Pseudospectral functions for canonical differential systems. II

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To the memory of Israel Gohberg.

Abstract. A spectral theory is constructed for canonical differential systems whose Hamiltonians have selfadjoint matrix values. In contrast with the case of nonnegative Hamiltonians, eigenvalues in general can be complex, and root functions as well as eigenfunctions come into play. Eigentransforms are defined and turn out to be isometric on the span of root functions with respect to a suitably defined indefinite inner product on entire functions.

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1. Introduction

We are concerned with the spectral theory of canonical differential systems

$$\begin{aligned} \frac{dY}{dx} &= izJH(x)Y, \qquad 0 \le x \le \ell, \\ \begin{bmatrix} I_m & 0 \end{bmatrix} & Y(0,z) = 0. \end{aligned} \tag{1.1}$$

In (1.1), we assume that

$$J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \qquad Y(x,z) = \begin{bmatrix} Y_1(x,z) \\ Y_2(x,z) \end{bmatrix}, \tag{1.2}$$

where $Y_1(x,z)$ and $Y_2(x,z)$ are m-dimensional vector-valued functions, and z is a complex parameter. As in [3], the Hamiltonian $H(x)$ is assumed to be a measurable $2m \times 2m$ matrix-valued function such that

$$H(x)^* = H(x) \text{ a.e.} \quad \text{and} \quad \int_0^\ell \|H(x)\|\, dx < \infty. \tag{1.3}$$

For technical reasons, we also assume throughout that

$$H(x)\begin{bmatrix}0\\ g\end{bmatrix} = 0 \text{ a.e. on } [0,\ell] \quad \Longrightarrow \quad g = 0. \tag{1.4}$$

For any such system we define $L^2(Hdx)$ as a Kreĭn space of (equivalence classes of) $2m$-dimensional vector-valued functions on $[0,\ell]$. Write

$$H(x) = H_+(x) - H_-(x)$$

where $H_\pm(x)$ are measurable, $H_\pm(x) \geq 0$, and $H_+(x)H_-(x) = 0$ a.e. As a linear space, $L^2(Hdx)$ is the set of measurable $2m$-dimensional vector-valued functions f on $[0,\ell]$ such that

$$\int_0^\ell f(t)^*[H_+(t) + H_-(t)]f(t)\,dt < \infty.$$

Two functions f_1 and f_2 in $L^2(Hdx)$ are identified if $H(x)[f_1(x) - f_1(x)] = 0$ a.e. Taken with the inner product

$$\langle f_1, f_2\rangle_H = \int_0^\ell f_2(x)^* H(x) f_1(x)\,dx, \qquad f_1, f_2 \in L^2(Hdx),$$

$L^2(Hdx)$ is a Kreĭn space. In a natural way we can view $L^2(H_\pm dx)$ as closed subspaces, and then

$$L^2(Hdx) = L^2(H_+dx) \oplus L^2(H_-dx)$$

is a fundamental decomposition.

Define an eigentransform $F = Vf$ for any f in $L^2(Hdx)$ by

$$F(z) = \int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix} W(x,\bar z)^*\, H(x) f(x)\,dx, \tag{1.5}$$

where $W(x,z)$ is the unique $2m \times 2m$ matrix-valued function such that

$$\begin{aligned} \frac{dW}{dx} &= izJH(x)W, \qquad && 0 \leq x \leq \ell, \\ W(0,z) &= I_{2m}, && z \in \mathbb{C}\,. \end{aligned} \tag{1.6}$$

The function $W(x,z)$ is continuous on $[0,\ell] \times \mathbb{C}$ and entire in z for each fixed x. For each f in $L^2(Hdx)$, $F = Vf$ is an m-dimensional vector-valued entire function. Throughout we write

$$W(\ell,\bar z)^* = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z)\end{bmatrix}. \tag{1.7}$$

Here $a(z), b(z), c(z), d(z)$ are $m \times m$ matrix-valued entire functions.

Consider first the definite case, that is, $H(x) \geq 0$ a.e. Then $L^2(Hdx)$ is a Hilbert space. In this case, by a spectral function for (1.1) is meant a nondecreasing $m \times m$ matrix-valued function $\tau(x)$ of real x such that the eigentransform V acts as an isometry from $L^2(Hdx)$ into $L^2(d\tau)$. We call $\tau(x)$ a pseudospectral function for (1.1) if V is a partial isometry from $L^2(Hdx)$ into $L^2(d\tau)$. Pseudospectral functions can be constructed using a boundary condition at the right endpoint of the interval $[0, \ell]$. The boundary condition has the form

$$\begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell, z) = 0,$$

where R and Q are $m \times m$ matrices such that $R^*Q + Q^*R = 0$, and such that the entire function $c(z)R+d(z)Q$ has invertible values except at isolated points. Then

$$v(z) = i[a(z)R + b(z)Q][c(z)R + d(z)Q]^{-1}$$

is meromorphic in the complex plane, $v(z) = v(\bar{z})^*$ at all points of analyticity, and $v(z)$ has nonnegative imaginary part in the upper half-plane. In particular, $v(z)$ has only real and simple poles and a representation

$$v(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \frac{t}{1 + t^2} \right] d\tau(t), \tag{1.8}$$

where $\tau(x)$ is a nondecreasing $m \times m$ matrix-valued step function with jumps at the poles of $v(z)$, and $\alpha = \alpha^*$ and $\beta \geq 0$ are constant $m \times m$ matrices. The function $\tau(x)$ is a pseudospectral function. The isometric set for the eigentransform V is the closed span of eigenfunctions. See [4, Chapter 4] and Theorems 4.2.2, 4.2.4, and 4.2.5 in [3].

In this paper we generalize the preceding constructions to Hamiltonians such that $H(x) = H(x)^*$ a.e. We introduce a meromorphic function $v(z)$ in the same way as before. Now, however, $v(z)$ can have nonreal and nonsimple poles, and in general there is no representation of $v(z)$ in the form (1.8). In place of eigenfunctions, we have to deal now with eigenchains of root functions. The role of a pseudospectral function is replaced by a notion of pseudospectral data, which consists of the collection of poles and principal parts of the meromorphic function $v(z)$. The poles and principal parts of $v(z)$ are used to construct an inner product $\langle \cdot, \cdot \rangle$ on vector-valued entire functions. According to our main result, Theorem 4.7, the identity

$$\int_0^\ell f_2(t)^* H(t) f_1(t)\, dt = \langle F_1, F_2 \rangle$$

holds whenever f_1 and f_2 are finite linear combinations of root functions and F_1 and F_2 are their eigentransforms. This agrees with Theorem 4.1.11 of [3] for the special case when $v(z)$ has only simple poles. The general case turns out to be quite a bit more involved.

In Section 2 of the paper, we expand the function $W(x, z)$ in a Taylor series about a point $z = w$. The higher-order coefficients in this expansion do not arise in the definite theory, but they are important in the general case considered here. In Section 3 we derive explicit formulas for the root functions and their eigentransforms. These formulas are needed for the main results of the paper, which appear in Section 4.

Remark. We thank the referee for the comment that the construction of a related linear operator and its resolvent might yield insights into our main results. We leave this as an open question. Concerning such related linear operators, see the remark preceding Proposition 3.1. See also Section 3 of [3], where resolvent operators for canonical differential systems are investigated.

2. Taylor expansions and their coefficients

Assume given a system (1.1)–(1.4). Define $W(x,z)$ and $a(z), b(z), c(z), d(z)$ as in (1.6) and (1.7). By (1.6),

$$\frac{d}{dt} W(t,\bar{z})^* J W(t,w) = i(w-z) W(t,\bar{z})^* H(t) W(t,w)$$

a.e. on $[0,\ell]$ for all complex w and z. We deduce that

$$W(x,\bar{z})^* J W(x,z) = W(x,z) J W(x,\bar{z})^* = J, \tag{2.1}$$

and

$$\int_0^\ell W(t,\bar{z})^* H(t) W(t,w)\, dt = \frac{W(\ell,\bar{z})^* J W(\ell,w) - J}{i(w-z)}. \tag{2.2}$$

Set

$$W(x,z) = \sum_{j=0}^\infty W_j(x,w)(z-w)^j, \tag{2.3}$$

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = \sum_{j=0}^\infty \begin{bmatrix} a_j(w) & b_j(w) \\ c_j(w) & d_j(w) \end{bmatrix} (z-w)^j, \tag{2.4}$$

for all x in $[0,\ell]$ and w in $\mathbb{C}$. Using the values $x=0$ and $x=\ell$, we get

$$\begin{aligned} W_0(0,w) &= I_{2m}, \qquad W_j(0,w) = 0, \quad j \geq 1, \\ W_j(\ell,w) &= \begin{bmatrix} a_j(\bar{w})^* & c_j(\bar{w})^* \\ b_j(\bar{w})^* & d_j(\bar{w})^* \end{bmatrix}, \qquad j \geq 0. \end{aligned} \tag{2.5}$$

For each $j \geq 0$, $W_j(x,w)$ is continuous on $[0,\ell]$ and entire in w for fixed x. To prove this, represent the coefficients as Cauchy integrals as in (2.8) below and use the corresponding properties for $W(x,z)$.

Proposition 2.1. *For every $w \in \mathbb{C}$,*

$$\begin{aligned} \frac{d}{dx} W_0(x,w) &= iwJH(x)W_0(x,w), \\ \frac{d}{dx} W_j(x,w) &= iwJH(x)W_j(x,w) + iJH(x)W_{j-1}(x,w), \qquad j \geq 1, \end{aligned} \tag{2.6}$$

a.e. on $[0,\ell]$.

Proof. The first equation in (2.6) holds by (1.6) since $W_0(x,z) = W(x,z)$. Since $W_j(0,w) = 0$ for $j \geq 1$, the second equation in (2.6) is equivalent to

$$W_j(x,w) = iwJ\int_0^x H(t)W_j(t,w)\, dt + iJ\int_0^x H(t)W_{j-1}(t,w)\, dt, \tag{2.7}$$

$0 \le x \le \ell$. Let Γ be a circular path around w in the counterclockwise direction. For each x in $[0,\ell]$ and $k \ge 0$,

$$W_k(x,w) = \frac{1}{2\pi i}\int_\Gamma \frac{W(x,\zeta)}{(\zeta - w)^{k+1}}\, d\zeta. \tag{2.8}$$

To prove (2.7), first write (1.6) in the form

$$W(x,\zeta) = I_{2m} + i\zeta J \int_0^x H(t)W(t,\zeta)\, dt.$$

Since we assume $j \ge 1$, $\int_\Gamma d\zeta/(\zeta - w)^{j+1} = 0$. Thus

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{W(x,\zeta)}{(\zeta - w)^{j+1}}\, d\zeta &= \frac{1}{2\pi i}\int_\Gamma \left[i\zeta J \int_0^x H(t)W(t,\zeta)\, dt \right] \frac{d\zeta}{(\zeta - w)^{j+1}} \\
&= \frac{1}{2\pi i}\int_\Gamma \Bigg[iwJ \int_0^x H(t)W(t,\zeta)\, dt \\
&\qquad + i(\zeta - w)J \int_0^x H(t)W(t,\zeta)\, dt \Bigg] \frac{d\zeta}{(\zeta - w)^{j+1}} \\
&= iwJ \int_0^x H(t)\, \frac{1}{2\pi i}\int_\Gamma \frac{W(t,\zeta)}{(\zeta - w)^{j+1}}\, d\zeta\, dt \\
&\qquad + iJ \int_0^x H(t)\, \frac{1}{2\pi i}\int_\Gamma \frac{W(t,\zeta)}{(\zeta - w)^{j}}\, d\zeta\, dt.
\end{aligned}$$

By (2.8), this is the same as (2.7). The interchange in order of integration is justified because $\|H(t)\|\|W(t,\zeta)\|$ is integrable over $[0,\ell] \times \Gamma$. □

Proposition 2.2. *For all $w \in \mathbb{C}$, $x \in [0,\ell]$, and $n \ge 0$,*

$$\begin{aligned}
\sum_{p+q=n} W_p(x,\bar{w})^* J W_q(x,w) &= \sum_{p+q=n} W_p(x,w) J W_q(x,\bar{w})^* \\
&= \begin{cases} J, & n = 0, \\ 0, & n \ge 1. \end{cases}
\end{aligned} \tag{2.9}$$

Proof. By (2.1) and (2.3),

$$\begin{aligned}
&\sum_{p=0}^\infty W_p(x,\bar{w})^*(z-w)^p\; J \sum_{q=0}^\infty W_q(x,w)(z-w)^q \\
&\qquad = \sum_{p=0}^\infty W_p(x,w)(z-w)^p\; J \sum_{q=0}^\infty W_q(x,\bar{w})^*(z-w)^q = J.
\end{aligned}$$

The relations (2.9) follow on expanding the products and collecting powers of $z - w$: the constant terms equal J, and all other coefficients are zero. □

Corollary 2.3. *For every $w \in \mathbb{C}$,*

$$\begin{aligned} &a_0(w)b_0(\bar w)^* + b_0(w)a_0(\bar w)^* = 0, && a_0(\bar w)^*c_0(w) + c_0(\bar w)^*a_0(w) = 0,\\ &a_0(w)d_0(\bar w)^* + b_0(w)c_0(\bar w)^* = I_m, && a_0(\bar w)^*d_0(w) + c_0(\bar w)^*b_0(w) = I_m,\\ &c_0(w)d_0(\bar w)^* + d_0(w)c_0(\bar w)^* = 0, && b_0(\bar w)^*d_0(w) + d_0(\bar w)^*b_0(w) = 0. \end{aligned} \tag{2.10}$$

For all $n \geq 1$,

$$\begin{aligned} &\sum_{p+q=n} [a_p(w)b_q(\bar w)^* + b_p(w)a_q(\bar w)^*] = 0,\\ &\sum_{p+q=n} [a_p(w)d_q(\bar w)^* + b_p(w)c_q(\bar w)^*] = 0,\\ &\sum_{p+q=n} [c_p(w)d_q(\bar w)^* + d_p(w)c_q(\bar w)^*] = 0, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\sum_{p+q=n} [a_p(\bar w)^*c_q(w) + c_p(\bar w)^*a_q(w)] = 0,\\ &\sum_{p+q=n} [a_p(\bar w)^*d_q(w) + c_p(\bar w)^*b_q(w)] = 0,\\ &\sum_{p+q=n} [b_p(\bar w)^*d_q(w) + d_p(\bar w)^*b_q(w)] = 0. \end{aligned} \tag{2.12}$$

Proof. These identities follow on choosing $x = \ell$ in (2.9) and expanding using (1.7). The relations (2.10) follow from the case $n = 0$ and coincide with the formulas (2.1.5) of [3]. Suppose $n \geq 1$. Then by (2.9),

$$\begin{aligned} \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix} &= \sum_{p+q=n} W_p(\ell,\bar w)^* J W_q(\ell,w)\\ &= \sum_{p+q=n} \begin{bmatrix} a_p(w) & b_p(w)\\ c_p(w) & d_p(w) \end{bmatrix} \begin{bmatrix} b_q(\bar w)^* & d_q(\bar w)^*\\ a_q(\bar w)^* & c_q(\bar w)^* \end{bmatrix}\\ &= \sum_{p+q=n} \begin{bmatrix} a_p(w)b_q(\bar w)^* + b_p(w)a_q(\bar w)^* & a_p(w)d_q(\bar w)^* + b_p(w)c_q(\bar w)^*\\ c_p(w)b_q(\bar w)^* + d_p(w)a_q(\bar w)^* & c_p(w)d_q(\bar w)^* + d_p(w)c_q(\bar w)^* \end{bmatrix}, \end{aligned}$$

yielding (2.11). We prove (2.12) in a similar way using (2.9). □

Proposition 2.4. *For all $w, z \in \mathbb{C}$ and $n \geq 0$,*

$$\begin{aligned} &\int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* H(t) W_n(t,w) \begin{bmatrix} 0\\ I_m \end{bmatrix} dt\\ &\qquad = i \sum_{p+q=n} \frac{c(z)d_p(\bar w)^* + d(z)c_p(\bar w)^*}{(z-w)^{q+1}} = \sum_{k=0}^\infty \Delta_{nk}(w)(z-w)^k. \end{aligned} \tag{2.13}$$

In (2.13)*, for all $n, k \geq 0$,*

$$\Delta_{nk}(w) = i \sum_{p+q=n} [c_{q+k+1}(w)d_p(\bar w)^* + d_{q+k+1}(w)c_p(\bar w)^*], \tag{2.14}$$

and the middle expression is interpreted by continuity for $z = w$. Moreover,

$$\Delta_{nk}(w) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W_k(t,\bar{w})^* H(t) W_n(t,w) \begin{bmatrix} 0 \\ I_m \end{bmatrix} dt \tag{2.15}$$

and

$$\Delta_{nk}(\bar{w})^* = \Delta_{kn}(w). \tag{2.16}$$

Proof. By (2.2) and (1.7),

$$\begin{aligned} &\int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar{z})^* H(t) W(t,\lambda) \begin{bmatrix} 0 \\ I_m \end{bmatrix} dt \\ &= \begin{bmatrix} 0 & I_m \end{bmatrix} \frac{W(\ell,\bar{z})^* J W(\ell,\lambda) - J}{i(\lambda - z)} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = i\, \frac{c(z)d(\bar{\lambda})^* + d(z)c(\bar{\lambda})^*}{z - \lambda}. \end{aligned} \tag{2.17}$$

Using the expansions

$$W(t,\lambda) = \sum_{n=0}^\infty W_n(t,w)(\lambda - w)^n, \tag{2.18}$$

$$\begin{bmatrix} d(\bar{\lambda})^* \\ c(\bar{\lambda})^* \end{bmatrix} = \sum_{p=0}^\infty \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} (\lambda - w)^p, \tag{2.19}$$

$$\frac{i}{z-\lambda} = \frac{i}{z-w}\, \frac{1}{1 - \dfrac{\lambda - w}{z - w}} = i \sum_{q=0}^\infty \frac{(\lambda - w)^q}{(z-w)^{q+1}}, \tag{2.20}$$

we obtain

$$\begin{aligned} &\int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar{z})^* H(t) \sum_{n=0}^\infty W_n(t,w) \begin{bmatrix} 0 \\ I_m \end{bmatrix} (\lambda - w)^n \, dt \\ &\qquad = \begin{bmatrix} c(z) & d(z) \end{bmatrix} \begin{bmatrix} d(\bar{\lambda})^* \\ c(\bar{\lambda})^* \end{bmatrix} \frac{i}{z - \lambda} \\ &\qquad = \sum_{p=0}^\infty \begin{bmatrix} c(z) & d(z) \end{bmatrix} \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} (\lambda - w)^p \, i \sum_{q=0}^\infty \frac{(\lambda - w)^q}{(z-w)^{q+1}} \\ &\qquad = \sum_{n=0}^\infty (\lambda - w)^n \, i \sum_{p+q=n} \frac{c(z)d_p(\bar{w})^* + d(z)c_p(\bar{w})^*}{(z-w)^{q+1}}. \end{aligned} \tag{2.21}$$

In fact, in (2.21) the first equality is identical to (2.17) by the Taylor expansion for $W(t,\lambda)$ in (2.18); the second equality substitutes the two Taylor expansions in (2.19) and (2.20); the third equality collects powers of $\lambda - w$. The first equality in (2.13) follows from (2.21) on interchanging the order of integration and summation on the left and comparing coefficients.

To prove the second equality in (2.13), expand $c(z)$ and $d(z)$ in Taylor series, and write

$$\begin{aligned}
i \sum_{p+q=n} &\frac{c(z)d_p(\bar{w})^* + d(z)c_p(\bar{w})^*}{(z-w)^{q+1}} \\
&= i \sum_{p+q=n} \sum_{k=0}^{\infty} \begin{bmatrix} c_k(w) & d_k(w) \end{bmatrix} (z-w)^k \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} \frac{1}{(z-w)^{q+1}} \\
&= i \sum_{p+q=n} \sum_{k=-q-1}^{\infty} \begin{bmatrix} c_{k+q+1}(w) & d_{k+q+1}(w) \end{bmatrix} \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} (z-w)^k \\
&= i \sum_{p+q=n} \sum_{k=0}^{\infty} \begin{bmatrix} c_{k+q+1}(w) & d_{k+q+1}(w) \end{bmatrix} \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} (z-w)^k \\
&\qquad + i \sum_{p+q=n} \sum_{k=-q-1}^{-1} \begin{bmatrix} c_{k+q+1}(w) & d_{k+q+1}(w) \end{bmatrix} \begin{bmatrix} d_p(\bar{w})^* \\ c_p(\bar{w})^* \end{bmatrix} (z-w)^k \\
&= \sum_{k=0}^{\infty} \Delta_{nk}(w)(z-w)^k + \text{Term 2.} \qquad (2.22)
\end{aligned}$$

Here Term 2 = 0, since by the first equality in (2.13), proved above, the left side of (2.22) is entire. Thus (2.22) yields the second equality in (2.13) with $\Delta_{nk}(w)$ defined by (2.14).

The identity (2.15) follows from (2.13) on expanding $W(t,\bar{z})^*$ in a Taylor series about $z = w$ and comparing coefficients. Then (2.16) follows from (2.15). □

3. Root spaces and eigenchains

We now add a boundary condition at the right endpoint of the interval $[0,\ell]$. Thus we consider a system

$$\begin{aligned}
\frac{dY}{dx} &= izJH(x)Y, \qquad 0 \le x \le \ell, \\
\begin{bmatrix} I_m & 0 \end{bmatrix} Y(0,z) &= 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell,z) = 0,
\end{aligned} \qquad (3.1)$$

subject to the conditions (1.2)–(1.4). Define $a(z), b(z), c(z), d(z)$ by (1.7) as before. We assume two conditions:

(1°) R and Q are $m \times m$ matrices such that $R^*Q + Q^*R = 0$;
(2°) the values of $c(z)R + d(z)Q$ are invertible except at isolated points.

There are many choices of matrices meeting these conditions because $c(0) = 0$ and $d(0) = I_m$.

The operator $R^*R + Q^*Q$ is invertible, since otherwise $c(z)R + d(z)Q$ has no invertible value, in violation of (2°).

Notice that (2°) assures that the function

$$v(z) = i[a(z)R + b(z)Q][c(z)R + d(z)Q]^{-1} \tag{3.2}$$

is defined except at isolated points. This function is meromorphic on $\mathbb{C}$, and it satisfies $v(z) = v(\bar{z})^*$ by [3, Proposition 2.3.1]. The poles and principal parts of $v(z)$ contain important information for the spectral theory of the system (3.1).

For each $\zeta \in \mathbb{C}$, let $\mathfrak{L}_\zeta^{(0)}$ be the set of all solutions $Y = Y(x)$ of (3.1) with $z = \zeta$. If $\mathfrak{L}_\zeta^{(0)}, \dots, \mathfrak{L}_\zeta^{(k)}$ have been defined, let $\mathfrak{L}_\zeta^{(k+1)}$ be the set of all $Y = Y(x)$ such that

$$\begin{aligned} \frac{dY}{dx} &= i\zeta JH(x)Y + JH(x)Y^{(k)}, \\ \begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) &= 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0, \end{aligned} \tag{3.3}$$

for some $Y^{(k)} \in \mathfrak{L}_\zeta^{(k)}$. We call $\mathfrak{L}_\zeta^{(0)}, \mathfrak{L}_\zeta^{(1)}, \dots$ **root spaces**. Elements of these spaces are **root functions**. Root spaces are linear spaces which we view as subspaces of $L^2(Hdx)$. By Proposition 3.2 below there is a largest root space

$$\mathfrak{L}_\zeta = \bigcup_{j=0}^{\infty} \mathfrak{L}_\zeta^{(j)} = \mathfrak{L}_\zeta^{(\mu)}. \tag{3.4}$$

We say that ζ is an **eigenvalue** for (3.1) if $\mathfrak{L}_\zeta \neq \{0\}$ as a subspace of $L^2(Hdx)$.

Remark. Following [2, 4], we work directly with canonical differential systems and make no use of underlying operators on $L^2(Hdx)$. Nevertheless, it may be noted that our definitions of eigenvalue and root space are equivalent to standard operator definitions. The root subspaces $\mathfrak{K}_0, \mathfrak{K}_1, \dots$ for a bounded linear operator T and eigenvalue ζ are defined recursively by $\mathfrak{K}_0 = \ker(T - \zeta I)$ and $\mathfrak{K}_{j+1} = \{f \colon (T - \zeta I)f \in \mathfrak{K}_j\}$ for all $j = 0, 1, \dots$. With due attention to domains, the same definition is used for an unbounded operator. If $H(x)$ has invertible values, we can take $T = -iH(x)^{-1}J\, d/dx$ with domain specified by boundary values [4, p. 49]. The two notions of eigenvalue and root space then coincide. In principle, one can reduce to the case of invertible Hamiltonian with a transformation given in [4, p.143] that replaces $W(x,z)$ with $\widetilde{W}(x,z) = e^{-iz\gamma x}W(x,z)$ for some $\gamma > 0$. This yields a new system with selfadjoint Hamiltonian $\widetilde{H}(x) = H(x) - \gamma J$. If $H(x)$ is bounded and γ is sufficiently large, $\widetilde{H}(x)$ has invertible values. The transformation is well behaved with respect to eigenvalues and root spaces. We do not use these constructions and therefore omit details.

Proposition 3.1. *For any complex number ζ, the following are equivalent:*

(i) *ζ is an eigenvalue of* (3.1)*;*

(ii) *$c(\zeta)R + d(\zeta)Q$ is not invertible;*

(iii) *ζ is a pole of $v(z)$.*

The eigenvalues of (3.1) *are isolated points in the complex plane and occur in conjugate pairs $\zeta, \bar{\zeta}$.*

As a preliminary to the proof, consider an $m \times m$ matrix-valued analytic function $F(z)$ on a region Ω which has invertible values except at isolated points. If $\zeta \in \Omega$ and $F(\zeta)$ is not invertible, there is an $r \geq 1$ such that

$$F(z) = F_r(z)P(z), \tag{3.5}$$

where $F_r(z)$ is analytic on Ω, $F_r(\zeta)$ is invertible, and $P(z)$ is a polynomial of the form

$$P(z) = \big[I - P_r + P_r(z-\zeta)\big] \cdots \big[I - P_2 + P_2(z-\zeta)\big]\big[I - P_1 + P_1(z-\zeta)\big] \tag{3.6}$$

for some rank-one projections $P_1, P_2, \ldots, P_r$.

To see this, let Let r be the order of ζ as a zero of $\det F(z)$. Since $F(\zeta)$ is not invertible, there is a $g_1 \neq 0$ in $\mathbb{C}^m$ such that $F(\zeta)g_1 = 0$. Let P_1 be the projection on the span of g_1, and set

$$F_1(z) = F(z)\left[I - P_1 + \frac{P_1}{z-\zeta}\right].$$

Since $F(\zeta)P_1 = 0$, we can define $F_1(\zeta)$ so that $F_1(z)$ is analytic on Ω. We have

$$F(z) = F_1(z)\big[I - P_1 + P_1(z-\zeta)\big],$$
$$\det F_1(z) = \frac{\det F(z)}{z-\zeta}.$$

If $r = 1$, then $\det F_1(\zeta) \neq 0$ because then ζ is a zero of $\det F(z)$ of order 1. The assertion follows in the case $r = 1$. In general, we proceed in the same way but repeat the procedure r times.

Proof of Proposition 3.1. Everything here is in [3, Proposition 4.1.8] except for the equivalence of (ii) and (iii). Clearly (iii) implies (ii), so what remains is to show that (ii) implies (iii). We argue by contradiction, assuming that (ii) holds but (iii) fails. Write

$$v(z) = iu_1(z)u_2(z)^{-1},$$

where $u_1(z) = a(\zeta)R + b(\zeta)Q$ and $u_2(z) = c(\zeta)R + d(\zeta)Q$. Here $u_2(z)$ has invertible values except at isolated points, $u_2(\zeta)$ is not invertible, and ζ is a removable singularity of $v(z)$. Applying (3.5) to $F(z) = u_2(z)$, we obtain

$$u_2(z) = \tilde{u}_2(z)P(z),$$

where $\tilde{u}_2(z)$ is entire, $\tilde{u}_2(\zeta)$ is invertible, and $P(z)$ has the form (3.6). Set

$$\tilde{u}_1(z) = u_1(z)P(z)^{-1}, \qquad z \neq \zeta.$$

Then for all $z \neq \zeta$,

$$v(z) = iu_1(z)P(z)^{-1}\tilde{u}_2(z)^{-1} = i\tilde{u}_1(z)\tilde{u}_2(z)^{-1}.$$

Since ζ is a removable singularity of $v(z)$ and $\tilde{u}_2(\zeta)$ is invertible, ζ is a removable singularity of $\tilde{u}_1(z)$. Therefore we can define $\tilde{u}_1(\zeta)$ so that $\tilde{u}_1(z)$ is entire. By (1.7),

$$\begin{bmatrix} \tilde{u}_1(z)P(z) \\ \tilde{u}_2(z)P(z) \end{bmatrix} = \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix} = W(\ell, \bar{z})^* \begin{bmatrix} R \\ Q \end{bmatrix}.$$

We can choose $g \neq 0$ in $\mathbb{C}^m$ such that $P(\zeta)g = 0$, and then we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = W(\ell, \bar{\zeta})^* \begin{bmatrix} R \\ Q \end{bmatrix} g.$$

Since $W(\ell, \bar{\zeta})^*$ is invertible, $Rg = Qg = 0$. The desired contradiction follows because $R^*R + Q^*Q$ is invertible under our assumptions. □

Proposition 3.2. *The root spaces for* (3.1) *are finite dimensional. Moreover, for every eigenvalue* ζ *of* (3.1), *there is a* $\mu \geq 0$ *such that*

$$\mathfrak{L}_\zeta^{(0)} \subsetneq \mathfrak{L}_\zeta^{(1)} \subsetneq \cdots \subsetneq \mathfrak{L}_\zeta^{(\mu)} = \mathfrak{L}_\zeta^{(\mu+1)} = \cdots . \tag{3.7}$$

Proof. By Proposition 4.1.4(ii) of [3] the root spaces for (3.1) coincide with the root spaces for a nonzero eigenvalue of a compact operator. The assertions thus follow from well-known properties of compact operators (see e.g. [1, Chapter I]). □

We call $Y^{(0)}(x), Y^{(1)}(x), \ldots, Y^{(\nu)}(x)$ an **eigenchain** for the system (3.1) for an eigenvalue ζ if

$$\frac{dY^{(0)}}{dx} = i\zeta JH(x)Y^{(0)},$$
$$\begin{bmatrix} I_m & 0 \end{bmatrix} Y^{(0)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y^{(0)}(\ell) = 0,$$

and for each $j = 1, \ldots, \nu$,

$$\frac{dY^{(j)}}{dx} = i\zeta JH(x)Y^{(j)} + JH(x)Y^{(j-1)},$$
$$\begin{bmatrix} I_m & 0 \end{bmatrix} Y^{(j)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y^{(j)}(\ell) = 0.$$

Every root function $Y(x)$ is the last member $Y(x) = Y^{(\nu)}(x)$ of some eigenchain. We use this fact to prove the following orthogonality relation, which generalizes Proposition 4.1.1 of [3].

Proposition 3.3. *For any complex* ζ_1 *and* ζ_2, *if* $Y \in \mathfrak{L}_{\zeta_1}$ *and* $Z \in \mathfrak{L}_{\zeta_2}$, *then*

$$i(\zeta_1 - \bar{\zeta}_2) \int_0^\ell Z(t)^* H(t) Y(t)\, dt = 0. \tag{3.8}$$

Hence if ζ *is a nonreal eigenvalue for* (3.1), *the root space* $\mathfrak{L}_\zeta$ *is a neutral subspace of* $L^2(Hdx)$.

A subspace of an indefinite inner product space is called **neutral** if the inner product of any two of its elements is zero.

Lemma 3.4. *Let* $M = \operatorname{ran} \begin{bmatrix} R \\ Q \end{bmatrix}$.

(1) *If* $h, k \in \mathbb{C}^m$*, the following are equivalent:*

(i) $R^*h + Q^*k = 0$; (ii) $\begin{bmatrix} k \\ h \end{bmatrix} \in M$; (iii) $\begin{bmatrix} h \\ k \end{bmatrix} \in M^\perp$.

(2) *If* A *and* B *are* $m \times m$ *matrices such that* $AR + BQ = 0$*, then* $\begin{bmatrix} A & B \end{bmatrix} M = \{0\}$.

Proof of Lemma 3.4. Since $R^*R + Q^*Q$ is invertible, M is the range of a one-to-one operator from $\mathbb{C}^m$ into $\mathbb{C}^{2m}$ and hence $\dim M = m$. Since $R^*Q + Q^*R = 0$, $JM \subseteq M^\perp$. By a dimension argument $JM = M^\perp$, and so $M = JM^\perp$. The assertions in (1) follow.

To prove (2), consider any $\xi \in M$ and $u \in \mathbb{C}^m$. Since $AR + BQ = 0$ by assumption, $R^*A^*u + Q^*B^*u = 0$. By part (1),

$$\begin{bmatrix} A^*u \\ B^*u \end{bmatrix} \in M^\perp.$$

Therefore $\begin{bmatrix} A^*u \\ B^*u \end{bmatrix}^* \xi = 0$. By the arbitrariness of u, $\begin{bmatrix} A & B \end{bmatrix} \xi = 0$. □

Proof of Proposition 3.3. The assertion is trivial if $\zeta_1 = \bar{\zeta}_2$, so assume that $\zeta_1 \neq \bar{\zeta}_2$. We must show that in this case,

$$\int_0^\ell Z(t)^* H(t) Y(t)\, dt = 0. \tag{3.9}$$

Let $Y^{(0)}(x), Y^{(1)}(x), \ldots, Y^{(\nu_1)}(x)$ and $Z^{(0)}(x), Z^{(1)}(x), \ldots, Z^{(\nu_2)}(x)$ be eigenchains with $Y^{(\nu_1)}(x) = Y(x)$ and $Z^{(\nu_2)}(x) = Z(x)$. Set

$$Y^{(-1)}(x) = Z^{(-1)}(x) = 0.$$

Then

$$\frac{dY^{(j+1)}}{dx} = i\zeta_1 JH(x) Y^{(j+1)} + JH(x) Y^{(j)},$$
$$\frac{dZ^{(k+1)}}{dx} = i\zeta_2 JH(x) Z^{(k+1)} + JH(x) Z^{(k)},$$

and

$$\begin{bmatrix} I_m & 0 \end{bmatrix} Y^{(j)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y^{(j)}(\ell) = 0,$$
$$\begin{bmatrix} I_m & 0 \end{bmatrix} Z^{(k)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Z^{(k)}(\ell) = 0,$$

for all $j = -1, 0, \ldots, \nu_1$ and $k = -1, 0, \ldots, \nu_2$. By the boundary conditions and Lemma 3.4(1),

$$Z^{(k)}(0)^* J Y^{(j)}(0) = Z^{(k)}(\ell)^* J Y^{(j)}(\ell) = 0$$

for all $j = -1, 0, \dots, \nu_1$ and $k = -1, 0, \dots, \nu_2$. Hence for the same values of j, k,

$$\int_0^\ell \Big[Z^{(k)}(t)^* J \frac{dY^{(j)}}{dt} + \frac{dZ^{(k)\,*}}{dt} JY^{(j)}(t) \Big] \, dt = Z^{(k)}(t)^* JY(t) \Big|_0^\ell = 0. \quad (3.10)$$

We show that

$$i(\zeta_1 - \bar\zeta_2) \Big\langle Y^{(j)}, Z^{(k)} \Big\rangle_H = -\Big\langle Y^{(j-1)}, Z^{(k)} \Big\rangle_H - \Big\langle Y^{(j)}, Z^{(k-1)} \Big\rangle_H, \quad (3.11)$$
$$j = 0, 1, \dots, \nu_1, \qquad k = 0, 1, \dots, \nu_2,$$

where $\langle \cdot, \cdot \rangle_H$ is the inner product of $L^2(H dx)$. In fact,

$$\begin{aligned} i(\zeta_1 - \bar\zeta_2) &\int_0^\ell Z^{(k)}(t)^* H(t) Y^{(j)}(t) \, dt \\ &= \int_0^\ell Z^{(k)}(t)^* J \big[i\zeta_1 JH(t) Y^{(j)}(t) \big] \, dt \\ &\quad + \int_0^\ell \big[i\zeta_2 JH(t) Z^{(k)}(t) \big]^* JY^{(j)}(t) \, dt \\ &= \int_0^\ell Z^{(k)}(t)^* J \Big[\frac{dY^{(j)}}{dt} - JH(t) Y^{(j-1)} \Big] \, dt \\ &\quad + \int_0^\ell \Big[\frac{dZ^{(k)}}{dt} - JH(t) Z^{(k-1)} \Big]^* JY^{(j)}(t) \, dt. \end{aligned}$$

By (3.10),

$$\begin{aligned} i(\zeta_1 - \bar\zeta_2) \int_0^\ell Z^{(k)}(t)^* H(t) Y^{(j)}(t) \, dt = &- \int_0^\ell Z^{(k)}(t)^* H(t) Y^{(j-1)}(t) \, dt \\ &- \int_0^\ell Z^{(k-1)}(t)^* H(t) Y^{(j)}(t) \, dt, \end{aligned}$$

which proves (3.11).

The proof is completed by repeated application of (3.11). Start by choosing $j = \nu_1$ and $k = \nu_2$ in (3.11). For each term on the right, multiply by $\zeta_1 - \bar\zeta_2$, and repeat. Eventually we reach $j = 0$ or $k = 0$ for each term, and then $Y^{(j-1)}(x) = Y^{(-1)}(x) = 0$ or $Z^{(j-1)}(x) = Z^{(-1)}(x) = 0$ accordingly. In the end, we arrive at (3.9), as was to be shown. □

We shall need explicit formulas for eigenchains. Such formulas can be derived from the Taylor expansions (2.3) and (2.4). Set

$$K(z) = c(z) R + d(z) Q \quad \text{and} \quad K_j(z) = c_j(z) R + d_j(z) Q, \quad j \geq 0. \quad (3.12)$$

Proposition 3.5. *The general form of an eigenchain $Y^{(0)}(x), Y^{(1)}(x), \dots, Y^{(\nu)}(x)$ for* (3.1) *for an eigenvalue ζ is*

$$\begin{aligned} Y^{(0)}(x) &= W_0(x,\zeta)\begin{bmatrix} 0 \\ g_0 \end{bmatrix}, \\ Y^{(1)}(x) &= (-i)W_1(x,\zeta)\begin{bmatrix} 0 \\ g_0 \end{bmatrix} + W_0(x,\zeta)\begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \\ Y^{(2)}(x) &= (-i)^2 W_2(x,\zeta)\begin{bmatrix} 0 \\ g_0 \end{bmatrix} + (-i)W_1(x,\zeta)\begin{bmatrix} 0 \\ g_1 \end{bmatrix} + W_0(x,\zeta)\begin{bmatrix} 0 \\ g_2 \end{bmatrix}, \\ &\dots \\ Y^{(\nu)}(x) &= (-i)^\nu W_\nu(x,\zeta)\begin{bmatrix} 0 \\ g_0 \end{bmatrix} + (-i)^{\nu-1}W_{\nu-1}(x,\zeta)\begin{bmatrix} 0 \\ g_1 \end{bmatrix} \\ &\qquad + \dots + W_0(x,\zeta)\begin{bmatrix} 0 \\ g_\nu \end{bmatrix}, \end{aligned} \tag{3.13}$$

where $g_0, g_1, \dots, g_\nu$ are vectors in $\mathbb{C}^m$ satisfying

$$\begin{aligned} K_0(\bar\zeta)^* g_0 &= 0, \\ (-i)K_1(\bar\zeta)^* g_0 + K_0(\bar\zeta)^* g_1 &= 0, \\ (-i)^2 K_2(\bar\zeta)^* g_0 + (-i)K_1(\bar\zeta)^* g_1 + K_0(\bar\zeta)^* g_2 &= 0, \\ &\dots \\ (-i)^\nu K_\nu(\bar\zeta)^* g_0 + (-i)^{\nu-1}K_{\nu-1}(\bar\zeta)^* g_1 + \dots + K_0(\bar\zeta)^* g_\nu &= 0. \end{aligned} \tag{3.14}$$

Proof. The case $\nu = 0$ follows from [3, Proposition 3.1.2]. We proceed by induction for the general case. Assume that the assertion is known up to the k-th stage for some $k \ge 0$. Consider an eigenchain $Y^{(0)}(x), \dots, Y^{(k)}(x)$, $Y^{(k+1)}(x)$. In particular,

$$\begin{gathered} \frac{dY^{(k+1)}}{dx} = i\zeta JH(x)Y^{(k+1)} + JH(x)Y^{(k)}, \\ \begin{bmatrix} I_m & 0 \end{bmatrix} Y^{(k+1)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y^{(k+1)}(\ell) = 0. \end{gathered} \tag{3.15}$$

By the inductive assumption, we can represent $Y^{(0)}(x), \dots, Y^{(k)}(x)$ in the form (3.13)–(3.14) with $\nu = k$. Set

$$\begin{aligned} \widetilde{Y}^{(k+1)}(x) = -i \Bigg\{ &(-i)^k W_{k+1}(x,\zeta)\begin{bmatrix} 0 \\ g_0 \end{bmatrix} + (-i)^{k-1}W_k(x,\zeta)\begin{bmatrix} 0 \\ g_1 \end{bmatrix} \\ &+ \dots + (-i)W_2(x,\zeta)\begin{bmatrix} 0 \\ g_{k-1} \end{bmatrix} + W_1(x,\zeta)\begin{bmatrix} 0 \\ g_k \end{bmatrix} \Bigg\}. \end{aligned}$$

By (2.6),

$$\begin{aligned}
\frac{d\widetilde{Y}^{(k+1)}}{dx} &= -i\Bigg\{(-i)^k\bigg(i\zeta JH(x)W_{k+1}(x,\zeta)\begin{bmatrix}0\\ g_0\end{bmatrix} + iJH(x)W_k(x,\zeta)\begin{bmatrix}0\\ g_0\end{bmatrix}\bigg)\\
&\qquad + (-i)^{k-1}\bigg(i\zeta JH(x)W_k(x,\zeta)\begin{bmatrix}0\\ g_1\end{bmatrix} + iJH(x)W_{k-1}(x,\zeta)\begin{bmatrix}0\\ g_1\end{bmatrix}\bigg)\\
&\qquad + \cdots\\
&\qquad + (-i)\bigg(i\zeta JH(x)W_2(x,\zeta)\begin{bmatrix}0\\ g_{k-1}\end{bmatrix} + iJH(x)W_1(x,\zeta)\begin{bmatrix}0\\ g_{k-1}\end{bmatrix}\bigg)\\
&\qquad + \bigg(i\zeta JH(x)W_1(x,\zeta)\begin{bmatrix}0\\ g_k\end{bmatrix} + iJH(x)W_0(x,\zeta)\begin{bmatrix}0\\ g_k\end{bmatrix}\bigg)\Bigg\}\\
&= i\zeta JH(x)(-i)\Bigg\{(-i)^k W_{k+1}(x,\zeta)\begin{bmatrix}0\\ g_0\end{bmatrix} + (-i)^{k-1}W_k(x,\zeta)\begin{bmatrix}0\\ g_1\end{bmatrix}\\
&\qquad + \cdots + (-i)W_2(x,\zeta)\begin{bmatrix}0\\ g_{k-1}\end{bmatrix} + W_1(x,\zeta)\begin{bmatrix}0\\ g_k\end{bmatrix}\Bigg\}\\
&\qquad + JH(x)\Bigg\{(-i)^k W_k(x,\zeta)\begin{bmatrix}0\\ g_0\end{bmatrix} + (-i)^{k-1}W_{k-1}(x,\zeta)\begin{bmatrix}0\\ g_1\end{bmatrix}\\
&\qquad + \cdots + (-i)W_1(x,\zeta)\begin{bmatrix}0\\ g_{k-1}\end{bmatrix} + W_0(x,\zeta)\begin{bmatrix}0\\ g_k\end{bmatrix}\Bigg\}.
\end{aligned}$$

Thus

$$\frac{d\widetilde{Y}^{(k+1)}}{dx} = i\zeta JH(x)\widetilde{Y}^{(k+1)} + JH(x)Y^{(k)}.$$

In view of (3.15), it follows that

$$\frac{d}{dx}(Y^{(k+1)} - \widetilde{Y}^{(k+1)}) = i\zeta JH(x)(Y^{(k+1)} - \widetilde{Y}^{(k+1)}).$$

By (2.5), $\begin{bmatrix}I_m & 0\end{bmatrix}(Y^{(k+1)}(0) - \widetilde{Y}^{(k+1)}(0)) = 0 - 0 = 0$. Therefore

$$Y^{(k+1)}(x) - \widetilde{Y}^{(k+1)}(x) = W_0(x,\zeta)\begin{bmatrix}0\\ g_{k+1}\end{bmatrix}$$

for some $g_{k+1} \in \mathbb{C}^m$. By the definition of $\widetilde{Y}^{(k+1)}(x)$,

$$\begin{aligned}
Y^{(k+1)}(x) &= (-i)^{k+1}W_{k+1}(x,\zeta)\begin{bmatrix}0\\ g_0\end{bmatrix} + (-i)^k W_k(x,\zeta)\begin{bmatrix}0\\ g_1\end{bmatrix}\\
&\qquad + \cdots + (-i)W_1(x,\zeta)\begin{bmatrix}0\\ g_k\end{bmatrix} + W_0(x,\zeta)\begin{bmatrix}0\\ g_{k+1}\end{bmatrix}.
\end{aligned}$$

The boundary condition $\begin{bmatrix}I_m & 0\end{bmatrix}Y^{(k+1)}(0) = 0$ imposes no condition on g_{k+1}. A restriction on g_{k+1} is imposed by the condition $\begin{bmatrix}R^* & Q^*\end{bmatrix}Y^{(k+1)}(\ell) = 0$. By the second equation in (2.5) and the identity $R^*c_j(\bar{\zeta})^* + Q^*d_j(\bar{\zeta})^* = K_j(\bar{\zeta})^*$, the restriction on g_{k+1} is that

$$(-i)^{k+1}K_{k+1}(\bar{\zeta})^*g_0 + (-i)^k K_k(\bar{\zeta})^*g_1 + \cdots + (-i)K_1(\bar{\zeta})^*g_k + K_0(\bar{\zeta})^*g_{k+1} = 0.$$

Thus the eigenchain $Y^{(0)}(x), \dots, Y^{(k)}(x), Y^{(k+1)}(x)$ has the required form. The steps are reversible, and the inductive step follows. □

We also need formulas for the eigentransforms (1.5) of an eigenchain. These are given in the next result in both explicit and recursive forms.

Proposition 3.6. *Let* $Y^{(0)}(x), Y^{(1)}(x), \dots, Y^{(\nu)}(x)$ *be an eigenchain for* (3.1) *of the form* (3.13)*, and let* $F^{(0)}(z), F^{(1)}(z), \dots, F^{(\nu)}(z)$ *be the corresponding eigentransforms.*

(1) *For each* $r = 0, \dots, \nu$,

$$\begin{aligned} F^{(r)}(z) &= \sum_{n=0}^{r} (-i)^n \, i \sum_{p+q=n} \frac{c(z)d_p(\bar{\zeta})^* + d(z)c_p(\bar{\zeta})^*}{(z-\zeta)^{q+1}} \, g_{r-n} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{r} (-i)^n \Delta_{nk}(\zeta)\, g_{r-n} \right) (z-\zeta)^k \,, \end{aligned} \tag{3.16}$$

where the coefficients in the last expression are as in Proposition 2.4.

(2) *The functions* $F^{(0)}(z), F^{(1)}(z), \dots, F^{(\nu)}(z)$ *in* (1) *are given recursively by*

$$F^{(0)}(z) = i\, \frac{c(z)d_0(\bar{\zeta})^* + d(z)c_0(\bar{\zeta})^*}{z-\zeta}\, g_0, \tag{3.17}$$

and

$$F^{(k)}(z) = \frac{F^{(k-1)}(z) - \begin{bmatrix} c(z) & d(z) \end{bmatrix} JY^{(k)}(\ell)}{i(z-\zeta)}\,, \qquad k = 1, \dots, \nu. \tag{3.18}$$

Moreover, for all $k = 0, \dots, \nu$,

$$\begin{aligned} F^{(k)}(z) = -\begin{bmatrix} c(z) & d(z) \end{bmatrix} \Bigg\{ & \frac{JY^{(k)}(\ell)}{i(z-\zeta)} + \frac{JY^{(k-1)}(\ell)}{i^2(z-\zeta)^2} \\ & + \cdots + \frac{JY^{(0)}(\ell)}{i^{k+1}(z-\zeta)^{k+1}} \Bigg\}. \end{aligned} \tag{3.19}$$

Notice that if we set $F^{(-1)}(z) \equiv 0$, then (3.18) agrees with (3.17) when $k = 0$.

Proof. (1) By (3.13),

$$Y^{(r)}(x) = \sum_{n=0}^{r} (-i)^n W_n(x,\zeta) \begin{bmatrix} 0 \\ g_{r-n} \end{bmatrix}.$$

Hence by (2.13),

$$\begin{aligned}
F^{(r)}(z) &= \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* H(t) Y^{(r)}(t)\, dt \\
&= \sum_{n=0}^{r} (-i)^n \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* H(t) W_n(t,\zeta) \begin{bmatrix} 0 \\ I_m \end{bmatrix} dt\, g_{r-n} \\
&= \sum_{n=0}^{r} (-i)^n\, i \sum_{p+q=n} \frac{c(z) d_p(\bar\zeta)^* + d(z) c_p(\bar\zeta)^*}{(z-\zeta)^{q+1}}\, g_{r-n} \\
&= \sum_{n=0}^{r} (-i)^n \sum_{k=0}^{\infty} \Delta_{nk}(\zeta) g_{r-n} (z-\zeta)^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{r} (-i)^n \Delta_{nk}(\zeta)\, g_{r-n} \right) (z-\zeta)^k .
\end{aligned}$$

The two equalities in (3.16) follow.

(2) Consider an eigenchain $Y^{(0)}(x), Y^{(1)}(x), \dots, Y^{(\nu)}(x)$ of the form (3.13), and let $F^{(0)}(z), F^{(1)}(z), \dots, F^{(\nu)}(z)$ be the corresponding eigentransforms. The identity (3.17) is a special case of (3.16). Suppose $k = 1, \dots, \nu$. Then

$$\frac{dY^{(k)}}{dx} = i\zeta J H(x) Y^{(k)} + J H(x) Y^{(k-1)},$$

$$\begin{bmatrix} I_m & 0 \end{bmatrix} Y^{(k)}(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y^{(k)}(\ell) = 0.$$

Thus

$$\begin{aligned}
\int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* J \frac{dY^{(k)}}{dt}\, dt &= \int_0^\ell i\zeta \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* H(t) Y^{(k)}(t)\, dt \\
&\quad + \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* H(t) Y^{(k-1)}(t)\, dt \\
&= i\zeta F^{(k)}(z) + F^{(k-1)}(z).
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
i\zeta F^{(k)}(z) + F^{(k-1)}(z) &= \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar z)^* J Y^{(k)}(t) \Big|_{t=0}^{\ell} \\
&\quad - \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} \Big(\frac{d}{dt} W(t,\bar z)^* \Big) J Y^{(k)}(t)\, dt \\
&= \begin{bmatrix} 0 & I_m \end{bmatrix} W(\ell,\bar z)^* J Y^{(k)}(\ell) \\
&\quad - \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} \Big(-iz W(t,\bar z)^* H(t) J \Big) J Y^{(k)}(t)\, dt \\
&= \begin{bmatrix} 0 & I_m \end{bmatrix} W(\ell,\bar z)^* J Y^{(k)}(\ell) + iz F^{(k)}(z).
\end{aligned}$$

Since $\begin{bmatrix} 0 & I_m \end{bmatrix} W(\ell,\bar z)^* = \begin{bmatrix} c(z) & d(z) \end{bmatrix}$ by (1.7), we obtain (3.18).

We prove (3.19) by iterating (3.17) and (3.18). By (3.17),

$$F^{(0)}(z) = -\big[c(z) \quad d(z)\big] J \begin{bmatrix} c_0(\bar\zeta)^* \\ d_0(\bar\zeta)^* \end{bmatrix} \frac{1}{i(z-\zeta)} = -\big[c(z) \quad d(z)\big] \frac{JY^{(0)}(\ell)}{i(z-\zeta)},$$

which is the case $k=0$ of (3.19). By (3.18) with $k=1$,

$$\begin{aligned} F^{(1)}(z) &= \Big\{ F^{(0)}(z) - \big[c(z) \quad d(z)\big] JY^{(1)}(\ell) \Big\} \frac{1}{i(z-\zeta)} \\ &= \Big\{ -\big[c(z) \quad d(z)\big] \frac{JY^{(0)}(\ell)}{i(z-\zeta)} - \big[c(z) \quad d(z)\big] JY^{(1)}(\ell) \Big\} \frac{1}{i(z-\zeta)} \\ &= -\big[c(z) \quad d(z)\big] \Big\{ \frac{JY^{(1)}(\ell)}{i(z-\zeta)} + \frac{JY^{(0)}(\ell)}{i^2(z-\zeta)^2} \Big\}, \end{aligned}$$

proving (3.19) for $k=1$. The general case follows by a straightforward induction. □

4. Main results

We assume given a system (3.1) satisfying (1.2)–(1.4), with operators R and Q satisfying (1°) and (2°). Let $W(x,z)$ be the unique solution of (1.6). As before, we set

$$v(z) = i[a(z)R + b(z)Q][c(z)R + d(z)Q]^{-1}, \tag{4.1}$$

where

$$W(\ell,\bar z)^* = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}. \tag{4.2}$$

Recall that $v(z) = v(\bar z)^*$, and the only singularities of $v(z)$ are poles, which occur at the points where $c(z)R + d(z)Q$ is not invertible (see Proposition 3.1).

By Proposition 3.1, the eigenvalues of (3.1) coincide with the poles of $v(z)$. In Definition 4.2 we use the poles of $v(z)$ to introduce an inner product space $\mathfrak{H}_0(v)$ whose elements are m-dimensional vector-valued entire functions. Our main result, Theorem 4.7, asserts that the eigentransform (1.5) acts an isometry on the span of root functions in $L^2(Hdx)$ to $\mathfrak{H}_0(v)$.

For each $w \in \mathbb{C}$, write

$$v(z) = -\frac{\gamma_\varkappa(w)}{(z-w)^\varkappa} - \cdots - \frac{\gamma_1(w)}{z-w} + \tilde v(z), \tag{4.3}$$

where $\tilde v(z)$ is analytic at $z=w$. Here $\varkappa = \varkappa_w \geq 1$ is chosen large enough that such a representation exists. The value of $\varkappa$ is not important, and zero coefficients can be added at will. Such a representation is nontrivial only for poles, but it is notationally convenient to also allow w to be a point of analyticity for $v(z)$, in which case all coefficients are zero. Since $v(z) = v(\bar{z})^*$,

$$v(z) = -\frac{\gamma_\varkappa(w)^*}{(z-\bar{w})^\varkappa} - \cdots - \frac{\gamma_1(w)^*}{z-\bar{w}} + \tilde{v}(\bar{z})^*, \tag{4.4}$$

where $\tilde{v}(\bar{z})^*$ is analytic at $z = \bar{w}$. Hence $\gamma_j(\bar{w}) = \gamma_j(w)^*$, $j = 1, \dots, \varkappa$.

Proposition 4.1. *Let ζ be an eigenvalue for* (3.1), *and write $v(z)$ as in* (4.3) *for $w = \zeta$. Let $u \in \mathbb{C}^m$ be any vector, and define $Y^{(0)}(x), Y^{(1)}(x), \dots, Y^{(\varkappa-1)}(x)$ by* (3.13) *with $\nu = \varkappa - 1$ and*

$$g_j = (-i)^j \gamma_{\varkappa-j}(\zeta) u, \qquad j = 0, \dots, \varkappa - 1. \tag{4.5}$$

Then $Y^{(0)}(x), Y^{(1)}(x), \dots, Y^{(\varkappa-1)}(x)$ is an eigenchain for (3.1).

Proof. We must show that $g_0, \dots, g_{\varkappa-1}$ satisfy (3.14). By (4.1) and (3.12),

$$i[a(z)R + b(z)Q] = v(z)K(z) = v(z)\sum_{j=0}^{\infty} K_j(w)(z-w)^j.$$

Hence by (4.3),

$$i[a(z)R+b(z)Q] = -\left[\frac{\gamma_\varkappa(w)}{(z-w)^\varkappa} + \cdots + \frac{\gamma_1(w)}{z-w}\right]\sum_{j=0}^{\infty} K_j(w)(z-w)^j + \tilde{v}(z)K(z).$$

Since the left side is analytic at w, we deduce $\varkappa$ relations by expanding the first term on the right side and equating coefficients of negative powers of $z - w$ to zero:

$$\begin{aligned} \gamma_\varkappa(w)K_0(w) &= 0, \\ \gamma_\varkappa(w)K_1(w) + \gamma_{\varkappa-1}(w)K_0(w) &= 0, \\ \gamma_\varkappa(w)K_2(w) + \gamma_{\varkappa-1}(w)K_1(w) + \gamma_{\varkappa-2}(w)K_0(w) &= 0, \\ &\cdots \\ \gamma_\varkappa(w)K_{\varkappa-1}(w) + \gamma_{\varkappa-1}(w)K_{\varkappa-2}(w) + \cdots + \gamma_1(w)K_0(w) &= 0. \end{aligned} \tag{4.6}$$

On replacing w by $\bar{w}$ and taking adjoints, we deduce (3.14). □

We introduce an inner product space that will be used in Theorem 4.7 to describe the action of the eigentransform (1.5) on root functions.

Definition 4.2. Let $\mathfrak{H}_0(v)$ be the set of entire functions $F(z)$ with values in $\mathbb{C}^m$ such that $v(z)F(z)$ has finitely many poles. For $F, G \in \mathfrak{H}_0(v)$ and $w \in \mathbb{C}$, set

$$\langle F, G\rangle_w = -\frac{1}{2\pi i}\int_{\Gamma_w} G(\bar{z})^* v(z) F(z)\, dz, \tag{4.7}$$

where Γ_w is a counterclockwise circle about w, chosen small enough that $v(z)$ is analytic on Γ_w and its interior except perhaps at $z = w$. Set

$$\langle F, G\rangle = \sum_{w\in\mathbb{C}} \langle F, G\rangle_w. \tag{4.8}$$

We identify entire functions F and G in $\mathfrak{H}_0(v)$ such that $v(z)[F(z)-G(z)]$ is entire (or, more precisely, has only removable singularities).

The integral in (4.7) is independent of the choice of Γ_w. All but finitely many terms of the sum in (4.8) are finite, and hence $\langle F, G\rangle$ is well defined.

Lemma 4.3. *Let $F, G \in \mathfrak{H}_0(v)$, $w \in \mathbb{C}$, and let*

$$F(z) = \sum_{p=0}^{\infty} F_p(w)(z-w)^p \quad \text{and} \quad G(z) = \sum_{q=0}^{\infty} G_q(\bar{w})(z-\bar{w})^q \tag{4.9}$$

be Taylor expansions about w and $\bar{w}$, respectively. If $v(z)$ is given by (4.3)*, then*

$$\langle F, G\rangle_w = \sum_{j=1}^{\varkappa} \sum_{p+q=j-1} G_q(\bar{w})^* \gamma_j(w) F_p(w), \tag{4.10}$$

or, equivalently,

$$\langle F, G\rangle_w = \begin{bmatrix} G_0(\bar{w}) \\ G_1(\bar{w}) \\ \vdots \\ G_{\varkappa-1}(\bar{w}) \end{bmatrix}^* \begin{bmatrix} \gamma_1(w) & \gamma_2(w) & \cdots & \gamma_{\varkappa-1}(w) & \gamma_\varkappa(w) \\ \gamma_2(w) & \gamma_3(w) & \cdots & \gamma_\varkappa(w) & 0 \\ & & \cdots & & \\ \gamma_\varkappa(w) & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} F_0(w) \\ F_1(w) \\ \vdots \\ F_{\varkappa-1}(w) \end{bmatrix}.$$

Proof. By (4.3) and (4.9),

$$\begin{aligned} -G(\bar{z})^* v(z) F(z) &= \sum_{q=0}^{\infty} G_q(\bar{w})^* (z-w)^q \sum_{j=1}^{\varkappa} \frac{\gamma_j(w)}{(z-w)^j} \sum_{p=0}^{\infty} F_p(w)(z-w)^p + \varphi(z) \\ &= \sum_{j=1}^{\varkappa} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} G_q(\bar{w})^* \gamma_j(w) F_p(w)(z-w)^{p+q-j} + \varphi(z), \end{aligned}$$

where $\varphi(z)$ is analytic at $z = w$. Only the terms with $p+q-j = -1$ make a contribution to the integral (4.7). Therefore

$$\langle F, G\rangle_w = -\frac{1}{2\pi i} \int_{\Gamma_w} G(\bar{z})^* v(z) F(z)\, dz = \sum_{j=1}^{\varkappa} \sum_{p+q-j=-1} G_q(\bar{w})^* \gamma_j(w) F_p(w),$$

which is equivalent to (4.10). □

Proposition 4.4. *The inner product* (4.8) *is linear and symmetric.*

Proof. Linearity in the first variable is clear from (4.10). Fix $F, G \in \mathfrak{H}_0(v)$. For any $w \in \mathbb{C}$, $\gamma_j(w)^* = \gamma_j(\bar{w})$, $j = 1, \dots, \varkappa$. Hence by (4.9) and (4.10),

$$\langle G, F\rangle_{\bar{w}} = \sum_{j=1}^{\varkappa} \sum_{p+q=j-1} F_p(w)^* \gamma_j(\bar{w}) G_q(\bar{w}) = \overline{\langle F, G\rangle}_w \, .$$

Therefore

$$\langle G, F\rangle = \sum_{w\in\mathbb{C}} \langle G, F\rangle_w = \sum_{w\in\mathbb{C}} \langle G, F\rangle_{\overline{w}} = \sum_{w\in\mathbb{C}} \overline{\langle F, G\rangle_w} = \overline{\langle F, G\rangle}.$$

This proves symmetry. □

We come now to a critical property of eigentransforms of root functions. In the case of simple poles, Lemma 4.1.10 in [3] provides what is needed. The next result is a generalization to arbitrary poles, which is stated in different language but essentially accomplishes the same thing.

Proposition 4.5. (1) *If* $Y(x) \in \mathfrak{L}_\zeta^{(\mu-1)}$ *for some* $\mu \geq 1$ *and* $F = VY$, *then* $v(z)F(z)$ *is analytic in the complex plane except perhaps for a pole at* ζ *of order at most* μ.

(2) *If* $F = Vf$ *where* f *is a finite linear combination of root functions of* (3.1), *then* $F \in \mathfrak{H}_0(v)$.

Proof. (1) By (3.19),

$$F(z) = -\big[c(z) \quad d(z)\big]\left\{ \frac{JY^{(\mu-1)}(\ell)}{i(z-\zeta)} + \frac{JY^{(\mu-2)}(\ell)}{i^2(z-\zeta)^2} + \cdots + \frac{JY^{(0)}(\ell)}{i^\mu(z-\zeta)^\mu} \right\}.$$

Here the boundary conditions $\big[R^* \quad Q^*\big]Y^{(j)}(\ell) = 0$ together with Lemma 3.4 imply that

$$JY^{(j)}(\ell) \in M = \operatorname{ran}\begin{bmatrix} R \\ Q \end{bmatrix}, \qquad j = 0, \dots, \mu - 1.$$

It follows that

$$F(z) = \big[c(z) \quad d(z)\big]\begin{bmatrix} R \\ Q \end{bmatrix}\varphi(z), \tag{4.11}$$

where

$$\varphi(z) = \frac{\varphi_1}{z-\zeta} + \frac{\varphi_2}{(z-\zeta)^2} + \cdots + \frac{\varphi_\mu}{(z-\zeta)^\mu} \tag{4.12}$$

for some vectors $\varphi_1, \varphi_2, \dots, \varphi_\mu$ in $\mathbb{C}^m$.

By (4.1), $v(z)[c(z)R + d(z)Q] = i[a(z)R + b(z)Q]$. Hence

$$v(z)\big[c(z) \quad d(z)\big]\begin{bmatrix} R \\ Q \end{bmatrix} = i[a(z)R + b(z)Q]. \tag{4.13}$$

By (4.11) and (4.13),

$$v(z)F(z) = v(z)\big[c(z) \quad d(z)\big]\begin{bmatrix} R \\ Q \end{bmatrix}\varphi(z) = i[a(z)R + b(z)Q]\varphi(z). \tag{4.14}$$

Since $a(z)$ and $b(z)$ are entire functions, (4.14) and (4.12) show that $v(z)F(z)$ is analytic in $\mathbb{C}$ except perhaps for a pole at ζ of order at most μ.

(2) This is immediate from (1). □

Corollary 4.6. *Suppose* $Y(x) \in \mathfrak{L}_\zeta$ *and*

$$F(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar{z})^* H(t) Y(t)\, dt.$$

Then for any $G(z)$ *in* $\mathfrak{H}_0(v)$, $\langle F, G \rangle = \langle F, G \rangle_\zeta$.

Proof. By (4.8), the problem is to show that $\langle F, G \rangle_w = 0$ for all $w \neq \zeta$. Fix $w \neq \zeta$. Write $F(z) = \sum_{p=0}^\infty F_p(w)(z-w)^p$. Let $v(z)$ be given by (4.3). By Proposition 4.5(1), $v(z)F(z)$ is analytic at $z = w$, and so

$$\begin{aligned} \gamma_\varkappa(w)F_0(w) &= 0 \\ \gamma_{\varkappa-1}(w)F_0(w) + \gamma_\varkappa(w)F_1(w) &= 0 \\ &\cdots \\ \gamma_1(w)F_0(w) + \gamma_2(w)F_1(w) + \cdots + \gamma_\varkappa(w)F_{\varkappa-1}(w) &= 0. \end{aligned}$$

These relations say that

$$\begin{bmatrix} \gamma_1(w) & \gamma_2(w) & \cdots & \gamma_{\varkappa-1}(w) & \gamma_\varkappa(w) \\ \gamma_2(w) & \gamma_3(w) & \cdots & \gamma_\varkappa(w) & 0 \\ & & \cdots & & \\ \gamma_\varkappa(w) & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} F_0(w) \\ F_1(w) \\ \vdots \\ F_{\varkappa-1}(w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and hence $\langle F, G \rangle_w = 0$ by Lemma 4.3. □

We are now ready to state and prove our main result, which generalizes Theorem 4.1.11 of [3].

Theorem 4.7. (1) *Let* $Y(x)$ *and* $Z(x)$ *be finite linear combinations of root functions for the system* (3.1), *and let* $F(z), G(z)$ *be their eigentransforms. Then*

$$\int_0^\ell Z(t)^* H(t) Y(t)\, dt = \langle F, G \rangle . \tag{4.15}$$

(2) *Suppose* $f \in L^2(Hdx)$, *and let* F *be its eigentransform. If* f *is orthogonal to every root function of* (3.1), *then* $F = 0$ *as an element of* $\mathfrak{H}_0(v)$.

The definite case is treated in [3], and in this case more can be said. In the definite case, $v(z)$ is a Nevanlinna function which is meromorphic on the complex plane. Its poles are real and simple and coincide with the eigenvalues $\lambda_1, \lambda_2, \ldots$ of (3.1). In the Nevanlinna representation

$$v(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\tau(t),$$

the nondecreasing function $\tau(t)$ is constant on the intervals between poles, and the jump in $\tau(t)$ at a pole λ_j is

$$\tau_j = - \operatorname*{Res}_{z=\lambda_j} v(z) = \gamma_1(\lambda_j).$$

In this case, the inner product (4.8) on $\mathfrak{H}_0(v)$ is the inner product of $L^2(d\tau)$, and Theorem 4.7 is subsumed in the more precise Theorem 4.2.2 of [3]. In the terminology of Definition 4.2.3 of [3], $\tau(t)$ is a pseudospectral function for (1.1).

In general, the inner product $\langle F,G\rangle$ in (4.15) depends on the collection of poles w of $v(z)$ and coefficients $\gamma_1(w),\gamma_2(w),\dots$ in (4.3). Because of (4.15), we call this collection **pseudospectral data** for (1.1).

Proof of Theorem 4.7*, Part* (1)*.* By linearity and symmetry, it is sufficient to prove (4.15) when $Y(x)$ and $Z(x)$ are root functions, say $Y(x)\in\mathfrak{L}_{\zeta_1}$ and $Z(x)\in\mathfrak{L}_{\zeta_2}$.

Case 1: $\zeta_1\neq\bar\zeta_2$

In this case, the left side of (4.15) is zero by Proposition 3.3. By Corollary 4.6,

$$\langle F,G\rangle=\langle F,G\rangle_{\zeta_1}.$$

Since $\bar\zeta_1\neq\zeta_2$, $v(z)G(z)$ is analytic at $z=\bar\zeta_1$ by Proposition 4.5(1). Therefore

$$\begin{aligned}\gamma_\varkappa(\bar\zeta_1)G_0(\bar\zeta_1)&=0\\ \gamma_{\varkappa-1}(\bar\zeta_1)G_0(\bar\zeta_1)+\gamma_\varkappa(\bar\zeta_1)G_1(\bar\zeta_1)&=0\\ &\cdots\\ \gamma_1(\bar\zeta_1)G_0(\bar\zeta_1)+\gamma_2(\bar\zeta_1)G_1(\bar\zeta_1)+\cdots+\gamma_\varkappa(\bar\zeta_1)G_{\varkappa-1}(\bar\zeta_1)&=0.\end{aligned}$$

Since $\gamma_j(\zeta_1)=\gamma_j(\bar\zeta_1)^*$, $j=1,\dots,\varkappa$,

$$\begin{bmatrix}G_0(\bar\zeta_1)\\ G_1(\bar\zeta_1)\\ \vdots\\ G_{\varkappa-1}(\bar\zeta_1)\end{bmatrix}^*\begin{bmatrix}\gamma_1(\zeta_1)&\gamma_2(\zeta_1)&\cdots&\gamma_{\varkappa-1}(\zeta_1)&\gamma_\varkappa(\zeta_1)\\ \gamma_2(\zeta_1)&\gamma_3(\zeta_1)&\cdots&\gamma_\varkappa(\zeta_1)&0\\ &&\cdots&&\\ \gamma_\varkappa(\zeta_1)&0&\cdots&0&0\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}^*.$$

Hence $\langle F,G\rangle_{\zeta_1}=0$ by Lemma 4.3.

Case 2: $\zeta_1=\bar\zeta_2$

Put $\zeta_1=\zeta$ and $\zeta_2=\bar\zeta$. As a first step we derive the formula (4.18) for the left side of (4.15). Suppose

$$v(z)=-\frac{\gamma_\varkappa(w)}{(z-w)^\varkappa}-\cdots-\frac{\gamma_1(w)}{z-w}+\tilde v(z),\tag{4.16}$$

By adding zero terms in (4.16), we can choose $\varkappa$ as large as we wish. Hence in view of the inclusions (3.7), we can assume without loss of generality that $Y(x)$ and $Z(x)$ are root functions of the same order $\varkappa$, that is, they belong to eigenchains

$$\begin{aligned}Y^{(0)}(x),Y^{(1)}(x),\dots,Y^{(\varkappa-1)}(x)&=Y(x),\\ Z^{(0)}(x),Z^{(1)}(x),\dots,Z^{(\varkappa-1)}(x)&=Z(x).\end{aligned}$$

Thus

$$Y(x) = (-i)^{\varkappa-1} W_{\varkappa-1}(x,\zeta) \begin{bmatrix} 0 \\ g_0 \end{bmatrix} + (-i)^{\varkappa-2} W_{\varkappa-2}(x,\zeta) \begin{bmatrix} 0 \\ g_1 \end{bmatrix} + \cdots + (-i) W_1(x,\zeta) \begin{bmatrix} 0 \\ g_{\varkappa-2} \end{bmatrix} + W_0(x,\zeta) \begin{bmatrix} 0 \\ g_{\varkappa-1} \end{bmatrix}$$

and

$$Z(x) = (-i)^{\varkappa-1} W_{\varkappa-1}(x,\bar\zeta) \begin{bmatrix} 0 \\ h_0 \end{bmatrix} + (-i)^{\varkappa-2} W_{\varkappa-2}(x,\bar\zeta) \begin{bmatrix} 0 \\ h_1 \end{bmatrix} + \cdots + (-i) W_1(x,\bar\zeta) \begin{bmatrix} 0 \\ h_{\varkappa-2} \end{bmatrix} + W_0(x,\bar\zeta) \begin{bmatrix} 0 \\ h_{\varkappa-1} \end{bmatrix},$$

where the conditions on $g_0, \dots, g_{\varkappa-1}$ and $h_0, \dots, h_{\varkappa-1}$ in Proposition 3.5 are met. In the former case, by (3.14) these conditions can be written:

$$\begin{bmatrix} R^* & Q^* \end{bmatrix} \begin{bmatrix} c_0(\bar\zeta)^* \\ d_0(\bar\zeta)^* \end{bmatrix} g_0 = 0,$$

$$\begin{bmatrix} R^* & Q^* \end{bmatrix} \left\{ (-i) \begin{bmatrix} c_1(\bar\zeta)^* \\ d_1(\bar\zeta)^* \end{bmatrix} g_0 + \begin{bmatrix} c_0(\bar\zeta)^* \\ d_0(\bar\zeta)^* \end{bmatrix} g_1 \right\} = 0,$$

$$\begin{bmatrix} R^* & Q^* \end{bmatrix} \left\{ (-i)^2 \begin{bmatrix} c_2(\bar\zeta)^* \\ d_2(\bar\zeta)^* \end{bmatrix} g_0 + (-i) \begin{bmatrix} c_1(\bar\zeta)^* \\ d_1(\bar\zeta)^* \end{bmatrix} g_1 + \begin{bmatrix} c_0(\bar\zeta)^* \\ d_0(\bar\zeta)^* \end{bmatrix} g_2 \right\} = 0,$$

$$\cdots$$

$$\begin{bmatrix} R^* & Q^* \end{bmatrix} \left\{ (-i)^{\varkappa-1} \begin{bmatrix} c_{\varkappa-1}(\bar\zeta)^* \\ d_{\varkappa-1}(\bar\zeta)^* \end{bmatrix} g_0 + (-i)^{\varkappa-2} \begin{bmatrix} c_{\varkappa-2}(\bar\zeta)^* \\ d_{\varkappa-2}(\bar\zeta)^* \end{bmatrix} g_1 + \cdots + (-i) \begin{bmatrix} c_1(\bar\zeta)^* \\ d_1(\bar\zeta)^* \end{bmatrix} g_{\varkappa-2} + \begin{bmatrix} c_0(\bar\zeta)^* \\ d_0(\bar\zeta)^* \end{bmatrix} g_{\varkappa-1} \right\} = 0. \tag{4.17}$$

By (2.15),

$$\int_0^\ell Z(t)^* H(t) Y(t)\, dt = \int_0^\ell \sum_{q=0}^{\varkappa-1} i^q h_{\varkappa-1-q}^* \begin{bmatrix} 0 & I_m \end{bmatrix} W_q(t,\bar\zeta)^* H(t) \cdot \cdot \sum_{p=0}^{\varkappa-1} (-i)^p W_p(t,\zeta) \begin{bmatrix} 0 \\ I_m \end{bmatrix} g_{\varkappa-1-p}\, dt$$

$$= \sum_{p,q=0}^{\varkappa-1} i^q (-i)^p h_{\varkappa-1-q}^* \Delta_{pq}(\zeta) g_{\varkappa-1-p}\,. \tag{4.18}$$

We next derive the formula (4.26) for the right side of (4.15). By Corollary 4.6 and Lemma 4.3,

$$\begin{aligned}
\langle F,G\rangle &= \langle F,G\rangle_\zeta \\
&= \sum_{j=1}^{\varkappa}\sum_{p+q=j-1} G_q(\bar\zeta)^*\gamma_j(\zeta)F_p(\zeta) \\
&= G_0(\bar\zeta)^*\gamma_1(\zeta)F_0(\zeta) \\
&\quad + G_1(\bar\zeta)^*\gamma_2(\zeta)F_0(\zeta) + G_0(\bar\zeta)^*\gamma_2(\zeta)F_1(\zeta) \\
&\quad + \cdots \\
&\quad + G_{\varkappa-1}(\bar\zeta)^*\gamma_\varkappa(\zeta)F_0(\zeta) + G_{\varkappa-2}(\bar\zeta)^*\gamma_\varkappa(\zeta)F_1(\zeta) \\
&\qquad\qquad + \cdots + G_0(\bar\zeta)^*\gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta) \\
&= G_0(\bar\zeta)^*\Big[\gamma_1(\zeta)F_0(\zeta) + \gamma_2(\zeta)F_1(\zeta) + \cdots + \gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta)\Big] \\
&\quad + G_1(\bar\zeta)^*\Big[\gamma_2(\zeta)F_0(\zeta) + \gamma_3(\zeta)F_1(\zeta) + \cdots + \gamma_\varkappa(\zeta)F_{\varkappa-2}(\zeta)\Big] \\
&\quad + \cdots \\
&\quad + G_{\varkappa-2}(\bar\zeta)^*\Big[\gamma_{\varkappa-1}(\zeta)F_0(\zeta) + \gamma_\varkappa(\zeta)F_1(\zeta)\Big] \\
&\quad + G_{\varkappa-1}(\bar\zeta)^*\gamma_\varkappa(\zeta)F_0(\zeta)\,,
\end{aligned} \tag{4.19}$$

where

$$F(z) = \sum_{p=0}^{\infty} F_p(w)(z-w)^p \quad\text{and}\quad G(z) = \sum_{q=0}^{\infty} G_q(\bar w)(z-\bar w)^q.$$

By (3.16),

$$F(z) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\varkappa-1}(-i)^j\Delta_{jk}(\zeta)\,g_{\varkappa-1-j}\right)(z-\zeta)^k\,,$$

$$G(z) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\varkappa-1}(-i)^j\Delta_{jk}(\bar\zeta)\,g_{\varkappa-1-j}\right)(z-\bar\zeta)^k\,,$$

and so

$$F_k(\zeta) = \sum_{j=0}^{\varkappa-1}(-i)^j\Delta_{jk}(\zeta)\,g_{\varkappa-1-j}\,, \tag{4.20}$$

$$G_k(\bar\zeta) = \sum_{j=0}^{\varkappa-1}(-i)^j\Delta_{jk}(\bar\zeta)\,h_{\varkappa-1-j}\,, \tag{4.21}$$

for all $k = 0, 1, \ldots, \varkappa-1$.

Claim: *For every* $\xi \in \operatorname{ran} \begin{bmatrix} R \\ Q \end{bmatrix}$,

$$v(z)\begin{bmatrix} c(z) & d(z) \end{bmatrix}\xi = i\begin{bmatrix} a(z) & b(z) \end{bmatrix}\xi. \tag{4.22}$$

The claim follows on writing (4.1) in the form

$$v(z)\begin{bmatrix} c(z) & d(z) \end{bmatrix}\begin{bmatrix} R \\ G \end{bmatrix} = i\begin{bmatrix} a(z) & b(z) \end{bmatrix}\begin{bmatrix} R \\ G \end{bmatrix}.$$

Now by (3.19),

$$F(z) = -\begin{bmatrix} c(z) & d(z) \end{bmatrix}\left\{(-i)\frac{JY^{(\varkappa-1)}(\ell)}{z-\zeta} + (-i)^2\frac{JY^{(\varkappa-2)}(\ell)}{(z-\zeta)^2} + \cdots + (-i)^{\varkappa}\frac{JY^{(0)}(\ell)}{(z-\zeta)^{\varkappa}}\right\}. \tag{4.23}$$

By (4.17) and Lemma 3.4(1),

$$JY^{(0)}(\ell) = \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_0 \in \operatorname{ran}\begin{bmatrix} R \\ Q \end{bmatrix},$$

$$JY^{(1)}(\ell) = (-i)\begin{bmatrix} d_1(\bar\zeta)^* \\ c_1(\bar\zeta)^* \end{bmatrix} g_0 + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_1 \in \operatorname{ran}\begin{bmatrix} R \\ Q \end{bmatrix},$$

$$JY^{(2)}(\ell) = (-i)^2\begin{bmatrix} d_2(\bar\zeta)^* \\ c_2(\bar\zeta)^* \end{bmatrix} g_0 + (-i)\begin{bmatrix} d_1(\bar\zeta)^* \\ c_1(\bar\zeta)^* \end{bmatrix} g_1 + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_2 \in \operatorname{ran}\begin{bmatrix} R \\ Q \end{bmatrix},$$

$$\cdots$$

$$JY^{(\varkappa-1)}(\ell) = (-i)^{\varkappa-1}\begin{bmatrix} d_{\varkappa-1}(\bar\zeta)^* \\ c_{\varkappa-1}(\bar\zeta)^* \end{bmatrix} g_0 + (-i)^{\varkappa-2}\begin{bmatrix} d_{\varkappa-2}(\bar\zeta)^* \\ c_{\varkappa-2}(\bar\zeta)^* \end{bmatrix} g_1 + \cdots + (-i)\begin{bmatrix} d_1(\bar\zeta)^* \\ c_1(\bar\zeta)^* \end{bmatrix} g_{\varkappa-2} + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_{\varkappa-1} \in \operatorname{ran}\begin{bmatrix} R \\ Q \end{bmatrix}.$$

Therefore by (4.23) and the claim,

$$\begin{aligned} v(z)F(z) &= -v(z)\begin{bmatrix} c(z) & d(z) \end{bmatrix}\left\{(-i)\frac{JY^{(\varkappa-1)}(\ell)}{z-\zeta} + (-i)^2\frac{JY^{(\varkappa-2)}(\ell)}{(z-\zeta)^2} + \cdots + (-i)^{\varkappa}\frac{JY^{(0)}(\ell)}{(z-\zeta)^{\varkappa}}\right\} \\ &= -i\begin{bmatrix} a(z) & b(z) \end{bmatrix}\left\{(-i)\frac{JY^{(\varkappa-1)}(\ell)}{z-\zeta} + (-i)^2\frac{JY^{(\varkappa-2)}(\ell)}{(z-\zeta)^2} + \cdots + (-i)^{\varkappa}\frac{JY^{(0)}(\ell)}{(z-\zeta)^{\varkappa}}\right\}. \end{aligned} \tag{4.24}$$

For the left side of (4.24), the series expansions of $v(z)$ and $F(z)$ yield

$$v(z)F(z) = -\frac{\gamma_\varkappa(\zeta)F_0(\zeta)}{(z-\zeta)^\varkappa} - \frac{\gamma_{\varkappa-1}(\zeta)F_0(\zeta)+\gamma_\varkappa(\zeta)F_1(\zeta)}{(z-\zeta)^{\varkappa-1}} - \cdots - \frac{\gamma_1(\zeta)F_0(\zeta)+\gamma_2(\zeta)F_1(\zeta)+\cdots+\gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta)}{z-\zeta} + \text{holomorphic part}\,. \tag{4.25}$$

The numerators here are key to calculating (4.19). We next show that these numerators are very simple expressions. In fact, by (4.24) and (4.25),

$$\begin{aligned}
&-\frac{\gamma_\varkappa(\zeta)F_0(\zeta)}{(z-\zeta)^\varkappa} - \frac{\gamma_{\varkappa-1}(\zeta)F_0(\zeta)+\gamma_\varkappa(\zeta)F_1(\zeta)}{(z-\zeta)^{\varkappa-1}} \\
&\qquad - \cdots - \frac{\gamma_1(\zeta)F_0(\zeta)+\gamma_2(\zeta)F_1(\zeta)+\cdots+\gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta)}{z-\zeta} \\
&\qquad + \text{holomorphic part} \\
&= -i\big[a(z) \quad b(z)\big]\Big\{(-i)\frac{JY^{(\varkappa-1)}(\ell)}{z-\zeta} + (-i)^2\frac{JY^{(\varkappa-2)}(\ell)}{(z-\zeta)^2} + \cdots + (-i)^\varkappa\frac{JY^{(0)}(\ell)}{(z-\zeta)^\varkappa}\Big\} \\
&= -\Big\{\big[a_0(\zeta) \quad b_0(\zeta)\big] + \big[a_1(\zeta) \quad b_1(\zeta)\big](z-\zeta) + \big[a_2(\zeta) \quad b_2(\zeta)\big](z-\zeta)^2 + \cdots\Big\} \cdot \\
&\qquad \cdot\Big\{\frac{JY^{(\varkappa-1)}(\ell)}{z-\zeta} + (-i)\frac{JY^{(\varkappa-2)}(\ell)}{(z-\zeta)^2} + \cdots + (-i)^{\varkappa-1}\frac{JY^{(0)}(\ell)}{(z-\zeta)^\varkappa}\Big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\gamma_\varkappa(\zeta)F_0(\zeta) &= \big[a_0(\zeta) \quad b_0(\zeta)\big](-i)^{\varkappa-1}JY^{(0)}(\ell) \\
&= (-i)^{\varkappa-1}\big[a_0(\zeta) \quad b_0(\zeta)\big]\begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_0 \\
&= (-i)^{\varkappa-1} g_0\,,
\end{aligned}$$

the last equality holding by (2.10). By (2.10) and (2.11),

$$\begin{aligned}
&\gamma_{\varkappa-1}(\zeta)F_0(\zeta)+\gamma_\varkappa(\zeta)F_1(\zeta) \\
&\quad = \big[a_1(\zeta) \quad b_1(\zeta)\big](-i)^{\varkappa-1}JY^{(0)}(\ell) + \big[a_0(\zeta) \quad b_0(\zeta)\big](-i)^{\varkappa-2}JY^{(1)}(\ell) \\
&\quad = (-i)^{\varkappa-1}\big[a_1(\zeta) \quad b_1(\zeta)\big]\begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^* \end{bmatrix} g_0
\end{aligned}$$

$$
\begin{aligned}
&\quad + (-i)^{\varkappa-2}\begin{bmatrix} a_0(\zeta) & b_0(\zeta)\end{bmatrix}\left\{(-i)\begin{bmatrix} d_1(\bar\zeta)^* \\ c_1(\bar\zeta)^*\end{bmatrix} g_0 + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^*\end{bmatrix} g_1\right\} \\
&= (-i)^{\varkappa-2} g_1 \ .
\end{aligned}
$$

We continue in this way, obtaining at the last stage

$$
\begin{aligned}
&\gamma_1(\zeta)F_0(\zeta) + \gamma_2(\zeta)F_1(\zeta) + \cdots + \gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta) \\
&\quad = \begin{bmatrix} a_{\varkappa-1}(\zeta) & b_{\varkappa-1}(\zeta)\end{bmatrix}(-i)^{\varkappa-1} JY^{(0)}(\ell) \\
&\qquad + \begin{bmatrix} a_{\varkappa-2}(\zeta) & b_{\varkappa-2}(\zeta)\end{bmatrix}(-i)^{\varkappa-2} JY^{(1)}(\ell) \\
&\qquad + \cdots \\
&\qquad + \begin{bmatrix} a_0(\zeta) & b_0(\zeta)\end{bmatrix} JY^{(\varkappa-1)}(\ell) \\
&\quad = (-i)^{\varkappa-1}\begin{bmatrix} a_{\varkappa-1}(\zeta) & b_{\varkappa-1}(\zeta)\end{bmatrix}\begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^*\end{bmatrix} g_0 \\
&\qquad + (-i)^{\varkappa-2}\begin{bmatrix} a_{\varkappa-2}(\zeta) & b_{\varkappa-2}(\zeta)\end{bmatrix}\left\{(-i)\begin{bmatrix} d_1(\bar\zeta)^* \\ c_1(\bar\zeta)^*\end{bmatrix} g_0 + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^*\end{bmatrix} g_1\right\} \\
&\qquad\quad + \cdots \\
&\qquad + \begin{bmatrix} a_0(\zeta) & b_0(\zeta)\end{bmatrix}\left\{(-i)^{\varkappa-1}\begin{bmatrix} d_{\varkappa-1}(\bar\zeta)^* \\ c_{\varkappa-1}(\bar\zeta)^*\end{bmatrix} g_0 + (-i)^{\varkappa-2}\begin{bmatrix} d_{\varkappa-2}(\bar\zeta)^* \\ c_{\varkappa-2}(\bar\zeta)^*\end{bmatrix} g_1 \right. \\
&\qquad\qquad\qquad \left. + \cdots + \begin{bmatrix} d_0(\bar\zeta)^* \\ c_0(\bar\zeta)^*\end{bmatrix} g_{\varkappa-1}\right\} \\
&\quad = g_{\varkappa-1}.
\end{aligned}
$$

Thus (4.19) yields

$$
\begin{aligned}
\langle F, G\rangle &= G_0(\bar\zeta)^* g_{\varkappa-1} + G_1(\bar\zeta)^*(-i) g_{\varkappa-2} + \cdots \\
&\qquad + G_{\varkappa-2}(\bar\zeta)^*(-i)^{\varkappa-2} g_1 + G_{\varkappa-1}(\bar\zeta)^*(-i)^{\varkappa-1} g_0 \\
&= \sum_{p=0}^{\varkappa-1} G_p(\bar\zeta)^*(-i)^p g_{\varkappa-1-p}
\end{aligned}
\tag{4.26}
$$

The final step is to compare (4.18) and (4.26). By (4.21),

$$
G_p(\bar\zeta) = \sum_{q=0}^{\varkappa-1} (-i)^q \Delta_{qp}(\bar\zeta) h_{\varkappa-1-q}
$$

and so by (2.16),

$$
G_p(\bar\zeta)^* = \sum_{q=0}^{\varkappa-1} i^q h_{\varkappa-1-q}^* \Delta_{qp}(\bar\zeta)^* = \sum_{q=0}^{\varkappa-1} i^q h_{\varkappa-1-q}^* \Delta_{pq}(\bar\zeta) \ .
$$

Therefore (4.26) yields

$$\begin{aligned}\langle F, G\rangle &= \sum_{p=0}^{\varkappa-1}\Bigg(\sum_{q=0}^{\varkappa-1} i^q h^*_{\varkappa-1-q}\Delta_{pq}(\bar\zeta)\Bigg)(-i)^p g_{\varkappa-1-p}\\ &= \sum_{p,q=0}^{\varkappa-1} i^q(-i)^p h^*_{\varkappa-1-q}\Delta_{pq}(\zeta) g_{\varkappa-1-p} = \int_0^\ell Z(t)^* H(t) Y(t)\, dt,\end{aligned}$$

where the last equality is by (4.18). We have verified (4.15), and this completes the proof. □

Proof of Theorem 4.7*, Part* (2)*.* According to Definition 4.2, to show that $F=0$ as an element of $\mathfrak{H}_0(v)$, we must show that $v(z)F(z)$ is entire, that is, it is analytic at every pole of $v(z)$.

Let ζ be a pole of $v(z)$, and represent $v(z)$ as in (4.3) for $w=\zeta$ and $w=\bar\zeta$. The coefficients in these representations satisfy

$$\gamma_k(\bar\zeta)^* = \gamma_k(\zeta), \qquad k=1,\dots,\varkappa. \tag{4.27}$$

Write $F(z)=\sum_{j=0}^\infty F_j(\zeta)(z-\zeta)^j$. Since

$$\begin{aligned} v(z)F(z) = &\left\{-\left[\frac{\gamma_\varkappa(\zeta)}{(z-\zeta)^\varkappa} + \frac{\gamma_{\varkappa-1}(\zeta)}{(z-\zeta)^{\varkappa-1}} + \cdots + \frac{\gamma_1(\zeta)}{z-\zeta}\right] + \mathcal{O}(1)\right\}\cdot\\ &\cdot\left\{F_0(\zeta) + F_1(\zeta)(z-\zeta) + \cdots + F_{\varkappa-1}(\zeta)(z-\zeta)^{\varkappa-1} + \cdots\right\},\end{aligned}$$

the problem is to show that

$$\begin{aligned} \gamma_\varkappa(\zeta)F_0(\zeta) &= 0,\\ \gamma_\varkappa(\zeta)F_1(\zeta) + \gamma_{\varkappa-1}(\zeta)F_0(\zeta) &= 0,\\ &\cdots\\ \gamma_\varkappa(\zeta)F_{\varkappa-1}(\zeta) + \gamma_{\varkappa-1}(\zeta)F_{\varkappa-2}(\zeta) + \cdots + \gamma_1(\zeta)F_0(\zeta) &= 0.\end{aligned} \tag{4.28}$$

By (1.5) and (2.3),

$$\begin{aligned} F(z) &= \int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix} W(x,\bar z)^*\, H(x) f(x)\, dx\\ &= \sum_{j=0}^\infty (z-\zeta)^j \int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix} W_j(x,\bar\zeta)^*\, H(x) f(x)\, dx,\end{aligned}$$

and so for all $j=0,1,2,\dots,$

$$F_j(\zeta) = \int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix} W_j(x,\bar\zeta)^*\, H(x) f(x)\, dx\,. \tag{4.29}$$

By Proposition 3.1, $\bar\zeta$ is an eigenvalue of (3.1). Since f is orthogonal to all root functions of (3.1), f is orthogonal to the root functions for the eigenvalue $\bar\zeta$ provided by Proposition 4.1. Denote these functions

$$Y^{(0)}(x), \dots, Y^{(\varkappa-1)}(x). \tag{4.30}$$

Explicit formulas for the functions (4.30) are given by (3.13) and (4.5) with ζ replaced by $\bar{\zeta}$. Thus for each $j = 0, 1, \dots, \varkappa - 1$,

$$\begin{aligned} Y^{(j)}(x) &= (-i)^j W_j(x,\bar{\zeta}) \begin{bmatrix} 0 \\ g_0 \end{bmatrix} + (-i)^{j-1} W_{j-1}(x,\bar{\zeta}) \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \\ &\qquad + \cdots + (-i) W_1(x,\bar{\zeta}) \begin{bmatrix} 0 \\ g_{j-1} \end{bmatrix} + W_0(x,\bar{\zeta}) \begin{bmatrix} 0 \\ g_j \end{bmatrix} \\ &= (-i)^j W_j(x,\bar{\zeta}) \begin{bmatrix} 0 \\ \gamma_\varkappa(\bar{\zeta}) u \end{bmatrix} + (-i)^{j-1} W_{j-1}(x,\bar{\zeta}) \begin{bmatrix} 0 \\ (-i)\gamma_{\varkappa-1}(\bar{\zeta}) u \end{bmatrix} \\ &\qquad + \cdots + (-i) W_1(x,\bar{\zeta}) \begin{bmatrix} 0 \\ (-i)^{j-1}\gamma_{\varkappa-j+1}(\bar{\zeta}) u \end{bmatrix} \\ &\qquad + W_0(x,\bar{\zeta}) \begin{bmatrix} 0 \\ (-i)^j \gamma_{\varkappa-j}(\bar{\zeta}) u \end{bmatrix}, \end{aligned}$$

where u is an arbitrary vector in $\mathbb{C}^m$. We obtain

$$\begin{aligned} 0 &= (-i)^j \int_0^\ell Y^{(j)}(t)^* H(t) f(t)\, dt \\ &= \int_0^\ell \begin{bmatrix} 0 & u^* \gamma_\varkappa(\bar{\zeta})^* \end{bmatrix} W_j(t,\bar{\zeta}) H(t) f(t)\, dt \\ &\qquad + \int_0^\ell \begin{bmatrix} 0 & u^* \gamma_{\varkappa-1}(\bar{\zeta})^* \end{bmatrix} W_{j-1}(t,\bar{\zeta}) H(t) f(t)\, dt \\ &\qquad + \cdots + \int_0^\ell \begin{bmatrix} 0 & u^* \gamma_{\varkappa-j+1}(\bar{\zeta})^* \end{bmatrix} W_1(t,\bar{\zeta}) H(t) f(t)\, dt \\ &\qquad + \int_0^\ell \begin{bmatrix} 0 & u^* \gamma_{\varkappa-j}(\bar{\zeta})^* \end{bmatrix} W_0(t,\bar{\zeta}) H(t) f(t)\, dt. \end{aligned}$$

In view of (4.27) and (4.29) and the arbitrariness of u, we conclude that

$$\gamma_\varkappa(\zeta)^* F_j(\zeta) + \gamma_{\varkappa-1}(\zeta)^* F_{j-1}(\zeta) + \cdots + \gamma_{\varkappa-j+1}(\zeta)^* F_1(\zeta) + \gamma_{\varkappa-j}(\zeta)^* F_0(\zeta) = 0,$$

which is equivalent to the system (4.28).

We have shown that $v(z)F(z)$ is analytic at every pole ζ of $v(z)$. Therefore $F = 0$ as an element of $\mathfrak{H}_0(v)$, and the proof is complete. □

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