HILBERT SPACES OF ENTIRE FUNCTIONS: EARLY HISTORY

JAMES ROVNYAK

University of Virginia
Charlottesville, VA
rovnyak@virginia.edu

Abstract. The theory of Hilbert spaces of entire functions was conceived as a generalization of Fourier analysis by its founder, Louis de Branges. The Paley-Wiener spaces provided the motivating example. This chapter outlines the early development of the theory, showing how key steps were guided by the Hamburger moment problem, matrix differential equations, and eigenfunction expansions.

1. Introduction

The theory of Hilbert spaces of entire functions was initiated in de Branges, 1959a, 1959b, and completed in a remarkable series of papers: de Branges, 1960, 1961a, 1961b, 1962. Fourier analysis and other classical subjects motivated the development. The book that followed, de Branges, 1968, gives a complete account and includes improvements and many additional examples and applications. This chapter is an introduction to the theory as it unfolded in the original works.

Hilbert spaces of entire functions are implicit in de Branges, 1959a, and formally introduced in de Branges, 1959b. They are defined as Hilbert spaces $\mathcal{H}$ whose elements are entire functions which satisfy three axioms:

(H1) Whenever $F(z)$ is in the space and w is a nonreal zero of $F(z)$, the function $F(z)(z-\bar{w})/(z-w)$ is in the space and has the same norm.

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(H2) Whenever w is any nonreal complex number, the linear functional defined on the space by $F(z) \to F(w)$, which gives each function in the space its value at w, is continuous.

(H3) Whenever $F(z)$ is in the space, the function $F^*(z) = \overline{F}(\bar{z})$ is in the space and has the same norm.

The axioms imply that the transformation multiplication by z in $\mathcal{H}$ is symmetric, has deficiency indices $(1,1)$, and is real with respect to the conjugation $F(z)$ into $F^*(z)$. The main result of de Branges, 1959b, is a characterization of spaces that satisfy the axioms.

Theorem 1 (de Branges, 1959b)**.** *If $\mathcal{H}$ is a Hilbert space of entire functions satisfying* (H1)*,* (H2)*, and* (H3) *and containing a nonzero element, there is an entire function $E(z)$ such that for $y > 0$, $|E(\bar{z})| < |E(z)|$ and the Hilbert space consists exactly of the entire functions $F(z)$ such that*

$$\|F(t)\|_E^2 = \int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt < \infty$$

and $(z = x + iy)$

$$|F(z)|^2 \leq (4\pi y)^{-1} \|F(t)\|_E^2 \,(|E(z)|^2 - |E(\bar{z})|^2).$$

Furthermore, $E(z)$ can be so chosen that $\| \cdot \|_E$ agrees with the Hilbert space norm.

Theorem statements that appear here are either direct quotes or close paraphrases from the originals.

The Hilbert space in Theorem 1 is denoted $\mathcal{H}(E)$. The reproducing kernel for $\mathcal{H}(E)$, which exists by (H2), is given in de Branges, 1959b, as

$$K(w,z) = \frac{\overline{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}. \tag{1}$$

Thus for every complex number w, $K(w,z)$ belongs to $\mathcal{H}(E)$ as a function of z, and the identity

$$\langle F(t), K(w,t) \rangle_E = F(w)$$

holds for every function $F(z)$ in $\mathcal{H}(E)$, where $\langle \cdot, \cdot \rangle_E$ is the inner product of $\mathcal{H}(E)$.

The converse to Theorem 1 is stated without proof in de Branges, 1959b: every entire function $E(z)$ such that $|E(\bar{z})| < |E(z)|$ for $y > 0$ occurs as in Theorem 1 for a unique Hilbert space of entire functions satisfying (H1), (H2), and (H3).

The theory of the de Branges spaces $\mathcal{H}(E)$ is based on the classical theory of entire functions as presented in Boas, 1954. The form of the theory, however, depends more directly on other subjects:

- *Paley-Wiener spaces.* A well-known summation formula in the Paley-Wiener spaces led to the discovery of the spaces $\mathcal{H}(E)$ and a far-reaching program (de Branges, 1968, Preface): "I conjectured that a generalization of Fourier analysis was associated with these spaces. I spent the years 1958–1961 verifying this conjecture."
- *The Hamburger moment problem.* Connections with the Hamburger moment problem came at an early stage. They provided new examples and motivated important steps in the general theory. One is a characterization of all measures μ on the real line such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$.
- *Matrix differential equations.* Special two-by-two matrix-valued entire functions characterize when one space $\mathcal{H}(E)$ is contained isometrically in another. Matrix-valued entire functions of the required type occur as special solutions of matrix differential equations. Such equations determine the structure of totally ordered families of spaces.
- *Eigenfunction expansions.* An eigenfunction expansion that generalizes the Fourier transformation is associated with any given totally ordered family of spaces.

The four sections that follow explain the connections between these areas and Hilbert spaces of entire functions.

For parallel work of M. G. Kreĭn, see Gorbachuk and Gorbachuk, 1997, and the excellent review by Arov and Dym, 1999.

2. Paley-Wiener spaces

The theory of Hilbert spaces of entire functions has one dominant example. The Paley-Wiener space of type c, $0 < c < \infty$, is the Hilbert space $\mathcal{H}_c$ of entire functions $F(z)$ of exponential type at most c such that

$$\|F(t)\|^2 = \int_{-\infty}^{\infty} |F(t)|^2 \, dt < \infty.$$

By a theorem of Paley and Wiener, $\mathcal{H}_c$ coincides with the set of entire functions of the form

$$F(z) = \frac{1}{2\pi} \int_{-c}^{c} e^{-izt} \varphi(t) \, dt \qquad (2)$$

with $\varphi(t)$ in $L^2(-c, c)$ (Paley and Wiener, 1934, pp. 12–13). If $F(z)$ has this form, then by Plancherel's formula,

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt = \frac{1}{2\pi} \int_{-c}^{c} |\varphi(t)|^2\, dt. \tag{3}$$

The space $\mathcal{H}_c$ is equal isometrically to $\mathcal{H}(E(c))$, where

$$E(c, z) = e^{-icz}.$$

To see this, first use (2) to argue that $\mathcal{H}_c$ has reproducing kernel

$$K_c(w, z) = \frac{1}{2\pi} \int_{-c}^{c} e^{-izt} e^{it\bar{w}}\, dt.$$

Since

$$\frac{1}{2\pi} \int_{-c}^{c} e^{-izt} e^{it\bar{w}}\, dt = \frac{\sin(cz - c\bar{w})}{\pi(z - \bar{w})} = \frac{e^{-icz} e^{ic\bar{w}} - e^{icz} e^{-ic\bar{w}}}{2\pi i(\bar{w} - z)}$$

is also the reproducing kernel for $\mathcal{H}(E(c))$ by (1), $\mathcal{H}_c$ and $\mathcal{H}(E(c))$ are isometrically equal.

A well-known identity states that for every function $F(z)$ in a Paley-Wiener space $\mathcal{H}(E(c))$,

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt = \frac{\pi}{c} \sum_{n=-\infty}^{\infty} \left| F\left(\frac{n\pi}{c}\right) \right|^2. \tag{4}$$

This is easily proved by writing $F(z)$ in the form (2), and expanding $\varphi(t)$ in the complete orthogonal set $\{e^{i\pi n x/c}\}_{-\infty}^{\infty}$ in $L^2(-c, c)$:

$$\varphi(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{i\pi n t/c}. \tag{5}$$

The coefficients in this expansion are given by

$$a_n = \frac{\pi}{c} F\left(\frac{n\pi}{c}\right)$$

for all n. Thus (4) follows from Parseval's formula for (5).

The paper de Branges, 1959a, generalizes (4) using the classical theory of entire functions in place of Fourier analysis. The spaces $\mathcal{H}(E)$ do not appear explicitly in this work, but in the form presented in de Branges, 1968 (Theorem 22), the main result of de Branges, 1959a, states that for any space $\mathcal{H}(E)$ and any function $F(z)$ in $\mathcal{H}(E)$,

$$\int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 dt = \sum \left| \frac{F(t_n)}{E(t_n)} \right|^2 \frac{\pi}{\varphi'(t_n)}. \tag{6}$$

Here $\varphi(t)$ is a continuous real-valued function on the real line such that $E(t)e^{i\varphi(t)}$ is real for all real t. Summation is over all points t_n such that

$\varphi(t_n) \equiv \alpha$ modulo π for a fixed real number α. The identity is valid for all but at most one value of α modulo π. An exceptional value of α occurs when $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$. There are no exceptional values in the Paley-Wiener case $E(z) = e^{-icz}$; in this case the identity (6), taken with $\varphi(t) = ct$ and $\alpha = 0$, reduces to (4).

It is a leap to see evidence in (6) for a generalization of Fourier analysis, yet this is how the idea for the theory of Hilbert spaces of entire functions came about (de Branges, 1968, Preface). The spaces $\mathcal{H}(E)$ would replace the Paley-Wiener spaces in the generalization. A striking feature of the Paley-Wiener spaces is that they form a one-parameter family which is totally ordered by isometric inclusion and contained isometrically in $L^2(-\infty, \infty)$. The general theory of spaces $\mathcal{H}(E)$ ultimately shows that this situation is typical.

3. The Hamburger moment problem

Examples of totally ordered families of polynomial spaces that are contained isometrically in a space $L^2(\mu)$ arise in the Hamburger moment problem. Given real numbers $s_0, s_1, s_2, \dots$, the Hamburger moment problem is to find all nonnegative measures μ on the real line having finite moments $\int_{-\infty}^{\infty} t^{2n} d\mu(t)$ of all orders such that

$$s_k = \int_{-\infty}^{\infty} t^k d\mu(t), \qquad k \geq 0. \tag{7}$$

It is further required that μ not reduce to a finite number of point masses. By a theorem of Hamburger, the problem admits a solution if and only if $\det \Delta_n > 0$ for all $n \geq 0$, where $\Delta_n = [s_{j+k}]_{j,k=0}^n$. See Akhiezer, 1965, p. 30, and Shohat and Tamarkin, 1943, p. 5.

Polynomial spaces satisfying the axioms (H1), (H2), and (H3) arise naturally in this setting. Let $s_0, s_1, s_2, \dots$ be given real numbers, and let μ satisfy (7) and not reduce to a finite number of point masses. Let $\mathcal{P}$ be the space of polynomials with complex coefficients in the inner product defined by

$$\langle F(t), G(t) \rangle = \sum_{j=0}^{m} \sum_{k=0}^{n} s_{j+k}\, a_j \bar{b}_k,$$

where $F(z) = a_0 + a_1 z + \cdots a_m z^m$ and $G(z) = b_0 + b_1 z + \cdots + b_n z^n$. The inner product is linear, symmetric, and strictly positive. By (7), the associated norm is given by

$$\|F\|^2 = \int_{-\infty}^{\infty} |F(t)|^2 d\mu(t)$$

for every $F(z)$ in $\mathcal{P}$. Thus $\mathcal{P}$ is contained isometrically in $L^2(\mu)$. Let $\mathcal{P}_n$ be the set of polynomials of degree at most n. Then for each $n \geq 0$, $\mathcal{P}_n$ is a Hilbert space in the inner product of $\mathcal{P}$. It is a straightforward exercise to show that $\mathcal{P}_n$ satisfies the axioms (H1), (H2), and (H3). Therefore by Theorem 1, $\mathcal{P}_n$ is a space $\mathcal{H}(E_n)$ for some entire function $E_n(z)$ such that $|E_n(\bar{z})| < |E_n(z)|$ for $y > 0$. It can be shown that $E_n(z)$ is a polynomial of degree $n + 1$ having no zeros in the closed upper half-plane. The spaces $\mathcal{H}(E_n)$, $n \geq 0$, are totally ordered by isometric inclusion and each space is contained isometrically in $L^2(\mu)$.

A Hamburger moment problem (7) is called *indeterminate* if it has more than one solution. In the indeterminate case, the set of all solutions is described by a theorem of R. Nevanlinna. There are certain entire functions $A(z), B(z), C(z), D(z)$ constructed from the data of the problem such that the formula

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{t - z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}$$

establishes a one-to-one correspondence between the set of solutions μ of (7) and the set of functions $\varphi(z)$ of the class N augmented by the constant ∞. Here N is the Nevanlinna class of analytic functions $\varphi(z)$ of nonreal z such that $\bar{\varphi}(\bar{z}) = \varphi(z)$ and $\operatorname{Im} \varphi(z)/\operatorname{Im} z \geq 0$ for $\operatorname{Im} z \neq 0$. This result is given in Akhiezer, 1965, p. 98, and Shohat and Tamarkin, 1943, p. 57.

A related problem in the theory of Hilbert spaces of entire functions is to determine all measures μ on the line such that a given space $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. The idea for the following result is credited in de Branges, 1960, to a study of polynomial spaces and comparison with the accounts of the Hamburger moment problem by Shohat and Tamarkin, 1943, and Stone, 1932.

Theorem 2 (de Branges, 1960, Theorem V.A). *Let $E(z)$ be an entire function such that $|E(\bar{z})| < |E(z)|$ for $y > 0$. Let ν be a nonnegative measure on the Borel sets of the real line. A necessary and sufficient condition that*

$$\|F(t)\|_E^2 = \int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 d\nu(t) \tag{8}$$

for every $F(z)$ in $\mathcal{H}(E)$ is that

$$\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(t - x)^2 + t^2} = \operatorname{Re} \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)} \tag{9}$$

for $y > 0$, where $A(z)$ is defined and analytic for $y > 0$ and $|A(z)| \leq 1$.

In Section 4 the Hamburger moment problem is credited as part of the motivation to introduce matrix differential equations in the study of families of spaces.

4. Matrix differential equations

The first step in the study of totally ordered families of spaces $\mathcal{H}(E)$ is to characterize when one space is contained isometrically in another. Certain two-by-two matrices (12) of entire functions are used in the characterization. Similar matrices occur in the study of differential and difference equations, as shown in de Branges, 1960, Theorems X.A and X.B. The idea for these results, which separate the continuous and discrete cases, was suggested by the "discussion of Sturm-Liouville differential equations by [Stone, 1932], and of the Hamburger moment problem by [Shohat and Tamarkin, 1943], and [Stone, 1932]." The continuous and discrete cases are combined into a single formulation using matrix differential equations in de Branges, 1961a, 1961b.

The basic underlying differential equation has the form

$$\frac{d}{dt}(A(t,z),B(t,z))\,I = z\,(A(t,z),B(t,z))\,m'(t), \quad t>0, \tag{10}$$

or, in equivalent integral form,

$$(A(b,z),B(b,z))I-(A(a,z),B(a,z))I = z\int_a^b (A(t,z),B(t,z))\,dm(t),$$

where $0<a<b<\infty$. Here

$$I=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{11}$$

and

$$m(t)=\begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

is a nondecreasing matrix-valued function of $t>0$ with absolutely continuous real-valued entries. The constant matrix I can be thought of as a matrix counterpart of the complex imaginary unit. These assumptions follow the papers de Branges, 1960, 1961a, 1961b, 1962; the book de Branges, 1968, uses somewhat different conventions. The functions $A(t,z)$ and $B(t,z)$ are absolutely continuous functions of $t>0$ for each fixed z, and they are entire in z for each fixed t.

The condition that one space $\mathcal{H}(E)$ is isometrically contained in another is expressed in terms of matrix-valued entire functions

$$M(z)=\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{12}$$

such that $A(z), B(z), C(z), D(z)$ are entire functions which are real for real z and satisfy

$$\begin{aligned} A(z)D(z) - B(z)C(z) &= 1, \\ \operatorname{Re}\,[A(z)\overline{D}(z) - B(z)\overline{C}(z)] &\geq 1, \\ [B(z)\overline{A}(z) - A(z)\overline{B}(z)]/(z-\bar{z}) &\geq 0, \\ [D(z)\overline{C}(z) - C(z)\overline{D}(z)]/(z-\bar{z}) &\geq 0. \end{aligned} \tag{13}$$

The meaning of these conditions is found in Lemma 1 of de Branges, 1962. The conditions (13) imply that

$$\frac{M(z)IM(w)^* - I}{2\pi(z-\bar{w})} \tag{14}$$

is an entire function of z for every fixed w, and

$$\frac{M(z)IM(z)^* - I}{2\pi(z-\bar{z})} \geq 0 \tag{15}$$

for all complex z. The matrix inequality (15) in turn can be used to show that (14) is a nonnegative kernel, and therefore (14) is the reproducing kernel for a Hilbert space $\mathcal{H}(M)$ whose elements are vector-valued entire functions

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}.$$

Conversely, if $M(z)$ is a matrix-valued entire function of the form (12) such that (14) is the reproducing kernel for a space $\mathcal{H}(M)$, then the functions $A(z), B(z), C(z), D(z)$ satisfy (13).

Conditions for the isometric inclusion of one space $\mathcal{H}(E)$ in another appear in various forms in de Branges, 1960, 1961a, 1968.

Theorem 3 (de Branges, 1968, Theorem 33)**.** *Assume that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a,z)/E(b,z)$ has no real zeros. Write*

$$E(a,z) = A(a,z) - iB(a,z), \qquad E(b,z) = A(b,z) - iB(b,z),$$

where $A(a,z), B(a,z)$ and $A(b,z), B(b,z)$ are entire functions which are real for real z. Then there exists a matrix-valued entire function $M(a,b,z)$ such that a space $\mathcal{H}(M(a,b))$ exists and such that

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z). \tag{16}$$

The transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[A(a,z)F_+(z) + B(a,z)F_-(z)]$$

takes the space $\mathcal{H}(M(a,b))$ isometrically onto the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$.

A converse result is given in Theorem 34 of de Branges, 1968.

Conditions on real zeros as in Theorem 3 are generally not serious restrictions. For if $\mathcal{H}(E)$ is any given space, it is possible to write $E(z) = S(z)E_0(z)$, where $S(z)$ is a Weierstrass canonical product formed from the real zeros of $E(z)$, $|E_0(\bar{z})| < |E_0(z)|$ for $y > 0$, and $E_0(z)$ has no real zeros. Then multiplication by $S(z)$ is an isometry from $\mathcal{H}(E_0)$ onto $\mathcal{H}(E)$ (de Branges, 1968, Problem 44).

If, for example, $\mathcal{H}(E(t))$, $t > 0$, is a totally ordered family of spaces, then by Theorem 3 there is an associated matrix-valued entire function

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

satisfying (16) such that a space $\mathcal{H}(M(a,b))$ exists whenever $0 < a < b < \infty$. On the other hand, matrix-valued entire functions $M(a,b,z)$ of the required type occur as solutions of equations

$$M(a,b,z)I - I = z \int_a^b M(a,t,z)\, dm(t), \qquad a \le b,$$

where $m(t)$ is as in (10). In fact, the identity (de Branges, 1968, p. 126)

$$\frac{M(a,b,z)IM(a,b,w)^* - I}{z - \bar{w}} = \int_a^b M(a,t,z)\, dm(t)\, M(a,t,w)^*$$

shows that

$$\frac{M(a,b,z)IM(a,b,w)^* - I}{2\pi(z - \bar{w})}$$

is a nonnegative kernel, and therefore a space $\mathcal{H}(M(a,b))$ exists whenever $0 < a < b < \infty$. Assume that the entries of

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix} \tag{17}$$

are real-valued absolutely continuous functions of $t > 0$ such that

$$\alpha'(t) \ge 0, \quad \gamma'(t) \ge 0, \quad \beta'(t)^2 \le \alpha'(t)\gamma'(t) \tag{18}$$

a.e. for $t > 0$,

$$\alpha(t) > 0 \quad \text{for } t > 0 \quad \text{and} \quad \lim_{t \downarrow 0} \alpha(t) = 0, \tag{19}$$

and

$$\lim_{t \to \infty} [\alpha(t) + \gamma(t)] = \infty. \tag{20}$$

A number $b > 0$ is called *singular* for $m(t)$ if it belongs to an interval (a, c) in which $\alpha'(t), \beta'(t), \gamma'(t)$ are constant multiples of a single function and $\beta'(t)^2 = \alpha'(t)\gamma'(t)$ a.e. Otherwise b is called *regular* for $m(t)$.

The next result shows what a family of spaces associated with $m(t)$ should look like, if one exists. Write

$$E(t, z) = A(t, z) - iB(t, z),$$

where $A(t, z)$ and $B(t, z)$ are real for real z, and

$$K(t, w, z) = \frac{\overline{E}(t, w)E(t, z) - E(t, \bar{w})E^*(t, z)}{2\pi i(\bar{w} - z)}.$$

Theorem 4 (de Branges, 1961b, Theorem I)**.** *Let $m(t)$ be a matrix-valued function of $t > 0$ as in* (17)–(20)*. Suppose there exist spaces $\mathcal{H}(E(t))$, $t > 0$, such that $E(t, z)$ has no real zeros and $E(t, 0) = 1$ for each t, and such that for each complex z, $E(t, z)$ is a continuous function of $t > 0$,*

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))\, dm(t)$$

whenever $0 < a < b < \infty$, and

$$\lim_{a\downarrow 0} K(a, z, z) = 0.$$

Then when $a < b$ are regular points with respect to $m(t)$, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$. For all nonreal numbers z,

$$\lim_{b\to\infty} K(b, z, z) = \infty.$$

There is a unique measure μ on the real line such that

$$\int_{-\infty}^{\infty} \frac{|E(a, z)|^2}{1 + t^2}\, d\mu(t) < \infty$$

for each regular $a > 0$, and

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|E(a, t)|^2 d\mu(t)}{(t - x)^2 + y^2} = \lim_{b\to\infty} \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|E(a, t)|^2|E(b, t)|^{-2}}{(t - x)^2 + y^2}\, dt$$

for $y > 0$. When $a > 0$ is regular with respect to $m(t)$, $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$. The union of the spaces $\mathcal{H}(E(a))$, with a regular, is dense in $L^2(\mu)$.

There are many such families. In fact, every space $\mathcal{H}(E)$ such that $E(z)$ has no real zeros and $E(0) = 1$ is contained in such a family. Moreover, the measure μ can be chosen in any way such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Recall that Theorem 2 characterizes all such measures. It may occur that the positive real line is the union of a sequence of open intervals of singular points together with their endpoints. This situation occurs with the polynomial spaces associated with a Hamburger moment problem.

Theorem 5 (de Branges, 1961b, Theorem II)**.** *Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $E(0) = 1$. Let ν be a measure on the real line such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\nu)$. Then $E(z) = E(c, z)$ and $\nu = \mu$ for some choice of $m(t)$ and $E(t, z)$ as in Theorem 4, and some $c > 0$ which is regular with respect to $m(t)$.*

The paper de Branges, 1962, is devoted to uniqueness questions for families of spaces. Results take the form of ordering theorems which give conditions on two spaces $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ that one contains the other. Here is a special case:

Theorem 6 (de Branges, 1962, Theorem I)**.** *Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(E(c))$ be spaces such that $E(a, z)$, $E(b, z)$, and $E(c, z)$ have no real zeros. If $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are contained isometrically in $\mathcal{H}(E(c))$, then either $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ or $\mathcal{H}(E(b))$ contains $\mathcal{H}(E(a))$.*

The final form of the ordering theorem appears in Theorem 35 of de Branges, 1968. The proof of the ordering theorem draws on the classical theory of entire functions, work of Heins, 1959, and ideas from the earlier papers of de Branges, 1958, 1959c, on local operators on Fourier transforms (see de Branges, 1962, p. 53).

5. Eigenfunction expansions

The analogy with Fourier analysis and the Paley-Wiener spaces is completed with an eigenfunction expansion that generalizes the Fourier transformation. Eigenfunction expansions for differential operators are classical and consist of series or integral representations of a given function in terms of eigenfunctions; for example, see Titchmarsh, 1946, Kodaira, 1949, and Coddington and Levinson, 1955. The expansion in de Branges, 1961b, includes the traditional features of an eigenfunction expansion and additionally relates them to families of Hilbert spaces of entire functions.

Let $m(t)$ be a matrix-valued function as in (17)–(20). Define $L^2(m)$ as the space of measurable vector-valued functions $(f(t), g(t))$ of $t > 0$

such that

$$\|(f(t),g(t))\|^2 = \int_0^\infty (f(t),g(t))\,dm(t)\begin{pmatrix}\bar f(t)\\ \bar g(t)\end{pmatrix} < \infty.$$

The integral is defined by writing $dm(t) = m'(t)dt$ and integrating in the Lebesgue sense. Pairs $(f_1(t),g_1(t))$ and $(f_2(t),g_2(t))$ of measurable vector-valued functions are said to be equivalent in an interval (a,b) if

$$\int_a^b (f_1(t)-f_2(t),g_1(t)-g_2(t))\,dm(t)\begin{pmatrix}\bar f_1(t)-\bar f_2(t)\\ \bar g_1(t)-\bar g_2(t)\end{pmatrix} = 0.$$

Pairs in $L^2(m)$ which are equivalent on $(0,\infty)$ are identified. Let $L^2_0(m)$ be the subspace of $L^2(m)$ consisting of all pairs which are equivalent to constants in intervals containing only singular points.

Let $\chi_c(t)$ be the characteristic function of $(0,c]$, that is, $\chi_c(t)$ is 1 or 0 according as $0 < t \le c$ or $t > c$.

Theorem 7 (de Branges, 1961b, Theorem III)**.** *Assume given a family of spaces $\mathcal{H}(E(t))$, $0 < t < \infty$, and associated matrix-valued function $m(t)$ and measure μ as in Theorem 4. Write $E(t,z) = A(t,z) - iB(t,z)$, where $A(t,z)$ and $B(t,z)$ are entire functions which are real for real z for each $t > 0$.*

(1) *Let $c > 0$ be a regular point with respect to $m(t)$. Then for every complex number z, $\chi_c(t)(A(t,z),B(t,z))$ belongs to $L^2_0(m)$ as a function of $t > 0$. For every pair $(f(t),g(t))$ in $L^2_0(m)$ which is supported on $(0,c]$, define an eigentransform*

$$F(z) = \frac{1}{\pi}\int_0^c (f(t),g(t))dm(t)\begin{pmatrix}A(t,z)\\ B(t,z)\end{pmatrix}. \tag{21}$$

Then $F(z)$ is an entire function which belongs to $\mathcal{H}(E(c))$, and

$$\int_{-\infty}^{\infty} |F(t)|^2 d\mu(t) = \frac{1}{\pi}\int_0^c (f(t),g(t))\,dm(t)\begin{pmatrix}\bar f(t)\\ \bar g(t)\end{pmatrix}. \tag{22}$$

Every function in $\mathcal{H}(E(c))$ arises in this way. If $F(z)$ is the eigentransform of $(f(t),g(t))$, then $F^*(z)$ is the eigentransform of $(\bar f(t),\bar g(t))$.

(2) *Every pair $(f(t),g(t))$ in $L^2_0(m)$ has an eigentransform*

$$F(x) = \lim_{c\to\infty}\frac{1}{\pi}\int_0^c (f(t),g(t))dm(t)\begin{pmatrix}A(t,x)\\ B(t,x)\end{pmatrix}$$

which exists in the metric of $L^2(\mu)$ and satisfies

$$\int_{-\infty}^{\infty} |F(t)|^2 d\mu(t) = \frac{1}{\pi}\int_0^\infty (f(t),g(t))\,dm(t)\begin{pmatrix}\bar f(t)\\ \bar g(t)\end{pmatrix}.$$

Every function in $L^2(\mu)$ arises in this way.

Part of Theorem III of de Branges, 1961b, is omitted in Theorem 7. The eigenfunction expansion diagonalizes a differential operator H on $L^2_0(m)$. The operator H is defined by its graph, which consists of all pairs $\big((f_1(x), g_1(x)), (f_2(x), g_2(x))\big)$ in $L^2_0(m) \times L^2_0(m)$ such that $f_1(x)$ and $g_1(x)$ are absolutely continuous on $(0, \infty)$, $g_1(x)$ is continuous on $[0, \infty)$, $g_1(0) = 0$, and

$$\frac{d}{dt}\,(f_1(t), g_1(t))I = (f_2(t), g_2(t))m'(t)$$

a.e. on $(0, \infty)$. For details see the full statement of the eigenfunction expansion in Theorem III of de Branges, 1961b.

Paley-Wiener spaces. The classical Fourier transform is recovered as a special case. Let $\mathcal{H}(E(t))$, $0 < t < \infty$, be the Paley-Wiener spaces. Then the identities (21) and (22) in Theorem 7 take the form (de Branges, 1961b, p. 74)

$$F(z) = \frac{1}{\pi}\int_0^c f(t)\,\cos(tz)\,dt + \frac{1}{\pi}\int_0^c g(t)\,\sin(tz)\,dt \tag{23}$$

and

$$\int_{-\infty}^{\infty} |F(t)|^2 dt = \frac{1}{\pi}\int_0^c |f(t)|^2 dt + \frac{1}{\pi}\int_0^c |g(t)|^2 dt. \tag{24}$$

For, in the Paley-Wiener case,

$$E(t,z) = e^{-itz}, \qquad A(t,z) = \cos(tz), \qquad B(t,z) = \sin(tz),$$

and μ is Lebesgue measure. In this case,

$$m(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \qquad t > 0,$$

and so $L^2_0(m) = L^2(0,\infty) \oplus L^2(0,\infty)$.

The identities (23) and (24) can also be derived directly. Recall that the Paley-Wiener space $\mathcal{H}(E(c))$ of type c is the set of all entire functions (2) in the norm (3). Suppose $\varphi(x) \in L^2(-c,c)$. Introduce

$$f(x) = \frac{\varphi(x) + \varphi(-x)}{2}, \qquad g(x) = \frac{\varphi(x) - \varphi(-x)}{2i}, \tag{25}$$

$-c < x < c$. Then $f(x)$ is even and $g(x)$ is odd, and

$$\varphi(x) = f(x) + ig(x). \tag{26}$$

The relations (25) and (26) establish a natural correspondence between $L^2(-c,c)$ and $L^2(0,c) \oplus L^2(0,c)$. A short calculation shows that

$$\begin{aligned}\frac{1}{2\pi}\int_{-c}^{c} e^{-izt}\varphi(t)\,dt &= \frac{1}{2\pi}\int_{-c}^{c}[\cos(tz) - i\sin(tz)]\,[f(t)+ig(t)]\,dt \\ &= \frac{1}{\pi}\int_{0}^{c} f(t)\cos(tz)\,dt + \frac{1}{\pi}\int_{0}^{c} g(t)\sin(tz)\,dt.\end{aligned}$$

Similarly,

$$\frac{1}{2\pi}\int_{-c}^{c} |\varphi(t)|^2\,dt = \frac{1}{\pi}\int_{0}^{c} |f(t)|^2 dt + \frac{1}{\pi}\int_{0}^{c} |g(t)|^2 dt.$$

The steps are reversible, and thus (23) and (24) are equivalent forms of (2) and (3).

The differential operator H on $L_0^2(m) = L^2(0,\infty) \oplus L^2(0,\infty)$ can be mapped to $L^2(-\infty,\infty)$ using the correspondence with $L^2(0,\infty) \oplus L^2(0,\infty)$ defined by the same relations (25) and (26). The operator on $L^2(-\infty,\infty)$ corresponding to H is the classical operator

$$H_0 = -i\,\frac{d}{dx}$$

acting on the set of absolutely continuous functions $\varphi(x)$ in $L^2(-\infty,\infty)$ such that $\varphi'(x)$ belongs to $L^2(-\infty,\infty)$.

6. Conclusion

The theory of Hilbert spaces of entire functions by Louis de Branges is a highly original and remarkable achievement. The path was not clear at the outset, and difficult problems had to be overcome. The Hamburger moment problem and matrix differential equations guided the way at critical stages. The result is a theory of families of spaces that are totally ordered by isometric inclusion and contained isometrically in a space $L^2(\mu)$. Such families are shown to have properties analogous to the Paley-Wiener spaces. In particular, they are associated with eigenfunction expansions that generalize the Fourier transformation.

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