On Indefinite Cases of Operator Identities Which Arise in Interpolation Theory. II

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Abstract. This paper studies operator identities from interpolation theory that are related to the generalized Carathéodory class of matrix-valued functions on the interior and exterior of the unit circle. A key tool is a Kreĭn-Langer integral representation for generalized Carathéodory functions that generalizes the Herglotz representation in the classical case. Every generalized Carathéodory function meeting certain conditions induces an operator identity. The main results of the paper characterize the class of identities which arise in this way. An application to a tangential interpolation problem is given. Parallel results for generalized Nevanlinna functions were obtained by the authors in a previous work.

1. Introduction

In this paper we study operator identities of the form

$$\begin{gathered} S - ASA^* = \Phi_1\Phi_2^* + \Phi_2\Phi_1^* \, , \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}), \end{gathered} \tag{1.1}$$

where $S = S^*$. Throughout $\mathfrak{H}$ denotes some Hilbert space. We take $\mathfrak{G} = \mathbb{C}^m$ for some positive integer m and identify $\mathfrak{L}(\mathfrak{G})$ with $m \times m$ matrices. A well-known example from interpolation theory is

$$A = \begin{bmatrix} z_1 I_m & 0 & \cdots & 0 \\ 0 & z_2 I_m & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & z_n I_m \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} I_m \\ \vdots \\ I_m \end{bmatrix}, \tag{1.2}$$

and

$$S = \left[\frac{G_j + G_k^*}{1 - z_j \bar{z}_k} \right]_{j,k=1}^{n}, \qquad \Phi_1 = \begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix}, \tag{1.3}$$

where $\mathfrak{H} = \mathbb{C}^{nm}$, $z_1, \dots, z_n$ are points in the open unit disk, and $G_1, \dots, G_n$ are $m \times m$ matrices.

In Chapter 2 of [11] the theory of identities (1.1) is related to Carathéodory functions, that is, $m \times m$ matrix-valued analytic functions $F(z)$ on the interior and exterior of the unit circle $\mathbb{T}$ satisfying $F(1/\bar{z})^* = -F(z)$ such that $\operatorname{Re} F(z) \geq 0$ for $|z| < 1$. Every such function admits a Herglotz representation

$$F(z) = iC + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta)\,, \tag{1.4}$$

where $C = C^*$ is a constant matrix and σ is a nonnegative matrix-valued measure on the circle. Let $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ be given, and suppose that the spectrum of A does not meet the unit circle. Define $S = S_F \in \mathfrak{L}(\mathfrak{H})$ by

$$S_F = 2 \int_{\mathbb{T}} (I + \zeta A)^{-1} \Phi_2 d\sigma(\zeta) \Phi_2^* (I + \bar{\zeta} A^*)^{-1}, \tag{1.5}$$

and $\Phi_1 = \Phi_{1,F} \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ by

$$\Phi_{1,F} = -i\Phi_2 C + \int_{\mathbb{T}} (I - \zeta A)(I + \zeta A)^{-1} \Phi_2\, d\sigma(\zeta). \tag{1.6}$$

It is easy to show that A, S, Φ_1, Φ_2 satisfy (1.1). The operator interpolation problem is to determine when a given identity (1.1) has the form $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some Carathéodory function $F(z)$, and to determine all such functions. Since any operator of the form (1.5) satisfies $S_F \geq 0$, a necessary condition for existence is that $S \geq 0$. This is the definite case, which is treated in Chapter 2 of [11].

We are concerned with the indefinite case. Here the condition $S \geq 0$ is replaced by the weaker condition $\varkappa_S < \infty$, that is, the negative spectrum of S consists of a finite number of eigenvalues having finite total multiplicity $\varkappa_S$. We extend the formulas (1.4), (1.5), and (1.6) to generalized Carathéodory functions and study the operator interpolation problem in this setting. Our main results characterize all operator identities (1.1) such that $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some generalized Carathéodory function. A similar program was carried out in [8, 9] for generalized Nevanlinna functions and operator identities of the form

$$\begin{gathered} AS - SA^* = i\left[\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*\right], \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}). \end{gathered} \tag{1.7}$$

Part 1 of the paper has four sections and contains statements of results. A Kreĭn-Langer integral representation for generalized Carathéodory functions is presented in Section 2. This is used in Section 3 to define operators S_F and $\Phi_{1,F}$ in the indefinite case. The operator interpolation problem is taken up in Section 4. Solutions are parametrized as linear fractional expressions whose coefficients are determined by the data. The abstract theory is illustrated with an application to a tangential interpolation problem.

Part 2 of the paper has two sections. Sections 5 reviews background material. Proofs of the results in Part 1 are given in Section 6.

Part 1. Results

2. Kreĭn-Langer integral representation

We work in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Write

$$\mathbb{C}_\infty = \mathbb{D} \cup \mathbb{T} \cup \mathbb{D}^c,$$

where $\mathbb{D} = \{z\colon |z| < 1\}$, $\mathbb{T} = \{z\colon |z| = 1\}$, and $\mathbb{D}^c = \{z\colon |z| > 1\} \cup \{\infty\}$. We use the standard conformal mapping

$$w = \varphi(z) = i\,\frac{1+z}{1-z}, \qquad z = \varphi^{-1}(w) = \frac{w-i}{w+i}\,. \tag{2.1}$$

Thus $\mathbb{C}_+ = \varphi(\mathbb{D})$ and $\mathbb{C}_- = \varphi(\mathbb{D}^c)$, where $\mathbb{C}_\pm$ are the upper and lower half-planes.

The **generalized Carathéodory class $\mathbf{C}_\varkappa$** is the set of meromorphic $m \times m$ matrix-valued functions $F(z)$ on $\mathbb{D} \cup \mathbb{D}^c$ satisfying $F(1/\bar z)^* = -F(z)$ such that the kernel $[F(z) + F(\zeta)^*]/(1 - \bar\zeta z)$ has κ negative squares, $\varkappa \geq 0$. In the simplest case $\varkappa = 0$, $\mathbf{C_0}$ is the classical Carathéodory class of analytic $m \times m$ matrix-valued functions $F(z)$ on $\mathbb{D} \cup \mathbb{D}^c$ such that $F(1/\bar z)^* = -F(z)$ and $\operatorname{Re} F(z) \geq 0$ on $\mathbb{D}$.

We shall derive a Kreĭn-Langer integral representation for the class $\mathbf{C}_\varkappa$ from the corresponding half-plane result. The **generalized Nevanlinna class $\mathbf{N}_\varkappa$** is the set of $m \times m$ matrix-valued functions $v(w)$ which are meromorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ such that $v(w) = v(\bar w)^*$ and the kernel $[v(w) - v(\xi)^*]/(w - \bar\xi)$ has $\varkappa$ negative squares. Set

$$\frac{1}{t-w} - S(\alpha,\rho;t,w) = \begin{cases} \dfrac{1}{t-w}\left(\dfrac{t-\alpha}{w-\alpha}\right)^{2\rho}, & \alpha \in \mathbb{R}, \\[2ex] \dfrac{1+tw}{t-w}\,\dfrac{1}{1+t^2}\left(\dfrac{1+w^2}{1+t^2}\right)^{\rho}, & \alpha = \infty, \end{cases} \tag{2.2}$$

where ρ is a nonnegative integer, $\alpha \in \mathbb{R} \cup \{\infty\}$, $t \in \mathbb{R}$, and $w \in \mathbb{C}$. Here

$$S(\alpha,\rho;t,w) = -\sum_{k=0}^{2\rho-1} \frac{(t-\alpha)^k}{(w-\alpha)^{k+1}}\,, \qquad \alpha \in \mathbb{R},$$

and

$$S(\infty,\rho;t,w) = (t+w)\sum_{k=0}^{\rho-1} \frac{(1+w^2)^k}{(1+t^2)^{k+1}} + t\,\frac{(1+w^2)^\rho}{(1+t^2)^{\rho+1}}\,.$$

In order to apply the method of operator identities, we need versions of the Kreĭn-Langer integral representation for the classes $\mathbf{N}_\varkappa$ and $\mathbf{C}_\varkappa$ in the forms given in Theorems 2.1 and 2.3. The scalar case of Theorem 2.1 is due to Kreĭn and Langer [5]. The general case of Theorem 2.1 appears in [8]; the proof in [8] uses a different integral representation for matrix-valued functions in the generalized Nevanlinna class, due to Daho and Langer [1]. Theorem 2.3, the corresponding result for the circle, is derived from Theorem 2.1 by a change of variable (see Section 6).

Theorem 2.1. *If $v(w) \in \mathbf{N}_\varkappa$, $\varkappa \geq 0$, there exist an integer $r \geq 0$, distinct points $\alpha_0 = \infty$ and $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, and integers $\rho_0 \geq 0$ and $\rho_1, \dots, \rho_r \geq 1$ such that $\sum_0^\infty \rho_j \leq \varkappa$ and*

$$v(w) = \sum_{j=0}^{r} \int_{\Delta(\alpha_j)\setminus\{\alpha_j\}} \left[\frac{1}{t-w} - S(\alpha_j, \rho_j; t, w) \right] d\tau(t) + R(w)\,, \tag{2.3}$$

where $\tau(t)$ is a selfadjoint $m \times m$ matrix-valued function which is nondecreasing on the $r+1$ intervals of $\mathbb{R}$ determined by $\alpha_1, \dots, \alpha_r$, and such that the integral

$$\int_{\mathbb{R}\setminus\{\alpha_1,\dots,\alpha_r\}} \frac{(t-\alpha_1)^{2\rho_1}, \dots, (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\dots+\rho_r}} \, \frac{d\tau(t)}{(1+t^2)^{\rho_0+1}} \tag{2.4}$$

is convergent. The sets $\Delta(\alpha_1), \dots, \Delta(\alpha_r)$ are bounded open intervals containing $\alpha_1, \dots, \alpha_r$ and having disjoint closures; $\Delta(\alpha_0)$ is the complement of $\bigcup_{j=0}^\infty \Delta(\alpha_j)$ in $\mathbb{R} \cup \{\infty\}$. Such intervals can be chosen in any way. The term $R(w)$ has the form

$$\begin{aligned} R(w) = R_0(w) &- \sum_{j=1}^{r} R_j \left(\frac{1}{w-\alpha_j} \right) \\ &- \sum_{k=1}^{s} \left[M_k \left(\frac{1}{w-\beta_k} \right) + M_k^* \left(\frac{1}{\bar{w}-\beta_k} \right) \right], \end{aligned} \tag{2.5}$$

where $\beta_1, \dots, \beta_s$ are the poles of $v(w)$ in $\mathbb{C}_+$, and

(i) *for each $j = 0, \dots, r$, $R_j(w)$ is a polynomial of the form*

$$R_j(w) = \sum_{p=0}^{2\rho_j+1} R_{jp} w^p,$$

where $R_{j0}, \dots, R_{j,2\rho_j+1}$ are selfadjoint $m \times m$ matrices and $R_{j,2\rho_j+1} \geq 0$;

(ii) *$R_1(0) = \dots = R_r(0) = 0$;*

(iii) *for each $k = 1, \dots, s$, $M_k(w)$ is a polynomial $\not\equiv 0$ such that $M_k(0) = 0$.*

Conversely, every function of the form (2.3) belongs to some class $\mathbf{N}_\varkappa$, $\varkappa \geq 0$.

The classes $\mathbf{N}_\varkappa$ and $\mathbf{C}_\varkappa$ are connected by the relation

$$F(z) = \frac{1}{i}\, v(w), \qquad w = \varphi(z). \tag{2.6}$$

If $w = \varphi(z)$ and $\xi = \varphi(\zeta)$, then

$$\frac{F(z) + F(\zeta)^*}{1 - \bar{\zeta} z} = \frac{(w+i)(\bar{\xi}-i)}{2} \, \frac{v(w) - v^*(\xi)}{w - \bar{\xi}}. \tag{2.7}$$

Thus $F(z) \in \mathbf{C}_\varkappa$ if and only if $v(w) \in \mathbf{N}_\varkappa$.

The counterpart to Theorem 2.1 for $\mathbf{C}_\varkappa$ uses a generalization of the Herglotz kernel. Fix $a \in \mathbb{T}$ and $\rho = 0, 1, 2, \dots$. Define $U(a, \rho; \zeta, z)$ for $\zeta \in \mathbb{T}$, $z \in \mathbb{C}$ by

$$\frac{\zeta + z}{\zeta - z} - U(a, \rho; \zeta, z) = \begin{cases} \dfrac{2\zeta}{\zeta - z}\, \dfrac{z-1}{\zeta - 1} \left(\dfrac{\zeta - a}{\zeta - 1}\right)^{2\rho} \left(\dfrac{z-1}{z-a}\right)^{2\rho}, & a \neq 1, \\[2ex] \dfrac{\zeta + z}{\zeta - z}\, \dfrac{(\zeta - 1)^{2\rho}}{\zeta^\rho}\, \dfrac{z^\rho}{(z-1)^{2\rho}}\,, & a = 1. \end{cases} \tag{2.8}$$

Thus $U(a, \rho; \zeta, z)$ is a rational function of z and ζ such that

$$\overline{U(a, \rho; \zeta, 1/\bar{z})} = -U(a, \rho; \zeta, z). \tag{2.9}$$

For fixed $z \notin \mathbb{T}$,

$$\frac{\zeta + z}{\zeta - z} - U(a, \rho; \zeta, z) = \mathcal{O}\big((\zeta - a)^{2\rho}\big), \qquad \zeta \to a. \tag{2.10}$$

By straightforward algebra, for $a \neq 1$,

$$U(a, \rho; \zeta, z) = \frac{\zeta + 1}{\zeta - 1} - \frac{2\zeta(a-1)}{(\zeta - 1)^2} \sum_{k=0}^{2\rho - 1} \left(\frac{\zeta - a}{\zeta - 1}\right)^k \left(\frac{z - 1}{z - a}\right)^{k+1}, \tag{2.11}$$

and for $a = 1$,

$$\begin{aligned} U(1, \rho; \zeta, z) &= \frac{(\zeta + z)(1 - \zeta z)}{\zeta (1 - z)^2} \sum_{k=0}^{\rho - 1} \frac{z^k}{(z-1)^{2k}}\, \frac{(\zeta - 1)^{2k}}{\zeta^k} \\ &= \frac{\zeta + 1}{\zeta - 1} + \frac{2(1 - \zeta z)}{(\zeta - 1)(z - 1)} \sum_{k=0}^{\rho - 1} \frac{(\zeta - 1)^{2k}}{\zeta^k}\, \frac{z^k}{(z-1)^{2k}} \\ &\qquad - \frac{\zeta + 1}{\zeta - 1}\, \frac{(\zeta - 1)^{2\rho}}{\zeta^\rho}\, \frac{z^\rho}{(z-1)^{2\rho}}\,. \end{aligned} \tag{2.12}$$

Empty sums are interpreted as zero.

Lemma 2.2. *For all $\alpha \in \mathbb{R} \cup \infty$,*

$$\frac{1 + t^2}{i} \left(\frac{1}{t - w} - S(\alpha, \rho; t, w) \right) = \frac{\zeta + z}{\zeta - z} - U(a, \rho; \zeta, z), \tag{2.13}$$

where $a = \varphi(\alpha)$, $t = \varphi(\zeta)$, and $w = \varphi(z)$.

Theorem 2.3. *If $F(z) \in \mathbf{C}_\varkappa$, $\varkappa \geq 0$, then there exist an integer $r \geq 0$, distinct points $a_0 = 1, a_1, \dots, a_r$ on $\mathbb{T}$, and integers $\rho_0 \geq 0$ and $\rho_1, \dots, \rho_r \geq 1$, such that $\sum_0^r \rho_j \leq \varkappa$ and*

$$F(z) = \sum_{j=0}^{r} \int_{E(a_j) \setminus \{a_j\}} \left[\frac{\zeta + z}{\zeta - z} - U(a_j, \rho_j; \zeta, z) \right] d\sigma(\zeta) + T(z)\,, \tag{2.14}$$

where σ is a nonnegative $m \times m$ matrix-valued measure on $\mathbb{T} \setminus \{a_0, \dots, a_r\}$ such that the integral

$$\int_{\mathbb{T}\setminus\{a_0,\dots,a_r\}} |\zeta - a_0|^{2\rho_0} \cdots |\zeta - a_r|^{2\rho_r} \, d\sigma(\zeta) \tag{2.15}$$

is convergent. The sets $E(a_1), \dots, E(a_r)$ are open arcs of $\mathbb{T}$ containing $a_1, \dots, a_r$ whose closures are disjoint and do not contain the point $a_0 = 1$, and $E(a_0) = \mathbb{T} \setminus \bigcup_{j=1}^{r} E(a_j)$. Such arcs can be chosen in any way. The term $T(z)$ has the form

$$\begin{aligned} T(z) = \frac{1}{i}\, T_0\left(i\, \frac{1+z}{1-z}\right) - \sum_{j=1}^{r} \frac{1}{i}\, T_j\left(i\, \frac{z+a_j}{z-a_j}\right) \\ + \sum_{k=1}^{s} \left[N_k\left(\frac{1}{z-b_k}\right) - N_k^*\left(\frac{\bar z}{1-b_k \bar z}\right)\right], \end{aligned} \tag{2.16}$$

where $b_1, \dots, b_s$ are the poles of $F(z)$ in $\mathbb{D}$, and

(i) *for each $j = 0, \dots, r$, $T_j(z)$ is a polynomial of the form*

$$T_j(z) = \sum_{p=0}^{2\rho_j+1} T_{jp} z^p,$$

where $T_{j0}, \dots, T_{j,2\rho_j+1}$ are selfadjoint $m \times m$ matrices and $T_{j,2\rho_j+1} \geq 0$;

(ii) *$T_1(0) = \cdots = T_r(0) = 0$;*

(iii) *for each $k = 1, \dots, s$, $N_k(z)$ is a polynomial $\not\equiv 0$ such that $N_k(0) = 0$.*

Conversely, every function of the form (2.14) belongs to some class $\mathbf{C}_\varkappa$, $\varkappa \geq 0$.

A different representation for the rational term $T(z)$ in (2.14) is worth noting:

$$\begin{aligned} T(z) = \frac{1}{i}\, R_0(\varphi(z)) - \frac{1}{i} \sum_{j=1}^{r} R_j\left(\frac{1}{\varphi(z) - \varphi(a_j)}\right) \\ - \frac{1}{i} \sum_{k=1}^{s} \left[M_k\left(\frac{1}{\varphi(z) - \varphi(b_k)}\right) + M_k^*\left(\frac{1}{\overline{\varphi(z)} - \varphi(b_k)}\right)\right]. \end{aligned} \tag{2.17}$$

Here $R_j(w)$ and $M_k(w)$ are polynomials as in Theorem 2.1. See the proof of Theorem 2.3 in Section 6.

The formula (2.14) reduces to the Herglotz representation (1.4) in the classical case $\varkappa = 0$. To see this, take $r = s = 0$ and $\rho_0 = 0$, so that (2.14) becomes

$$F(z) = \frac{1}{i}\, T_{00} + \frac{1}{i}\, T_{01}\, i\, \frac{1+z}{1-z} + \int_{\mathbb{T}\setminus\{1\}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta)\,,$$

where $T_{00} = T_{00}^*$ and $T_{01} \geq 0$. Setting $C = -T_{00}$ and redefining σ to include a point mass T_{01} at $\zeta = 1$, we obtain

$$F(z) = iC + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta)\,.$$

Conversely, any Herglotz representation (1.4) can be written in the form (2.14).

Let μ be normalized Lebesgue measure on $\mathbb{T}$:

$$d\mu(\zeta) = \frac{d\theta}{2\pi}, \qquad \zeta = e^{i\theta}. \tag{2.18}$$

Theorem 2.4 (Stieltjes inversion formula). *If $F(z) \in \mathbf{C}_\varkappa$ is represented in the form* (2.14)*, then*

$$\lim_{r\uparrow 1} \int_\gamma g(\zeta)^* \operatorname{Re} F(r\zeta)\, f(\zeta)\, d\mu(\zeta) = \int_\gamma g(\zeta)^*\, d\sigma(\zeta)\, f(\zeta)\,, \tag{2.19}$$

where $\gamma = \{e^{i\theta} : c \le \theta \le d\}$ is any closed arc in $\mathbb{T}\backslash\{a_0,\dots,a_r\}$ whose endpoints are not point masses for σ, and $f(\zeta)$ and $g(\zeta)$ are any continuous $\mathbb{C}^m$-valued functions on γ.

We characterize the case in which $\rho_0 = 0$ and $T(z)$ is analytic at $z = 1$.

Theorem 2.5. *Suppose $F(z) \in \mathbf{C}_\varkappa$. Then the representation* (2.14) *can be chosen such that $\rho_0 = 0$ and the rational term $T(z)$ is analytic at $z = 1$ if and only if $(1-r)F(r) \to 0$ as $r \uparrow 1$.*

3. Operator identities

We turn to the study of operator identities

$$\begin{gathered} S - ASA^* = \Phi_1\Phi_2^* + \Phi_2\Phi_1^*\,, \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G},\mathfrak{H}), \end{gathered} \tag{3.1}$$

where $S = S^*$ and $\varkappa_S < \infty$. As a first step we generalize the formulas (1.5) and (1.6). The new formulas (3.3) and (3.4) provide a large class of operator identities in the indefinite case (Theorem 3.4).

Let $F(z) \in \mathbf{C}_\varkappa$ have Kreĭn-Langer representation (2.14). Let $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$ be given operators. *We assume that the spectrum of A is a finite subset of $\mathbb{D} \cup \mathbb{D}^c \cup \{-1\}$. We assume that $F(z)$ is analytic at the singularities of $(I + zA)^{-1}$ and $(A^* + zI)^{-1}$. If $-1 \in \sigma(A)$, we further assume that the rational term $T(z)$ in* (2.14) *is analytic at $z = 1$, $\rho_0 = 0$, and the integral*

$$\int_{E(1)\backslash\{1\}} (I + \zeta A)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(I + \bar\zeta A^*)^{-1} \tag{3.2}$$

is weakly convergent.

When the preceding assumptions are met, we write

$$\begin{aligned} S_F = \sum_{j=0}^{r} \int_{E(a_j)\backslash\{a_j\}} \Big\{ &2(I + \zeta A)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(I + \bar\zeta A^*)^{-1} \\ &- d\sigma(a_j, \rho_j, \zeta; A, \Phi_2)\Big\} + \sum_{\lambda\in\Lambda} \mathcal{T}_\lambda\,, \end{aligned} \tag{3.3}$$

$$\Phi_{1,F} = \sum_{j=0}^{r} \int_{E(a_j)\setminus\{a_j\}} \left\{ \frac{2\zeta}{\zeta-1}\,(I+A)(I+\zeta A)^{-1} - \mathcal{V}(a_j,\rho_j,\zeta;A) \right\} \Phi_2\, d\sigma(\zeta) + \sum_{\lambda\in\Lambda} \widehat{\mathcal{T}}_\lambda\,, \tag{3.4}$$

where $\Lambda = \{a_j, b_k, 1/\bar{b}_k \colon 0 \le j \le r, 1 \le k \le s\}$, and the terms in these expressions are defined as follows.

Case 1: $\sigma(A) \subseteq \mathbb{D}\cup\mathbb{D}^c$**.** For all $a \in \mathbb{T}$ and $\rho \ge 0$, define

$$d\sigma(a,\rho,\zeta;A,\Phi_2) = \operatorname*{Res}_{z=a} \left[\left(\frac{\zeta+z}{\zeta-z} - U(a,\rho;\zeta,z) \right) \cdot \cdot\, (I+zA)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(A^*+zI)^{-1} \right], \tag{3.5}$$

$$\mathcal{V}(a,\rho,\zeta;A) = \operatorname*{Res}_{z=a} \left[\left(\frac{\zeta+z}{\zeta-z} - U(a,\rho;\zeta,z) \right) \frac{(I+A)(I+zA)^{-1}}{z-1} \right]. \tag{3.6}$$

In the discrete terms, for all $\lambda\in\Lambda$ we set

$$\mathcal{T}_\lambda = -\operatorname*{Res}_{z=\lambda} \left[(I+zA)^{-1}\Phi_2 T(z)\Phi_2^*(A^*+zI)^{-1} \right], \tag{3.7}$$

$$\widehat{\mathcal{T}}_\lambda = -\operatorname*{Res}_{z=\lambda} \left[\frac{(I+A)(I+zA)^{-1}}{z-1}\,\Phi_2 T(z) \right]. \tag{3.8}$$

Case 2: $-1\in\sigma(A)$**.** We use the same definitions as in Case 1 for $a \ne 1$ and $\lambda \ne 1$, and for $a = 1$ and $\lambda = 1$ we define

$$d\sigma(1,0,\zeta;A,\Phi_2) = 0, \qquad \mathcal{V}(1,0,\zeta;A) = \frac{\zeta+1}{\zeta-1}\,I\,, \tag{3.9}$$

$$\mathcal{T}_1 = 0, \qquad \widehat{\mathcal{T}}_1 = -\Phi_2 T(1). \tag{3.10}$$

We give explicit formulas for the preceding expressions.

Theorem 3.1. *The following statements apply in both Case* 1 *and Case* 2*.*
(1) *For all* $a \in \mathbb{T}\setminus\{1\}$*,* $\rho \ge 1$*, and* $\zeta \in E(a)\setminus\{a\}$*,*

$$d\sigma(a,\rho,\zeta;A,\Phi_2) = 2\zeta \left(\frac{a-1}{\zeta-1}\right)^2 \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k \sum_{p+q=k} (I+A)^p(I+aA)^{-p-1} \cdot \cdot\, \Phi_2 d\sigma(\zeta)\Phi_2^*(A^*+aI)^{-q-1}(I+A^*)^q\,, \tag{3.11}$$

$$\mathcal{V}(a,\rho,\zeta;A) = \frac{2\zeta(a-1)}{(\zeta-1)^2} \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k (I+A)^{k+1}(I+aA)^{-k-1}\,. \tag{3.12}$$

(2) *For* $a = 1$ *and* $\rho = 0$*,* $d\sigma(1,0,\zeta;A,\Phi_2)$ *and* $\mathcal{V}(1,0,\zeta;A)$ *are given by* (3.9)*.*

The formulas (3.11) and (3.12) can be described in another way. For any $a \neq 1$, expand

$$2(I+\zeta A)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(I+\zeta^{-1}A^*)^{-1} \quad \text{and} \quad \frac{2\zeta}{\zeta-1}\,(I+A)(I+\zeta A)^{-1}$$

using the identity

$$(I+\zeta A)^{-1} = \frac{a-1}{\zeta-1}\sum_{k=0}^{\infty}\left(\frac{\zeta-a}{\zeta-1}\right)^k (I+A)^k(I+aA)^k.$$

When all terms that are $\mathcal{O}((\zeta-a)^{2\rho})$ are discarded, the terms that remain coincide with the expressions in (3.11) and (3.12).

There is one more possibility in Case 1.

Theorem 3.2. *Assume Case 1. For $a=1$ and $\rho \geq 1$,*

$$\begin{aligned} d\sigma(1,\rho,\zeta;A,\Phi_2) &= \sum_{k=0}^{\rho-1}\frac{2(\zeta-1)^{2k}}{\zeta^k}\,(I+A)\,dG_k(\zeta)\,(I+A^*) \\ &\quad + \sum_{k=0}^{\rho-2}\frac{2(\zeta-1)^{2k+1}}{\zeta^k}\,dG_k(\zeta)\,A^* - \sum_{k=0}^{\rho-2}\frac{2(\zeta-1)^{2k+1}}{\zeta^{k+1}}\,A\,dG_k(\zeta) \\ &\quad + (\zeta+1)\,\frac{(\zeta-1)^{2\rho-1}}{\zeta^\rho}\,\big[dG_{\rho-1}(\zeta)\,A^* - A\,dG_{\rho-1}(\zeta)\big]\,, \qquad (3.13) \\ \mathcal{V}(1,\rho,\zeta;A) &= 2(A+\zeta I)\sum_{k=0}^{\rho-1}\frac{(\zeta-1)^{2k-1}}{\zeta^k}\,(-A)^k(I+A)^{-2k-1} \\ &\quad + (\zeta+1)\frac{(\zeta-1)^{2\rho-1}}{\zeta^\rho}(-A)^\rho(I+A)^{-2\rho}\,, \qquad (3.14) \end{aligned}$$

where

$$dG_k(\zeta) = \sum_{\substack{p+q=k \\ p,q\geq 0}}(-A)^p(I+A)^{-2p-2}\Phi_2 d\sigma(\zeta)\Phi_2^*(I+A^*)^{-2q-2}(-A^*)^q, \qquad (3.15)$$

$k=0,1,2,\ldots$. *Empty sums are interpreted as zero.*

The formulas (3.11)–(3.14) are derived by evaluating the residues in (3.5) and (3.6). They can be computed in a different way, namely, by change of variables from the corresponding formulas for the line in [9, Theorem 3.2]. The results are the same, except in one case. In place of (3.13), we obtain the different but equivalent formula

$$
\begin{aligned}
&d\sigma(1,\rho,\zeta;A,\Phi_2)\\
&\quad = 2\sum_{\ell=0}^{\rho-1}\frac{(\zeta-1)^{2\ell}}{(-4\zeta)^{\ell}}\sum_{p=0}^{\ell}\binom{\ell}{p}\sum_{\substack{j+k=2p+2\\ j,k\geq 1}} Q_j(A)\,\Phi_2 d\sigma(\zeta)\Phi_2^*\,Q_k(A)^* \qquad (3.16)\\
&\qquad - 2i\sum_{\ell=0}^{\rho-1}(\zeta+1)\,\frac{(\zeta-1)^{2\ell+1}}{(-4\zeta)^{\ell+1}}\sum_{p=1}^{\ell+1}\binom{\ell+1}{p}\sum_{\substack{j+k=2p+1\\ j,k\geq 1}} Q_j(A)\,\Phi_2 d\sigma(\zeta)\Phi_2^*\,Q_k(A)^*,
\end{aligned}
$$

where $Q_k(A) = i^{-k}(I+A)^{-k}(I-A)^{k-1}$. We omit the details.

Theorem 3.3. *The definitions of S_F and $\Phi_{1,F}$ do not depend on the choice of Kreĭn-Langer representation of $F(z)$.*

Theorem 3.4. *Let $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some function $F(z) \in \mathbf{C}_\varkappa$ and operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$. Then A, S, Φ_1, Φ_2 satisfy* (3.1), *$S = S^*$, and $\varkappa_S < \infty$.*

4. Operator interpolation problem

Our main results provide converses to Theorem 3.4.

Operator Interpolation Problem. *Given an operator identity* (3.1) *with $S = S^*$ and $\varkappa_S < \infty$, find all generalized Carathéodory functions $F(z)$ such that*

$$S = S_F \qquad \text{and} \qquad \Phi_1 = \Phi_{1,F}. \tag{4.1}$$

If S is invertible, the problem is said to be nondegenerate, and otherwise it is called degenerate. We are mainly concerned with the nondegenerate case.

Let A, S, Φ_1, Φ_2 be operators that satisfy (3.1). Assume that S is invertible, $\varkappa_S < \infty$, and $\sigma(A)$ is a finite subset of $\mathbb{D} \cup \mathbb{D}^c \cup \{-1\}$. Set

$$\Pi = \begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_m \\ I_m & 0\end{bmatrix}, \tag{4.2}$$

and

$$\mathfrak{A}(z) = I_{2m} - (1+z)\Pi^*(A^* + zI)^{-1}S^{-1}(I-A)^{-1}\Pi J. \tag{4.3}$$

The operator-valued function $(1+z)(A^*+zI)^{-1}$ is analytic at infinity, and thus if $\sigma(A) = \{z_1,\dots,z_n\}$ then $\mathfrak{A}(z)$ is analytic on the set

$$\Omega_{\mathfrak{A}} = \mathbb{C}_\infty \setminus \{-\bar z_1,\dots,-\bar z_n\}.$$

Lemma 4.1. *If* $1/\bar{z}, 1/\bar{\zeta} \in \Omega_{\mathfrak{A}}$*, then*

$$\frac{J - \mathfrak{A}(1/\bar{\zeta})J\mathfrak{A}(1/\bar{z})^*}{1 - \bar{\zeta}z} = \Pi^*(I + \bar{\zeta}A^*)^{-1}S^{-1}(I + zA)^{-1}\Pi\,. \tag{4.4}$$

If $z, 1/\bar{z} \in \Omega_{\mathfrak{A}}$*, then* $\mathfrak{A}(z)$ *is invertible, and*

$$\mathfrak{A}(z)^{-1} = J\mathfrak{A}(1/\bar{z})^*J. \tag{4.5}$$

Write

$$\mathfrak{A}(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \tag{4.6}$$

where $a(z), b(z), c(z), d(z)$ are $m \times m$ matrix-valued functions analytic on $\Omega_{\mathfrak{A}}$.

Definition 4.2. By $N(\mathfrak{A})$ we mean the set of all functions

$$F(z) = [a(z)P(z) + b(z)Q(z)][c(z)P(z) + d(z)Q(z)]^{-1}, \tag{4.7}$$

where

(i) $P(z), Q(z)$ are $m \times m$ matrix-valued functions which are analytic on $\mathbb{D} \cup \mathbb{D}^c$ except at isolated points and satisfy

$$P(1/\bar{z})^*Q(z) + Q(1/\bar{z})^*P(z) = 0;$$

(ii) $c(z)P(z) + d(z)Q(z)$ is invertible on $\mathbb{D} \cup \mathbb{D}^c$ except at isolated points;

(iii) the kernel

$$D_{P,Q}(z,\zeta) = \frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{1 - \bar{\zeta}z}$$

has $\varkappa_{P,Q} < \infty$ negative squares.

Lemma 4.3. *Let* $E(z), F(z), P(z), Q(z)$ *be* $m \times m$ *matrix-valued functions which are analytic on* $\mathbb{D} \cup \mathbb{D}^c$ *except at isolated points, such that*

$$\mathfrak{A}(z)^{-1} \begin{bmatrix} F(z) \\ I \end{bmatrix} = \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} E(z). \tag{4.8}$$

Then

$$\frac{F(z) + F(\zeta)^*}{1 - \bar{\zeta}z} = E(\zeta)^* \, \frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{1 - \bar{\zeta}z} \, E(z) + B_F(\zeta)^*S^{-1}B_F(z),$$

where

$$B_F(z) = (I + zA)^{-1}[\Phi_1 + \Phi_2F(z)]\,. \tag{4.9}$$

The kernel

$$L_F(z,\zeta) = \begin{bmatrix} S & B_F(z) \\ B_F(\zeta)^* & \dfrac{F(z) + F(\zeta)^*}{1 - \bar{\zeta}z} \end{bmatrix} \tag{4.10}$$

plays an important role in the operator interpolation problem. A first application is to characterize functions in the class $N(\mathfrak{A})$.

Theorem 4.4. *Let A, S, Φ_1, Φ_2 satisfy* (3.1). *Assume S selfadjoint and invertible and $\varkappa_S < \infty$. Let $F(z)$ be an $m \times m$ matrix-valued function which is analytic on $\mathbb{D} \cup \mathbb{D}^c$ except at isolated points. Then*

(1) *$F(z) \in N(\mathfrak{A})$ if and only if $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa = \varkappa_F < \infty$ and the kernel $L_F(z,\zeta)$ has a finite number $\varkappa_{L_F}$ of negative squares;*

(2) *if $F(z) \in N(\mathfrak{A})$ has the form* (4.7)*, then $\varkappa_F \leq \varkappa_S + \varkappa_{P,Q} = \varkappa_{L_F}$.*

The kernel $L_F(z,\zeta)$ is also used in a necessary condition for $F(z)$ to be a solution of the interpolation problem. The condition is an abstract form of the Potapov fundamental matrix inequality and is valid in both the nondegenerate and degenerate cases.

Theorem 4.5. *Let A, S, Φ_1, Φ_2 satisfy* (3.1)*. If $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some $F(z) \in \mathbf{C}_\varkappa$, $\varkappa \geq 0$, then $L_F(z,\zeta)$ has a finite number of negative squares. Hence if S is invertible, then also $F(z) \in N(\mathfrak{A})$.*

Our main results for the operator interpolation problem follow.

Theorem 4.6. *Let A, S, Φ_1, Φ_2 satisfy* (3.1)*. Assume S selfadjoint and invertible and $\varkappa_S < \infty$. Assume that $\lambda\bar{\mu} \neq 1$ for all $\lambda, \mu \in \sigma(A)$.*

(1) *Let $F(z) \in N(\mathfrak{A})$. Assume that $F(z)$ and $(1-z)B_F(z)$ are analytic at every point $\lambda \in \mathbb{C}_\infty$ such that $-1/\lambda \in \sigma(A)$. Then $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa \geq 0$, and $S = S_F$ and $\Phi_1 = \Phi_{1,F}$.*

(2) *Conversely, if $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some $F(z) \in \mathbf{C}_\varkappa$, then $F(z) \in N(\mathfrak{A})$ and satisfies the conditions in* (1).

A function that has a removable singularity at a point is considered to be analytic at the point. The condition in (1) requires that $F(z)$ and $(1-z)B_F(z)$ are analytic at ∞ whenever $0 \in \sigma(A)$.

A practical way to apply Theorem 4.6 is supplied by a method of A. L. Sakhnovich (see [10, Theorem 13]), adapted to the circle case.

Theorem 4.7. *Let A, S, Φ_1, Φ_2 satisfy* (3.1)*. Assume S selfadjoint and invertible and $\varkappa_S < \infty$. Assume that $\lambda\bar{\mu} \neq 1$ for all $\lambda, \mu \in \sigma(A)$.*

(1) *Suppose $F(z) \in N(\mathfrak{A})$ and assume that $F(z)$ has a representation* (4.7) *satisfying the conditions* (i) $-$ (iii) *in Definition* 4.2 *and, in addition, the condition*

(iv) *the functions*

$$P(z)[c(z)P(z) + d(z)Q(z)]^{-1} \quad \text{and} \quad \Phi_1 Q(z)[c(z)P(z) + d(z)Q(z)]^{-1}$$

are analytic at every point λ such that $-1/\lambda \in \sigma(A)$.

Then $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa \geq 0$, and $S = S_F$ and $\Phi_1 = \Phi_{1,F}$.

(2) *Conversely, suppose $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa \geq 0$, and $S = S_F$ and $\Phi_1 = \Phi_{1,F}$. Then $F(z) \in N(\mathfrak{A})$, and every representation* (4.7) *of $F(z)$ satisfying* (i)-(iii) *also satisfies* (iv).

When $0 \in \sigma(A)$, the condition (iv) is understood to hold for $\lambda = \infty$. An obvious sufficient condition for (iv) to hold is that $P(z)$ and $Q(z)$ are analytic and $c(z)P(z) + d(z)Q(z)$ is invertible at every point λ such that $-1/\lambda \in \sigma(A)$.

Theorems 4.6 and 4.7 apply only in Case 1, that is, when $\sigma(A)$ contains no point of $\mathbb{T}$. We treat one instance of Case 2.

Theorem 4.8. *Let A, S, Φ_1, Φ_2 satisfy* (3.1). *Assume S selfadjoint and invertible and $\varkappa_S < \infty$. Assume that $\sigma(A) = \{-1\}$, and that for every $f \in \mathfrak{H}$, $f \neq 0$,*

$$(1 + r)\|(A - rI)^{-1} f\| \neq \mathcal{O}(1), \qquad r \to -1 \ (r \text{ real}). \tag{4.11}$$

(1) *Suppose $F(z) \in N(\mathfrak{A})$, and assume that for $r \to 1$ through real values:*

(i) $(1 - r)F(r) \to 0$ *;*

(ii) *for all $h \in \mathfrak{H}$ and $g \in \mathbb{C}^m$,*

$$(1 - r)\,\langle B_F(r)g, h\rangle = \mathcal{O}(1);$$

(iii) *for all $h, k \in \mathfrak{H}$,*

$$\big\langle [-S(I + A^*) + (1 - r)B_F(r)\Phi_2^*](A^* + rI)^{-1} h, k\big\rangle = \mathcal{O}(1).$$

Then $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa \geq 0$, and $S = S_F$ and $\Phi_1 = \Phi_{1,F}$.

(2) *Conversely, if $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ for some $F(z) \in \mathbf{C}_\varkappa$, $\varkappa \geq 0$, then $F(z) \in N(\mathfrak{A})$ and $F(z)$ satisfies the conditions* (i)–(iii) *in* (1).

Example. We illustrate the operator interpolation problem in a concrete situation. Fix distinct points $z_1, \dots, z_n$ in $\mathbb{D}$ and $p \times m$ matrices $e_1, \dots, e_n$ and $f_1, \dots, f_n$.

Tangential interpolation problem. *Find all generalized Carathéodory functions $G(z)$ such that*

$$f_j = e_j G(z_j), \qquad j = 1, \dots, n. \tag{4.12}$$

We apply the theorems of Section 4 for the indefinite case and classes $\mathbf{C}_\varkappa$ (cf. [11, Section 6.1]). Set

$$A = \begin{bmatrix} z_1 I_p & 0 & \cdots & 0 \\ 0 & z_2 I_p & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & z_n I_p \end{bmatrix}_{pn \times pn}, \qquad \Phi_2 = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}_{pn \times m}, \tag{4.13}$$

and

$$S = \left[\frac{f_j e_k^* + e_j f_k^*}{1 - z_j \bar{z}_k} \right]_{pn \times pn}, \qquad \Phi_1 = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}_{pn \times m}. \tag{4.14}$$

Then A, S, Φ_1, Φ_2 satisfy (3.1).

Theorem 4.9. *Let $F(z) \in \mathbf{C}_\varkappa$, and assume that $F(z)$ is analytic at the points $-\bar{z}_1, \dots, -\bar{z}_n$. Set $G(z) = F(-\bar{z})^*$. Then $G(z) \in \mathbf{C}_\varkappa$, and for A, Φ_2 as in* (4.13),

$$S_F = \left[e_j \, \frac{G(z_j) + G(z_k)^*}{1 - z_j \bar{z}_k} \, e_k^* \right]_{pn \times pn}, \tag{4.15}$$

$$\Phi_{1,F} = \begin{bmatrix} e_1 G(z_1) \\ \vdots \\ e_n G(z_n) \end{bmatrix}_{pn \times m}. \tag{4.16}$$

In view of Theorem 4.7, we obtain a solution to the tangential interpolation problem.

Corollary 4.10. *Let $z_1, \dots, z_n$ be distinct points in $\mathbb{D}$. Let $e_1, \dots, e_n$ and $f_1, \dots, f_n$ be given $p \times m$ matrices, and assume that the matrix*

$$\left[\frac{f_j e_k^* + e_j f_k^*}{1 - z_j \bar{z}_k} \right]_{j,k=1}^n$$

is invertible. Define $\mathfrak{A}(z)$ by (4.3) *for the operators* (4.13) *and* (4.14). *Then the set of solutions of the problem* (4.12) *coincides with the set of functions $G(z) = F(-\bar{z})$, where $F(z)$ has the form*

$$F(z) = [a(z)P(z) + b(z)Q(z)][c(z)P(z) + d(z)Q(z)]^{-1},$$

for some functions $P(z)$ and $Q(z)$ which satisfy the conditions (i)-(iii) *in Definition* 4.2 *and condition* (iv) *in Theorem* 4.7.

Klotz and Lasarow [4] treat the more general multiple-point interpolation problem for the generalized Carathéodory class and obtain a result in a different form by a different method. Multiple-point problems can be included in our formulation by choosing A to be a direct sum of Jordan blocks (cf. formulas (5.9)–(5.12) in [3]).

Part 2. Background and proofs

5. Operator identities and generalized Nevanlinna functions

We review results from [9] concerning operator identities of the form

$$\begin{gathered} A_\ell S_\ell - S_\ell A_\ell^* = i \left[\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^* \right], \\ A_\ell, S_\ell \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}), \end{gathered} \tag{5.1}$$

where $S_\ell = S_\ell^*$. The subscript ℓ is for the "line", as opposed to the "circle" case:

$$\begin{gathered} S - ASA^* = \Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*, \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}). \end{gathered} \tag{5.2}$$

The operator identities (5.1) and (5.2) are connected by

$$\begin{aligned} A_\ell &= i\,(I+A)(I-A)^{-1}\,, \qquad & A &= (A_\ell - iI)(A_\ell + iI)^{-1}\,, \\ S_\ell &= \frac{1}{2}\,(I-A)S(I-A^*)\,, \qquad & S &= 2\,(I-A)^{-1}S_\ell(I-A^*)^{-1}\,. \end{aligned} \tag{5.3}$$

Thus A, S, Φ_1, Φ_2 satisfy (5.2) if and only if $A_\ell, S_\ell, \Phi_1, \Phi_2$ satisfy (5.1). In our applications, $1 \notin \sigma(A)$, and so $-i \notin \sigma(A_\ell)$. Hence the inverses in (5.3) exist in all cases of interest.

Suppose we are given a function $v(w) \in \mathbf{N}_\varkappa$, $\varkappa \geq 0$, with Kreĭn-Langer representation (2.3). Let $A_\ell \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G})$ be given operators such that the spectrum of A_ℓ is a finite subset of $\mathbb{C}_+ \cup \mathbb{C}_- \cup \{0\}$. *We assume that $v(w)$ is analytic at the singularities of $(I - wA_\ell)^{-1}$ and $(I - wA_\ell^*)^{-1}$. If $0 \in \sigma(A_\ell)$, we assume further that the rational term $R(w)$ in (2.3) is analytic at $w = \infty$, $\rho_0 = 0$, and the integral*

$$\int_{\Delta(\infty)\setminus\{\infty\}} (I - A_\ell t)^{-1}\Phi_2 d\tau(t)\Phi_2^*(I - A_\ell^* t)^{-1}$$

is weakly convergent.

Operators $S_v \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_{1,v} \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$ are defined as in [9, (3.2)] and [9, (3.3)]. We use simplified but equivalent versions of these formulas. Set

$$\begin{aligned} S_v = \sum_{j=0}^{r} \int_{\Delta(\alpha_j)\setminus\{\alpha_j\}} \Big\{ &(I - A_\ell t)^{-1}\Phi_2 d\tau(t)\Phi_2^*(I - A_\ell^* t)^{-1} \\ &- d\tau(\alpha_j, \rho_j, t; A_\ell, \Phi_2)\Big\} + \sum_{\mu\in\Lambda_\ell} \mathfrak{R}_\mu\,, \end{aligned} \tag{5.4}$$

$$\begin{aligned} \Phi_{1,v} = -i \sum_{j=0}^{r} \int_{\Delta(\alpha_j)\setminus\{\alpha_j\}} &\Big\{ A_\ell(I - A_\ell t)^{-1} - \mathfrak{S}(\alpha_j, \rho_j, t; A_\ell)\Big\}\, \Phi_2\, d\tau(t) \\ &- i \sum_{\mu\in\Lambda_\ell} \widehat{\mathfrak{R}}_\mu\,, \end{aligned} \tag{5.5}$$

where $\Lambda_\ell = \{\alpha_j, \beta_k, \bar\beta_k \colon 0 \leq j \leq r, 1 \leq k \leq s\}$, and the terms are defined as follows.

Case 1: $0 \notin \sigma(A_\ell)$**.** Then for $\alpha \in \mathbb{R} \cup \{\infty\}$ and $\mu \in \Lambda_\ell$, we define

$$\begin{aligned} d\tau(\alpha, \rho, t; A_\ell, \Phi_2) = \operatorname*{Res}_{w=\alpha} \Bigg[&\bigg(\frac{1}{t-w} - S(\alpha, \rho; t, w) \bigg) \cdot \\ &\cdot (I - wA_\ell)^{-1}\Phi_2 d\tau(t)\Phi_2^*\,(I - wA_\ell^*)^{-1} \Bigg], \end{aligned} \tag{5.6}$$

$$\mathfrak{S}(\alpha, \rho, t; A_\ell) = \operatorname*{Res}_{w=\alpha} \Bigg[\bigg(\frac{1}{t-w} - S(\alpha, \rho; t, w) \bigg) A_\ell (I - wA_\ell)^{-1} \Bigg], \tag{5.7}$$

$$\mathfrak{R}_\mu = -\operatorname*{Res}_{w=\mu} \big[(I - wA_\ell)^{-1}\Phi_2 R(w)\Phi_2^*(I - wA_\ell^*)^{-1} \big], \tag{5.8}$$

$$\widehat{\mathfrak{R}}_\mu = -\operatorname*{Res}_{w=\mu}\big[A_\ell(I-wA_\ell)^{-1}\Phi_2 R(w)\big]. \tag{5.9}$$

Case 2: $0 \in \sigma(A_\ell)$**.** In this case use the same formulas for $\alpha \neq \infty$ and $\mu \neq \infty$, but for $\alpha = \infty$ and $\mu = \infty$ we set

$$d\tau(\infty, 0, t; A_\ell, \Phi_2) = 0, \qquad \mathfrak{S}(\infty, 0, t; A_\ell) = -\frac{tI}{1+t^2}\,, \tag{5.10}$$

$$\mathfrak{R}_\infty = 0, \qquad \widehat{R}_\infty = -\Phi_2 R(\infty). \tag{5.11}$$

The discrete terms (5.4) and (5.5) are defined differently in [9, (3.4)–(3.11)]; the versions here are equivalent to the original forms in [9].

The next result shows the connection between the circle and line cases.

Theorem 5.1. *Let $F(z) \in \mathbf{C}_\varkappa$ and $v(w) \in \mathbf{N}_\varkappa$ be connected by $F(z) = -iv(w)$, $w = \varphi(z)$. Then*

$$S_F = 2(I-A)^{-1}S_v(I-A^*)^{-1} \qquad \text{and} \qquad \Phi_{1,F} = \Phi_{1,v}. \tag{5.12}$$

In particular, $\varkappa_{S_F} = \varkappa_{S_v}$.

Proof. Let $S_F, \Phi_{1,F}$ be defined by (3.3)–(3.4) for given operators A, Φ_2, and define $S_v, \Phi_{1,v}$ by (5.4)–(5.5) for A_ℓ, Φ_2 where $A_\ell = i(I+A)(I-A)^{-1}$. Let $F(z)$ and $v(w)$ have Kreĭn-Langer representations (2.14) and (2.3). By (5.4)–(5.5),

$$S_v = \sum_{j=0}^{r} S_v^{(\alpha_j)} + \sum_{\mu\in\Lambda_\ell} \mathfrak{R}_\mu, \tag{5.13}$$

$$\Phi_{1,v} = \sum_{j=0}^{r} \Phi_{1,v}^{(\alpha_j)} + \sum_{\mu\in\Lambda_\ell} \frac{1}{i}\,\widehat{\mathfrak{R}}_\mu, \tag{5.14}$$

where $S_v^{(\alpha_j)}$ and $\Phi_{1,v}^{(\alpha_j)}$ denote the integral terms in (5.4)–(5.5). We can similarly write (3.3)–(3.4) in the form

$$S_F = \sum_{j=0}^{r} S_F^{(a_j)} + \sum_{\lambda\in\Lambda} \mathcal{T}_\lambda, \tag{5.15}$$

$$\Phi_{1,F} = \sum_{j=0}^{r} \Phi_{1,F}^{(a_j)} + \sum_{\lambda\in\Lambda} \widehat{\mathcal{T}}_\lambda. \tag{5.16}$$

Dropping subscripts, we must show that

$$2(I-A)^{-1}S_v^{(\alpha)}(I-A^*)^{-1} = S_F^{(a)}, \tag{5.17}$$

$$2(I-A)^{-1}\mathfrak{R}_\mu(I-A^*)^{-1} = \mathcal{T}_\lambda, \tag{5.18}$$

$$\Phi_{1,v}^{(\alpha)} = \Phi_{1,F}^{(a)}, \tag{5.19}$$

$$\frac{1}{i}\,\widehat{\mathfrak{R}}_\mu = \widehat{\mathcal{T}}_\lambda, \tag{5.20}$$

for all $\alpha \in \mathbb{R}\cup\{\infty\}$, $a \in \mathbb{T}$, $\mu \in \Lambda_\ell$, $\lambda \in \Lambda$ such that $\alpha = \varphi(a)$ and $\mu = \varphi(\lambda)$.

Case 1: $0 \notin \sigma(A_\ell)$ **and** $-1 \notin \sigma(A)$

Proof of (5.17): Let $\alpha = \varphi(a)$, $\alpha \in \mathbb{R} \cup \{\infty\}$, $a \in \mathbb{T}$. By (5.4),

$$\begin{aligned} &2(I-A)^{-1} S_v^{(\alpha)} (I-A^*)^{-1} \\ &\quad = \int_{\Delta(\alpha)\setminus\{\alpha\}} 2(I-A)^{-1} \Big\{ (I-A_\ell t)^{-1} \Phi_2 d\tau(t) \Phi_2^* (I-A_\ell^* t)^{-1} \\ &\qquad\qquad - d\tau(\alpha,\rho,t;A_\ell,\Phi_2) \Big\} (I-A^*)^{-1}, \end{aligned} \tag{5.21}$$

In the first term of (5.21) make the substitutions $\zeta = \varphi(t)$,

$$d\sigma(\zeta) = \frac{d\tau(t)}{1+t^2}, \qquad t = i\,\frac{1+\zeta}{1-\zeta}\,,$$
$$(I-A)^{-1}(I-A_\ell t)^{-1} = (1-it)^{-1}(I+\zeta A)^{-1},$$

to get

$$\begin{aligned} &2(I-A)^{-1}(I-A_\ell t)^{-1}\Phi_2 d\tau(t)\Phi_2^*(I-A_\ell^* t)^{-1}(I-A^*)^{-1} \\ &\qquad = 2(I+\zeta A)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(I+\bar{\zeta}A^*)^{-1}. \end{aligned} \tag{5.22}$$

The second term of (5.21) is computed from (5.6):

$$\begin{aligned} &2(I-A)^{-1} d\tau(\alpha,\rho,t;A_\ell,\Phi_2)(I-A^*)^{-1} \\ &\quad = \operatorname*{Res}_{w=\alpha} \Bigg[\left(\frac{1}{t-w} - S(\alpha,\rho;t,w) \right) \cdot \\ &\qquad \cdot 2(I-A)^{-1}(I-wA_\ell)^{-1}\Phi_2 d\tau(t)\Phi_2^*\,(I-wA_\ell^*)^{-1}(I-A^*)^{-1} \Bigg]. \end{aligned} \tag{5.23}$$

Since

$$\begin{aligned} (I-A)^{-1}(I-wA_\ell)^{-1} &= \frac{1-z}{2}\,(I+zA)^{-1}, \\ (I-A^*)^{-1}(I-wA_\ell^*)^{-1} &= \frac{1-z}{-2}\,(A^*+zI)^{-1}, \end{aligned} \tag{5.24}$$

we obtain

$$\begin{aligned} &2(I-A)^{-1} d\tau(\alpha,\rho,t;A_\ell,\Phi_2)(I-A^*)^{-1} \\ &\quad \overset{(5.23)}{=} \operatorname*{Res}_{w=\alpha} \Bigg[\frac{1+t^2}{i} \left(\frac{1}{t-w} - S(\alpha,\rho;t,w) \right) \cdot \\ &\qquad \cdot 2i\,\frac{1-z}{2}\,(I+zA)^{-1}\Phi_2 \frac{d\tau(t)}{1+t^2} \Phi_2^*\,\frac{1-z}{-2}\,(A^*+zI)^{-1} \Bigg] \\ &\quad \overset{(A.2)}{=} \operatorname*{Res}_{z=a} \Bigg[\frac{2i}{(z-1)^2} \left(\frac{\zeta+z}{\zeta-z} - U(a,\rho;\zeta,z) \right) \\ &\qquad \cdot 2i\,\frac{(z-1)^2}{-4} (I+zA)^{-1} d\sigma(\zeta)(A^*+zI)^{-1} \Bigg] \\ &\quad \overset{(3.5)}{=} d\sigma(a,\rho;\zeta,z). \end{aligned} \tag{5.25}$$

Thus by (3.3),

$$2(I-A)^{-1}S_v^{(\alpha)}(I-A^*)^{-1}$$
$$= \int_{E(a)\setminus\{a\}} \Big\{ 2(I+\zeta A)^{-1}\Phi_2 d\sigma(\zeta)\Phi_2^*(I+\bar\zeta A^*)^{-1} - d\sigma(a_j,\rho_j,\zeta;A,\Phi_2) \Big\} = S_F^{(a)}.$$

This proves (5.17).

Proof of (5.18): Let $\mu = \varphi(\lambda)$, $\mu \in \Lambda_\ell$, $\lambda \in \Lambda$. By (5.8) and (5.24),

$$2(I-A)^{-1}\mathfrak{R}_\mu(I-A^*)^{-1}$$
$$= -\operatorname*{Res}_{w=\mu}\Big[2(I-A)^{-1}(I-wA_\ell)^{-1}\Phi_2 R(w)\Phi_2^*(I-wA_\ell^*)^{-1}(I-A^*)^{-1}\Big]$$
$$= -\operatorname*{Res}_{w=\mu}\Big[2\,\frac{1-z}{2}\,(I+zA)^{-1}\Phi_2 R(w)\Phi_2^*\,\frac{1-z}{-2}\,(A^*+zI)^{-1}\Big].$$

By the proof of Theorem 2.3, $T(z) = -iR(\varphi(z))$. Hence by (A.2),

$$2(I-A)^{-1}\mathfrak{R}_\mu(I-A^*)^{-1}$$
$$= -\operatorname*{Res}_{z=\lambda}\left[\frac{2i}{(z-1)^2}\,\frac{(z-1)^2}{-2}\,(I+zA)^{-1}\Phi_2 R(\varphi(z))\Phi_2^*\,(A^*+zI)^{-1}\right]$$
$$= -\operatorname*{Res}_{z=\lambda}\left[(I+zA)^{-1}\Phi_2 T(z)\Phi_2^*\,(A^*+zI)^{-1}\right]$$
$$= \mathcal{T}_\lambda,$$

where the last equality is by (3.7). This proves (5.18).

The proofs of (5.19) and (5.20) are similar and omitted.

Case 2: $0 \in \sigma(A_\ell)$ **and** $-1 \in \sigma(A)$

We proceed as in Case 1, but now it is only necessary to verify (5.17)–(5.20) when $\alpha = \infty, a = 1$ and $\mu = \infty, \lambda = 1$, since otherwise the terms are unchanged from Case 1. Thus we must check that

$$2(I-A)^{-1}S_v^{(\infty)}(I-A^*)^{-1} = S_F^{(1)}, \tag{5.26}$$
$$2(I-A)^{-1}\mathfrak{R}_\infty(I-A^*)^{-1} = \mathcal{T}_1, \tag{5.27}$$
$$\Phi_{1,v}^{(\infty)} = \Phi_{1,F}^{(1)}, \tag{5.28}$$
$$\frac{1}{i}\,\widehat{\mathfrak{R}}_\infty = \widehat{\mathcal{T}}_1, \tag{5.29}$$

In Case 2, $\rho = 0$, $R(w)$ is analytic at $w = \infty$, and $T(z)$ is analytic at $z = 1$. Under these assumptions, the identities (5.26)–(5.28) are readily verified from (3.9)–(3.10) and (5.10)–(5.11).

The equality $\varkappa_{S_F} = \varkappa_{S_v}$ follows, for example, from the uniqueness of indices in the Bognár-Krámli factorization [2, Theorem 2.1]. □

6. Proofs of the theorems

Proof of Lemma 2.2. Suppose $\alpha \in \mathbb{R}$ and $a \in \mathbb{T} \setminus \{1\}$. By straightforward algebra,

$$\frac{1}{i}\,\frac{1+t^2}{t-w} = \frac{2\zeta}{\zeta - z}\,\frac{z-1}{\zeta-1} \quad \text{and} \quad \frac{t-\alpha}{w-\alpha} = \frac{\zeta - a}{\zeta - 1}\,\frac{z-1}{z-a}\,.$$

Hence

$$\frac{1+t^2}{i}\,\frac{1}{t-w}\left(\frac{t-\alpha}{w-\alpha}\right)^{2\rho} = \frac{2\zeta}{\zeta-z}\,\frac{z-1}{\zeta-1}\left(\frac{\zeta-a}{\zeta-1}\,\frac{z-1}{z-a}\right)^{2\rho},$$

and this is the same as (2.13) by (2.8) and (2.2). For the case $\alpha = \infty$ and $a = 1$,

$$\frac{1}{i}\,\frac{1+tw}{t-w} = \frac{\zeta+z}{\zeta-z} \quad \text{and} \quad \frac{1+w^2}{1+t^2} = \frac{z}{(z-1)^2}\,\frac{(\zeta-1)^2}{\zeta}\,,$$

and so

$$\frac{1+t^2}{i}\,\frac{1+tw}{t-w}\,\frac{1}{1+t^2}\left(\frac{1+w^2}{1+t^2}\right)^{\rho} = \frac{\zeta+z}{\zeta-z}\,\frac{z^\rho}{(z-1)^{2\rho}}\,\frac{(\zeta-1)^{2\rho}}{\zeta^\rho}\,.$$

By (2.13) and (2.8) with $\alpha = \infty$ and $a = 1$, we again obtain (2.13). □

Proof of Theorem 2.3. We take Theorem 2.1 as known. Let $F(z) \in \mathbf{C}_\varkappa$, and define $v(w) \in \mathbf{N}_\varkappa$ by (2.6). Represent $v(w)$ in the form (2.3). Define $a_0, a_1, \dots, a_r \in \mathbb{T}$ by $\alpha_j = \varphi(a_j)$, $j = 0, \dots, r$. Then

$$F(z) = \sum_{j=0}^{r} \int_{\Delta(\alpha_j)\setminus\{\alpha_j\}} \frac{1+t^2}{i}\left[\frac{1}{t-w} - S(\alpha_j, \rho_j; t, w)\right]\frac{d\tau(t)}{1+t^2} + T(z)\,, \qquad (6.1)$$

where $w = \varphi(z)$ and $T(z) = -iR(\varphi(z))$. Let $E(a_0), \dots, E(a_r)$ be the sets in $\mathbb{T}$ corresponding to $\Delta(\alpha_0), \dots, \Delta(\alpha_r)$ via φ. Define σ by

$$d\sigma(\zeta) = \frac{d\tau(t)}{1+t^2}, \qquad t = \varphi(\zeta)\,. \qquad (6.2)$$

By a change of variables using (6.2) and Lemma 2.2,

$$\int_{\Delta(\alpha_j)\setminus\{\alpha_j\}} \frac{1+t^2}{i}\left[\frac{1}{t-w} - S(\alpha_j, \rho_j; t, w)\right]\frac{d\tau(t)}{1+t^2} = \int_{E(a_j)\setminus\{a_j\}}\left[\frac{\zeta+z}{\zeta-z} - U(a_j, \rho_j; \zeta, z)\right] d\sigma(\zeta)\,,$$

$j = 0, 1, \dots, r$. Hence the integral terms in (6.1) coincide with the corresponding terms in (2.14).

In the discrete terms $T(z) = -iR(\varphi(z))$ of (6.1), observe that the poles $b_1, \dots, b_s$ of $F(z)$ in $\mathbb{D}$ are related to the poles of $v(w)$ in $\mathbb{C}_+$ by $\beta_k = \varphi(b_k)$, $k = 1, \dots, s$. Thus by (2.5),

$$T(z) = \frac{1}{i}\,R(\varphi(z))$$

$$= \frac{1}{i}\, R_0(\varphi(z)) - \sum_{j=1}^{r} \frac{1}{i}\, R_j \left(\frac{1}{\varphi(z) - \varphi(a_j)} \right)$$
$$- \frac{1}{i} \sum_{k=1}^{s} \left[M_k \left(\frac{1}{\varphi(z) - \varphi(b_k)} \right) + M_k^* \left(\frac{1}{\overline{\varphi(z) - \varphi(b_k)}} \right) \right]. \tag{6.3}$$

If we take $T_0(z) = R_0(z)$, the first terms in (6.3) and (2.16) coincide (below we change $T_0(z)$ by adding certain constants). In the second term of (6.3), for each $j = 1, \dots, r$, write $w = \varphi(z)$ and

$$-\frac{1}{i}\, R_j \left(\frac{1}{\varphi(z) - \varphi(a_j)} \right) = -\frac{1}{i}\, R_j \left(\frac{1}{1+\alpha_j^2}\, \frac{1+\alpha_j w}{w - \alpha_j} - \frac{\alpha_j}{1+\alpha_j^2} \right).$$

Here $R_j(w) = \sum_{p=0}^{2\rho_j+1} R_{jp}\, w^p$ has selfadjoint coefficients and $R_{j,2\rho_j+1} \geq 0$. We obtain

$$-\frac{1}{i}\, R_j \left(\frac{1}{\varphi(z) - \varphi(a_j)} \right) = -\frac{1}{i} \sum_{p=0}^{2\rho_j+1} R_{jp} \left(\frac{1}{1+\alpha_j^2}\, \frac{1+\alpha_j w}{w - \alpha_j} - \frac{\alpha_j}{1+\alpha_j^2} \right)^p$$
$$= -\frac{1}{i} \left[C_j + T_j \left(\frac{1+\alpha_j w}{w - \alpha_j} \right) \right] = -\frac{1}{i} \left[C_j + T_j \left(i\, \frac{z + a_j}{z - a_j} \right) \right],$$

where C_j is selfadjoint, $T_j(z) = \sum_{p=1}^{2\rho_j+1} T_{jp}\, z^p$ has selfadjoint coefficients, and $T_{j,2\rho_j+1} \geq 0$. By including $-C_j$ in the constant term in $T_0(z)$, we arrive at the second term in (2.16). In the third term of (6.3), for each $k = 1, \dots, s$, write

$$-\frac{1}{i} \left[M_k \left(\frac{1}{\varphi(z) - \varphi(b_k)} \right) + M_k^* \left(\frac{1}{\overline{\varphi(z) - \varphi(b_k)}} \right) \right]$$
$$= -\frac{1}{i} \left[M_k \left(\frac{(1-b_k)(1-z)}{2i(z-b_k)} \right) + M_k^* \left(\frac{(1-b_k)(1-\bar z)}{-2i(1-b_k \bar z)} \right) \right]$$
$$= H(z) - H(1/z)^*.$$

Here

$$H(z) = iM_k \left(\frac{1-b_k}{2i} \left(\frac{1-b_k}{z-b_k} - 1 \right) \right) = D_k + N_k \left(\frac{1}{z - b_k} \right),$$

where $N_k(z)$ is a polynomial $\not\equiv 0$ with $N_k(0) = 0$. We obtain the third term in (2.16) except for an imaginary constant, which we include in the first term of (2.16).

The converse follows from the corresponding part of Theorem 2.1. □

Proof of Theorem 2.4. The case $\varkappa = 0$ and f, g constant follows from the classical Stieltjes inversion formula [7, p. 12]. The extension to continuous f and g follows as in the proof of Theorem 3.1 in [8].

In the general case, we can choose the sets $E(a_0), \dots, E(a_r)$ in (2.14) so that γ is entirely contained in one of them, say $\gamma \subseteq E(a_j)$. Then only the corresponding integral term of (2.14) needs to be considered, since all other terms define functions that are analytic and take imaginary values on γ. Thus we assume that

$$F(z) = \int_{E(a)\setminus\{a\}} \left[\frac{\zeta+z}{\zeta-z} - U(a,\rho;\zeta,z)\right] d\sigma(\zeta), \tag{6.4}$$

where $a = a_j$, $\rho = \rho_j$, and $\gamma \subseteq E(a) \setminus \{a\}$. Without loss of generality we may suppose $a \neq 1$. Set

$$\psi(z) = \frac{a-1}{i}\,\frac{z-1}{z-a}\,. \tag{6.5}$$

By (2.8) and (2.15), we can write

$$\begin{aligned} F(z) &= \int_{E(a)\setminus\{a\}} \frac{2\zeta}{\zeta-z}\,\frac{z-1}{\zeta-1}\left(\frac{\zeta-a}{\zeta-1}\right)^{2\rho}\left(\frac{z-1}{z-a}\right)^{2\rho} d\sigma(\zeta) \\ &= \psi(z)^{2\rho}\int_{E(a)\setminus\{a\}} \frac{2\zeta}{\zeta-z}\,\frac{z-1}{\zeta-1}\,\frac{d\sigma(\zeta)}{\psi(\zeta)^{2\rho}} \\ &= \psi(z)^{2\rho}\int_{E(a)\setminus\{a\}} \left[\frac{\zeta+z}{\zeta-z} + \frac{1+\zeta}{1-\zeta}\right]\frac{d\sigma(\zeta)}{\psi(\zeta)^{2\rho}} \\ &= \psi(z)^{2\rho}\, G(z), \end{aligned}$$

where

$$G(z) = iC + \int_{E(a)\setminus\{a\}} \frac{\zeta+z}{\zeta-z}\,\frac{d\sigma(\zeta)}{\psi(\zeta)^{2\rho}}\,, \qquad C = C^*. \tag{6.6}$$

Notice that $\psi(\zeta)$ is real valued on $E(a)\setminus\{a\}$. Therefore $G(z) \in \mathbf{C_0}$, and (6.6) is a Herglotz representation. Since

$$\begin{aligned} \operatorname{Re} F(z) = \Psi(\zeta)^{2\rho}\operatorname{Re} G(z) &+ \left[\psi(z)^{2\rho} - \psi(\zeta)^{2\rho}\right]\operatorname{Re} G(z) \\ &+ \frac{1}{2}\left[\overline{\psi(z)}^{\,2\rho} - \psi(z)^{2\rho}\right] G(z)^*, \end{aligned}$$

we can write

$$\begin{aligned} &\int_\gamma g(\zeta)^* \operatorname{Re} F(r\zeta)\, f(\zeta)\, d\mu(\zeta) \\ &\quad = \int_\gamma \psi(\zeta)^{2\rho} g(\zeta)^* \operatorname{Re} G(r\zeta)\, f(\zeta)\, d\mu(\zeta) \\ &\qquad + \int_\gamma \left[\psi(r\zeta)^{2\rho} - \psi(\zeta)^{2\rho}\right] g(\zeta)^* \operatorname{Re} G(r\zeta)\, f(\zeta)\, d\mu(\zeta) \\ &\qquad + \frac{1}{2}\int_\gamma \left[\overline{\psi(r\zeta)}^{\,2\rho} - \psi(r\zeta)^{2\rho}\right] g(\zeta)^*\, G(r\zeta)^*\, f(\zeta)\, d\mu(\zeta) \\ &\quad = T_1(r) + T_2(r) + T_3(r)\,. \end{aligned} \tag{6.7}$$

By (6.6) and the known case $\varkappa = 0$ of the theorem,

$$\lim_{r\uparrow 1} T_1(r) = \int_\gamma \psi(\zeta)^{2\rho} g(\zeta)^* \, \frac{d\sigma(\zeta)}{\psi(\zeta)^{2\rho}} \, f(\zeta) = \int_\gamma g(\zeta)^* \, d\sigma(\zeta) \, f(\zeta).$$

It remains to show that $T_2(r) \to 0$ and $T_3(r) \to 0$. To see this, observe that $(1 - |z|)G(z)$ is bounded in $\mathbb{D}$ by the Herglotz repesentation. Also

$$\big|\psi(r\zeta) - \psi(\zeta)\big| \le K|1 - r|, \qquad \big|\overline{\psi(r\zeta)} - \psi(r\zeta)\big| \le K|1 - r|$$

for all $\zeta \in \gamma$ and $0 < r < 1$ and some some $K = K_{a,\gamma}$. Since $\operatorname{Re} G(z) \ge 0$ on $\mathbb{D}$, $G(z)$ has radial limits μ-a.e. It follows that the integrands in $T_1(r)$ and $T_2(r)$ are bounded functions that converge to zero μ-a.e. as $r \uparrow 1$. Therefore $T_2(r) \to 0$ and $T_3(r) \to 0$ as $r \uparrow 1$, and we obtain (2.19). □

Proof of Theorem 2.5. Define $v(w) \in \mathbf{N}_\varkappa$ as in (2.6). Then

$$(1 - r)F(r) = (1 + r)\, v\!\left(i\, \frac{1 + r}{1 - r} \right) \Big/ \, i\, \frac{1 + r}{1 - r},$$

and so $(1 - r)F(r) \to 0$ as $r \uparrow 1$ if and only if $v(iy)/y \to 0$ as as $y \to \infty$. By [8, Theorem 4.1], the latter condition holds if and only if the Kreĭn-Langer representation (2.3) for $v(w)$ can be chosen such that $\rho_0 = 0$ and the rational term $R(w)$ is constant. By the proof of Theorem 2.3, this is equivalent to the existence of a representation (2.14) for $F(z)$ such that $\rho_0 = 0$ and the rational term $T(z)$ is analytic at $z = 1$. □

Proof of Theorem 3.1. (1) Fix $a \in \mathbb{T} \setminus \{1\}$, $\rho \ge 1$, and $\zeta \in E(a) \setminus \{a\}$. As a preliminary, note that for any polynomial $P(z)$ of degree at most N,

$$\operatorname*{Res}_{z=a} \frac{P(z)}{(z - a)^{N+2}} = 0. \tag{6.8}$$

For brevity, set

$$dS = \Phi_2 d\sigma(\zeta) \Phi_2^*. \tag{6.9}$$

By (2.8) and (2.11),

$$\begin{aligned} &\frac{\zeta + z}{\zeta - z} - U(a, \rho; \zeta, z) \\ &\qquad = \frac{\zeta + z}{\zeta - z} - \frac{\zeta + 1}{\zeta - 1} + \frac{2\zeta(a - 1)}{(\zeta - 1)^2} \sum_{k=0}^{2\rho - 1} \left(\frac{\zeta - a}{\zeta - 1} \right)^k \left(\frac{z - 1}{z - a} \right)^{k+1}. \end{aligned} \tag{6.10}$$

Thus by (3.5),

$$\begin{aligned} &d\sigma(a, \rho, \zeta; A, \Phi_2) \\ &\quad = \operatorname*{Res}_{z=a} \left[\frac{2\zeta(a - 1)}{(\zeta - 1)^2} \sum_{k=0}^{2\rho - 1} \left(\frac{\zeta - a}{\zeta - 1} \right)^k \left(\frac{z - 1}{z - a} \right)^{k+1} (I + zA)^{-1} dS (A^* + zI)^{-1} \right] \end{aligned}$$

$$= \frac{2\zeta(a-1)}{(\zeta-1)^2} \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k \operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} (I+zA)^{-1}\, dS\, (A^*+zI)^{-1} \right].$$

To evaluate the residues in the last sum, write

$$(I+zA)^{-1} = \frac{a-1}{z-1} \sum_{p=0}^{\infty} \left(\frac{z-a}{z-1}\right)^p C_p, \tag{6.11}$$

$$(A^*+zI)^{-1} = \frac{a-1}{z-1} \sum_{q=0}^{\infty} \left(\frac{z-a}{z-1}\right)^q \widetilde{C}_q, \tag{6.12}$$

where for all $p \geq 0$ and $q \geq 0$,

$$C_p = (I+A)^p (I+aA)^{-p-1}, \qquad \widetilde{C}_q = (I+A^*)^q (A^*+aI)^{-q-1}. \tag{6.13}$$

Then

$$(I+zA)^{-1}\, dS\, (A^*+zI)^{-1} = \frac{(a-1)^2}{(z-1)^2} \sum_{n=0}^{\infty} \left(\frac{z-a}{z-1}\right)^n \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q\,,$$

and so

$$\operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} (I+zA)^{-1}\, dS\, (A^*+zI)^{-1} \right]$$

$$= \operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} \frac{(a-1)^2}{(z-1)^2} \sum_{n=0}^{k} \left(\frac{z-a}{z-1}\right)^n \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q \right]$$

$$= \sum_{n=0}^{k} \operatorname*{Res}_{z=a} \left[\frac{(a-1)^2 (z-1)^{k-n-1}}{(z-a)^{k-n+1}} \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q \right].$$

By (6.8), the only nontrivial term in the last sum is $n = k$:

$$\operatorname*{Res}_{z=a} \left[\frac{(a-1)^2/(z-1)}{z-a} \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q \right] = (a-1) \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q\,.$$

Hence

$$d\sigma(a,\rho,\zeta;A,\Phi_2) = \frac{2\zeta(a-1)}{(\zeta-1)^2} \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k (a-1) \sum_{p+q=n} C_p\, dS\, \widetilde{C}_q\,,$$

which is equivalent to (3.11).

By (3.6) and (6.10),

$$\mathcal{V}(a,\rho,\zeta;A)$$

$$= \operatorname*{Res}_{z=a} \left[\frac{2\zeta(a-1)}{(\zeta-1)^2} \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k \left(\frac{z-1}{z-a}\right)^{k+1} \frac{(I+A)(I+zA)^{-1}}{z-1} \right]$$

$$= \frac{2\zeta(a-1)}{(\zeta-1)^2} \sum_{k=0}^{2\rho-1} \left(\frac{\zeta-a}{\zeta-1}\right)^k \operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} \frac{(I+A)(I+zA)^{-1}}{z-1} \right].$$

By (6.11),

$$\operatorname*{Res}_{z=a}\left[\left(\frac{z-1}{z-a}\right)^{k+1} \frac{(I+A)(I+zA)^{-1}}{z-1} \right] = \operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} \frac{a-1}{(z-1)^2} \sum_{n=0}^{k} \left(\frac{z-a}{z-1}\right)^n (I+A)C_n \right].$$

By (6.8), only the term $n = k$ in the last sum contributes to the residue:

$$\operatorname*{Res}_{z=a} \left[\left(\frac{z-1}{z-a}\right)^{k+1} \frac{a-1}{(z-1)^2} \left(\frac{z-a}{z-1}\right)^k (I+A)C_k \right] = (I+A)C_k\,.$$

Putting these facts together, we obtain (3.12).

(2) Let $a = 1$, $\rho = 0$, and $\zeta \in E(1) \setminus \{1\}$. In Case 1, the operator-valued functions $(I + zA)^{-1}$ and $(A^* + zI)^{-1}$ are analytic at $z = 1$. Since

$$\frac{\zeta+z}{\zeta-z} - U(1,0;\zeta,z) = \frac{\zeta+z}{\zeta-z}$$

by (2.8), the residues in (3.5) and (3.6) are readily evaluated and yield (3.9). In Case 2, (3.9) holds by definition. □

Proof of Theorem 3.2. Fix $a = 1$, $\rho \geq 1$, and $\zeta \in E(1) \setminus \{1\}$. By (2.8) and (2.12),

$$\begin{aligned} &\frac{\zeta+z}{\zeta-z} - U(1,\rho;\zeta,z) \\ &\quad = \frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1} + \frac{2(\zeta z-1)}{(\zeta-1)(z-1)} \sum_{k=0}^{\rho-1} \frac{(\zeta-1)^{2k}}{\zeta^k} \frac{z^k}{(z-1)^{2k}} \\ &\quad + \frac{\zeta+1}{\zeta-1} \frac{(\zeta-1)^{2\rho}}{\zeta^\rho} \frac{z^\rho}{(z-1)^{2\rho}}\,. \end{aligned} \tag{6.14}$$

Therefore by (3.5)

$$\begin{aligned} &d\sigma(1,\rho,\zeta;A,\Phi_2) \\ &= \operatorname*{Res}_{z=1} \left[\frac{2(\zeta z-1)}{(\zeta-1)(z-1)} \sum_{k=0}^{\rho-1} \frac{(\zeta-1)^{2k}}{\zeta^k} \frac{z^k}{(z-1)^{2k}} (I+zA)^{-1} dS (A^*+zI)^{-1} \right] \\ &\quad + \operatorname*{Res}_{z=1} \left[\frac{\zeta+1}{\zeta-1} \frac{(\zeta-1)^{2\rho}}{\zeta^\rho} \frac{z^\rho}{(z-1)^{2\rho}} (I+zA)^{-1} dS (A^*+zI)^{-1} \right] \end{aligned} \tag{6.15}$$

$$= \sum_{k=0}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^k} \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+1}} \, (I+zA)^{-1} dS(A^*+zI)^{-1} \right]$$

$$+ \frac{\zeta+1}{\zeta-1} \, \frac{(\zeta-1)^{2\rho}}{\zeta^\rho} \operatorname*{Res}_{z=1} \left[\frac{z^\rho}{(z-1)^{2\rho}} \, (I+zA)^{-1} dS(A^*+zI)^{-1} \right]. \quad (6.16)$$

To compute the residues in (6.16), we use the expansions

$$(I+zA)^{-1} = \big(I + \frac{1}{z}\, A\big) \sum_{p=0}^{\infty} \frac{(z-1)^{2p}}{z^p}\, E_p, \quad (6.17)$$

$$(A^*+zI)^{-1} = \big(A^* + \frac{1}{z}\, I\big) \sum_{q=0}^{\infty} \frac{(z-1)^{2q}}{z^q}\, E_q^*, \quad (6.18)$$

where

$$E_n = (-A)^n (I+A)^{-2n-2}, \qquad n = 0,1,2,\dots . \quad (6.19)$$

In a neighborhood of $z=1$,

$$\begin{aligned} &(I+zA)^{-1} dS(A^*+zI)^{-1} \\ &\quad = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(z-1)^{2p+2q}}{z^{p+q}} \big(I + \frac{1}{z}\, A\big) E_p \, dS \, E_q^* \big(A^* + \frac{1}{z}\, I\big) \\ &\quad = \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{z^n}\, K_n + \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{z^{n+1}}\, L_n + \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{z^{n+2}}\, M_n \,, \end{aligned}$$

where for all $n = 0,1,2,\dots$,

$$K_n = \sum_{p+q=n} E_p \, dS \, E_q^* A^* \,, \quad (6.20)$$

$$L_n = \sum_{p+q=n} \big(AE_p \, dS \, E_q^* A^* + E_p \, dS \, E_q^* \big) \,, \quad (6.21)$$

$$M_n = \sum_{p+q=n} AE_p \, dS \, E_q^* \,. \quad (6.22)$$

Therefore for all $k = 0,\dots,\rho-1$,

$$\begin{aligned} &\operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+1}} \, (I+zA)^{-1} dS(A^*+zI)^{-1} \right] \\ &\quad = \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+1}} \, \sum_{n=0}^{k} \left(\frac{(z-1)^{2n}}{z^n}\, K_n + \frac{(z-1)^{2n}}{z^{n+1}}\, L_n + \frac{(z-1)^{2n}}{z^{n+2}}\, M_n \right) \right] \\ &\quad = \sum_{n=0}^{k} K_n \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^{k-n}}{(z-1)^{2(k-n)+1}} \right] + \sum_{n=0}^{k} L_n \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^{k-n-1}}{(z-1)^{2(k-n)+1}} \right] \end{aligned}$$

$$+\sum_{n=0}^{k} M_n \operatorname*{Res}_{z=1}\left[\frac{(\zeta z-1)z^{k-n-2}}{(z-1)^{2(k-n)+1}}\right].$$

Most of the residues in the last three sums are zero by (6.8): in the first we only need to consider $n = k-1, k$, in the second $n = k$, and in the third $n = k-1, k$. The residues are elementary, and we obtain

$$\begin{aligned}
&\operatorname*{Res}_{z=1}\left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+1}}\,(I+zA)^{-1}dS(A^*+zI)^{-1}\right] \\
&\qquad = \Big(\zeta K_{k-1} + (\zeta-1)K_k\Big) + (\zeta-1)L_k + \Big(-M_{k-1} + (\zeta-1)M_k\Big) \\
&\qquad = (\zeta-1)\big(K_k + L_k + M_k\big) + \zeta K_{k-1} - M_{k-1}\,,
\end{aligned}$$

where $K_{-1} = M_{-1} = 0$. Similarly,

$$\operatorname*{Res}_{z=1}\left[\frac{z^\rho}{(z-1)^{2\rho}}\,(I+zA)^{-1}dS(A^*+zI)^{-1}\right] = K_{\rho-1} - M_{\rho-1}\,.$$

Therefore by (6.16),

$$\begin{aligned}
&d\sigma(1,\rho,\zeta;A,\Phi_2) \\
&\quad = \sum_{k=0}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^k}\,\Big[(\zeta-1)\big(K_k+L_k+M_k\big) + \zeta K_{k-1} - M_{k-1}\Big] \\
&\qquad + \frac{\zeta+1}{\zeta-1}\,\frac{(\zeta-1)^{2\rho}}{\zeta^\rho}\,\Big[K_{\rho-1} - M_{\rho-1}\Big] \\
&\quad = \sum_{k=0}^{\rho-1} \frac{2(\zeta-1)^{2k}}{\zeta^k}\big(K_k+L_k+M_k\big) \\
&\qquad + \sum_{k=1}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^{k-1}} K_{k-1} - \sum_{k=1}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^k} M_{k-1} \\
&\qquad + \frac{\zeta+1}{\zeta-1}\,\frac{(\zeta-1)^{2\rho}}{\zeta^\rho}\,\big(K_{\rho-1} - M_{\rho-1}\big)\,.
\end{aligned}$$

Here by (6.20), (6.21), and (6.22),

$$\begin{gathered}
K_k + L_k + M_k = (I+A)dG_k(\zeta)(I+A^*), \\
K_k = dG_k(\zeta)A^*, \qquad M_k = A\,dG_k(\zeta),
\end{gathered}$$

where $dG_k(\zeta)$ is given by (3.15). In a few steps of algebra, we obtain (3.13).

By (3.6) and (6.14),

$$\begin{aligned}
&\mathcal{V}(1,\rho,\zeta;A) \\
&\quad = \operatorname*{Res}_{z=1}\left[\frac{2(\zeta z-1)}{(\zeta-1)(z-1)} \sum_{k=0}^{\rho-1} \frac{(\zeta-1)^{2k}}{\zeta^k}\,\frac{z^k}{(z-1)^{2k}}\,\frac{(I+A)(I+zA)^{-1}}{z-1}\right]
\end{aligned}$$

$$+ \operatorname*{Res}_{z=1} \left[\frac{\zeta+1}{\zeta-1} \, \frac{(\zeta-1)^{2\rho}}{\zeta^{\rho}} \frac{z^{\rho}}{(z-1)^{2\rho}} \, \frac{(I+A)(I+zA)^{-1}}{z-1} \right]$$

$$= \sum_{k=0}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^k} \, (I+A) \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+2}} \, (I+zA)^{-1} \right]$$

$$+ (\zeta+1) \, \frac{(\zeta-1)^{2\rho-1}}{\zeta^{\rho}} (I+A) \operatorname*{Res}_{z=1} \left[\frac{z^{\rho}}{(z-1)^{2\rho+1}} (I+zA)^{-1} \right].$$

By calculations similar to those above, using (6.17) we find that

$$\operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+2}} \, (I+zA)^{-1} \right] = \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+2}} \, \sum_{n=0}^{k} \frac{(z-1)^{2n}}{z^n} \, \big(I + \frac{1}{z} \, A\big) E_n \right]$$

$$= \operatorname*{Res}_{z=1} \left[\frac{(\zeta z-1)z^k}{(z-1)^{2k+2}} \, \frac{(z-1)^{2k}}{z^k} \, \big(I + \frac{1}{z} \, A\big) E_k \right]$$

$$= (\zeta I + A) E_k \, ,$$

and

$$\operatorname*{Res}_{z=1} \left[\frac{z^{\rho}}{(z-1)^{2\rho-1}} (I+zA)^{-1} \right] = \operatorname*{Res}_{z=1} \left[\frac{z^{\rho}}{(z-1)^{2\rho-1}} \, \sum_{n=0}^{\rho} \frac{(z-1)^{2n}}{z^n} \, \big(I + \frac{1}{z} \, A\big) E_n \right]$$

$$= \operatorname*{Res}_{z=1} \left[\frac{z^{\rho}}{(z-1)^{2\rho-1}} \, \frac{(z-1)^{2\rho}}{z^{\rho}} \, \big(I + \frac{1}{z} \, A\big) E_{\rho} \right]$$

$$= (I+A) E_{\rho} \, .$$

This yields

$$\mathcal{V}(1,\rho,\zeta;A) = \sum_{k=0}^{\rho-1} \frac{2(\zeta-1)^{2k-1}}{\zeta^k} \, (I+A) \, (\zeta I + A) E_k$$

$$+ (\zeta+1) \, \frac{(\zeta-1)^{2\rho-1}}{\zeta^{\rho}} (I+A) \, (I+A) E_{\rho}$$

$$= 2(A+\zeta I) \sum_{k=0}^{\rho-1} \frac{(\zeta-1)^{2k-1}}{\zeta^k} (-A)^k (I+A)^{-2k-1}$$

$$+ (\zeta+1) \, \frac{(\zeta-1)^{2\rho-1}}{\zeta^{\rho}} (-A)^{\rho} (I+A)^{-2\rho},$$

which is (3.14). □

Proofs of Theorems 3.3 *and* 3.4. These follow from Theorem 5.1 and the corresponding results for the line, which are given in Theorems 3.4 and 3.5 of [9]. □

Proof of Lemma 4.1. The identity can be proved by direct calculation, but the quickest way is to use the corresponding result for the line, which is given in

Theorem 4.1 of [9]. Consider the operator identity (5.1) corresponding to (3.1) by means of the relations (5.3). Following (4.8) in [9], set

$$\mathfrak{A}_\ell(w) = I_{2m} - iw\Pi^*(I - wA_\ell^*)^{-1} S_\ell^{-1} \Pi J. \tag{6.23}$$

A short calculation shows that

$$\mathfrak{A}(z) = \mathfrak{A}_\ell(w), \qquad w = i\,\frac{1+z}{1-z}. \tag{6.24}$$

By Theorem 4.1 of [9],

$$\frac{J - \mathfrak{A}_\ell(\bar\xi) J \mathfrak{A}_\ell(\bar w)^*}{i(\bar\xi - w)} = \Pi^*(I - \bar\xi A_\ell^*)^{-1} S_\ell^{-1} (I - wA_\ell)^{-1}\Pi\,. \tag{6.25}$$

Writing $\xi = i(1+\zeta)/(1-\zeta)$, we have $\mathfrak{A}_\ell(\bar\xi) = \mathfrak{A}(1/\bar\zeta)$ and $\mathfrak{A}_\ell(\bar w) = \mathfrak{A}(1/\bar z)$ by (6.24). Since

$$\begin{aligned} i(\bar\xi - w) &= 2\,\frac{1-\bar\zeta z}{(1-\bar\zeta)(1-z)}\,, \\ (I - \bar\xi A_\ell^*)^{-1} &= \frac{1}{2}\,(1-\bar\zeta)(I - A^*)(I + \bar\zeta A^*)^{-1}\,, \\ (I - wA_\ell)^{-1} &= \frac{1}{2}\,(1-z)(I-A)(I+zA)^{-1}\,, \end{aligned}$$

we easily bring (6.25) to the form (4.4). Then (4.5) follows from (4.4). □

Proof of Lemma 4.3. Write the function (4.9) in the form

$$B_F(z) = (I + zA)^{-1}\Pi J \Psi_F(z), \qquad \Psi_F(z) = \begin{bmatrix} F(z) \\ I \end{bmatrix}.$$

Then by (4.4),

$$\begin{aligned} &\frac{F(z) + F(\zeta)^*}{1-\bar\zeta z} - B_F(\zeta)^* S^{-1} B_F(z) \\ &= \frac{\Psi_F(\zeta)^* J \Psi_F(z)}{1-\bar\zeta z} - \Psi_F(\zeta)^* J \Pi^* (I - \bar\zeta A^*)^{-1} S^{-1} (I + zA)^{-1} \Pi J \Psi_F(z) \\ &= \frac{\Psi_F(\zeta)^* J \Psi_F(z)}{1-\bar\zeta z} - \Psi_F(\zeta)^* J\, \frac{J - \mathfrak{A}(1/\bar\zeta) J \mathfrak{A}(1/\bar z)^*}{1-\bar\zeta z}\, J\Psi_F(z) \\ &= \frac{\Psi_F(\zeta)^* J \mathfrak{A}(1/\bar\zeta) J \mathfrak{A}(1/\bar z)^* J \Psi_F(z)}{1-\bar\zeta z}\,. \end{aligned}$$

By (4.5), $\mathfrak{A}(1/\bar z)^* = J\mathfrak{A}(z)^{-1}J$ and $\mathfrak{A}(1/\bar\zeta) = J\mathfrak{A}(\zeta)^{*-1}J$. Thus by (4.8),

$$\frac{F(z) + F(\zeta)^*}{1-\bar\zeta z} - B_F(\zeta)^* S^{-1} B_F(z) = \frac{\Psi_F(\zeta)^* \mathfrak{A}(\zeta)^{*-1} J \mathfrak{A}(z)^{-1} \Psi_F(z)}{1-\bar\zeta z}$$

$$= \frac{E(\zeta)^* \begin{bmatrix} P(\zeta)^* & Q(\zeta)^* \end{bmatrix} J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} E(z)}{1-\bar\zeta z}$$

$$= E(\zeta)^* \, \frac{P(\zeta)^* Q(z) + Q(\zeta)^* P(z)}{1-\bar\zeta z} \, E(z)\,,$$

as was to be shown. □

Proof of Theorem 4.4. Suppose that $F(z) \in N(\mathfrak{A})$ and has the representation (4.7). Set

$$\begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \mathfrak{A}(z) \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} a(z)P(z) + b(z)Q(z) \\ c(z)P(z) + d(z)Q(z) \end{bmatrix}.$$

Then $F(z) = H(z)K(z)^{-1}$ and

$$\mathfrak{A}(z)^{-1} \begin{bmatrix} F(z) \\ I \end{bmatrix} = \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1}\,. \tag{6.26}$$

By (6.26) and Lemma 4.3 with $E(z) = K(z)^{-1}$,

$$\frac{F(z) + F(\zeta)^*}{1-\bar\zeta z} = K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} + B_F(\zeta)^* S^{-1} B_F(z). \tag{6.27}$$

Multiplying by $1-\bar\zeta z$ and letting $\zeta \to 1/\bar z$, we get $F(z) = -F(1/\bar z)^*$ by Definition 4.2(i). By Definition 4.2(iii) and our assumption that $\varkappa_S < \infty$, the kernel on the left side of (6.27) has a finite number of negative squares. It follows that $F(z) \in \mathbf{C}_\varkappa$ for some $\varkappa = \varkappa_F \le \varkappa_{P,Q} + \varkappa_S$.

By (4.10) and (6.27),

$$L_F(z,\zeta) = \begin{bmatrix} I & 0 \\ B_F(\zeta)^* S^{-1} & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & \dfrac{F(z)+F(\zeta)^*}{1-\bar\zeta z} - B_F(\zeta)^* S^{-1} B_F(z) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I & S^{-1} B_F(z) \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ B_F(\zeta)^* S^{-1} & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} I & S^{-1} B_F(z) \\ 0 & I \end{bmatrix}. \tag{6.28}$$

It follows that $L_F(z,\zeta)$ has $\varkappa_{L_F} = \varkappa_{P,Q} + \varkappa_S$ negative squares. This proves the necessity part of (1) and part (2) of the theorem.

It remains to prove sufficiency in part of (1) of the theorem. Suppose $F(z) \in \mathbf{C}_\varkappa$ and $\varkappa_{L_F} < \infty$. Set

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \mathfrak{A}(z)^{-1} \begin{bmatrix} F(z) \\ I \end{bmatrix}. \tag{6.29}$$

By Lemma 4.3 with $E(z) = I$,

$$\frac{F(z) + F(\zeta)^*}{1 - \bar{\zeta} z} = D_{P,Q}(z, \zeta) + B_F(\zeta)^* S^{-1} B_F(z). \tag{6.30}$$

Multiply (6.30) by $1 - \bar{\zeta} z$ and let $\zeta \to 1/\bar{z}$; since $F(z) \in \mathbf{C}_\varkappa$ and hence $F(z) = -F(1/\bar{z})^*$, we get

$$P(1/\bar{z})^* Q(z) + Q(1/\bar{z})^* P(z) = 0.$$

By (6.30) and the first equality in (6.28),

$$L_F(z,\zeta) = \begin{bmatrix} I & 0 \\ B(\zeta)^* S^{-1} & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D_{P,Q}(z,\zeta) \end{bmatrix} \begin{bmatrix} I & S^{-1} B_F(z) \\ 0 & I \end{bmatrix}.$$

Since $\varkappa_{L_F} < \infty$ by assumption, $\varkappa_{P,Q} < \infty$. By (6.29),

$$\begin{aligned} F(z) &= a(z)P(z) + b(z)Q(z), \\ I &= c(z)P(z) + d(z)Q(z), \end{aligned}$$

and so $F(z)$ has the representation (4.7) where P and Q meet the conditions (i)–(iii) in Definition 4.2. Therefore $F(z) \in N(\mathfrak{A})$. □

Proof of Theorem 4.5. Define $v(w) \in \mathbf{N}_\varkappa$ by (2.6), and construct S_v and $\Phi_{1,v}$ by (5.4) and (5.5) using $A_\ell = i(I + A)(I - A)^{-1}$ and the same Φ_2. By (5.12),

$$S = S_F = 2(I - A)^{-1} S_v (I - A^*)^{-1} \quad \text{and} \quad \Phi_1 = \Phi_{1,F} = \Phi_{1,v}.$$

By construction, $A_\ell, S_v, \Phi_1, \Phi_2$ satisfy the operator identity (5.1). As in (4.2) and (4.3) of [9], we set

$$L_{v,\ell}(w,\xi) = \begin{bmatrix} S_\ell & B_{v,\ell}(w) \\ B_{v,\ell}(\xi)^* & \dfrac{v(w) - v(\xi)^*}{w - \bar{\xi}} \end{bmatrix},$$

where

$$B_{v,\ell}(w) = (I - wA_\ell)^{-1} [\Phi_1 - i\Phi_2 v(w)].$$

By a routine calculation,

$$L_{v,\ell}(w,\xi) = \frac{1}{2} \begin{bmatrix} I - A & 0 \\ 0 & (1 - \bar{\zeta})I \end{bmatrix} L_F(z,\zeta) \begin{bmatrix} I - A^* & 0 \\ 0 & (1 - z)I \end{bmatrix}, \tag{6.31}$$

where $w, \xi \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $z, \zeta \in \mathbb{D} \cup \mathbb{D}^c$ are related by $w = \varphi(z)$ and $\xi = \varphi(\zeta)$. By Theorem 4.5 of [9], $L_{v,\ell}(w,\xi)$ has a finite number of negative squares. Hence $L_F(z,\zeta)$ has a finite number of squares by (6.31). □

Proof of Theorem 4.6. We shall apply [9, Theorem 5.1] to $A_\ell, S_\ell, \Phi_1, \Phi_2$, where $A_\ell = i(I+A)(I-A)^{-1}$ and $S_\ell = \frac{1}{2}(I-A)S(I-A^*)$. The hypotheses of [9, Theorem 5.1] are readily checked from the assumptions of the theorem.

Define $\mathfrak{A}_\ell(w)$ by (6.23), and write

$$\mathfrak{A}_\ell(w) = \begin{bmatrix} a_\ell(w) & b_\ell(w) \\ c_\ell(w) & d_\ell(w) \end{bmatrix}, \tag{6.32}$$

where $a_\ell(w), b_\ell(w), c_\ell(w), d_\ell(w)$ are $m \times m$ matrix-valued functions. Following [9, Definition 4.3], we define $N(\mathfrak{A}_\ell)$ as the set of functions

$$v(w) = i\ [a_\ell(w)P_\ell(w) + b_\ell(w)Q_\ell(w)]\,[c_\ell(w)P_\ell(w) + d_\ell(w)Q_\ell(w)]^{-1}, \tag{6.33}$$

where $P_\ell(w)$ and $Q_\ell(w)$ are $m \times m$ matrix-valued functions which are analytic on $\mathbb{C}_+ \cup \mathbb{C}_-$ except at isolated points, such that

(i′) $P_\ell(\bar{w})^* Q_\ell(w) + Q_\ell(\bar{w})^* P_\ell(w) \equiv 0$;
(ii′) $c(w)P_\ell(w) + d(w)Q_\ell(w)$ is invertible except at isolated points;
(iii′) the kernel $i[P_\ell(\xi)^* Q_\ell(w) + Q_\ell(\xi)^* P_\ell(w)]/(w - \bar{\xi})$ has a finite number of negative squares.

Claim. *If $F(z)$ and $v(w)$ are related by $F(z) = -iv(w)$, $w = i(1+z)/(1-z)$, then $F(z) \in N(\mathfrak{A})$ if and only if $v(w) \in N(\mathfrak{A}_\ell)$.*

The claim is proved by connecting the representations (4.7) and (6.33) using the relations

$$P_\ell(w) = \frac{\sqrt{2}}{1-z}\,P(z), \qquad Q_\ell(w) = \frac{\sqrt{2}}{1-z}\,Q(z)$$

and identity

$$\frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{1-\bar{\zeta}z} = i\,\frac{P_\ell(\xi)^*Q_\ell(w) + Q_\ell(\xi)^*P_\ell(w)}{w - \bar{\xi}}, \tag{6.34}$$

$w = i(1+z)/(1-z)$, $\xi = i(1+\zeta)/(1-\zeta)$. Notice also that $\mathfrak{A}(z) = \mathfrak{A}_\ell(w)$ by (6.24), and hence

$$a_\ell(w) = a(z), \quad b_\ell(w) = b(z), \quad c_\ell(w) = c(z), \quad d_\ell(w) = d(z),$$

by (4.6) and (6.32).

Proof of (1). By assumption, the spectrum of A is a finite subset of $\mathbb{C} \setminus \mathbb{T}$, say $\sigma(A) = \{z_1, \dots, z_n\}$, and $F(z)$ and $(1-z)B_F(z)$ are analytic on the set

$$\Big\{ -\frac{1}{z_1}, \dots, -\frac{1}{z_n} \Big\}.$$

Notice that this set contains ∞ when $0 \in \sigma(A)$. Define $v(w)$ by (2.6). Since $F(z) \in N(\mathfrak{A})$, $v(w) \in N(\mathfrak{A}_\ell)$ by the claim. Set

$$B_{v,\ell}(w) = (I - wA_\ell)^{-1}[\Phi_1 - i\Phi_2 v(w)].$$

By (2.6),

$$v(w) = iF(z) = iF\left(\frac{w-i}{w+i}\right), \tag{6.35}$$

and by (5.24),

$$\begin{aligned} B_{v,\ell}(w) &= \frac{1-z}{2}\,(I-A)(I+zA)^{-1}\Big[\Phi_1 + \Phi_2 F(z)\Big] \\ &= \frac{1-z}{2}\,(I-A)B_F(z) \\ &= \frac{i}{w+i}\,(I-A)B_F\bigg(\frac{w-i}{w+i}\bigg). \end{aligned} \tag{6.36}$$

The spectrum of A_ℓ is $\{\varphi(z_1),\dots,\varphi(z_n)\}$. To apply [9, Theorem 5.1(1)], we must show that $v(w)$ and $B_{v,\ell}(w)$ are analytic at every point of the set

$$\{w_1,\dots,w_n\} = \left\{\frac{1}{\varphi(z_1)},\dots,\frac{1}{\varphi(z_n)}\right\} = \left\{i\,\frac{z_1-1}{z_1+1},\dots,i\,\frac{z_n-1}{z_n+1}\right\}.$$

In fact, this follows from our assumptions on the analyticity of the functions $F(z)$ and $(1-z)B_F(z)$. For if $w_j = i(z_j-1)/(z_j+1)$ for some $j=1,\dots,n$, then

$$\frac{w_j-i}{w_j+i} = -\frac{1}{z_j} \qquad \text{and} \qquad \frac{i}{w_j+i} = \frac{1}{2}\bigg(1+\frac{1}{z_j}\bigg).$$

Thus by (6.35) and (6.36), $v(w)$ and $B_{v,\ell}(w)$ are analytic at each w_j, $j=1,\dots,n$, and the values at these points are

$$\begin{aligned} v(w_j) &= iF\bigg(-\frac{1}{z_j}\bigg), \\ B_{v,\ell}(w_j) &= \frac{1}{2}\bigg(1+\frac{1}{z_j}\bigg)(I-A)B_F\bigg(-\frac{1}{z_j}\bigg). \end{aligned}$$

Hence by [9, Theorem 5.1(1)], $v(w) \in \mathbf{N}_\varkappa$, $S_\ell = S_v$, and $\Phi_1 = \Phi_{1,v}$ are the operators defined by (5.4) and (5.5). Since (2.6) is a one-to-one correspondence between $\mathbf{C}_\varkappa$ and $\mathbf{N}_\varkappa$, $F(z) \in \mathbf{C}_\varkappa$. By (5.3) and (5.12),

$$\begin{gathered} S = 2\,(I-A)^{-1}S_\ell(I-A^*)^{-1} = 2\,(I-A)^{-1}S_v(I-A^*)^{-1} = S_F, \\ \Phi_1 = \Phi_{1,v} = \Phi_{1,F}, \end{gathered}$$

as was to be shown.

Proof of (2). Assume $S = S_F$ and $\Phi_1 = \Phi_{1,F}$ where $F(z) \in \mathbf{C}_\varkappa$. Define $v(w)$ by (2.6). By (5.3) and (5.12),

$$\begin{gathered} S_\ell = \frac{1}{2}\,(I-A)S(I-A^*) = \frac{1}{2}\,(I-A)S_F(I-A^*) = S_v, \\ \Phi_1 = \Phi_{1,F} = \Phi_{1,v}. \end{gathered}$$

By [9, Theorem 5.1(2)], $v \in N(\mathfrak{A}_\ell)$, and $v(w)$ and $B_{v,\ell}(w)$ are analytic at all points $\{w_1,\dots,w_n\}$. By reversing the steps in the proof of (1), we see $F(z)$ belongs to $N(\mathfrak{A})$ and $F(z)$ and $B_F(z)$ are analytic every $\lambda \in \mathbb{C}_\infty$ such that $-1/\lambda \in \sigma(A)$. $\square$

Proof of Theorem 4.7. (1) Let $\sigma(A) = \{z_1, \dots, z_n\}$. By Theorem 4.6(1), (1) will follow if we can show that $F(z)$ and $(1-z)B_F(z)$ are analytic at $\{-1/z_1, \dots, -1/z_n\}$. The functions $a(z), b(z), c(z), d(z)$ are analytic on $\Omega_{\mathfrak{A}} = \mathbb{C}_\infty \setminus \{-\bar{z}_1, \dots, -\bar{z}_n\}$, and the sets $\{-1/z_1, \dots, -1/z_n\}$ and $\{-\bar{z}_1, \dots, -\bar{z}_n\}$ are disjoint by our assumption that $\lambda\bar{\mu} \neq 1$ for all $\lambda, \mu \in \sigma(A)$. Therefore $a(z), b(z), c(z), d(z)$ are analytic at $\{-1/z_1, \dots, -1/z_n\}$.

Set

$$\begin{aligned} H(z) &= a(z)P(z) + b(z)Q(z), \\ K(z) &= c(z)P(z) + d(z)Q(z). \end{aligned}$$

Then $F(z) = H(z)K(z)^{-1}$ by (4.7), and

$$\begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \mathfrak{A}(z) \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}.$$

Thus

$$\begin{bmatrix} F(z) \\ I_m \end{bmatrix} = \begin{bmatrix} H(z) \\ K(z) \end{bmatrix} K(z)^{-1} = \mathfrak{A}(z) \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1}, \tag{6.37}$$

and so

$$F(z) = \begin{bmatrix} I_m & 0 \end{bmatrix} \mathfrak{A}(z) \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1}.$$

By (4.3),

$$\mathfrak{A}(z) = I_{2m} - M(z)\Pi J,$$

where

$$M(z) = (1+z)\Pi^*(A^* + zI)^{-1} S^{-1} (I - A)^{-1}$$

is analytic at $\{-1/z_1, \dots, -1/z_n\}$. This gives

$$\begin{aligned} F(z) &= \begin{bmatrix} I_m & 0 \end{bmatrix} \Big\{ I_{2m} - M(z)\Pi J \Big\} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} \\ &= P(z)K(z)^{-1} - \begin{bmatrix} I_m & 0 \end{bmatrix} M(z) \big[\Phi_2 P(z) + \Phi_1 Q(z) \big] K(z)^{-1}. \end{aligned} \tag{6.38}$$

By (iv), the right side is analytic at $\{-1/z_1, \dots, -1/z_n\}$. Therefore $F(z)$ is analytic at these points.

We can write (4.9) in the form

$$B_F(z) = (I + zA)^{-1} \Pi J \begin{bmatrix} F(z) \\ I_m \end{bmatrix}.$$

Hence by (6.37),

$$(1-z)B_F(z) = (1-z)(I + zA)^{-1} \Pi J \, \mathfrak{A}(z) \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1}. \tag{6.39}$$

By (4.3),

$$\begin{aligned} \Pi J \mathfrak{A}(z) &= \Pi J - (1+z)\Pi J \Pi^* (A^* + zI)^{-1} S^{-1} (I - A)^{-1} \Pi J \\ &= \Pi J - (1+z)(S - ASA^*)(A^* + zI)^{-1} S^{-1} (I - A)^{-1} \Pi J \\ &= \big\{ (I - A)S(A^* + zI) - (1+z)(S - ASA^*) \big\} \cdot \end{aligned}$$

$$\cdot (A^* + zI)^{-1} S^{-1} (I - A)^{-1} \Pi J$$
$$= (I + zA) S (A^* - I)(A^* + zI)^{-1} S^{-1} (I - A)^{-1} \Pi J.$$

Thus by (6.39),

$$\begin{aligned}(1 - z) B_F(z) &= (1 - z) \Big\{ (I + zA)^{-1} \Pi J \, \mathfrak{A}(z) \Big\} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} \\ &= (1 - z) \Big\{ S (A^* - I)(A^* + zI)^{-1} S^{-1} (I - A)^{-1} \Pi J \Big\} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} \\ &= N(z) \Pi J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1}, \end{aligned} \tag{6.40}$$

where $N(z)$ and $N(z)^{-1}$ are analytic at $\{-1/z_1, \dots, -1/z_n\}$. By condition (iv),

$$\Pi J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} = [\Phi_2 P(z) + \Phi_1 Q(z)] K(z)^{-1} \tag{6.41}$$

is analytic at $\{-1/z_1, \dots, -1/z_n\}$. Therefore $(1 - z) B_F(z)$ is analytic at these points, and (1) follows.

(2) By Theorem 4.6(2), $F(z) \in N(\mathfrak{A})$, and $F(z)$ and $(1-z) B_F(z)$ are analytic at $\{-1/z_1, \dots, -1/z_n\}$. Consider any representation (4.7) satisfying (i)–(iii). We show that (iv) holds.

Define $H(z)$ and $K(z)$ as in the proof of (1) and deduce (6.38), (6.40), and (6.41) as before. By (6.40) and (6.41),

$$[\Phi_2 P(z) + \Phi_1 Q(z)] K(z)^{-1} = \Pi J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} = N(z)^{-1} (1 - z) B_F(z).$$

Since $N(z)^{-1}$ and $(1 - z) B_F(z)$ are analytic at $\{-1/z_1, \dots, -1/z_n\}$, so is the function $[\Phi_2 P(z) + \Phi_1 Q(z)] K(z)^{-1}$. Then by (6.38), since $F(z)$ is analytic at $\{-1/z_1, \dots, -1/z_n\}$, so is $P(z) K(z)^{-1}$. Thus (iv) follows, and (2) is proved. □

Proof of Theorem 4.8. We proceed as in the proof of Theorem 4.6, but now we use [9, Theorem 5.3]. We readily verify the hypotheses of that result from the assumptions of the theorem.

(1) Suppose that $F \in N(\mathfrak{A})$ and (i)–(iii) hold. Define $v(w)$ by (2.6). Then $v(w) \in N(\mathfrak{A}_\ell)$, where $\mathfrak{A}_\ell(w)$ and $N(\mathfrak{A}_\ell)$ are as in the proof of Theorem 4.6. To apply [9, Theorem 5.3(1)], we must show that for $|y| \to \infty$,

(a) $v(iy)/y \to 0$;

(b) for all $h \in \mathfrak{H}$ and $g \in \mathbb{C}^m$, $\langle B_{v,\ell}(iy) g, h \rangle = \mathcal{O}(1)$;

(c) for all $h, k \in \mathfrak{H}$, $\langle B_{v,T,\ell}(iy) h, k \rangle = \mathcal{O}(1/|y|)$.

Here $B_{v,\ell}(w) = (I - wA_\ell)^{-1} [\Phi_1 - i \Phi_2 v(w)]$, and

$$B_{v,T,\ell}(w) = [S_\ell A_\ell^* + i B_{v,\ell}(w) \Phi_2](I - wA_\ell^*)^{-1}.$$

To verify these conditions, write

$$iy = i\,\frac{1+r}{1-r}, \qquad r = \frac{y-1}{y+1}\,.$$

Then $|y| \to \infty$ corresponds to $r \to 1$. Thus (a) follows from (i), since

$$\frac{v(iy)}{y} = i\,\frac{1-r}{1+r}\,F(r).$$

We obtain (b) from (ii) and the identity

$$B_{v,\ell}(iy) = \frac{1}{2}\,(1-r)(I-A)B_F(r). \tag{6.42}$$

To prove (c), use (6.42) and the identity

$$(I - iyA_\ell^*)^{-1} = \frac{1-r}{-2}\,(I-A^*)(A^*+rI)^{-1}$$

to write

$$\begin{aligned} B_{v,T,\ell}(iy) &= [S_\ell A_\ell^* + iB_{v,\ell}(iy)\Phi_2](I - iyA_\ell^*)^{-1} \\ &= \Bigg[\frac{1}{2}(I-A)S(I-A^*)(-i)(I-A^*)^{-1}(I+A^*) \\ &\qquad + i\,\frac{1}{2}\,(1-r)(I-A)B_F(r)\Bigg]\frac{1-r}{-2}\,(I-A^*)(A^*+rI)^{-1} \\ &= \frac{1-r}{4i}\,(I-A)\,[-S(I+A^*)+(1-r)B_F(r)\Phi_2]\,\cdot \\ &\qquad\qquad\qquad\qquad \cdot\,(A^*+rI)^{-1}(I-A^*). \end{aligned}$$

Therefore if $h, k \in \mathfrak{H}$,

$$\begin{aligned} y\,\langle B_{v,T,\ell}(iy)h, k\rangle &= \frac{1+r}{1-r}\,\frac{1-r}{4i}\,\cdot \\ &\qquad \cdot \left\langle [-S(I+A^*)+(1-r)B_F(r)\Phi_2](A^*+rI)^{-1}\tilde{h}, \tilde{k}\right\rangle, \end{aligned}$$

where $\tilde{h} = (I-A^*)h$ and $\tilde{k} = (I-A^*)k$. Since $I - A$ is invertible, (c) follows from (iii). By [9, Theorem 5.3(1)], $v(w) \in \mathbf{N}_\varkappa$ for some $\varkappa \geq 0$, $S = S_v$, and $\Phi_1 = \Phi_{1,v}$. Therefore $F(z) \in \mathbf{C}_\varkappa$. By (5.12),

$$S = 2(I-A)^{-1}S_\ell(I-A^*)^{-1} = 2(I-A)^{-1}S_v(I-A^*)^{-1} = S_F,$$

and $\Phi_1 = \Phi_{1,v} = \Phi_{1,F}$.

(2) This is the same as the corresponding part of Theorem 4.6, except that in place of [9, Theorem 5.1(2)] we use [9, Theorem 5.3(2)]. □

Proof of Theorem 4.9. It is enough to prove (4.16). For if (4.16) is known and $\widetilde{S}$ is the right side of (4.15), then

$$\widetilde{S} - A\widetilde{S}A^* = \Phi_{1,F}\Phi_2^* + \Phi_2\Phi_{1,F}^*.$$

By Theorem 3.4,

$$S_F - AS_F A^* = \Phi_{1,F}\Phi_2^* + \Phi_2\Phi_{1,F}^*.$$

Therefore $X = \widetilde{S} - S_F$ satisfies $AXA^* = X$. Since $z_1, \dots, z_n$ are in $\mathbb{D}$, the only solution of the operator equation is $X = 0$, and so $S_F = \widetilde{S}$, which is (4.15).

We deduce (4.16) from a corresponding result for the line. Define $v(w) \in \mathbf{N}_\varkappa$ by $F(z) = -iv(w)$, $w = \varphi(z)$. Then

$$F(-\bar{z})^* = i\,v(1/\bar{w})^*.$$

By Theorem 5.1, $\Phi_{1,F} = \Phi_{1,v}$ where $\Phi_{1,v}$ is calculated using (5.5) and

$$A_\ell = i(I+A)(I-A)^{-1} = \begin{bmatrix} w_1 I_p & 0 & \cdots & 0 \\ 0 & w_2 I_p & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & w_n I_p \end{bmatrix}_{pn\times pn},$$

$w_j = \varphi(z_j)$, $j = 1, \dots, n$. Calculating as in [9], we obtain

$$\Phi_{1,F} = \Phi_{1,v} = \begin{bmatrix} ie_1 v(1/\bar{w}_1)^* \\ \vdots \\ ie_n v(1/\bar{w}_n)^* \end{bmatrix} = \begin{bmatrix} e_1 F(-\bar{z}_1)^* \\ \vdots \\ e_n F(-\bar{z}_n)^* \end{bmatrix},$$

as required. We omit the details. □

Appendix A. Calculation of residues

At times it is convenient to transform residue formulas from one point of $\mathbb{C}_\infty$ to another. Here we define what is meant by a residue at infinity and give a general formula that works in all cases. Assume that $h(w)$ is a scalar- or operator-valued function which is analytic in a deleted neighborhood of a point $w_0 \in \mathbb{C}_\infty$. If w_0 is finite, the residue $\operatorname{Res}_{w=w_0} h(w)$ has its usual meaning as the coefficient of $(w-w_0)^{-1}$ in the Laurent expansion of $h(w)$ about w_0. Residues at infinity are defined by

$$\operatorname*{Res}_{w=\infty} h(w) = \operatorname*{Res}_{z=0} \Big[-z^{-2} h(z^{-1}) \Big]. \tag{A.1}$$

This yields the value $-a_{-1}$ if $h(w)$ has Laurent coefficients $\{a_n\}_{-\infty}^{\infty}$ at infinity, in agreement with e.g. Palka [6, p. 322].

Lemma A.1. *Let $w_0, z_0 \in \mathbb{C}_\infty$. If $w = \psi(z)$ is analytic and one-to-one in a neighborhood of z_0 and $w_0 = \psi(z_0)$, then*

$$\operatorname*{Res}_{w=w_0} h(w) = \operatorname*{Res}_{z=z_0} \big[\psi'(z) h(\psi(z)) \big]. \tag{A.2}$$

Proof. When w_0, z_0 are both finite, (A.2) can be proved by showing that

$$\frac{1}{2\pi i}\int_\Gamma h(w)\,dw = \frac{1}{2\pi i}\int_\gamma \psi'(z) h(\psi(z))\,dz$$

for suitable closed curves Γ and γ about w_0 and z_0. We take (A.2) as known in this case. It remains to check (A.2) when one or both of w_0, z_0 are infinite.

Case 1: $w_0 = \infty$, z_0 finite. By (A.1),

$$\operatorname*{Res}_{w=\infty} h(w) = \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} h(\lambda^{-1}) \big].$$

Apply (A.2) to $\lambda = 1/\psi(z)$ with the finite points $\lambda = 0$ and $z = z_0$. We get

$$\begin{aligned} \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} h(\lambda^{-1}) \big] &= \operatorname*{Res}_{z=z_0} \left[\frac{d\lambda}{dz} (-\psi(z)^2) h(\psi(z)) \right] \\ &= \operatorname*{Res}_{z=z_0} \big[\psi'(z) h(\psi(z)) \big]. \end{aligned}$$

Therefore (A.2) holds when $w_0 = \infty$ and z_0 is finite.

Case 2: w_0 finite, $z_0 = \infty$. By (A.1),

$$\operatorname*{Res}_{z=\infty} \big[\psi'(z) h(\psi(z)) \big] = \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} \psi'(\lambda^{-1}) h(\psi(\lambda^{-1})) \big].$$

Apply (A.2) to $w = \psi(1/\lambda)$ and the finite points $w = w_0$ and $\lambda = 0$. Thus

$$\begin{aligned} \operatorname*{Res}_{w=w_0} h(w) &= \operatorname*{Res}_{\lambda=0} \left[\frac{dw}{d\lambda} h(\psi(\lambda^{-1})) \right] \\ &= \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} \psi'(\lambda^{-1}) h(\psi(\lambda^{-1})) \big], \end{aligned}$$

again yielding the result.

Case 3: $w_0 = z_0 = \infty$. By (A.1),

$$\begin{aligned} \operatorname*{Res}_{w=\infty} h(w) &= \operatorname*{Res}_{\zeta=0} \big[-\zeta^{-2} h(\zeta^{-1}) \big], \\ \operatorname*{Res}_{z=\infty} \big[\psi'(z) h(\psi(z)) \big] &= \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} \psi'(\lambda^{-1}) h(\psi(\lambda^{-1})) \big]. \end{aligned}$$

Apply (A.2) with $\zeta = \psi(\lambda^{-1})^{-1}$ using the points $\zeta = 0$ and $\lambda = 0$. We get

$$\begin{aligned} \operatorname*{Res}_{\zeta=0} \big[-\zeta^{-2} h(\zeta^{-1}) \big] &= \operatorname*{Res}_{\lambda=0} \left[\frac{d\zeta}{d\lambda} (-\psi(\lambda^{-1})^2)\, h(\psi(\lambda^{-1})) \right] \\ &= \operatorname*{Res}_{\lambda=0} \big[-\lambda^{-2} \psi'(\lambda^{-1}) h(\psi(\lambda^{-1})) \big]. \end{aligned}$$

Thus (A.2) holds when $w_0 = z_0 = \infty$. □

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