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ORTHOGONAL SUMS IN KREĬN SPACES

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ABSTRACT. Infinite orthogonal families of regular subspaces of a Kreĭn space exhibit a wide range of behavior. An elementary method is used to show that conditions on sums of projections produce behavior similar to that of orthogonal closed subspaces of a Hilbert space.

1. Introduction and preliminaries

Orthogonality in Kreĭn spaces is defined as in Hilbert spaces but using the given Kreĭn space inner product. In this paper we study pairwise orthogonal families of regular subspaces of a Kreĭn space. We are concerned with such questions as, when is the closed span of the family regular? When are the elements of the closed span represented as sums? When is there a natural isomorphism between the closed span and the external orthogonal direct sum of the family? The papers of McEnnis [8, 9, 10] provide answers to such questions in the context of wandering subspaces of an isometry. McEnnis also shows by examples what can go wrong in general. In [10, Theorem 3.1], McEnnis obtains strong conclusions with hypotheses that include an assumption that a certain Boolean algebra of projections is bounded. In this paper we use an elementary method to derive similar results in a more general setting using boundedness assumptions on sums of projections.

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The basis for our method is a Hilbert space result that appears in Wermer [11, p. 356] and is attributed to lecture notes of Mackey [6, p. 147]. Its proof uses the identity

$$\sum_{i=1}^{n} \|y_i\|^2 = \frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n} \|\varepsilon_1 y_1 + \cdots + \varepsilon_n y_n\|^2, \tag{1.1}$$

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where $y_1, \dots, y_n$ are vectors in a Hilbert space and the sum is over all 2^n possible choices of $\varepsilon_1, \dots, \varepsilon_n$ as ± 1. The case $n = 2$ in (1.1) is the parallelogram law, and the general case can be proved by induction.

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space, M a positive number. Let $P_1, \dots, P_n$ be any operators in $\mathcal{L}(\mathcal{H})$ such that*

$$\Big\| \sum_{j=1}^{n} \delta_j P_j \Big\| \le M \tag{1.2}$$

for every set $\{\delta_1, \dots, \delta_n\}$ of zeros and ones. Then:
(1) *For all f in $\mathcal{H}$,*

$$\sum_{j=1}^{n} \|P_j f\|^2 \le 4M^2 \|f\|^2.$$

(2) *If $P_1, \dots, P_n$ satisfy both* (1.2) *and*

$$\sum_{i=1}^{n} P_i = 1, \quad P_i^2 = P_i \ \ (i = 1, \dots, n), \quad P_i P_j = 0 \ \ (i \ne j), \tag{1.3}$$

then for all f in $\mathcal{H}$,

$$\frac{1}{4M^2} \|f\|^2 \le \sum_{j=1}^{n} \|P_j f\|^2.$$

Proof. Our proof is a direct transcription of Wermer's argument. We include it because our statement of the lemma is stronger than Wermer's. The stronger version, however, has the same proof.

Fix f in $\mathcal{H}$. Write

$$\sum_{i=1}^{n} \|P_i f\|^2 = \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n} \|\varepsilon_1 P_1 f + \dots + \varepsilon_n P_n f\|^2.$$

as in (1.1). The 2^n terms in the sum on the right side cannot be all greater than $\sum_{i=1}^{n} \|P_i f\|^2$, or all less than $\sum_{i=1}^{n} \|P_i f\|^2$. Hence there exist choices $\varepsilon_1', \dots, \varepsilon_n'$ and $\varepsilon_1, \dots, \varepsilon_n$ in the sum such that

$$a_f = \|\varepsilon_1' P_1 f + \dots + \varepsilon_n' P_n f\|^2 \le \sum_{i=1}^{n} \|P_i f\|^2 \le \|\varepsilon_1 P_1 f + \dots + \varepsilon_n P_n f\|^2 = b_f.$$

Proof of (1). We can write b_f in the form

$$b_f = \Big\| \sum_{i=1}^{n} \delta_i^+ P_i f - \sum_{i=1}^{n} \delta_i^- P_i f \Big\|^2,$$

where the numbers $\delta_i^+ = (1 + \varepsilon_i)/2$ and $\delta_i^- = (1 - \varepsilon_i)/2$, $i = 1, \dots, n$, are all zeros and ones. Then by (1.2),

$$b_f \le \left(\Big\| \Big(\sum_{i=1}^{n} \delta_i^+ P_i \Big) f \Big\| + \Big\| \Big(\sum_{i=1}^{n} \delta_i^- P_i \Big) f \Big\| \right)^2 \le (2M\|f\|)^2 = 4M^2 \|f\|^2,$$

which implies (1).

Proof of (2). Write a_f in the form $a_f = \|P^+ f - P^- f\|^2$, where P^+ is the sum of the P_i's for which $\varepsilon'_i = 1$, and P^- is the sum of the P_i's for which $\varepsilon'_i = -1$. By (1.3), $(P^+ - P^-)^2 = P^+ + P^- = 1$. Therefore

$$\|f\|^2 = \|(P^+ - P^-)^2 f\|^2 \leq \Big(\|P^+ - P^-\| \, \|P^+ f - P^- f\|\Big)^2.$$

Hence by (1.2),

$$\|f\|^2 \leq 4M^2 a_f \leq 4M^2 \sum_{j=1}^{n} \|P_j f\|^2,$$

and (2) follows. □

2. Kreĭn space background

A Kreĭn space is a complex linear space $\mathcal{H}$, together with a linear and symmetric inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, which admits a **fundamental decomposition**, that is, a representation $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\mathcal{H}_+$ is a Hilbert space and $\mathcal{H}_-$ is the antispace of a Hilbert space in the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Standard accounts of Kreĭn spaces are given in Azizov and Iokhvidov [1] and Bognár [2]. We follow the terminology and notation in Dritschel and Rovnyak [3] but caution that the definition of a projection operator below is broader than that of [3].

Every fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of a Kreĭn space $\mathcal{H}$ induces a Hilbert space **norm** or **associated norm** on $\mathcal{H}$, which we denote $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$. It is defined by

$$\|f\|^2 = \langle f_+, f_+ \rangle_{\mathcal{H}} + |\langle f_-, f_- \rangle_{\mathcal{H}}|, \qquad f \in \mathcal{H}, \tag{2.1}$$

where $f_\pm$ are the components of f in $\mathcal{H}_\pm$. For any norm $\|\cdot\|$ on $\mathcal{H}$,

$$|\langle f, g \rangle_{\mathcal{H}}| \leq \|f\| \|g\|, \qquad f, g \in \mathcal{H}. \tag{2.2}$$

Any two associated norms on $\mathcal{H}$ are equivalent. Topological notions are understood to be with respect to the **strong topology** of $\mathcal{H}$, which is the norm topology induced by any choice of associated norm on $\mathcal{H}$.

If $\mathcal{H}$ and $\mathcal{K}$ are two Kreĭn spaces, $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}, \mathcal{K})$ are the spaces of continuous operators on $\mathcal{H}$ to itself and on $\mathcal{H}$ to $\mathcal{K}$. Any choices of norms on $\mathcal{H}$ and $\mathcal{K}$ induce an **operator norm** on $\mathcal{L}(\mathcal{H}, \mathcal{K})$ that we shall denote $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\mathcal{H},\mathcal{K})}$. Any two operator norms are equivalent. The identity operator $1 \in \mathcal{L}(\mathcal{H})$ has norm one, $\|1\| = 1$, no matter how the norm on $\mathcal{H}$ is chosen.

For any Kreĭn spaces $\mathcal{H}$ and $\mathcal{K}$, the **adjoint** $A^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ of an operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is defined using the Kreĭn space inner products of $\mathcal{H}$ and $\mathcal{K}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called **selfadjoint** or **normal** according as $A = A^*$ or $AA^* = A^*A$. A **projection operator** on $\mathcal{H}$ is an operator $P \in \mathcal{L}(\mathcal{H})$ such that $P^2 = P$. A **unitary operator** from $\mathcal{H}$ to $\mathcal{K}$ is an operator $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $U^*U = 1_{\mathcal{H}}$ and $UU^* = 1_{\mathcal{K}}$. A unitary operator $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a **Kreĭn space isomorphism**, that is, a one-to-one mapping from $\mathcal{H}$ onto

$\mathcal{K}$ that preserves the linear operations and the inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ of the spaces. Conversely, every Kreĭn space isomorphism is a unitary operator.

A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is called **regular** if (1) $\mathcal{M}$ is closed, and (2) $\mathcal{M}$ is a Kreĭn space in the inner product of $\mathcal{H}$. Every regular subspace $\mathcal{M}$ of $\mathcal{H}$ is ortho-complemented: $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. The set of regular subspaces $\mathcal{M}$ of $\mathcal{H}$ is in one-to-one correspondence with the set of selfadjoint projection operators $P \in \mathcal{L}(\mathcal{H})$ via the relation $\mathcal{M} = \operatorname{ran} P$. A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is said to be **neutral** if $\langle f, f\rangle_{\mathcal{H}} = 0$ for all f in $\mathcal{M}$.

We use a notion of summability for an indexed family $\{f_\lambda\}_{\lambda\in\Lambda}$ of vectors in a Kreĭn space, where the index set Λ can be finite or infinite of any cardinality. See Halmos [5, pp. 17-20, 25–27] for the Hilbert space case. To generalize this notion to a Kreĭn space $\mathcal{H}$, we simply adopt the definitions and results relative to the Hilbert space corresponding to any norm $\|\cdot\|_{\mathcal{H}}$ for $\mathcal{H}$. Since any two norms are equivalent, any choice of norm can be used.

Definition 2.1. Let $\mathcal{H}$ be a Kreĭn space with associated norm $\|\cdot\|_{\mathcal{H}}$. A family $\{f_\lambda\}_{\lambda\in\Lambda}$ of vectors in $\mathcal{H}$ is **summable** with **sum**

$$f = \sum\nolimits_{\lambda\in\Lambda}^{s} f_\lambda$$

if for every $\varepsilon > 0$ there is a finite subset Λ_0 of Λ such that

$$\Big\| f - \sum\nolimits_{\lambda\in\Lambda_1} f_\lambda \Big\|_{\mathcal{H}} < \varepsilon$$

whenever Λ_1 is a finite subset of Λ that contains Λ_0.

The notion of summability applies as well to scalars by choosing $\mathcal{H} = \mathbb{C}$ in the Euclidean metric. As a notational matter, we decree that for any nonnegative scalars a_λ, $\lambda \in \Lambda$, the relation

$$\sum\nolimits_{\lambda\in\Lambda} a_\lambda < \infty$$

shall mean that all finite sums of the a_λ, $\lambda \in \Lambda$, have a uniform bound.

Lemma 2.2. *Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be nonnegative scalars such that $\sum_{\lambda\in\Lambda} a_\lambda < \infty$. Then the family is summable, and its sum is the supremum of the set of all finite partial sums.*

We shall need the Cauchy Criterion.

Theorem 2.3. *Let $\mathcal{H}$ be a Kreĭn space with associated norm $\|\cdot\|_{\mathcal{H}}$. A family $\{f_\lambda\}_{\lambda\in\Lambda}$ of vectors in $\mathcal{H}$ is summable if and only if for every $\varepsilon > 0$, there is a finite subset Λ_0 of Λ such that*

$$\Big\| \sum\nolimits_{\lambda\in\Lambda_1} f_\lambda \Big\|_{\mathcal{H}} < \varepsilon$$

for every finite subset Λ_1 of Λ that is disjoint from Λ_0. In this case, the set of all $\lambda \in \Lambda$ for which $f_\lambda \neq 0$ is countable.

Theorem 2.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces.*

(1) *The finite partial sums of a summable family $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ are bounded in any associated norm on $\mathcal{H}$.*
(2) *If $f = \sum^s_{\lambda\in\Lambda} f_\lambda$ in $\mathcal{H}$, then f is the strong limit of some sequence in the linear span of the vectors f_λ, $\lambda \in \Lambda$.*
(3) *If $f = \sum^s_{\lambda\in\Lambda} f_\lambda$ in $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$, then $Af = \sum^s_{\lambda\in\Lambda} Af_\lambda$ in $\mathcal{K}$.*
(4) *If $f = \sum^s_{\lambda\in\Lambda} f_\lambda$ in $\mathcal{H}$, then*

$$\langle f, g\rangle_{\mathcal{H}} = \textstyle\sum^s_{\lambda\in\Lambda} \langle f_\lambda, g\rangle_{\mathcal{H}}, \qquad g \in \mathcal{H}.$$

Proofs of the preceding assertions can be obtained from Halmos [5] or by straightforward means.

3. Orthogonal sums in Kreĭn spaces

The **sum** $\sum^s_{\lambda\in\Lambda} \mathcal{M}_\lambda$ and **closed span** $\bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ of any family $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ of subspaces of a Kreĭn $\mathcal{H}$ space are defined as follows:

(i) $\sum^s_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is the set of all sums $f = \sum^s_{\lambda\in\Lambda} f_\lambda$ with $f_\lambda \in \mathcal{M}_\lambda$, $\lambda \in \Lambda$;
(ii) $\bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is the intersection of all closed subspaces of $\mathcal{H}$ that contain each $\mathcal{M}_\lambda$, $\lambda \in \Lambda$.

In general,

$$\textstyle\bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda = \overline{\sum^s_{\lambda\in\Lambda} \mathcal{M}_\lambda}\,. \tag{3.1}$$

By an example of McEnnis [8, Example 6.4], the subspace (3.1) may fail to be regular even when each $\mathcal{M}_\lambda$ is regular and $\mathcal{M}_\lambda \perp \mathcal{M}_\mu$ for $\lambda \neq \mu$.

Theorem 3.1. *Let $\mathcal{H}$ be a Kreĭn space with associated norm $\|\cdot\|_{\mathcal{H}}$. Let $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ be pairwise orthogonal regular subspaces of $\mathcal{H}$, and for each $\lambda \in \Lambda$ let $P_\lambda \in \mathcal{L}(\mathcal{H})$ be the selfadjoint projection onto $\mathcal{M}_\lambda$. Assume that there is a constant $C > 0$ such that*

$$\big\| \textstyle\sum_{\lambda\in\Lambda_0} P_\lambda \big\|_{\mathcal{L}(\mathcal{H})} \le C \tag{3.2}$$

for every finite subset Λ_0 of Λ. Then for all $f \in \mathcal{H}$,

$$\textstyle\sum^s_{\lambda\in\Lambda} \|P_\lambda f\|^2_{\mathcal{H}} \le 4C^2 \|f\|^2_{\mathcal{H}}. \tag{3.3}$$

There is a selfadjoint projection $P \in \mathcal{L}(\mathcal{H})$ such that for all $f \in \mathcal{H}$,

$$\textstyle\sum^s_{\lambda\in\Lambda} P_\lambda f = Pf. \tag{3.4}$$

Moreover,

$$\operatorname{ran} P = \textstyle\sum^s_{\lambda\in\Lambda} \mathcal{M}_\lambda = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda. \tag{3.5}$$

In particular, the subspace $\mathcal{M} = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is regular.

By the Uniform Boundedness Principle, the existence of a $C > 0$ satisfying (3.2) is equivalent to the following condition: *For each f in $\mathcal{H}$, there is a constant $M_f > 0$ such that*

$$\Big\| \sum_{\lambda\in\Lambda_0} P_\lambda f \Big\|_{\mathcal{H}} \le M_f$$

for every finite subset Λ_0 of Λ. Therefore Theorem 3.1 is best possible in the sense that, if (3.4) holds for each f in $\mathcal{H}$ and some operator $P \in \mathcal{L}(\mathcal{H})$, then a $C > 0$ exists satisfying (3.2). For then Theorem 2.4(1) assures that such a constant M_f exists for each f in $\mathcal{H}$.

Proof of Theorem 3.1. We shall apply Lemma 1.1 to $\mathcal{H}$, viewed as a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$. Consider any f in $\mathcal{H}$. By (3.2) and Lemma 1.1(1),

$$\sum_{\lambda\in\Lambda_0} \|P_\lambda f\|_{\mathcal{H}}^2 \le 4C^2 \|f\|_{\mathcal{H}}^2$$

for every finite subset Λ_0 of Λ. By Lemma 2.2, the family $\{\|P_\lambda f\|_{\mathcal{H}}^2\}_{\lambda\in\Lambda}$ is summable in the space $\mathbb{C}$ of scalars, and its sum satisfies (3.3).

Again consider any f in $\mathcal{H}$. We use the Cauchy Criterion (Theorem 2.3) to show that $\{P_\lambda f\}_{\lambda\in\Lambda}$ is summable in $\mathcal{H}$. Let $\varepsilon > 0$ be given. Set $M = 2C+1$. Since the scalar family $\{\|P_\lambda f\|_{\mathcal{H}}^2\}_{\lambda\in\Lambda}$ is summable, by the necessity part of the Cauchy Criterion in the scalar case, we can choose a finite subset Λ_0 of Λ such that

$$\sum_{\lambda\in\Lambda_1} \|P_\lambda f\|_{\mathcal{H}}^2 < \frac{\varepsilon^2}{4M^2} \tag{3.6}$$

whenever Λ_1 is a finite subset of Λ such that $\Lambda_1 \cap \Lambda_0 = \emptyset$. Now consider any finite subset Λ_1 of Λ such that $\Lambda_1 \cap \Lambda_0 = \emptyset$. We apply Lemma 1.1(2) to the family of operators

$$\mathcal{F} = \{P_\lambda \colon \lambda \in \Lambda_0 \cup \Lambda_1\} \cup \{Q\}, \qquad Q = 1 - \sum_{\lambda\in\Lambda_0\cup\Lambda_1} P_\lambda.$$

Our assumptions on the subspaces $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ and operators $\{P_\lambda\}_{\lambda\in\Lambda}$ imply the relations

$$P_\lambda^2 = P_\lambda \quad \text{and} \quad P_\lambda P_\mu = 0 \ \text{ for } \ \lambda \ne \mu. \tag{3.7}$$

By (3.7) and (3.2), $\mathcal{F}$ satisfies (1.2) and (1.3) with the constant $M = 2C+1$. Hence by Lemma 1.1(2), for every vector g in $\mathcal{H}$,

$$\frac{1}{4M^2}\|g\|_{\mathcal{H}}^2 \le \sum_{\lambda\in\Lambda_0\cup\Lambda_1} \|P_\lambda g\|_{\mathcal{H}}^2 + \|Qg\|_{\mathcal{H}}^2. \tag{3.8}$$

Choosing $g = \sum_{\mu\in\Lambda_1} P_\mu f$, we obtain

$$P_\lambda g = \begin{cases} P_\lambda f, & \lambda \in \Lambda_1, \\ 0, & \lambda \in \Lambda_0, \end{cases}$$

$$Qg = 0.$$

Therefore by (3.8) and (3.6),

$$\frac{1}{4M^2}\left\|\sum_{\lambda\in\Lambda_1} P_\lambda f\right\|_{\mathcal{H}}^2 \le \sum_{\lambda\in\Lambda_1} \|P_\lambda f\|_{\mathcal{H}}^2 < \frac{\varepsilon^2}{4M^2}\,.$$

The Cauchy Criterion is thus verified, and hence the family $\{P_\lambda f\}_{\lambda\in\Lambda}$ is summable to some vector in $\mathcal{H}$.

Define a mapping P from $\mathcal{H}$ into itself by

$$Pf = \sum_{\lambda\in\Lambda}^{s} P_\lambda f, \qquad f\in\mathcal{H}\,. \tag{3.9}$$

Clearly P is linear. Since the projections P_λ, $\lambda\in\Lambda$, are selfadjoint, by Theorem 2.4(4), for all f and g in $\mathcal{H}$,

$$\langle Pf,g\rangle_{\mathcal{H}} = \sum_{\lambda\in\Lambda}^{s}\langle P_\lambda f,g\rangle_{\mathcal{H}} = \sum_{\lambda\in\Lambda}^{s}\langle f,P_\lambda g\rangle_{\mathcal{H}} = \langle f,Pg\rangle_{\mathcal{H}}.$$

By the Closed Graph Theorem, $P\in\mathcal{L}(\mathcal{H})$ and $P^*=P$. By the definition of P, $PP_\lambda = P_\lambda$ for all $\lambda\in\Lambda$. Hence by Theorem 2.4(3), for all f in $\mathcal{H}$,

$$P^2 f = \sum_{\lambda\in\Lambda}^{s} PP_\lambda f = \sum_{\lambda\in\Lambda}^{s} P_\lambda f = Pf.$$

Therefore $P^2=P$, and P is a selfadjoint projection. By construction,

$$\operatorname{ran} P \subseteq \sum_{\lambda\in\Lambda}^{s}\mathcal{M}_\lambda \subseteq \bigvee_{\lambda\in\Lambda}\mathcal{M}_\lambda.$$

The second inclusion in the preceding relation follows from Theorem 2.4(2). The range of P is closed and contains every subspace $\mathcal{M}_\lambda$, $\lambda\in\Lambda$. Since $\bigvee_{\lambda\in\Lambda}\mathcal{M}_\lambda$ is the smallest such subspace, (3.5) follows. □

A stronger condition is needed to obtain conclusions fully analogous to the Hilbert space case. For Hilbert spaces, the closed span of an orthogonal family of closed subspaces is naturally isomorphic with its external orthogonal direct sum. We define this notion for Kreĭn spaces.

Let $\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}$ be a family of Kreĭn spaces. For each $\lambda\in\Lambda$, choose a fundamental decomposition $\mathcal{H}_\lambda = \mathcal{H}_\lambda^+\oplus\mathcal{H}_\lambda^-$, and let $\|\cdot\|_\lambda$ be the corresponding associated norm on $\mathcal{H}_\lambda$. The **external orthogonal direct sum**

$$\mathcal{H}_e = \textstyle\bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda \tag{3.10}$$

is defined as the space of all families $f=\{f_\lambda\}_{\lambda\in\Lambda}$ such that $f_\lambda\in\mathcal{H}_\lambda$, $\lambda\in\Lambda$, and $\sum_{\lambda\in\Lambda}\|f_\lambda\|_\lambda^2<\infty$, together with the termwise linear operations and inner product

$$\langle f,g\rangle_{\mathcal{H}_e} = \textstyle\sum_{\lambda\in\Lambda}^{s}\langle f_\lambda,g_\lambda\rangle_\lambda\,, \qquad f,g\in\mathcal{H}_e.$$

By the Hilbert space case [5, p. 21], $\mathcal{H}_e^+ = \bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda^+$ is a Hilbert space in the inner product of $\mathcal{H}_e$, and $\mathcal{H}_e^- = \bigoplus_{\lambda\in\Lambda}\mathcal{H}_\lambda^-$ is the antispace of a Hilbert space in the inner product of $\mathcal{H}_e$. Thus $\mathcal{H}_e = \mathcal{H}_e^+ \oplus \mathcal{H}_e^-$ is a fundamental decomposition, and hence $\mathcal{H}_e$ is a Kreĭn space with associated norm

$$\big\|\{f_\lambda\}_{\lambda\in\Lambda}\big\|^2_{\mathcal{H}_e} = \textstyle\sum^s_{\lambda\in\Lambda} \|f_\lambda\|^2_\lambda. \tag{3.11}$$

It should be noted that when Λ is an infinite set, different choices of fundamental decompositions of the spaces $\mathcal{H}_\lambda$, $\lambda \in \Lambda$, may lead to essentially different Kreĭn spaces (3.10).

Theorem 3.2. *In Theorem* 3.1*, define an external orthogonal direct sum*

$$\mathcal{M}_e = \textstyle\bigoplus_{\lambda\in\Lambda} \mathcal{M}_\lambda$$

by viewing each subspace $\mathcal{M}_\lambda$ as a Kreĭn space in the inner product of $\mathcal{H}$ and using some choice of fundamental decomposition $\mathcal{M}_\lambda = \mathcal{M}_\lambda^+ \oplus \mathcal{M}_\lambda^-$, $\lambda \in \Lambda$. Write $P_\lambda = P_\lambda^+ + P_\lambda^-$, where $P_\lambda^\pm \in \mathcal{L}(\mathcal{H})$ are the selfadjoint projections onto $\mathcal{M}_\lambda^\pm$, $\lambda \in \Lambda$. Assume there is a constant $C > 0$ such that

$$\big\| \textstyle\sum_{\lambda\in\Lambda_0} P_\lambda^\pm \big\|_{\mathcal{L}(\mathcal{H})} \le C \tag{3.12}$$

for every finite subset Λ_0 of Λ. Then the subspace $\mathcal{M} = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is regular, and the mapping

$$V \colon f \to \{P_\lambda f\}_{\lambda\in\Lambda} \tag{3.13}$$

is an isomorphism from $\mathcal{M}$ viewed as a Kreĭn space in the inner product of $\mathcal{H}$ onto $\mathcal{M}_e$. The inverse mapping is given by

$$V^* \colon \{f_\lambda\}_{\lambda\in\Lambda} \to \textstyle\sum^s_{\lambda\in\Lambda} f_\lambda\,. \tag{3.14}$$

Proof. Condition (3.12) implies (3.2) with C replaced by $2C$. With this change of constants, the conclusions of Theorem 3.1 hold for the subspaces $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$. Thus $\mathcal{M} = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is a regular subspace of $\mathcal{H}$, and by (3.4),

$$\textstyle\sum^s_{\lambda\in\Lambda} P_\lambda f = f, \qquad f \in \mathcal{M}. \tag{3.15}$$

Applying Theorem 3.1 to the subspaces $\mathcal{M}_\lambda^\pm = \operatorname{ran} P_\lambda^\pm$, $\lambda \in \Lambda$, and original constant C, we obtain

$$\textstyle\sum^s_{\lambda\in\Lambda} \|P_\lambda^\pm f\|^2_{\mathcal{H}} \le 4C^2 \|f\|^2_{\mathcal{H}}\,, \qquad f \in \mathcal{H}. \tag{3.16}$$

We show that for each $f \in \mathcal{M}$, $\{P_\lambda f\}_{\lambda\in\Lambda}$ belongs to $\mathcal{M}_e$, that is,

$$\textstyle\sum_{\lambda\in\Lambda} \|P_\lambda f\|^2_\lambda < \infty, \tag{3.17}$$

where $\|\cdot\|_\lambda$ is the norm on $\mathcal{M}_\lambda$ induced by the fundamental decomposition $\mathcal{M}_\lambda = \mathcal{M}_\lambda^+ \oplus \mathcal{M}_\lambda^-$, $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, by (2.1) and (2.2),

$$\begin{aligned} \|P_\lambda f\|^2_\lambda &= \big\langle P_\lambda^+ f, P_\lambda^+ f\big\rangle_{\mathcal{M}_\lambda} + |\big\langle P_\lambda^- f, P_\lambda^- f\big\rangle_{\mathcal{M}_\lambda}| \\ &= \big\langle P_\lambda^+ f, P_\lambda^+ f\big\rangle_{\mathcal{H}} + |\big\langle P_\lambda^- f, P_\lambda^- f\big\rangle_{\mathcal{H}}| \\ &\le \|P_\lambda^+ f\|^2_{\mathcal{H}} + \|P_\lambda^- f\|^2_{\mathcal{H}}. \end{aligned} \tag{3.18}$$

Thus (3.17) follows from (3.18) and (3.16), and hence V is well defined as a mapping from $\mathcal{M}$ into $\mathcal{M}_e$. It is easy to see that V is a linear mapping and $\operatorname{ran} V$ is dense in $\mathcal{M}_e$. By (3.11), (3.16), and (3.18),

$$\big\|\{P_\lambda f\}_{\lambda\in\Lambda}\big\|^2_{\mathcal{M}_e} = \textstyle\sum^s_{\lambda\in\Lambda} \|P_\lambda f\|^2_\lambda \le 8C^2\|f\|^2_{\mathcal{H}}, \qquad f\in\mathcal{M}.$$

Therefore V is continuous relative to the restriction of the strong topology of $\mathcal{H}$ to $\mathcal{M}$. This topology coincides with the strong topology on $\mathcal{M}$ induced by its Kreĭn space inner product. Therefore $V\in\mathcal{L}(\mathcal{M},\mathcal{M}_e)$. For any $f,g\in\mathcal{M}$,

$$\begin{aligned}\langle Vf,Vg\rangle_{\mathcal{M}_e} &= \sum_{\lambda\in\Lambda}^s \langle P_\lambda f, P_\lambda g\rangle_{\mathcal{M}_\lambda} = \sum_{\lambda\in\Lambda}^s \langle P_\lambda f, P_\lambda g\rangle_{\mathcal{H}} \\ &= \sum_{\lambda\in\Lambda}^s \langle P_\lambda f, g\rangle_{\mathcal{H}} = \langle f,g\rangle_{\mathcal{H}} = \langle f,g\rangle_{\mathcal{M}}.\end{aligned}$$

The second to last equality holds by (3.15) and Theorem 2.4(4). Thus V is a continuous isometry, and so $\operatorname{ran} V$ is a regular subspace of $\mathcal{M}_e$. Since $\operatorname{ran} V$ is dense in $\mathcal{M}_e$, V is unitary and hence a Kreĭn space isomorphism from $\mathcal{M}$ onto $\mathcal{M}_e$.

To prove (3.14), consider any $\{f_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{M}_e$. Since V is onto, there is an f in $\mathcal{M}$ such that $f_\lambda = P_\lambda f$ for each λ. By (3.15), the family $\{f_\lambda\}_{\lambda\in\Lambda} = \{P_\lambda f\}_{\lambda\in\Lambda}$ is summable with sum f. So (3.14) defines a mapping inverse to (3.13). Since V is unitary, this is the adjoint of V. □

Corollary 3.3. *In Theorem* 3.2*, a family of vectors* $\{f_\lambda\}_{\lambda\in\Lambda}$ *with* $f_\lambda\in\mathcal{M}_\lambda$*,* $\lambda\in\Lambda$*, is summable if and only if*

(3.19) $$\textstyle\sum_{\lambda\in\Lambda}\|f_\lambda\|^2_\lambda < \infty.$$

In this case, the sum $f = \sum^s_{\lambda\in\Lambda} f_\lambda$ *belongs to the subspace* $\mathcal{M} = \bigvee_{\lambda\in\Lambda}\mathcal{M}_\lambda$*, and* $f_\lambda = P_\lambda f$*,* $\lambda\in\Lambda$*.*

Proof. If (3.19) holds, then $\{f_\lambda\}_{\lambda\in\Lambda}$ is in $\mathcal{M}_e$, and so $\{f_\lambda\}_{\lambda\in\Lambda}$ is summable by the last statement in Theorem 3.2. Conversely, suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is summable, and set $f = \sum^s_{\lambda\in\Lambda} f_\lambda$. Then f is in $\mathcal{M}$ by Theorem 2.4(2), and $P_\lambda f = f_\lambda$ for all λ by Theorem 2.4(3). Therefore $\{f_\lambda\}_{\lambda\in\Lambda} = \{P_\lambda f\}_{\lambda\in\Lambda} = Vf\in\mathcal{M}_e$ by Theorem 3.2. Hence (3.19) holds. □

Corollary 3.4. *The regular subspace* $\mathcal{M} = \bigvee_{\lambda\in\Lambda}\mathcal{M}_\lambda$ *in Theorem* 3.2 *has fundamental decomposition* $\mathcal{M} = \mathcal{M}_+\oplus\mathcal{M}_-$*, where* $\mathcal{M}_\pm = \bigvee_{\lambda\in\Lambda}\mathcal{M}^\pm_\lambda$*.*

Proof. Let $\mathcal{M}^\pm_e$ be the external orthogonal direct sums of the subspaces $\mathcal{M}^\pm_\lambda$, $\lambda\in\Lambda$, viewed as Hilbert spaces/antispaces in the inner product of $\mathcal{H}$. Then $\mathcal{M}^+_e$ is a Hilbert space, and $\mathcal{M}^-_e$ is the antispace of a Hilbert space. Both are contained isometrically in $\mathcal{M}_e$, and $\mathcal{M}_e = \mathcal{M}^+_e\oplus\mathcal{M}^-_e$ is a fundamental decomposition. The corresponding fundamental decomposition of $\mathcal{M}$ via the isomorphism (3.14) is $\mathcal{M} = \mathcal{M}_+\oplus\mathcal{M}_-$. □

4. Pseudo-regular subspaces

A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is called **pseudo-regular** if it is closed and has the form $\mathcal{M} = \mathcal{R} \oplus \mathcal{N}$, where $\mathcal{R}$ is a regular subspace of $\mathcal{H}$ and $\mathcal{N}$ is a neutral subspace of $\mathcal{H}$. For example, every closed subspace of a Pontryagin space is pseudo-regular. Gheondea [4] gives a general account of pseudo-regular subspaces including a number of equivalent conditions that may be used to define a pseudo-regular subspace. See also [7, Corollary 4.4] for the case of a Pontryagin space, and [7, Theorem 4.1] for a list of equivalent conditions that characterize pseudo-regular subspaces.

Theorem 4.1 (Maestripieri and Martínez Pería)**.** *A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is pseudo-regular if and only if $\mathcal{M} = \operatorname{ran} Q$ for some normal projection operator Q in $\mathcal{L}(\mathcal{H})$.*

This result is stated in [7, Theorem 4.3] for closed subspaces, but the assumption there that $\mathcal{M}$ is closed can be omitted. For in both [7] and this work, a pseudo-regular subspace is closed by definition. In the other direction, the range of any (continuous) projection is closed. For suppose $Q \in \mathcal{L}(\mathcal{H})$ and $Q^2 = Q$. If $g \in \overline{\operatorname{ran} Q}$, then $Qf_n \to g$ for some $f_1, f_2, \dots$ in $\mathcal{H}$, and hence also $Qf_n = Q^2 f_n \to Qg$. Thus $g = Qg \in \operatorname{ran} Q$.

The correspondence between pseudo-regular subspaces and normal projection operators is not one-to-one, however. A pseudo-regular subspace that is not regular is the range of infinitely many normal projection operators.

We are concerned with families $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ and $\{\widetilde{\mathcal{M}}_\lambda\}_{\lambda \in \Lambda}$ of pseudo-regular subspaces of a Kreĭn space $\mathcal{H}$ of the form

$$\mathcal{M}_\lambda = \operatorname{ran} P_\lambda, \quad \widetilde{\mathcal{M}}_\lambda = \operatorname{ran} P_\lambda^*, \tag{4.1}$$

where $\{P_\lambda\}_{\lambda \in \Lambda}$ are normal projections in $\mathcal{L}(\mathcal{H})$. Thus for each $\lambda \in \Lambda$,

$$P_\lambda^2 = P_\lambda, \quad P_\lambda P_\lambda^* = P_\lambda^* P_\lambda\,. \tag{4.2}$$

We also require that for $\lambda \neq \mu$,

$$P_\mu^* P_\lambda = 0, \quad P_\mu P_\lambda^* = 0, \quad P_\mu P_\lambda = 0, \tag{4.3}$$

or, equivalently, for $\lambda \neq \mu$,

$$\mathcal{M}_\lambda \perp \mathcal{M}_\mu \quad \widetilde{\mathcal{M}}_\lambda \perp \widetilde{\mathcal{M}}_\mu, \quad \mathcal{M}_\lambda \perp \widetilde{\mathcal{M}}_\mu.$$

Wandering subspaces of isometries provide a large class of examples.

Example 4.2. Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry. Then $\mathcal{G} = \ker V^*$ is a regular subspace of $\mathcal{H}$, and $\mathcal{H} = \mathcal{G} \oplus V\mathcal{H}$. Define a normal projection $Q \in \mathcal{L}(\mathcal{H})$ by choosing any normal projection $Q_0 \in \mathcal{L}(\mathcal{G})$ and setting

$$Qf = Q_0 g \quad \text{if} \quad f = g + h,\ g \in \mathcal{G},\ h \in V\mathcal{H}.$$

Set $P_n = V^n Q V^{*n}$ and

$$\mathcal{M}_n = \operatorname{ran} P_n, \quad \widetilde{\mathcal{M}}_n = \operatorname{ran} P_n^*, \quad n = 0, 1, 2, \dots .$$

The identities $V^*V = 1$, $Q^2 = Q$, and $QQ^* = Q^*Q$ imply (4.2). Therefore each P_n is a normal projection, and $\mathcal{M}_n$, $\widetilde{\mathcal{M}}_n$ are pseudo-regular subspaces of $\mathcal{H}$. We deduce (4.3) from the relations $QV = Q^*V = 0$.

Theorem 4.3. *Let $\mathcal{H}$ be a Kreĭn space with associated norm $\|\cdot\|_{\mathcal{H}}$. Let $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ and $\{\widetilde{\mathcal{M}}_\lambda\}_{\lambda\in\Lambda}$ be pseudo-regular subspaces of $\mathcal{H}$, and let $\{P_\lambda\}_{\lambda\in\Lambda}$ be normal projections in $\mathcal{L}(\mathcal{H})$ satisfying (4.1)–(4.3). Assume that there is a constant $C > 0$ such that*

$$\Big\| \textstyle\sum_{\lambda\in\Lambda_0} P_\lambda \Big\|_{\mathcal{L}(\mathcal{H})} \le C \tag{4.4}$$

for every finite subset Λ_0 of Λ. Then for all f in $\mathcal{H}$,

$$\sum_{\lambda\in\Lambda}^{s} \|P_\lambda f\|_{\mathcal{H}}^2 \le 4C^2\, \|f\|_{\mathcal{H}}^2, \quad \sum_{\lambda\in\Lambda}^{s} \|P_\lambda^* f\|_{\mathcal{H}}^2 \le 4\widetilde{C}^2\, \|f\|_{\mathcal{H}}^2. \tag{4.5}$$

There is a normal projection $P \in \mathcal{L}(\mathcal{H})$ such that for all $f \in \mathcal{H}$,

$$\sum_{\lambda\in\Lambda}^{s} P_\lambda f = Pf, \quad \sum_{\lambda\in\Lambda}^{s} P_\lambda^* f = P^* f. \tag{4.6}$$

Moreover,

$$\operatorname{ran} P = \sum_{\lambda\in\Lambda}^{s} \mathcal{M}_\lambda = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda\,, \quad \operatorname{ran} P^* = \sum_{\lambda\in\Lambda}^{s} \widetilde{\mathcal{M}}_\lambda = \bigvee_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda. \tag{4.7}$$

Hence the subspaces $\mathcal{M} = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ and $\widetilde{\mathcal{M}} = \bigvee_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda$ and are pseudo-regular.

The constant $\widetilde{C}$ in (4.5) is any positive number such that

$$\Big\| \textstyle\sum_{\lambda\in\Lambda_0} P_\lambda^* \Big\|_{\mathcal{L}(\mathcal{H})} \le \widetilde{C} \tag{4.8}$$

for all finite subsets Λ_0 of Λ. The existence of such a number follows from (4.4), because a constant $K > 0$ can always be found such that $\|A^*\|_{\mathcal{L}(\mathcal{H})} \le K\|A\|_{\mathcal{L}(\mathcal{H})}$ for all $A \in \mathcal{L}(\mathcal{H})$, and then $\widetilde{C} = KC$ works.

Proof of Theorem 4.3. The proof of Theorem 3.1 up to (3.9) can be applied to each of the families (4.1). This yields linear mappings P and $\widetilde{P}$ from $\mathcal{H}$ into itself such that for all f in $\mathcal{H}$,

$$Pf = \sum_{\lambda\in\Lambda}^{s} P_\lambda f, \qquad \widetilde{P} f = \sum_{\lambda\in\Lambda}^{s} P_\lambda^* f.$$

By Theorem 2.4(4), for all f, g in $\mathcal{H}$,

$$\langle Pf, g\rangle_{\mathcal{H}} = \sum_{\lambda\in\Lambda}^{s} \langle P_\lambda f, g\rangle_{\mathcal{H}} = \sum_{\lambda\in\Lambda}^{s} \langle f, P_\lambda^* g\rangle_{\mathcal{H}} = \big\langle f, \widetilde{P} g\big\rangle_{\mathcal{H}}\,.$$

By the Closed Graph Theorem, $P, \widetilde{P} \in \mathcal{L}(\mathcal{H})$ and $\widetilde{P} = P^*$. For all f in $\mathcal{H}$,

$$P^2 f = \sum_{\lambda\in\Lambda}^{s} P P_\lambda f = \sum_{\lambda\in\Lambda}^{s} P_\lambda^2 f = \sum_{\lambda\in\Lambda}^{s} P_\lambda f = Pf,$$

and so $P^2 = P$. By (4.2) and (4.3), $P_\lambda P_\lambda^* = P_\lambda^* P_\lambda$ for all $\lambda \in \Lambda$, and $P_\mu P_\lambda^* = P_\mu^* P_\lambda = 0$ whenever $\lambda \neq \mu$. Therefore

$$PP^*f = P\sum_{\lambda\in\Lambda}^{s} P_\lambda^* f = \sum_{\lambda\in\Lambda}^{s} PP_\lambda^* f = \sum_{\lambda\in\Lambda}^{s}\left(\sum_{\mu\in\Lambda}^{s} P_\mu P_\lambda^* f\right)$$
$$= \sum_{\lambda\in\Lambda}^{s} P_\lambda P_\lambda^* f = \sum_{\lambda\in\Lambda}^{s} P_\lambda^* P_\lambda f = \widetilde{P}Pf = P^*Pf.$$

Thus $PP^* = P^*P$. So P is a normal projection, and hence $\operatorname{ran} P$ is a pseudo-regular subspace of $\mathcal{H}$. By construction,

$$\operatorname{ran} P \subseteq \sum_{\lambda\in\Lambda}^{s} \mathcal{M}_\lambda \subseteq \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda.$$

The range of P is closed and contains every subspace $\mathcal{M}_\lambda$, $\lambda \in \Lambda$. Since $\bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is the smallest such subspace, the first relation in (4.7) follows, and hence $\mathcal{M} = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$ is a pseudo-regular subspace of $\mathcal{H}$. Similarly, the second relation in (4.7) holds, and $\widetilde{\mathcal{M}} = \bigvee_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda$ is a pseudo-regular subspace of $\mathcal{H}$. □

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