $R = f(T)$.*

The **Schur class** is the set of analytic functions that are defined and bounded by one on the unit disk.

Clearly Theorem 1 in [11] implies Theorem 1.1. Conversely, assume Theorem 1.1. Let R be any nonzero bounded operator on $\mathcal{H}(C)$ that commutes with T. Then $R_1 = R/\|R\|$ is a contraction that commutes with T. By Theorem 1.1, $R_1 = g(T)$ for some Schur function g. Then

$$R = \|R\|R_1 = \|R\|g(T).$$

Thus $R = f(T)$, where $f(z) = \|R\|g(z)$ is in H^∞ and $\|f\|_\infty \leqslant \|R\|$. Since $R = f(T)$, the reverse inequality $\|R\| = \|f(T)\| \leqslant \|f\|_\infty$ follows from (1.5). Therefore $\|R\| = \|f\|_\infty$, which yields Theorem 1 in [11].

An indefinite generalization of Theorem 1.1. Throughout this work, the letter κ denotes a nonnegative integer. If H is a selfadjoint operator on a Hilbert space, we say that H has κ **negative squares** and write $\mathrm{sq}_-\, H = \kappa$, if the negative spectrum of H consists of a finite number of eigenvalues of total multiplicity κ. The following result is proved in [3] by an abstract method that uses a commutant lifting theorem.

Theorem 1.2 ([1, Th. 3.7])**.** *Let C be an inner function, and let R be a bounded operator on $\mathcal{H}(C)$ that commutes with T and satisfies*

$$\mathrm{sq}_-\,(1 - RR^*) = \kappa$$

for some nonnegative integer κ. Then there exist a Blaschke product B of degree κ and a Schur function f such that

$$B(T)R = f(T).$$

Conversely, if such f and B exist, $1 - RR^*$ has at most κ negative squares.

The case $\kappa = 0$ in Theorem 1.2 reduces to Theorem 1.1 since a Blaschke product of degree zero is a constant of modulus one.

In this paper we resolve a question left open in Theorem 1.2: what does the operator equation $B(T)R = f(T)$ actually say about R? The results needed to answer this question hold more generally in a broad class of spaces $\mathcal{H}(C)$. These are collected in Section 2, which is a study of the root subspaces of the operators T and T^* in any space $\mathcal{H}(C)$ that satisfies the identity for difference quotients.

In Section 3, the results on root subspaces are applied to the operator equation $B(T)R = f(T)$, yielding our main results, Theorems 3.2 and 3.4. Briefly, the operator R in Theorem 1.2 is determined on a subspace of $\mathcal{H}(C)$ of codimension at most κ, and on this subspace R is given by an explicit formula that depends on f and B.

Section 4 formulates dual results, Theorems 4.2 and 4.3, which hold for operators R that commute with T^*. The identity $B(T)R = f(T)$ is replaced by $B(T^*)R = f(T^*)$, and the condition $\operatorname{sq}_-(1 - RR^*) = \kappa$ is replaced by $\operatorname{sq}_-(1 - R^*R) = \kappa$. The dual results are derived from an interplay between the spaces $\mathcal{H}(C)$ and $\mathcal{H}(\widetilde{C})$, where $\widetilde{C}(z) = \overline{C(\bar{z})}$, which is described in Appendix A.

2. The root subspaces of T and T^*

For any space $\mathcal{H}(C)$, the operator T in (1.2) is also written in the form

$$\begin{aligned} &T\colon h(z) \to zh(z) - C(z)\widetilde{h}(0), \\ &\text{where } \widetilde{h}(0) = \left\langle h(z), \frac{C(z) - C(0)}{z} \right\rangle_C . \end{aligned} \tag{2.1}$$

More generally, for each $w \in \mathbb{D}$, the difference-quotient operator

$$R(w)\colon h(z) \to \frac{h(z) - h(w)}{z - w}$$

is everywhere defined and bounded on $\mathcal{H}(C)$, and

$$\begin{aligned} &R(w)^*\colon h(z) \to \frac{zh(z) - C(z)\widetilde{h}(\bar{w})}{1 - \bar{w}z}, \\ &\text{where } \widetilde{h}(\bar{w}) = \left\langle h(z), \frac{C(z) - C(w)}{z - w} \right\rangle_C . \end{aligned} \tag{2.2}$$

The formula (2.2) is given in [6, Prob. 85] and [2, p. 89]; we caution that in [2] the roles of T and T^* are reversed from their meaning here.

For any space $\mathcal{H}(C)$, the **difference-quotient inequality**

$$\left\| \frac{h(z) - h(0)}{z} \right\|_C^2 \leqslant \|h(z)\|_C^2 - |h(0)|^2 \tag{2.3}$$

holds for every h in $\mathcal{H}(C)$. The case of equality is characterized in [6, Th. 16], [12, p. 158], and [8, Cor. 25.15]:

The **difference-quotient identity**

$$\left\| \frac{h(z) - h(0)}{z} \right\|_C^2 = \|h(z)\|_C^2 - |h(0)|^2$$

holds for every h in $\mathcal{H}(C)$ if and only if $C \notin \mathcal{H}(C)$, or equivalently, C is an extreme point of the unit ball in H^∞.

The difference-quotient identity has an inner product form by the polarization identity. Less obvious inner product identities are known. For any space $\mathcal{H}(C)$ such that $C \notin \mathcal{H}(C)$, and for all $h \in \mathcal{H}(C)$ and $\alpha, \beta \in \mathbb{D}$,

$$\begin{aligned} \overline{C(\beta)}h(\alpha) = \bar{\beta} &\left\langle h(z), \frac{C(z) - C(\beta)}{z - \beta} \right\rangle_C \\ &- (1 - \alpha\bar{\beta}) \left\langle \frac{h(z) - h(\alpha)}{z - \alpha}, \frac{C(z) - C(\beta)}{z - \beta} \right\rangle_C \end{aligned} \tag{2.4}$$

and

$$\frac{1-\overline{C(\beta)}C(\alpha)}{1-\alpha\bar{\beta}} = \left\langle \frac{C(z)-C(\alpha)}{z-\alpha}, \frac{C(z)-C(\beta)}{z-\beta} \right\rangle_C . \tag{2.5}$$

See Problems 88 and 89 in [6] and Theorems 3.2.4 and 3.2.5 in [2]. We use these identities in a special case.

Lemma 2.1. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$. Then*

$$\left\langle \frac{h(z)-h(w)}{z-w}, \frac{C(z)-C(0)}{z} \right\rangle_C = -\overline{C(0)}h(w), \tag{2.6}$$

$$\left\langle \frac{C(z)-C(w)}{z-w}, \frac{C(z)-C(0)}{z} \right\rangle_C = 1-\overline{C(0)}C(w), \tag{2.7}$$

for all h in $\mathcal{H}(C)$ and w in $\mathbb{D}$.

The spectrum and resolvent of T^* in the extreme point case are described by Sarason [13, pp. 41–42]. The formulas in Lemma 2.1 provide another approach.

Theorem 2.2. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$. If $w \in \mathbb{D}$ and $C(w) \neq 0$, then $T-w$ is invertible and*

$$(T-w)^{-1}\colon h(z) \to \frac{h(z)-h(w)C(z)/C(w)}{z-w}. \tag{2.8}$$

In fact, the operator

$$h(z) \to \frac{h(z)-h(w)C(z)/C(w)}{z-w} \tag{2.9}$$

is a left inverse of $T-w$ whether or not $C \notin \mathcal{H}(C)$. Two-sided invertibility, however, requires the assumption that $C \notin \mathcal{H}(C)$.

Proof. We show that (2.9) is both a left and right inverse of $T-w$. For any $h(z)$ in $\mathcal{H}(C)$, the function

$$\begin{aligned} k(z) &= \frac{h(z)-h(w)C(z)/C(w)}{z-w} \\ &= \frac{h(z)-h(w)}{z-w} - \frac{h(w)}{C(w)}\,\frac{C(z)-C(w)}{z-w} \end{aligned}$$

is in $\mathcal{H}(C)$, and by (2.6) and (2.7),

$$\begin{aligned}(T-w)k(z) &= (z-w)k(z) - C(z)\left\langle k(z), \frac{C(z)-C(0)}{z}\right\rangle_C \\ &= (z-w)k(z) - C(z)\left\langle \frac{h(z)-h(w)}{z-w}, \frac{C(z)-C(0)}{z}\right\rangle_C \\ &\qquad + C(z)\frac{h(w)}{C(w)}\left\langle \frac{C(z)-C(w)}{z-w}, \frac{C(z)-C(0)}{z}\right\rangle_C \\ &= (z-w)k(z) - C(z)\Big[-\overline{C(0)}h(w)\Big] \\ &\qquad + C(z)\frac{h(w)}{C(w)}\Big[1-\overline{C(0)}C(w)\Big] \\ &= h(z).\end{aligned}$$

Thus (2.9) is a right inverse of $T-w$. If $h(z)$ is in $\mathcal{H}(C)$ and

$$\begin{aligned}k(z) &= (T-w)h(z) \\ &= (z-w)h(z) - C(z)\left\langle h(z), \frac{C(z)-C(0)}{z}\right\rangle_C,\end{aligned}$$

then

$$\begin{aligned}&\frac{k(z)-k(w)C(z)/C(w)}{z-w} \\ &\qquad = \frac{1}{z-w}\left[(z-w)h(z) - C(z)\left\langle h(z), \frac{C(z)-C(0)}{z}\right\rangle_C \right. \\ &\qquad\qquad \left. + C(w)\left\langle h(z), \frac{C(z)-C(0)}{z}\right\rangle_C \frac{C(z)}{C(w)}\right] \\ &\qquad = h(z).\end{aligned}$$

Therefore (2.9) is a left inverse of $T-w$. □

The **geometric multiplicity** of an eigenvalue α of any Hilbert space operator A is the dimension of $\ker(A-\alpha)$. Any vector $f \neq 0$ such that $(A-\alpha)^n f = 0$ for some $n \geqslant 1$ is called a **root vector**. If the subspace consisting of all root vectors plus the zero vector has finite dimension, it is called the **root subspace**, and its dimension is the **algebraic multiplicity** of α.

In a space $\mathcal{H}(C)$ with C inner, the root subspaces for T and T^* are described by Nikol′skiĭ [10, Lecture IV]. Here we consider spaces $\mathcal{H}(C)$ that satisfy the identity for difference quotients (the extreme point case).

Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and suppose $\alpha \in \mathbb{D}$ is a zero of C. Then

(1) α is an eigenvalue of T, and $\ker(T-\alpha) = \left[\dfrac{C(z)}{z-\alpha}\right]$;

(2) $\bar{\alpha}$ is an eigenvalue of T^*, and $\ker(T^*-\bar{\alpha}) = \left[\dfrac{1}{1-\bar{\alpha}z}\right]$.

Square brackets around any set of vectors denote the linear span of the vectors. See [13, Ch. V], [8, Th. 26.1 and Cor. 26.3], and for the inner case also [9, Th. 9.22 and Cor. 9.25]. We extend these results to arbitrary root subspaces.

Theorem 2.3. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let α be a zero of C of order n. Then α is an eigenvalue for T with geometric multiplicity one and algebraic multiplicity n. Moreover:*

(1) *The functions*

$$Q_j(z) = \frac{C(z)}{(z-\alpha)^j}, \qquad j = 1, \dots, n, \tag{2.10}$$

are root vectors for the operator T and eigenvalue α such that

$$\begin{aligned} &(T-\alpha)Q_1 = 0, \\ &(T-\alpha)Q_j = Q_{j-1}, \quad j = 2, \dots, n. \end{aligned} \tag{2.11}$$

(2) *The subspaces $\mathcal{R}_k = \ker(T-\alpha)^k$, $k \geqslant 1$, are given by*

$$\mathcal{R}_k = \begin{cases} [Q_1, \dots, Q_k], & k = 1, \dots, n, \\ \mathcal{R}_n, & k > n. \end{cases}$$

Thus the root subspace for T and eigenvalue α is $\mathcal{R}_n$.

Proof. By (1.2), for all $u \in \mathcal{H}(C)$,

$$(T-\alpha)u(z) = (z-\alpha)u(z) - C(z)\left\langle u(z), \frac{C(z)-C(0)}{z} \right\rangle_C. \tag{2.12}$$

(1) Since C has a zero of order n at α, $C(\alpha) = C'(\alpha) = \cdots = C^{(j-1)}(\alpha) = 0$ and hence $Q_1, \cdots, Q_n$ are given recursively by

$$\begin{aligned} Q_1(z) &= \frac{C(z)-C(\alpha)}{z-\alpha}, \\ Q_j(z) &= R(\alpha)Q_{j-1}(\alpha), \qquad j = 2, \dots, n. \end{aligned} \tag{2.13}$$

Therefore $Q_1, \dots, Q_n$ belong to $\mathcal{H}(C)$. By (2.1) and (2.7),

$$\begin{aligned} (T-\alpha)Q_1(z) &= (z-\alpha)Q_1(z) - C(z)\left\langle \frac{C(z)-C(\alpha)}{z-\alpha}, \frac{C(z)-C(0)}{z} \right\rangle_C \\ &= C(z) - C(z)\big[1 - \overline{C(0)}C(\alpha)\big] \\ &= 0, \end{aligned}$$

which is the first relation in (2.11). For $j = 2, \ldots, n$, by (2.6),

$$\begin{aligned}(T-\alpha)Q_j(z) &= (z-\alpha)Q_j(z) \\ &\qquad - C(z)\left\langle \frac{Q_{j-1}(z)-Q_{j-1}(\alpha)}{z-\alpha}, \frac{C(z)-C(0)}{z} \right\rangle_C \\ &= \frac{C(z)}{(z-\alpha)^{j-1}} + C(z)\overline{C(0)}Q_{j-1}(\alpha) \\ &= Q_{j-1}(z)\end{aligned}$$

because $Q_{j-1}(\alpha) = 0$, yielding the second relation in (2.11). By (2.11), $(T-\alpha)^k Q_k = 0$ for each $k = 1, \ldots, n$. Thus $Q_1, \ldots, Q_n$ are root vectors for T, and (1) follows.

(2) Set $\mathcal{R}_k = \ker(T-\alpha)^k$, $k \geqslant 1$. For all $k = 1, \ldots, n$, $\mathcal{R}_k \supseteq [Q_1, \ldots, Q_k]$ by part (1). Equality holds when $k = 1$, because if $u(z)$ is in $\ker(T-\alpha)$, then $(z-\alpha)u(z) - C(z)\widetilde{u}(0) = 0$ by (2.1), and hence $u(z)$ is a constant multiple of $Q_1(z) = C(z)/(z-\alpha)$. We show that equality also holds for each $k = 2, \ldots, n$.

Assume that for some $\mathcal{R}_k = [Q_1, \ldots, Q_k]$ for some $k = 1, \ldots, n-1$. Suppose $u \in \mathcal{R}_{k+1}$. Then

$$(T-\alpha)^k (T-\alpha)u = 0, \tag{2.14}$$

and therefore $(T-\alpha)u \in \mathcal{R}_k = [Q_1, \ldots, Q_k]$, say $(T-\alpha)u = \sum_{j=1}^k \gamma_j Q_j$, where $\gamma_1, \ldots, \gamma_k$ are constants. Then

$$(z-\alpha)u(z) - C(z)\widetilde{u}(0) = \sum_{j=1}^k \gamma_j \frac{C(z)}{(z-\alpha)^j},$$

and hence

$$u(z) = \frac{C(z)}{z-\alpha}\widetilde{u}(0) + \sum_{j=1}^k \gamma_j \frac{C(z)}{(z-\alpha)^{j+1}}. \tag{2.15}$$

Thus $u \in [Q_1, \ldots, Q_k, Q_{k+1}]$, and so $\mathcal{R}_{k+1} \subseteq [Q_1, \ldots, Q_k, Q_{k+1}]$. The reverse inclusion holds since $k+1 \leqslant n$, and hence equality holds. Therefore by induction, $\mathcal{R}_k = [Q_1, \ldots, Q_k]$ for all $k = 1, \ldots, n$.

We show that $\mathcal{R}_k = \mathcal{R}_n$ for all $k > n$. First let $k = n+1$. The inclusion $\mathcal{R}_{n+1} \supseteq \mathcal{R}_n$ is trivial. Let $u \in \mathcal{R}_{n+1}$, and repeat the steps (2.14) through (2.15) with $k = n$ to obtain

$$u(z) = \frac{C(z)}{z-\alpha}\widetilde{u}(0) + \sum_{j=1}^n \gamma_j \frac{C(z)}{(z-\alpha)^{j+1}}.$$

Then $\gamma_n = 0$ because u is analytic at $z = \alpha$ and $C^{(n)}(\alpha) \neq 0$ by our assumption that C has a zero at α of order n. Therefore u is in $[Q_1, \ldots, Q_n] = \mathcal{R}_n$, and hence $\mathcal{R}_{n+1} = \mathcal{R}_n$.

We proceed by induction. Assume that $\mathcal{R}_k = \mathcal{R}_n$ for some $k > n$. The inclusion $\mathcal{R}_{k+1} \supseteq \mathcal{R}_n$ is again trivial. Let $u \in \mathcal{R}_{k+1}$. Then

$$(T - \alpha)^k (T - \alpha) u = 0,$$

and so $(T - \alpha)u \in \mathcal{R}_k$. By our inductive assumption, this implies that $(T - \alpha)u \in \mathcal{R}_n$ and hence $u \in \mathcal{R}_{n+1}$. We showed above that $\mathcal{R}_{n+1} = \mathcal{R}_n$, and therefore $u \in \mathcal{R}_n$. Thus $\mathcal{R}_{k+1} \subseteq \mathcal{R}_n$ and hence $\mathcal{R}_{k+1} = \mathcal{R}_n$. This completes the inductive step. Therefore $\mathcal{R}_k = \mathcal{R}_n$ for all $k > n$. Part (2) of the theorem follows.

By (2.11), α is an eigenvalue for T. By part (2) its geometric multiplicity is one, and its algebraic multiplicity is n. □

Corollary 2.4. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let $Q_1, \dots, Q_n$ be the functions* (2.10) *corresponding to some zero α of C of order n. Suppose $f \in H^\infty$ and*

$$f(T)Q_k = 0$$

for some k, $1 \leqslant k \leqslant n$. Then f has a zero at α of order at least k.

Proof. Define $g \in H^\infty$ by

$$f(z) = f(\alpha) + f'(\alpha)(z - \alpha) + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}(z - \alpha)^{k-1} + (z - \alpha)^k g(z).$$

Then

$$f(T) = f(\alpha) + f'(\alpha)(T - \alpha) + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}(T - \alpha)^{k-1} + (T - \alpha)^k g(T).$$

By assumption, $f(T)Q_k = 0$. Hence by Theorem 2.3,

$$\begin{aligned} 0 &= f(\alpha)Q_k + f'(\alpha)Q_{k-1} + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}Q_1 + (T - \alpha)^k g(T) Q_k \\ &= f(\alpha)Q_k + f'(\alpha)Q_{k-1} + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}Q_1, \end{aligned}$$

because $(T - \alpha)^k g(t) Q_k = g(T)(T - \alpha)^k Q_k = 0$. The functions $Q_1, \dots, Q_k$ are linearly independent, and therefore

$$f(\alpha) = f'(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0.$$

Thus f has a zero at α of order at least k. □

Theorem 2.3 has a companion for T^*.

Theorem 2.5. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let α be a zero of C of order n. Then $\bar{\alpha}$ is an eigenvalue for T^* with geometric multiplicity one and algebraic multiplicity n. Moreover:*

(1) *The functions*

$$P_j(z) = \frac{z^{\,j-1}}{(1-\bar{\alpha}z)^{\,j}}, \qquad j = 1, \dots, n, \tag{2.16}$$

are root vectors for the operator T^* and eigenvalue $\bar{\alpha}$ such that

$$\begin{aligned} &(T^* - \bar{\alpha})P_1 = 0, \\ &(T^* - \bar{\alpha})P_j = P_{j-1}, \quad j = 2, \dots, n. \end{aligned} \tag{2.17}$$

(2) *The subspaces $\widetilde{\mathcal{R}}_k = \ker(T^* - \bar{\alpha})^k$, $k \geqslant 1$, are given by*

$$\widetilde{\mathcal{R}}_k = \begin{cases} [P_1, \dots, P_k], & k = 1, \dots, n, \\ \widetilde{\mathcal{R}}_n, & k > n. \end{cases}$$

Thus the root subspace for T^* and eigenvalue $\bar{\alpha}$ is $\widetilde{\mathcal{R}}_n$.

Theorem 2.5 can be deduced from Theorem 2.3 by means of the unitary equivalence in Theorem A.1 of Appendix A. Instead we give a direct proof, which also has some points of interest.

Lemma 2.6. *Let $B(z) = (z-\alpha)/(1-\bar{\alpha}z)$ for some $\alpha \in \mathbb{D}$. Then for any positive integer k, $\dim \mathcal{H}(B^k) = k$ and*

$$\mathcal{H}(B^k) = [P_1, \dots, P_k], \tag{2.18}$$

where $P_1, \dots, P_k$ are the functions (2.16). Moreover

$$\mathcal{H}(B^k) = \left\{ \frac{p(z)}{(1-\bar{\alpha}z)^k} : p \text{ is a polynomial of degree } \leqslant k-1 \right\}. \tag{2.19}$$

Lemma 2.7. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let α be a zero of C of order n. Then the functions (2.16) belong to $\mathcal{H}(C)$, but $z^n/(1-\bar{\alpha}z)^{n+1}$ is not in $\mathcal{H}(C)$.*

Proof of Lemma 2.6. As a preliminary, first note that the rational functions $P_1, P_2, P_3, \dots$ are linearly independent over the complex plane and hence over $\mathbb{D}$. This is clear if $\alpha = 0$. For $\alpha \neq 0$, linear independence follows from the fact that the P_j's have poles of different orders at $1/\bar{\alpha}$.

We prove the first assertion by induction. When $k = 1$,

$$\frac{1 - B(z)\overline{B(w)}}{1 - z\bar{w}} = \frac{1 - |\alpha|^2}{(1-\bar{\alpha}z)(1-\alpha\bar{w})}\,.$$

Hence $\mathcal{H}(B)$ is one-dimensional and satisfies (2.18). Suppose $\mathcal{H}(B^k)$ has dimension k and (2.18) holds for some $k \geqslant 1$. Since $B^{k+1} = B^k B$, by [6, Problems 48, 52],

$$\begin{aligned} \mathcal{H}(B^{k+1}) &= \mathcal{H}(B^k) + B^k \mathcal{H}(B) \\ &= [P_1, \dots, P_k] + B^k [P_1] \end{aligned} \tag{2.20}$$

as linear spaces. Therefore $\dim \mathcal{H}(B^{k+1}) \leqslant k+1$ and the function $(z-\alpha)^k/(1-\bar{\alpha}z)^{k+1}$ belongs to $\mathcal{H}(B^{k+1})$. Since $\mathcal{H}(B^{k+1})$ is invariant under $R(\alpha)$, it contains the $k+1$ functions

$$\frac{1}{(1-\bar{\alpha}z)^{k+1}}, \frac{z-\alpha}{(1-\bar{\alpha}z)^{k+1}}, \dots, \frac{(z-\alpha)^k}{(1-\bar{\alpha}z)^{k+1}}.$$

The linear span of these functions includes P_{k+1}, and hence

$$\mathcal{H}(B^{k+1}) \supseteq [P_1, \dots, P_k, P_{k+1}].$$

Since $P_1, \dots, P_k, P_{k+1}$ are linearly independent, $\dim \mathcal{H}(B^{k+1}) = k+1$ and $\mathcal{H}(B^{k+1}) = [P_1, \dots, P_k, P_{k+1}]$. The first assertion of the lemma follows by induction.

Let $\mathcal{M}$ denote the right side of (2.19). Clearly $\dim \mathcal{M} = k$ and $\mathcal{M}$ contains $P_1, \dots, P_k$. Hence $\mathcal{H}(B^k) \subseteq \mathcal{M}$ by (2.18). Since $\mathcal{H}(B^k)$ and $\mathcal{M}$ both have dimension k, (2.19) follows. □

Proof of Lemma 2.7. Set $B(z) = (z-\alpha)/(1-\bar{\alpha}z)$, Since C has a zero of order n at α,

$$C(z) = B(z)^n C_1(z),$$

where a space $\mathcal{H}(C_1)$ exists and $C_1(\alpha) \neq 0$. By [6, Problems 48, 52],

$$\mathcal{H}(C) = \mathcal{H}(B^n) + B^n \mathcal{H}(C_1) \tag{2.21}$$

as linear spaces. Therefore $P_1, \dots, P_n$ belong to $\mathcal{H}(C)$ by Lemma 2.6.

To see that $z^n/(1-\bar{\alpha}z)^{n+1} \notin \mathcal{H}(C)$, we argue by contradiction. Assume that $z^n/(1-\bar{\alpha}z)^{n+1} \in \mathcal{H}(C)$. Then $z^j/(1-\bar{\alpha}z)^{n+1} \in \mathcal{H}(C)$ for all $j = 0, \dots, n$ because $\mathcal{H}(C)$ is invariant under the formation of difference quotients at the origin. Hence

$$\frac{(z-\alpha)^n}{(1-\bar{\alpha}z)^{n+1}} \in \mathcal{H}(C).$$

By (2.21),

$$\frac{(z-\alpha)^n}{(1-\bar{\alpha}z)^{n+1}} = h(z) + B(z)^n k(z). \tag{2.22}$$

for some $h(z)$ in $\mathcal{H}(B^n)$ and $k(z) \in \mathcal{H}(C_1)$. By Lemma 2.6,

$$h(z) = \frac{p(z)}{(1-\bar{\alpha}z)^n},$$

where $p(z)$ is a polynomial of degree at most $n-1$. If $p(z)$ is not identically zero, it has a zero at α of order at least n, which is impossible because $\deg p \leqslant n-1$. Therefore $p(z) \equiv 0$, so $h(z) \equiv 0$ and

$$k(z) = \frac{1}{1-\bar{\alpha}z} \in \mathcal{H}(C_1).$$

Let T_1 and T_1^* be the operators (1.2) and (1.3) on $\mathcal{H}(C_1)$. Then $k(z)$ is an eigenfunction for T_1^* and eigenvalue $\bar{\alpha}$. Hence $T_1 - \alpha$ is not invertible. By Theorem 2.2, this implies $C_1(\alpha) = 0$, a contradiction. Therefore $z^n/(1-\bar{\alpha}z)^{n+1} \notin \mathcal{H}(C)$. □

Proof of Theorem 2.5. For every $u \in \mathcal{H}(C)$,

$$(T^* - \bar{\alpha})u(z) = \frac{(1-\bar{\alpha}z)u(z) - u(0)}{z}, \tag{2.23}$$

by (1.3).

(1) The functions $P_1, \dots, P_n$ belong to $\mathcal{H}(C)$ by Lemma 2.7, and (2.17) follows by routine algebra using (2.23). The relations (2.17) imply that $(T^* - \bar{\alpha})^k P_k = 0$ for all $k = 1, \dots, n$, and so $P_1, \dots, P_n$ are root vectors for T^*.

(2a) Set $\widetilde{\mathcal{R}}_k = \ker(T^* - \bar{\alpha})^k$, $k = 1, 2, 3, \dots$. We show that $\widetilde{\mathcal{R}}_k = [P_1, \dots, P_k]$ for $k = 1, \dots, n$. By part (1),

$$\widetilde{\mathcal{R}}_k \supseteq [P_1, \dots, P_k], \qquad k = 1, \dots, n, \tag{2.24}$$

Equality holds in (2.24) when $k = 1$. For if $u \in \widetilde{\mathcal{R}}$, then $(T^* - \bar{\alpha})u = 0$, so by (2.23), $u(z) = u(0)/(1-\bar{\alpha}z) = u(0)P_1(z)$. We show that equality also holds for $k = 2, \dots, n$.

Suppose $\widetilde{\mathcal{R}}_k = [P_1, \dots, P_k]$ for some $k = 1, \dots, n-1$. By (2.24) with k replaced by $k+1$, $\widetilde{\mathcal{R}}_{k+1} \supseteq [P_1, \dots, P_k, P_{k+1}]$. For any $u \in \mathcal{R}_{k+1}$,

$$(T^* - \bar{\alpha})^k (T^* - \bar{\alpha})u = 0.$$

Therefore $(T^* - \bar{\alpha})u \in \widetilde{\mathcal{R}}_k = [P_1, \dots, P_k]$, say $(T^* - \bar{\alpha})u = \sum_{j=1}^k \gamma_j P_j$, where $\gamma_1, \dots, \gamma_k$ are scalars. Then by (2.23),

$$\frac{(1-\bar{\alpha}z)u(z) - u(0)}{z} = \sum_{j=1}^k \gamma_j P_j(z),$$

and hence

$$\begin{aligned} u(z) &= \frac{u(0)}{1-\bar{\alpha}z} + \sum_{j=1}^k \gamma_j \frac{zP_j(z)}{1-\bar{\alpha}z} \\ &= u(0)P_1(z) + \sum_{j=1}^k \gamma_j P_{j+1}(z). \end{aligned}$$

Therefore $u \in [P_1, \dots, P_k, P_{k+1}]$, and so $\widetilde{\mathcal{R}}_{k+1} = [P_1, \dots, P_k, P_{k+1}]$. Hence by induction, $\widetilde{\mathcal{R}}_k = [P_1, \dots, P_k]$ for all $k = 1, \dots, n$.

(2b) We show by induction that $\widetilde{\mathcal{R}}_{n+k} = \widetilde{\mathcal{R}}_n$ for all $k \geqslant 1$. For the case $k = 1$, the inclusion $\widetilde{\mathcal{R}}_{n+1} \supseteq \widetilde{\mathcal{R}}_n$ is obvious. Let $u \in \widetilde{\mathcal{R}}_{n+1}$. Then

$$(T^* - \bar{\alpha})^n (T^* - \bar{\alpha}) u = 0.$$

Hence $(T^* - \bar{\alpha}) u \in \widetilde{\mathcal{R}}_n$, and so $(T^* - \bar{\alpha}) u = \sum_{j=1}^n \gamma_j P_j$, for some constants $\gamma_1, \dots, \gamma_n$ by part (1). By (2.23),

$$\frac{(1 - \bar{\alpha} z) u(z) - u(0)}{z} = \sum_{j=1}^n \gamma_j \frac{z^{j-1}}{(1 - \bar{\alpha} z)^j},$$

hence

$$\begin{aligned} u(z) &= \frac{u(0)}{1 - \bar{\alpha} z} + \sum_{j=1}^n \gamma_j \frac{z^j}{(1 - \bar{\alpha} z)^{j+1}} \\ &= u(0) P_1(z) + \sum_{j=1}^{n-1} \gamma_j P_{j+1}(z) + \gamma_n \frac{z^n}{(1 - \bar{\alpha} z)^{n+1}} . \end{aligned}$$

Here $\gamma_n = 0$ because otherwise $z^n/(1 - \bar{\alpha} z)^{n+1} \in \mathcal{H}(C)$, contradicting Lemma 2.7. Therefore u is in the span of $P_1, \dots, P_n$, that is, $u \in \mathcal{R}_n$, and thus $\widetilde{\mathcal{R}}_{n+1} = \widetilde{\mathcal{R}}_n$.

For the inductive step, assume that $\widetilde{\mathcal{R}}_{n+k} = \widetilde{\mathcal{R}}_n$ for some $k \geqslant 1$. Trivially, $\widetilde{\mathcal{R}}_{n+k+1} \supseteq \widetilde{\mathcal{R}}_n$. Let $u \in \widetilde{\mathcal{R}}_{n+k+1}$. Then

$$(T^* - \bar{\alpha})^{n+k} (T^* - \bar{\alpha}) u = 0,$$

so $(T^* - \bar{\alpha}) u \in \widetilde{\mathcal{R}}_{n+k}$. By our inductive assumption, $(T^* - \bar{\alpha}) u \in \widetilde{\mathcal{R}}_n$. Then $(T^* - \bar{\alpha})^{n+1} u = (T^* - \bar{\alpha})^n (T^* - \bar{\alpha}) u = 0$, and hence $u \in \widetilde{\mathcal{R}}_{n+1}$. Since we already showed that $\widetilde{\mathcal{R}}_{n+1} = \widetilde{\mathcal{R}}_n$, $u \in \widetilde{\mathcal{R}}_n$. Thus $\widetilde{\mathcal{R}}_{n+k} = \widetilde{\mathcal{R}}_n$. Therefore by induction, $\widetilde{\mathcal{R}}_{n+k} = \widetilde{\mathcal{R}}_n$ for all $k \geqslant 1$.

By (2.17), $\bar{\alpha}$ is an eigenvalue for T^*. By part (2) its geometric multiplicity is one, and its algebraic multiplicity is n. □

Theorem 2.8. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let α be a zero of C of order n. The functions $P_1, \dots, P_n$ in Theorem* 2.5 *are also given by*

$$P_k(z) = R(\alpha)^{* \, k-1} K_C(\alpha, z), \qquad k = 1, \dots, n. \tag{2.25}$$

Hence for each $k = 1, \dots, n$,

$$\begin{aligned} &\left[\ker (T^* - \bar{\alpha})^k \right]^\perp \\ &\qquad = \left\{ h \in \mathcal{H}(C) \colon h(\alpha) = h'(\alpha) = \cdots = h^{k-1}(\alpha) = 0 \right\}. \end{aligned} \tag{2.26}$$

Proof. Since $C(\alpha) = 0$, $K_C(\alpha, z) = 1/(1-\bar{\alpha}z)$, and so (2.25) holds for $k = 1$. Assume (2.25) holds for some $k = 1, \dots, n-1$. Then by (2.2),

$$\begin{aligned} R(\alpha)^{*\,k} K_C(\alpha, z) &= R(\alpha)^* P_k(z) \\ &= \frac{z}{1-\bar{\alpha}z} P_k(z) - \frac{C(z)}{1-\bar{\alpha}z} \widetilde{P}_k(\bar{\alpha}). \end{aligned}$$

By our inductive assumption and (2.13),

$$\begin{aligned} \widetilde{P}_k(\bar{\alpha}) &= \left\langle P_k(z), \frac{C(z) - C(\alpha)}{z - \alpha} \right\rangle_C \\ &= \left\langle R(\alpha)^{*\,k-1} K_C(\alpha, z), \frac{C(z) - C(\alpha)}{z - \alpha} \right\rangle_C \\ &= \left\langle K_C(\alpha, z), R(\alpha)^{k-1} Q_1(z) \right\rangle_C \\ &= \langle K_C(\alpha, z), Q_k(z) \rangle_C \\ &= \overline{Q_k(\alpha)} \\ &= 0. \end{aligned}$$

The last equality holds since $C(z)$ has a zero of order n at α and $Q_k(z) = C(z)/(z-\alpha)^k$ where $k \leqslant n-1$. Thus

$$R(\alpha)^{*\,k} K_C(\alpha, z) = \frac{z}{1-\bar{\alpha}z} P_k(z) = P_{k+1}(z).$$

The identity (2.25) follows by induction.

To prove (2.26), fix some $k = 1, \dots, n$. By Theorem 2.5, $\ker(T^* - \bar{\alpha})^k$ is spanned by $P_1, \dots, P_k$. Therefore by (2.25) a function h in $\mathcal{H}(C)$ is orthogonal to $\ker(T^* - \bar{\alpha})^k$ if and only if

$$\left\langle K_C(\alpha, z), R(\alpha)^{k-1} h(z) \right\rangle_C = 0, \qquad j = 1, \dots, n,$$

that is, $h(\alpha) = h'(\alpha) = \cdots = h^{k-1}(\alpha) = 0$. □

Theorem 2.9. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$. Let $\alpha_1, \dots, \alpha_r$ be distinct zeros of C. For any positive integers $m_1, \dots, m_r$,*

$$\ker \left[(T^* - \bar{\alpha}_1)^{m_1} \cdots (T^* - \bar{\alpha}_r)^{m_r} \right] = \sum_{j=1}^{r} \ker(T^* - \bar{\alpha}_j)^{m_j}. \tag{2.27}$$

Lemma 2.10. *For any space $\mathcal{H}(C)$ and $\alpha \in \mathbb{D}$, if $u \in \mathcal{H}(C)$ and*

$$(T^* - \bar{\alpha}) u(z) = v(z),$$

then

$$u(z) = \frac{u(0)}{1-\bar{\alpha}z} + \frac{z}{1-\bar{\alpha}z} v(z).$$

Lemma 2.11. *Let $\alpha, \beta \in \mathbb{D}$, $\alpha \neq \beta$. Then for all $m \geqslant 1$,*

$$\frac{z}{1-\bar{\alpha}z}\left[\frac{1}{1-\bar{\beta}z}, \frac{z}{(1-\bar{\beta}z)^2}, \dots, \frac{z^{m-1}}{(1-\bar{\beta}z)^m}\right] \subseteq \left[\frac{1}{1-\bar{\alpha}z}\right] + \left[\frac{1}{1-\bar{\beta}z}, \frac{z}{(1-\bar{\beta}z)^2}, \dots, \frac{z^{m-1}}{(1-\bar{\beta}z)^m}\right]. \tag{2.28}$$

Proof of Lemma 2.10. Solve $[u(z)-u(0)]/z - \bar{\alpha}u(z) = v(z)$ for $u(z)$. □

Proof of Lemma 2.11. When $m = 1$, this follows from the identity

$$\frac{z}{1-\bar{\alpha}z}\,\frac{1}{1-\bar{\beta}z} = \frac{1}{\bar{\alpha}-\bar{\beta}}\left[\frac{1}{1-\bar{\alpha}z} - \frac{1}{1-\bar{\beta}z}\right].$$

Assume (2.28) is known for some $m \geqslant 1$. Then to show that it holds with m replaced by $m+1$, it is sufficient to show that

$$\frac{z}{1-\bar{\alpha}z}\,\frac{z^m}{(1-\bar{\beta}z)^{m+1}} \in \left[\frac{1}{1-\bar{\alpha}z}\right] + \left[\frac{1}{1-\bar{\beta}z}, \dots, \frac{z^{m-1}}{(1-\bar{\beta}z)^m}, \frac{z^m}{(1-\bar{\beta}z)^{m+1}}\right].$$

In fact,

$$\begin{aligned} \frac{z}{1-\bar{\alpha}z}\,\frac{z^m}{(1-\bar{\beta}z)^{m+1}} &= \frac{z}{(1-\bar{\alpha}z)(1-\bar{\beta}z)}\,\frac{z^m}{(1-\bar{\beta}z)^m} \\ &= \frac{1}{\bar{\alpha}-\bar{\beta}}\left[\frac{1}{1-\bar{\alpha}z} - \frac{1}{1-\bar{\beta}z}\right]\frac{z^m}{(1-\bar{\beta}z)^m} \\ &= \frac{1}{\bar{\alpha}-\bar{\beta}}\,\frac{z}{1-\bar{\alpha}z}\,\frac{z^{m-1}}{(1-\bar{\beta}z)^m} - \frac{1}{\bar{\alpha}-\bar{\beta}}\,\frac{z^m}{(1-\bar{\beta}z)^{m+1}}\,. \end{aligned}$$

Both terms on the right side belong to the required set. □

Proof of Theorem 2.9. In the proof we denote by $n_1, \dots, n_r$ the orders of $\alpha_1, \dots, \alpha_r$ as zeros of C. The numbers $m_1, \dots, m_r$ are arbitrary positive integers. Set

$$k_j = \min(m_j, n_j), \qquad j = 1, \dots, r.$$

By Theorem 2.8(2), for all $j = 1, \dots, r$,

$$\begin{aligned} \ker(T^* - \bar{\alpha}_j)^{m_j} &= \ker(T^* - \bar{\alpha}_j)^{k_j} \\ &= \left[\frac{1}{1-\bar{\alpha}_j z}, \frac{z}{(1-\bar{\alpha}_j z)^2}, \dots, \frac{z^{k_j-1}}{(1-\bar{\alpha}_j z)^{k_j}}\right]. \end{aligned} \tag{2.29}$$

We prove (2.27) by induction on the number $M = m_1 + \dots + m_r$ of factors in the product. The equality is obvious when $M = 1$. Assume (2.27) holds for some $\alpha_1, \dots, \alpha_r$ and $m_1, \dots, m_r$. We show that (2.27) also holds if one new factor is added to the product.

Case 1. *We add a new zero α_{r+1} of C to the list $\alpha_1, \dots, \alpha_r$.*

We must show that

$$\ker\Big[(T^* - \bar\alpha_1)^{m_1} \cdots (T^* - \bar\alpha_r)^{m_r}(T^* - \bar\alpha_{r+1})\Big] = \sum_{j=1}^r \ker(T^* - \bar\alpha_j)^{m_j} + \ker(T^* - \bar\alpha_{r+1}). \tag{2.30}$$

The inclusion $\supseteq$ is clear. Let

$$u \in \ker\Big[(T^* - \bar\alpha_1)^{m_1} \cdots (T^* - \bar\alpha_r)^{m_r}(T^* - \bar\alpha_{r+1})\Big].$$

Then

$$(T^* - \bar\alpha_{r+1})u \in \ker\Big[(T^* - \bar\alpha_1)^{m_1} \cdots (T^* - \bar\alpha_r)^{m_r}\Big].$$

By our inductive assumption,

$$(T^* - \bar\alpha_{r+1})u = \sum_{j=1}^r v_j(z),$$

where $v_j \in \ker(T^* - \bar\alpha_j)^{m_j}$, $j = 1, \dots, r$. By (2.29),

$$v_j(z) \in \left[\frac{1}{1 - \bar\alpha_j z}, \frac{z}{(1 - \bar\alpha_j z)^2}, \dots, \frac{z^{k_j - 1}}{(1 - \bar\alpha_j z)^{k_j}}\right], \qquad j = 1, \dots, r.$$

By Lemma 2.10,

$$u(z) = \frac{u(0)}{1 - \bar\alpha_{r+1} z} + \sum_{j=1}^r \frac{z}{1 - \bar\alpha_{r+1} z}\, v_j(z).$$

By Lemma 2.11, for all $j = 1, \dots, r$,

$$\frac{z}{1 - \bar\alpha_{r+1} z}\, v_j(z) \in \left[\frac{1}{1 - \bar\alpha_{r+1} z}\right] + \left[\frac{1}{1 - \bar\alpha_j z}, \frac{z}{(1 - \bar\alpha_j z)^2}, \dots, \frac{z^{k_j - 1}}{(1 - \bar\alpha_j z)^{k_j}}\right].$$

Therefore

$$u(z) \in \left[\frac{1}{1 - \bar\alpha_{r+1} z}\right] + \sum_{j=1}^r \left[\frac{1}{1 - \bar\alpha_j z}, \frac{z}{(1 - \bar\alpha_j z)^2}, \dots, \frac{z^{k_j - 1}}{(1 - \bar\alpha_j z)^{k_j}}\right].$$

Hence

$$u \in \ker(T^* - \bar\alpha_{r+1}) + \sum_{j=1}^r \ker(T^* - \bar\alpha_j)^{m_j}$$

by (2.29), and this proves (2.30).

Case 2A. *For some $\ell \in \{1, \dots, r\}$ such that $m_\ell < n_\ell$, we increase m_ℓ by one.*

Without loss of generality, we can suppose that $\ell = r$. Thus $m_r < n_r$, and we must show that

$$\ker\Big[(T^* - \bar{\alpha}_1)^{m_1} \cdots (T^* - \bar{\alpha}_{r-1})^{m_{r-1}}(T^* - \bar{\alpha}_r)^{m_r+1}\Big] = \sum_{j=1}^{r-1} \ker(T^* - \bar{\alpha}_j)^{m_j} + \ker(T^* - \bar{\alpha}_r)^{m_r+1}. \tag{2.31}$$

The inclusion $\supseteq$ is obvious. Suppose

$$u \in \ker\Big[(T^* - \bar{\alpha}_1)^{m_1} \cdots (T^* - \bar{\alpha}_{r-1})^{m_{r-1}}(T^* - \bar{\alpha}_r)^{m_r+1}\Big].$$

Then

$$(T^* - \bar{\alpha}_r)u \in \Big[(T^* - \bar{\alpha}_1)^{m_1} \cdots (T^* - \bar{\alpha}_r)^{m_r}\Big].$$

By our inductive assumption, then

$$(T^* - \bar{\alpha}_r)u = \sum_{j=1}^{r} v_j(z),$$

where by (2.29), for each $j = 1, \dots, r$,

$$v_j(z) \in \ker(T^* - \bar{\alpha}_j)^{m_j} = \left[\frac{1}{1 - \bar{\alpha}_j z}, \frac{z}{(1 - \bar{\alpha}_j z)^2}, \dots, \frac{z^{k_j - 1}}{(1 - \bar{\alpha}_j z)^{k_j}}\right]$$

Here $k_r = m_r$ because $m_r < n_r$ by assumption. Also $m_r + 1 \leqslant n_r$, and therefore

$$\ker(T^* - \bar{\alpha}_r)^{m_r+1} = \left[\frac{1}{1 - \bar{\alpha}_j z}, \frac{z}{(1 - \bar{\alpha}_j z)^2}, \dots, \frac{z^{m_r}}{(1 - \bar{\alpha}_j z)^{m_r+1}}\right].$$

By Lemma 2.10,

$$\begin{aligned} u(z) &= \frac{u(0)}{1 - \bar{\alpha}_r z} + \sum_{j=1}^{r} \frac{z}{1 - \bar{\alpha}_r z}\, v_j(z) \\ &= \frac{u(0)}{1 - \bar{\alpha}_r z} + \sum_{j=1}^{r-1} \frac{z}{1 - \bar{\alpha}_r z}\, v_j(z) + \frac{z}{1 - \bar{\alpha}_r z}\, v_r(z). \end{aligned}$$

The first and third terms on the right are in $\ker(T^* - \bar{\alpha}_r)^{m_r+1}$. Since $\alpha_1, \dots, \alpha_r$ are distinct, by Lemma 2.11 the second term belongs to

$$\left[\frac{1}{1 - \bar{\alpha}_r z}\right] + \sum_{j=1}^{r-1} \left[\frac{1}{1 - \bar{\alpha}_j z}, \frac{z}{(1 - \bar{\alpha}_j z)^2}, \dots, \frac{z^{k_j - 1}}{(1 - \bar{\alpha}_j z)^{k_j}}\right] = \ker(T^* - \bar{\alpha}_r) + \sum_{j=1}^{r-1} \ker(T^* - \bar{\alpha}_j)^{m_j}.$$

Therefore $u(z) \in \sum_{j=1}^{r-1} \ker(T^* - \bar\alpha_j)^{m_j} + \ker(T^* - \bar\alpha_r)^{m_r+1}$, which proves (2.31).

Case 2B. *For some $\ell \in \{1, \dots, r\}$ such that $m_\ell \geqslant n_\ell$, we increase m_ℓ by one.*

Suppose first that $\ell = 1$, so $m_1 \geqslant n_1$. By Theorem 2.5,

$$\ker(T^* - \bar\alpha_1)^{m_1+1} = \ker(T^* - \bar\alpha_1)^{m_1}.$$

Therefore both the left side and the right side of (2.27) are unchanged if we replace m_1 by $m_1 + 1$. For the general case, by commutivity we can move any factor $(T^* - \bar\alpha_\ell)^{m_\ell}$ to the left in the product, and the same argument applies.

We have shown in all cases that (2.27) holds for $M + 1$ factors if it holds for M factors. The theorem follows by induction. □

3. The operator equation $B(T)R = f(T)$

We now consider the operator equation $B(T)R = f(T)$ in an arbitrary space $\mathcal{H}(C)$ that satisfies the identity for difference quotients, assuming that R commutes with T. In Theorem 3.2 we solve the operator equation for R, given any Blaschke product B of degree κ and any Schur function f. When specialized to the inner case and combined with Theorem 1.2, this result implies Theorem 3.4, which may be viewed as an indefinite analog of Sarason's generalized interpolation theorem.

Theorem 3.1. *Let R be a bounded operator on a space $\mathcal{H}(C)$ such that $C \notin \mathcal{H}(C)$. Assume that R commutes with T and satisfies $B(T)R = f(T)$, where f is a Schur function and B is a Blaschke product of degree κ. If B and C have no common zero, then $B(T)$ is invertible, and*

$$R = B(T)^{-1} f(T).$$

Proof. The problem is to show that $B(T)$ is invertible. Since no zero of B is a zero of C, $B(T)$ is a finite product of factors $(1 - \bar\gamma T)^{-1}(T - \gamma)$ such that $C(\gamma) \neq 0$. Each such factor is invertible by Theorem 2.2, and hence $B(T)$ is invertible. □

For the general case, let $\{\text{Zeros of } B\} \cap \{\text{Zeros of } C\} = \{\alpha_1, \dots, \alpha_r\}$, where $\alpha_1, \dots, \alpha_r$ are distinct points of $\mathbb{D}$. Set

$$\begin{aligned} m_1, \dots, m_r &= \text{orders of } \alpha_1, \dots, \alpha_r \text{ as zeros of } B, \\ n_1, \dots, n_r &= \text{orders of } \alpha_1, \dots, \alpha_r \text{ as zeros of } C. \end{aligned}$$

Factor $B(z)$ in the form

$$B(z) = B_1(z) B_0(z) B_2(z), \tag{3.1}$$

where

$$B_1(z) = \prod_{B(\gamma)=0,\, C(\gamma)\neq 0} \frac{z-\gamma}{1-\bar{\gamma}z},$$

$$B_0(z) = \prod_{j=1}^{r} \left(\frac{z-\alpha_j}{1-\bar{\alpha}_j z}\right)^{k_j}, \qquad k_j = \min(m_j, n_j),$$

$$B_2(z) = \prod_{m_j > n_j} \left(\frac{z-\alpha_j}{1-\bar{\alpha}_j z}\right)^{m_j - n_j}.$$

An empty product may be viewed as 1 or simply not present.

Theorem 3.2. *Let R be a bounded operator on a space $\mathcal{H}(C)$ such that $C \notin \mathcal{H}(C)$. Assume that R commutes with T and satisfies*

$$B(T)R = f(T), \tag{3.2}$$

where f is a Schur function and B is a Blaschke product of degree κ. Let B be factored as in (3.1)*. Then the subspace $\mathcal{K}$ of all h in $\mathcal{H}(C)$ such that*

$$h(\alpha_j) = h'(\alpha_j) = \cdots = h^{(k_j-1)}(\alpha_j) = 0, \qquad j = 1, \ldots, r, \tag{3.3}$$

has codimension at most κ in $\mathcal{H}(C)$, and $\mathcal{K}$ is invariant under T and R. Moreover, $g = f/B_0$ is a Schur function, and

$$R|_{\mathcal{K}} = B_1\big(T|_{\mathcal{K}}\big)^{-1} g\big(T|_{\mathcal{K}}\big) B_2\big(T|_{\mathcal{K}}\big)^{-1}. \tag{3.4}$$

The invertibility of $B_1\big(T|_{\mathcal{K}}\big)$ and $B_2\big(T|_{\mathcal{K}}\big)$ are shown in the proof.

Lemma 3.3. *Let $A \in \mathcal{L}(\mathcal{H})$ an invertible operator on a Hilbert space $\mathcal{H}$. Let $\mathcal{K}$ be a closed invariant subspace of A, and set $A_{\mathcal{K}} = A|_{\mathcal{K}}$. Then $A_{\mathcal{K}}$ is invertible in $\mathcal{L}(\mathcal{K})$ if and only if $\mathcal{K}$ is invariant under A^{-1}. In this case, $A_{\mathcal{K}}^{-1} = A^{-1}|_{\mathcal{K}}$.*

Proof of Lemma 3.3. Assume $A^{-1}\mathcal{K} \subseteq \mathcal{K}$. For any $v \in \mathcal{K}$, $A^{-1}v \in \mathcal{K}$ and

$$A_{\mathcal{K}}(A^{-1}v) = A(A^{-1}v) = v. \tag{3.5}$$

So $A_{\mathcal{K}}$ maps $\mathcal{K}$ onto itself. Since A is one-to-one, so is $A_{\mathcal{K}} = A|_{\mathcal{K}}$. Therefore $A_{\mathcal{K}}$ is invertible in $\mathcal{L}(\mathcal{K})$. By (3.5),

$$A_{\mathcal{K}}^{-1}v = A^{-1}v, \qquad v \in \mathcal{K},$$

hence $A_{\mathcal{K}}^{-1} = A^{-1}|_{\mathcal{K}}$. Conversely, let $A_{\mathcal{K}}$ be invertible in $\mathcal{L}(\mathcal{K})$. Given $v \in \mathcal{K}$, let $u = A_{\mathcal{K}}^{-1}v$. Then $u \in \mathcal{K}$ and $Au = A_{\mathcal{K}}A_{\mathcal{K}}^{-1}v = v$. Hence $A^{-1}v = u$ belongs to $\mathcal{K}$. Thus $A^{-1}\mathcal{K} \subseteq \mathcal{K}$. □

Proof of Theorem 3.2. Since the functional calculus is an algebra homomorphism, $B(T) = B_1(T)B_0(T)B_2(T)$ and

$$B_1(T) = \prod_{C(\gamma)\neq 0} (1-\bar{\gamma}T)^{-1}(T-\gamma), \tag{3.6}$$

$$B_0(T) = \prod_{j=1}^{r} \Big[(1-\bar{\alpha}_j T)^{-1}(T-\alpha_j)\Big]^{k_j}, \tag{3.7}$$

$$B_2(T) = \prod_{m_j>n_j} \Big[(1-\bar{\alpha}_j T)^{-1}(T-\alpha_j)\Big]^{m_j-n_j}. \tag{3.8}$$

All factors in (3.6)–(3.8) commute with R and with each other. Each factor $(1-\bar{\gamma}T)^{-1}(T-\gamma)$ in $B_1(T)$ is invertible by Theorem 2.2, and hence $B_1(T)$ is invertible. Therefore by (3.2),

$$RB_2(T)B_0(T) = B_1(T)^{-1}f(T), \tag{3.9}$$

Claim 1: *$f(T) = g(T)B_0(T)$ where g is a Schur function.*

By (3.9), the kernel of $f(T)$ contains the kernel of $B_0(T)$. For each $j = 1, \dots, r$, the kernel of $B_0(T)$ contains

$$\ker(T-\alpha_j)^{k_j} = [Q_1, \dots, Q_{k_j}]$$

by (3.7) and Theorem 2.3(2). Thus $B_0(T)Q_{k_j} = 0$, and hence

$$f(T)Q_{k_j} = 0.$$

By Corollary 2.4, f has a zero at α_j of order at least k_j. Since this holds for all $j = 1, \dots, r$, by standard function theory $f(z)/B_0(z)$ is the restriction of a Schur function. Claim 1 follows.

Define $\mathcal{K}$ as the closure of the range of $B_0(T)$. By (3.9) and Claim 1,

$$RB_2(T)B_0(T) = B_1(T)^{-1}g(T)B_0(T),$$

and hence

$$RB_2(T)|_{\mathcal{K}} = B_1(T)^{-1}g(T)\big|\mathcal{K}. \tag{3.10}$$

Clearly $\mathcal{K}$ is invariant under T. We show that $\mathcal{K}$ has codimension at most κ and is characterized by (3.3). By (3.7) and Theorem 2.9,

$$\begin{aligned}
\mathcal{K} &= \Big[\ker B_0(T)^*\Big]^{\perp} \\
&= \Big[\ker\big[(T^*-\bar{\alpha}_1)^{k_1}\cdots(T^*-\bar{\alpha}_r)^{k_r}\big]\Big]^{\perp} \\
&= \Big[\sum_{j=1}^{r} \ker(T^*-\bar{\alpha}_j)^{k_j}\Big]^{\perp}.
\end{aligned}$$

Therefore by Theorem 2.8, $\mathcal{K}$ is the set of all h in $\mathcal{H}(C)$ such that

$$h(\alpha_j) = h'(\alpha_j) = \cdots = h^{(k_j-1)}(\alpha_j) = 0$$

for all $j = 1, \ldots, r$. It also follows $\mathcal{K}^\perp$ is spanned by the sum of the kernels $\ker(T^* - \bar{\alpha}_j)^{k_j}$, $j = 1, \ldots, r$. Hence by Theorem 2.3, $\mathcal{K}^\perp$ is spanned by $k_1 + \cdots + k_r$ linearly independent functions, and so

$$\operatorname{codim} \mathcal{K} = k_1 + \cdots + k_r \leqslant \kappa.$$

Claim 2: $B_2(T)|_\mathcal{K} = B_2\big(T|_\mathcal{K}\big)$ *is an invertible operator on* $\mathcal{K}$.

It is enough to prove this when $B_2(z)$ consists of a single linear fractional factor. For definiteness, let this factor correspond to α_1, so

$$B_2(T) = (1 - \bar{\alpha}_1 T)^{-1}(T - \alpha_1). \tag{3.11}$$

By (3.8), $m_1 > n_1$. Therefore $k_1 = \min(m_1, n_1) = n_1$, and hence

$$C(\alpha_1) = C'(\alpha_1) = \cdots = C^{(k_1-1)}(\alpha_1) = 0 \quad \text{and} \quad C^{k_1}(\alpha_1) \neq 0. \tag{3.12}$$

We treat the two factors in (3.11) separately.

1°) $(T - \alpha_1)|_\mathcal{K} = T|_\mathcal{K} - \alpha_1$ *is an invertible operator on* $\mathcal{K}$.

We show that $(T - \alpha_1)|_\mathcal{K}$ is a one-to-one mapping of $\mathcal{K}$ onto itself. If $(T - \alpha_1)u = 0$ for some u in $\mathcal{K}$, then by (2.1),

$$u(z) = \frac{C(z)}{z - \alpha_1}\widetilde{u}(0). \tag{3.13}$$

Since $u \in \mathcal{K}$, $u^{(k_1-1)}(\alpha_1) = 0$. Then by (3.13), $C^{(k_1)}(\alpha_1)\widetilde{u}(0) = 0$. Hence by (3.12), $\widetilde{u}(0) = 0$ and so $u = 0$. Thus $(T - \alpha_1)|_\mathcal{K}$ is one-to-one.

To see that $(T - \alpha_1)\mathcal{K} = \mathcal{K}$, consider an arbitrary $v \in \mathcal{K}$. We seek a solution $u \in \mathcal{K}$ of

$$(T - \alpha_1)u = v. \tag{3.14}$$

We first find a solution u_0 of (3.14) in $\mathcal{H}(C)$. Since $v \in \mathcal{K}$,

$$v(\alpha_j) = v'(\alpha_j) = \cdots = v^{(k_j-1)}(\alpha_j) = 0, \qquad j = 1, \ldots, r. \tag{3.15}$$

In particular, $v(\alpha_1) = 0$. Define $u_0 \in \mathcal{H}(C)$ by

$$u_0(z) = \frac{v(z) - v(\alpha_1)}{z - \alpha_1} = \frac{v(z)}{z - \alpha_1}.$$

By (2.1),

$$(T - \alpha_1)u_0 = (z - \alpha_1)\,\frac{v(z)}{z - \alpha_1} - C(z)\widetilde{u}_0(0) = v(z) - C(z)\widetilde{u}_0(0),$$

where by (2.6),

$$\widetilde{u}_0(0) = \left\langle \frac{v(z) - v(\alpha_1)}{z - \alpha_1}, \frac{C(z) - C(0)}{z} \right\rangle_C = -\overline{C(0)}\,v(\alpha_1) = 0.$$

Thus $(T-\alpha_1)u_0 = v$ and u_0 is a solution of (3.14) in $\mathcal{H}(C)$. To find a solution u of (3.14) in $\mathcal{K}$, we modify u_0 by writing

$$u(z) = \frac{v(z)}{z-\alpha_1} + \gamma\,\frac{C(z)}{z-\alpha_1}, \tag{3.16}$$

where γ is a constant to be determined. No matter how γ is chosen, u satisfies (3.14) because $C(z)/(z-\alpha_1)$ is in the kernel of $T-\alpha_1$ by Theorem 2.3. For arbitrary γ and $2 \leqslant j \leqslant r$, the function (3.16) satisfies

$$u(\alpha_j) = u'(\alpha_j) = \cdots = u^{(k_j-1)}(\alpha_j) = 0, \tag{3.17}$$

because both $v(z)$ and $C(z)$ have a zero at α_j of order at least k_j and $\alpha_j \neq \alpha_1$. It remains to choose γ so that (3.17) holds for $j=1$. We use the Taylor expansions of $v(z)$ and $C(z)$ at α_1. By (3.15) and (3.12),

$$u(z) = \left[\frac{v^{(k_1)}(\alpha_1)}{k!} + \gamma\,\frac{C^{(k_1)}(\alpha_1)}{k!}\right](z-\alpha_1)^{k_1-1} + \text{higher powers.}$$

Since $C^{(k_1)}(\alpha_1) \neq 0$ by (3.12), we can choose γ such that

$$\frac{v^{(k_1)}(\alpha_1)}{k!} + \gamma\,\frac{C^{(k_1)}(\alpha_1)}{k!} = 0.$$

Then u satisfies (3.17) for $j=1$. Hence u is a solution of (3.14) belonging to $\mathcal{K}$, and so $(T-\alpha_1)\mathcal{K} = \mathcal{K}$. We have shown that $(T-\alpha_1)|_{\mathcal{K}}$ maps $\mathcal{K}$ one-to-one onto itself, and hence it is an invertible operator in $\mathcal{L}(\mathcal{K})$. The relation $(T-\alpha_1)|_{\mathcal{K}} = T|_{\mathcal{K}} - \alpha_1$ is clear, and so 1°) follows.

2°) $(1-\bar\alpha_1 T)^{-1}|_{\mathcal{K}} = (1-\bar\alpha_1 T|_{\mathcal{K}})^{-1}$ *is an invertible operator on* $\mathcal{K}$.

For any $u \in \mathcal{K}$,

$$(1-\bar\alpha_1 T)^{-1}u = \sum_{j=0}^{\infty} \bar\alpha_1^j\, T^j u = \sum_{j=0}^{\infty} \bar\alpha_1^j\, T|_{\mathcal{K}}^j u = (1-\bar\alpha_1 T|_{\mathcal{K}})^{-1}u,$$

so $(1-\bar\alpha_1 T)^{-1}|_{\mathcal{K}} = (1-\bar\alpha_1 T|_{\mathcal{K}})^{-1}$. Obviously $(1-\bar\alpha_1 T|_{\mathcal{K}})^{-1}$ is an invertible operator in $\mathcal{L}(\mathcal{K})$. This proves 2°), and Claim 2 follows.

It was shown above that $B_1(T)$ is an invertible operator on $\mathcal{H}(C)$.

Claim 3: $B_1\big(T|_{\mathcal{K}}\big)$ is invertible and $B_1\big(T|_{\mathcal{K}}\big)^{-1} = B_1(T)^{-1}|_{\mathcal{K}}$.

It is enough to prove this when $B_1(T)$ consists of just one linear fractional factor, say

$$B_1(T) = (1-\bar\gamma T)^{-1}(T-\gamma),$$

where $\gamma \in \mathbb{D}$ and $C(\gamma) \neq 0$. We shall apply Lemma 3.3 to the operator $B_1(T)$ on $\mathcal{H}(C)$ and invariant subspace $\mathcal{K}$. We show that

$$B_1(T)^{-1}\mathcal{K} \subseteq \mathcal{K}. \tag{3.18}$$

This reduces to showing that $(T-\gamma)^{-1}\mathcal{K} \subseteq \mathcal{K}$. Suppose $u \in \mathcal{K}$. Since $C(\gamma) \neq 0$, by Theorem 2.2,

$$(T-\gamma)^{-1}u(z) = \frac{u(z) - u(\gamma)C(z)/C(\gamma)}{z-\gamma}.$$

For any α_j, $j = 1, \ldots, r$, the first k_j Taylor coefficients of $u(z)$ at α_j are zero by the characterization (3.3) of $\mathcal{K}$, which was proved above. The first k_j Taylor coefficients of $C(z)$ at α_j are zero because $C(z)$ has a zero at α_j of order at least k_j by assumption. Since $1/(z-\gamma)$ is analytic at α_j, the same is true for $(T-\gamma)^{-1}u(z($, and hence $(T-\gamma)^{-1}u(z)$ belongs to $\mathcal{K}$. Thus (3.18) follows. By Lemma 3.3, $B_1(T)|_{\mathcal{K}}$ is invertible and

$$\big(B_1(T)|_{\mathcal{K}}\big)^{-1} = B_1(T)^{-1}|_{\mathcal{K}}.$$

Claim 3 follows because $B_1(T)|_{\mathcal{K}} = B_1\big(T|_{\mathcal{K}}\big)$ by the functional calculus.

We can now complete the proof. In (3.10) we showed that

$$RB_2(T)|_{\mathcal{K}} = B_1(T)^{-1}g(T)\big|\mathcal{K}$$

as operators from $\mathcal{K}$ into $\mathcal{H}(C)$. Here since $\mathcal{K}$ is invariant under T, by the functional calculus (1.4),

$$B_2(T)|_{\mathcal{K}} = B_2\big(T|_{\mathcal{K}}\big) \quad \text{and} \quad g(T)|_{\mathcal{K}} = g\big(T|_{\mathcal{K}}\big).$$

Thus

$$RB_2\big(T|_{\mathcal{K}}\big) = B_1(T)^{-1}g\big(T|_{\mathcal{K}}\big).$$

By Claim 2, $B_2\big(T|_{\mathcal{K}}\big)$ is invertible in $\mathcal{L}(\mathcal{K})$, and so

$$R|_{\mathcal{K}} = B_1(T)^{-1}g\big(T|_{\mathcal{K}}\big)B_2\big(T|_{\mathcal{K}}\big)^{-1}.$$

By Claim 3, $B_1(T)^{-1}|_{\mathcal{K}} = B_1\big(T|_{\mathcal{K}}\big)^{-1}$. Therefore $R\mathcal{K} \subseteq \mathcal{K}$ and

$$R|_{\mathcal{K}} = B_1\big(T|_{\mathcal{K}}\big)^{-1}g\big(T|_{\mathcal{K}}\big)B_2\big(T|_{\mathcal{K}}\big)^{-1},$$

which is (3.4).

By the functional calculus, $B_1\big(T|_{\mathcal{K}}\big), g\big(T|_{\mathcal{K}}\big), B_2\big(T|_{\mathcal{K}}\big)$ commute with $T|_{\mathcal{K}}$ and with each other. By Claims 2 and 3, $B_1\big(T|_{\mathcal{K}}\big)$ and $B_2\big(T|_{\mathcal{K}}\big)$ are invertible. Hence $B_1\big(T|_{\mathcal{K}}\big)^{-1}, g\big(T|_{\mathcal{K}}\big), B_2\big(T|_{\mathcal{K}}\big)^{-1}$ commute with $T|_{\mathcal{K}}$ and with each other. □

We can now state the form of Theorem 1.2 promised in Section 1.

Theorem 3.4. *Let C be an inner function, and let R be a bounded operator on $\mathcal{H}(C)$ that commutes with T and satisfies*

$$\operatorname{sq}_-(1 - RR^*) = \kappa$$

for some nonnegative integer κ. Then there is a Schur function f and a Blaschke product B of degree κ such that $B(T)R = f(T)$. Let B be factored as in (3.1)*. Then the subspace $\mathcal{K}$ of all h in $\mathcal{H}(C)$ such that*

$$h(\alpha_j) = h'(\alpha_j) = \cdots = h^{(k_j-1)}(\alpha_j) = 0, \qquad j = 1, \ldots, r,$$

has codimension at most κ in $\mathcal{H}(C)$, and $\mathcal{K}$ is invariant under T and R. Moreover, $g = f/B_0$ is a Schur function, and

$$R|_{\mathcal{K}} = B_1\big(T|_{\mathcal{K}}\big)^{-1} g\big(T|_{\mathcal{K}}\big) B_2\big(T|_{\mathcal{K}}\big)^{-1}.$$

Proof. By Theorem 1.2, there exist a Schur function f and a Blaschke product B of degree κ such that $B(T)R = f(T)$. The remaining assertions in the theorem then follow from Theorem 3.2. □

4. Dual results

The isomorphism in Theorem A.1 of Appendix A allows us to formulate dual versions of our main theorems with T replaced by T^*. Theorem 1.2 has the following dual version.

Theorem 4.1. *Let C be an inner function, and let R be a bounded operator on $\mathcal{H}(C)$ that commutes with T^* and satisfies*

$$\operatorname{sq}_-(1 - R^*R) = \kappa$$

for some nonnegative integer κ. Then there exist a Blaschke product B of degree κ and a Schur function f such that

$$B(T^*)R = f(T^*).$$

Conversely, if such f and B exist, $1 - R^*R$ has at most κ negative squares.

Proof. Set $R_\times = R^*$. Since R commutes with T^* and $\operatorname{sq}_-(1 - R^*R) = \kappa$, $R_\times$ commutes with T and

$$\operatorname{sq}_-(1 - R_\times R_\times^*) = \kappa.$$

By Theorem 1.2, there is a Schur function $f_\times$ and Blaschke product $B_\times$ of degree κ such that $B_\times(T)R_\times = f_\times(T)$, and hence

$$B_\times(T)^* R = f_\times(T)^*.$$

By the functional calculus (1.4),

$$B_\times(T)^* = \widetilde{B}_\times(T^*) \qquad \text{and} \qquad f_\times(T)^* = \widetilde{f}_\times(T^*),$$

where $\widetilde{B}_\times(z) = \overline{B_\times(\bar z)}$ is a Blaschke product of degree κ and $\widetilde{f}_\times(z) = \overline{f_\times(\bar z)}$ is a Schur function. Thus

$$\widetilde{B}_\times(T^*)R = \widetilde{f}_\times(T^*).$$

Setting $f = \widetilde{f}_\times$ and $B = \widetilde{B}_\times$, we obtain $B(T^*)R = f(T^*)$, as required.

The converse direction is similarly deduced from the converse part of Theorem 1.2. □

The dual version of Theorem 3.2 uses an analog of the factorization (3.1). Assume given a space $\mathcal{H}(C)$, $C \notin \mathcal{H}(C)$, and a Blaschke product B of degree κ. Set

$$\widetilde{C}(z) = \overline{C(\bar z)}.$$

Let $\{\text{Zeros of } B\} \cap \{\text{Zeros of } \widetilde{C}\} = \{\beta_1, \dots, \beta_s\}$, and

$$m_1, \dots, m_s = \text{orders of } \beta_1, \dots, \beta_s \text{ as zeros of } B,$$
$$n_1, \dots, n_s = \text{orders of } \beta_1, \dots, \beta_s \text{ as zeros of } \widetilde{C}.$$

Write

$$B(z) = B_1(z)B_0(z)B_2(z), \tag{4.1}$$

where

$$B_1(z) = \prod_{B(\gamma)=0,\, \widetilde{C}(\gamma)\neq 0} \frac{z-\gamma}{1-\bar{\gamma}z},$$
$$B_0(z) = \prod_{j=1}^{s} \left(\frac{z-\beta_j}{1-\bar{\beta}_j z}\right)^{k_j}, \qquad k_j = \min(m_j, n_j),$$
$$B_2(z) = \prod_{m_j>n_j} \left(\frac{z-\beta_j}{1-\bar{\beta}_j z}\right)^{m_j-n_j}.$$

For any h in $\mathcal{H}(C)$, in accordance with (2.2) we set

$$\widetilde{h}(w) = \left\langle h(z), \frac{C(z)-C(\bar{w})}{z-\bar{w}} \right\rangle_C, \qquad w \in \mathbb{D}.$$

Theorem 4.2. *Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$, and let R be a bounded operator on $\mathcal{H}(C)$ that commutes with T^*. Assume that*

$$B(T^*)R = f(T^*),$$

where f is a Schur function and B is a Blaschke product of degree κ. Let B be factored as in (4.1). Then the subspace $\mathcal{K}$ of all h in $\mathcal{H}(C)$ such that

$$\widetilde{h}(\beta_j) = \widetilde{h}\,'(\beta_j) = \cdots = \widetilde{h}^{(k_j-1)}(\beta_j) = 0, \qquad j = 1, \dots, s, \tag{4.2}$$

has codimension at most κ in $\mathcal{H}(C)$, and $\mathcal{K}$ is invariant under T^* and R. Moreover, $g = f/B_0$ is a Schur function, and

$$R|_{\mathcal{K}} = B_1\big(T^*|_{\mathcal{K}}\big)^{-1} g\big(T^*|_{\mathcal{K}}\big) B_2\big(T^*|_{\mathcal{K}}\big)^{-1}. \tag{4.3}$$

Proof. The mapping $U \colon h \to \widetilde{h}$ is a unitary operator from $\mathcal{H}(C)$ onto $\mathcal{H}(\widetilde{C})$ by Theorem A.1. Let $\widetilde{T}, \widetilde{T}^*$ be the operators on $\mathcal{H}(\widetilde{C})$ corresponding to T, T^* on $\mathcal{H}(C)$. By Theorem A.1,

$$\widetilde{T} = UT^*U^{-1}.$$

Set $\widetilde{R} = URU^{-1}$. Then since $T^*R = RT^*$,

$$\widetilde{T}\widetilde{R} = UT^*RU^{-1} = URT^*U^{-1} = \widetilde{R}\widetilde{T}. \tag{4.4}$$

Since $B(T^*)R = f(T^*)$,

$$UB(T^*)U^{-1}URU^{-1} = Uf(T^*)U^{-1}.$$

By the functional calculus (1.4),

$$UB(T^*)U^{-1} = B(UT^*U^{-1}) = B(\widetilde{T}),$$
$$Uf(T^*)U^{-1} = f(UT^*U^{-1}) = f(\widetilde{T}).$$

Hence

$$B(\widetilde{T})\widetilde{R} = f(\widetilde{T}). \tag{4.5}$$

By (4.4) and (4.5), we can apply Theorem 3.2 with the same f, B but with $\mathcal{H}(C), R$ replaced by $\mathcal{H}(\widetilde{C}), \widetilde{R}$. Let $\widetilde{\mathcal{K}}$ be the set of all $\widetilde{h}$ in $\mathcal{H}(\widetilde{C})$ that satisfy (4.2). By Theorem 3.2, $\widetilde{\mathcal{K}}$ has codimension at most κ and is invariant under $\widetilde{T}$. Moreover, $\widetilde{\mathcal{K}}$ is invariant under $\widetilde{R}$, and

$$\widetilde{R}|_{\widetilde{\mathcal{K}}} = B_1\big(\widetilde{T}|_{\widetilde{\mathcal{K}}}\big)^{-1} g\big(\widetilde{T}|_{\widetilde{\mathcal{K}}}\big) B_2\big(\widetilde{T}|_{\widetilde{\mathcal{K}}}\big)^{-1} \tag{4.6}$$

where $g(z) = f(z)/B_0(z)$ is a Schur function. Let $U_{\mathcal{K}}$ be the restriction of U to the subspace $\mathcal{K} = U^{-1}\widetilde{\mathcal{K}}$ of $\mathcal{H}(C)$, so $U_{\mathcal{K}}$ is a unitary operator from $\mathcal{K}$ onto $\widetilde{\mathcal{K}}$. By (1.4), for any φ in H^∞,

$$\varphi\big(\widetilde{T}|_{\widetilde{\mathcal{K}}}\big) = U_{\mathcal{K}}\, \varphi\big(T^*|_{\mathcal{K}}\big) U_{\mathcal{K}}^{-1}.$$

Since also $\widetilde{R}|_{\widetilde{\mathcal{K}}} = U_{\mathcal{K}}\, R|_{\mathcal{K}} U_{\mathcal{K}}^{-1}$, the relation (4.3) follows from (4.6). The theorem follows. □

Theorem 3.4 also has a dual version.

Theorem 4.3. *Let C be an inner function, and let R be a bounded operator on $\mathcal{H}(C)$ that commutes with T^* and satisfies*

$$\operatorname{sq}_-(1 - R^*R) = \kappa$$

for some nonnegative integer κ. Then there is a Schur function f and a Blaschke product B of degree κ such that $B(T^*)R = f(T^*)$. Let B be factored as in (4.1). Then the subspace $\mathcal{K}$ of all h in $\mathcal{H}(C)$ such that

$$\widetilde{h}(\beta_j) = \widetilde{h}'(\beta_j) = \cdots = \widetilde{h}^{(k_j-1)}(\beta_j) = 0, \qquad j = 1, \dots, s, \tag{4.7}$$

has codimension at most κ in $\mathcal{H}(C)$, and $\mathcal{K}$ is invariant under T^* and R. Moreover, $g = f/B_0$ is a Schur function, and

$$R|_{\mathcal{K}} = B_1\big(T^*|_{\mathcal{K}}\big)^{-1} g\big(T^*|_{\mathcal{K}}\big) B_2\big(T^*|_{\mathcal{K}}\big)^{-1}. \tag{4.8}$$

Proof. By Theorem 4.1, there exist a Schur function f and a Blaschke product B of degree κ such that $B(T^*)R = f(T^*)$. Hence the result follows from Theorem 4.2. □

Appendix A. An isomorphism

Let $\mathcal{H}(C)$ be a given space such that $C \notin \mathcal{H}(C)$. Set

$$\widetilde{C}(z) = \overline{C(\bar{z})}.$$

Then a space $\mathcal{H}(\widetilde{C})$ exists, and $\widetilde{C} \notin \mathcal{H}(\widetilde{C})$ by (1.1). We denote by $\widetilde{T}$ and $\widetilde{T}^*$ the operators on $\mathcal{H}(\widetilde{C})$ corresponding to (1.2) and (1.3).

Theorem A.1. *The mapping* $U\colon h(z) \to \widetilde{h}(z)$ *defined by*

$$\widetilde{h}(w) = \left\langle h(z), \frac{C(z) - C(\bar{w}}{z - \bar{w}} \right\rangle_C, \qquad w \in \mathbb{D}, \tag{A.1}$$

is a unitary operator from $\mathcal{H}(C)$ *onto* $\mathcal{H}(\widetilde{C})$ *such that*

$$UTU^{-1} = \widetilde{T}^*. \tag{A.2}$$

The inverse of U *is the corresponding mapping from* $\mathcal{H}(\widetilde{C})$ *onto* $\mathcal{H}(C)$*, that is, if* h *and* $\widetilde{h}$ *are connected by* (A.1)*, then also*

$$h(w) = \left\langle \widetilde{h}(z), \frac{\widetilde{C}(z) - \widetilde{C}(\bar{w}}{z - \bar{w}} \right\rangle_{\widetilde{C}}, \qquad w \in \mathbb{D}. \tag{A.3}$$

This result is given in [6, Th. 18] (see also [2, Th. 3.4.2(C)]). We sketch a proof by reproducing kernel methods.

Proof. If $h(z) = K_C(\alpha, z)$, $\alpha \in \mathbb{D}$, is any kernel function in $\mathcal{H}(C)$, then

$$\widetilde{h}(w) = \left\langle K_C(\alpha, z), \frac{C(z) - C(\bar{w}}{z - \bar{w}} \right\rangle_C = \frac{\widetilde{C}(w) - \widetilde{C}(\bar{\alpha})}{w - \bar{\alpha}},$$

which belongs to $\mathcal{H}(\widetilde{C})$. By (2.5) applied to $\mathcal{H}(\widetilde{C})$, for any $\alpha, \beta \in \mathbb{D}$,

$$\begin{aligned} \left\langle \frac{\widetilde{C}(z) - \widetilde{C}(\bar{\alpha})}{z - \bar{\alpha}}, \frac{\widetilde{C}(z) - \widetilde{C}(\bar{\beta})}{z - \bar{\beta}} \right\rangle_{\widetilde{C}} &= K_C(\alpha, \beta) \\ &= \langle K_C(\alpha, z), K_C(\beta, z) \rangle_C. \end{aligned}$$

Therefore the restriction of U to the span of kernel functions is an isometric mapping from a dense subspace of $\mathcal{H}(C)$ into $\mathcal{H}(\widetilde{C})$. The restriction has a unique extension by continuity to an isometry from $\mathcal{H}(C)$ into $\mathcal{H}(\widetilde{C})$. A short argument shows that the extension coincides with the mapping U defined by (A.1). Therefore U is an isometry from $\mathcal{H}(C)$ into $\mathcal{H}(\widetilde{C})$. To see that U is onto, we use (2.5) again, to show that

$$U\colon \frac{C(z) - C(\bar{\alpha})}{z - \bar{\alpha}} \to K_{\widetilde{C}}(\alpha, z), \qquad \alpha \in \mathbb{D}. \tag{A.4}$$

Thus the range of U contains all kernel functions in $\mathcal{H}(\widetilde{C})$ and hence is dense in $\mathcal{H}(\widetilde{C})$. Therefore U is unitary.

To prove (A.2), consider any h in $\mathcal{H}(C)$, and compute: for $w \in \mathbb{D}\backslash\{0\}$,

$$\begin{aligned}(UTh)(w) &= \left\langle Th(z), \frac{C(z)-C(\bar{w})}{z-\bar{w}} \right\rangle_C \\ &= \left\langle h(z), T^* \frac{C(z)-C(\bar{w})}{z-\bar{w}} \right\rangle_C \\ &= \left\langle h(z), \frac{1}{z}\left[\frac{C(z)-C(\bar{w})}{z-\bar{w}} - \frac{C(0)-C(\bar{w})}{-\bar{w}} \right] \right\rangle_C \\ &= \left\langle h(z), \frac{1}{\bar{w}}\left[\frac{C(z)-C(\bar{w})}{z-\bar{w}} - \frac{C(z)-C(0)}{z} \right] \right\rangle_C \\ &= \frac{\widetilde{h}(w)-\widetilde{h}(0)}{\bar{w}}.\end{aligned}$$

Therefore $UT = \widetilde{T}^* U$, which is equivalent to (A.2).

For the last statement, it is enough to check (A.3) when $\widetilde{h}(z)$ is a kernel function for $\mathcal{H}(\widetilde{C})$. By (A.4) we may take

$$h(z) = \frac{C(z)-C(\bar{\alpha})}{z-\bar{\alpha}} \qquad \text{and} \qquad \widetilde{h}(z) = K_{\widetilde{C}}(\alpha, z)$$

in (A.4). Then (A.3) reduces to the identity

$$\left\langle K_{\widetilde{C}}(\alpha, z), \frac{\widetilde{C}(z)-\widetilde{C}(\bar{w})}{z-\bar{w}} \right\rangle_{\widetilde{C}} = \frac{C(w)-C(\bar{\alpha})}{w-\bar{\alpha}}, \tag{A.5}$$

and the result follows. □

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