Pseudospectral functions for canonical differential systems

J. Rovnyak and L. A. Sakhnovich

This paper is dedicated to the memory of the great mathematician M. G. Kreĭn.

Abstract. A pseudospectral function for a canonical differential system is a nondecreasing function on the real line relative to which the eigentransform for the system is a partial isometry. Pseudospectral functions are constructed by means of eigenfunctions and resolvent operators which depend on boundary conditions for the system. Many results hold for Hamiltonians which have selfadjoint matrix values. The most complete results require the definite case, in which it is assumed that the Hamiltonian is nonnegative.

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Contents

1. Introduction

We are concerned with the spectral theory of canonical differential systems, which we write in the form

$$\begin{gathered}\frac{dY}{dx} = izJH(x)Y, \qquad 0 \le x \le \ell, \\ Y_1(0,z) = 0.\end{gathered} \tag{1.0.1}$$

We assume that $H(x)$, the Hamiltonian, has $2m \times 2m$ selfadjoint matrix values,

$$J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \qquad Y(x,z) = \begin{bmatrix} Y_1(x,z) \\ Y_2(x,z) \end{bmatrix}, \tag{1.0.2}$$

where $Y_1(x,z)$ and $Y_2(x,z)$ are m-dimensional vector-valued functions, and z is a complex parameter. It is assumed throughout that $\|H(x)\|$ is integrable on $[0,\ell]$. In a natural way, we shall define $L^2(Hdx)$ as a Kreĭn space of (equivalence classes of) $2m$-dimensional vector-valued functions with inner product

$$\langle f_1, f_2 \rangle_H = \int_0^\ell f_2^*(t)H(t)f_1(t)\,dt.$$

Let $W(x,z)$ be the unique $2m \times 2m$ matrix-valued function satisfying

$$\begin{gathered}\frac{dW}{dx} = izJH(x)W, \qquad 0 \le x \le \ell, \\ W(0,z) = I_{2m}\,.\end{gathered} \tag{1.0.3}$$

The eigentransform for (1.0.1) is defined by $Vf = F$,

$$F(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W^*(x,\bar z)\,H(x)f(x)\,dx,$$

for any f in $L^2(Hdx)$. For fixed f, $F = Vf$ is an m-dimensional vector-valued entire function.

Our purpose here is to construct inner products $\langle \cdot, \cdot \rangle_{\boldsymbol\tau}$ on vector-valued entire functions such that $\langle f_1, f_2 \rangle_H = \langle F_1, F_2 \rangle_{\boldsymbol\tau}$ for suitable transform pairs $F_1 = Vf_1$, $F_2 = Vf_2$. The quantities $\boldsymbol\tau$ used to define such inner products are constructed with the aid of boundary conditions at the right endpoint of $[0,\ell]$, and they are called pseudospectral data for the system (1.0.1). In general, we allow Hamiltonians satisfying $H(x) = H^*(x)$ a.e., and the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol\tau}$ is indefinite. In the definite case, that is, when $H(x) \ge 0$ a.e., $L^2(Hdx)$ is a Hilbert space, and the inner product identity becomes

$$\int_0^\ell f_2^*(x)H(x)f_1(x)\,dx = \int_{-\infty}^\infty F_2^*(x)\,d\tau(x)\,F_1(x), \tag{1.0.4}$$

where $\tau(x)$ is a nondecreasing $m \times m$ matrix-valued function of real x. However, (1.0.4) is not asserted for all f_1, f_2 in $L^2(Hdx)$, and in general V is a partial isometry. For this reason we call $\tau(x)$ a pseudospectral function for (1.0.1). We call $\tau(x)$ a spectral function if V is an isometry. In some cases, the pseudospectral functions that we construct are spectral functions. These results appear in §4 and depend on properties of eigenfunctions and resolvent operators for constant boundary conditions.

Basic notions of eigenfunction and resolvent operators relative to variable boundary conditions are introduced in §3. With variable boundary conditions, in the definite case $H(x) \geq 0$ a.e., in place of (1.0.4) we have the weaker result that

$$\int_{-\infty}^{\infty} F^*(x)\, d\tau(x)\, F(x) \leq \int_0^{\ell} f^*(x)H(x)f(x)\, dx \tag{1.0.5}$$

whenever $F = Vf$ for some f in $L^2(Hdx)$.

This paper is a continuation of our study of indefinite generalizations of some results of [15]. It differs in two key ways from our previous work. Whereas operator identities and the inverse problem are central in [10, 11, 12], here operator identities do not appear, and we are concerned now with the direct problem. The approach using eigenfunctions is similar in spirit to Atkinson [2, Chapter 9] but is technically different. Our methods are most closely related to A. L. Sakhnovich [13]. The study of canonical differential equations is a large and old one and owes much to fundamental work of L. de Branges and M. G. Kreĭn. For different approaches, historical remarks, and many additional references, see Arov and Dym [1], de Branges [3, 4], Gohberg and Kreĭn [5], Kaltenbäck and Woracek [8], and the second author [14, 15].

2. Preliminaries

Assume given a system (1.0.1) where $H(x)$ is a measurable $2m \times 2m$ matrix-valued function satisfying

(i) $H(x) = H^*(x)$ a.e. on $[0, \ell]$;

(ii) $\int_0^{\ell} \|H(x)\|\, dx < \infty$;

(iii) the only g in $\mathbb{C}^m$ such that $H(x)\begin{bmatrix} 0 \\ g \end{bmatrix} = 0$ a.e. on $[0, \ell]$ is $g = 0$.

In the definite case, that is, when $H(x) \geq 0$ a.e., (iii) is equivalent to:

(iii′) $\int_0^{\ell} H_{22}(t)\, dt \geq \delta I_m$ for some $\delta > 0$, where $H(x) = \begin{bmatrix} H_{11}(x) & H_{12}(x) \\ H_{12}^*(x) & H_{22}(x) \end{bmatrix}$.

In fact, if (iii′) is false, we can find $g \neq 0$ in $\mathbb{C}^m$ such that $H_{22}(x)g = 0$ a.e. on $[0, \ell]$. Hence for any x such that $H(x) \geq 0$ and any $u \in \mathbb{C}^m$ and $z \in \mathbb{C}$,

$$0 \leq \begin{bmatrix} zu \\ g \end{bmatrix}^* H(x) \begin{bmatrix} zu \\ g \end{bmatrix} = r^2 A + re^{-i\theta} B + re^{i\theta} \bar{B},$$

where $z = re^{i\theta}$, $A = u^* H_{11}(x)u$, and $B = u^* H_{12}(x)g$. This is only possible if $B = 0$. Since u is arbitrary, $H_{12}(x)g = 0$, and hence (iii) is false. Thus (iii) implies (iii′). The reverse implication is easy and omitted.

2.1. Fundamental solution

Let $W(x,z)$ be the unique solution of (1.0.3). In standard terminology, this is the fundamental solution of (1.0.1) whose value for $x=0$ is the identity matrix. For fixed x in $[0,\ell]$, $W(x,z)$ is an entire function of z satisfying

$$W^*(x,\bar z)JW(x,z) = W(x,z)JW^*(x,\bar z) = J \tag{2.1.1}$$

for all complex z. The Lagrange identity

$$\int_0^x W^*(u,w)H(u)W(u,z)\,du = \frac{W^*(x,w)JW(x,z)-J}{i(z-\bar w)} \tag{2.1.2}$$

holds for all x in $[0,\ell]$ and all complex z and w. When $w=\bar z$, (2.1.2) becomes

$$\begin{aligned}\int_0^x W^*(u,\bar z)H(u)W(u,z)\,du &= i\,W_1(x,\bar z)^*JW(x,z)\\ &= -i\,W(x,\bar z)^*JW_1(x,z),\end{aligned} \tag{2.1.3}$$

where $W_1(x,z) = d\,W(x,z)/dz$. For by (2.1.2) and (2.1.1),

$$\begin{aligned}\int_0^x W^*(u,\bar z)H(u)W(u,z)\,du &= \lim_{w\to\bar z}\int_0^x W^*(u,w)H(u)W(u,z)\,du\\ &= \lim_{w\to\bar z}\frac{W(x,w)^*JW(x,z)-W(x,\bar z)^*JW(x,z)}{i(z-\bar w)}\\ &= i\left[\lim_{w\to\bar z}\frac{W(x,w)-W(x,\bar z)}{w-\bar z}\right]^* JW(x,z)\\ &= i\,W_1(x,\bar z)^*JW(x,z),\end{aligned}$$

which is the first equality in (2.1.3). The second equality follows from the first on taking adjoints and replacing z by $\bar z$.

Throughout the paper we write

$$\mathfrak{A}(z) = W^*(\ell,\bar z) = \begin{bmatrix} a(z) & b(z)\\ c(z) & d(z)\end{bmatrix}. \tag{2.1.4}$$

Here $a(z), b(z), c(z), d(z)$ are $m\times m$ matrix-valued entire functions. By (2.1.1) and (2.1.3), for all complex z,

$$\begin{aligned} a(z)b^*(\bar z)+b(z)a^*(\bar z) &= 0, & a^*(\bar z)c(z)+c^*(\bar z)a(z) &= 0,\\ a(z)d^*(\bar z)+b(z)c^*(\bar z) &= I_m, & a^*(\bar z)d(z)+c^*(\bar z)b(z) &= I_m,\\ c(z)d^*(\bar z)+d(z)c^*(\bar z) &= 0, & b^*(\bar z)d(z)+d^*(\bar z)b(z) &= 0,\end{aligned} \tag{2.1.5}$$

and

$$\begin{aligned}&\int_0^\ell W^*(u,\bar z)H(u)W(u,z)\,du\\ &\qquad = i\begin{bmatrix} a'(z)b(\bar z)^*+b'(z)a(\bar z)^* & a'(z)d(\bar z)^*+b'(z)c(\bar z)^*\\ c'(z)b(\bar z)^*+d'(z)a(\bar z)^* & c'(z)d(\bar z)^*+d'(z)c(\bar z)^*\end{bmatrix}\end{aligned}$$

$$= -i \begin{bmatrix} a(z)b'(\bar{z})^* + b(z)a'(\bar{z})^* & a(z)d'(\bar{z})^* + b(z)c'(\bar{z})^* \\ c(z)b'(\bar{z})^* + d(z)a'(\bar{z})^* & c(z)d'(\bar{z})^* + d(z)c'(\bar{z})^* \end{bmatrix}. \tag{2.1.6}$$

2.2. Transform V

Given a $2m$-dimensional vector-valued function f, we define its transform $F = Vf$ as the m-dimensional vector-valued function

$$F(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W^*(x, \bar{z})\, H(x) f(x)\, dx. \tag{2.2.1}$$

The functions f which we consider here are assumed to belong to the Kreĭn space $L^2(Hdx)$ which is defined below. For each f in $L^2(Hdx)$, the transform $F = Vf$ is an entire function with values in $\mathbb{C}^m$.

In the definite case, $L^2(Hdx)$ is the well-known Hilbert space of (equivalence classes) of $2m$-dimensional vector-valued functions f on $[0, \ell]$ with

$$\|f\|_H^2 = \int_0^\ell f^*(t) H(t) f(t)\, dt < \infty.$$

To define $L^2(Hdx)$ in the general case, we write $H(x) = H_+(x) - H_-(x)$, where $H_\pm(x)$ are measurable functions on $[0, \ell]$ such that $H_\pm(x) \geq 0$ and $H_+(x)H_-(x) = 0$ a.e. As a linear space $L^2(Hdx)$ is defined to be $L^2((H_+ + H_-)dx)$. This is a Kreĭn space in the inner product

$$\langle f_1, f_2 \rangle_H = \int_0^\ell f_2^*(x) H(x) f_1(x)\, dx, \qquad f_1, f_2 \in L^2(Hdx).$$

We have $L^2(Hdx) = L^2(H_+dx) \oplus L^2(H_-dx)$, and this direct sum is a fundamental decomposition. Two elements f_1 and f_2 of the space are considered identical if $H(x)[f_1(x) - f_1(x)] = 0$ a.e. The elements of $L^2(Hdx)$ are thus cosets, but in the usual abuse of terminology we treat $L^2(Hdx)$ as a space of functions. We use standard notions of orthogonality, continuity, and boundedness for operators on a Kreĭn space.

Proposition 2.2.1. *Let $G(z) = [F(z) - F(z_0)]/(z - z_0)$ where $F = Vf$ for some $f \in L^2(Hdx)$ and some $z_0 \in \mathbb{C}$. Then $G = iVg$, where $g(x) = g(x, z_0)$ is the unique solution of*

$$\begin{gathered} \frac{dg}{dx} = iz_0 JH(x) g + JH(x) f, \qquad 0 \leq x \leq \ell, \\ g(\ell) = 0. \end{gathered} \tag{2.2.2}$$

The equation (2.2.2) is solved by setting $g(x) = W(x, z_0)U(x)$. We get

$$g(x, z_0) = -W(x, z_0) \int_x^\ell W(t, z_0)^{-1} JH(t) f(t)\, dt. \tag{2.2.3}$$

Proof. The Lagrange identity (2.1.2) can be used to show that

$$\frac{W^*(x,\bar z)-W^*(x,\bar z_0)}{z-z_0} = -i\int_0^x W^*(u,\bar z)H(u)W(u,z_0)\,du\;JW^*(x,\bar z_0).$$

Thus by (2.2.1), $G(z)=[F(z)-F(z_0)]/(z-z_0)$ is given by

$$\begin{aligned} G(z) &= \int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix}\frac{W^*(t,\bar z)-W^*(t,\bar z_0)}{z-z_0}H(t)f(t)\,dt \\ &= -i\begin{bmatrix}0 & I_m\end{bmatrix}\int_0^\ell\int_0^t W^*(u,\bar z)H(u)W(u,z_0)\,du\;JW^*(t,\bar z_0)\;H(t)f(t)\,dt. \end{aligned}$$

On interchanging the order of integration and using the second equality in (2.1.1), we obtain

$$G(z) = -i\int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix}W^*(u,\bar z)H(u)W(u,z_0)\int_u^\ell W(t,z_0)^{-1}JH(t)f(t)\,dt\,du.$$

By (2.2.3), $G(z)=i\int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix}W^*(u,\bar z)H(u)g(u,z_0)\,du$, that is, $G=iVg$. □

2.3. Nevanlinna pairs

By a **Nevanlinna pair** we mean a pair $R(z),Q(z)$ of $m\times m$ matrix-valued functions which are analytic on a region $\Omega_{R,Q}$ containing $\mathbb{C}_+\cup\mathbb{C}_-$ such that

(i) $R^*(\bar z)Q(z)+Q^*(\bar z)R(z)\equiv 0$ on $\Omega_{R,Q}$;
(ii) the kernel $i\big[R^*(\zeta)Q(z)+Q^*(\zeta)R(z)\big]/(z-\bar\zeta)$ is nonnegative on $\Omega_{R,Q}$.

When $R(z)\equiv R$ and $Q(z)\equiv Q$ are constant, $\Omega_{R,Q}=\mathbb{C}$ is the complex plane.

Proposition 2.3.1. *Let $R(z),Q(z)$ be a Nevanlinna pair of functions analytic on $\Omega_{R,Q}$ such that $c(z)R(z)+d(z)Q(z)$ is invertible except at isolated points. Then the meromorphic function*

$$v(z)=i\,[a(z)R(z)+b(z)Q(z)]\,[c(z)R(z)+d(z)Q(z)]^{-1} \tag{2.3.1}$$

satisfies $v(z)=v^*(\bar z)$ at all points of analyticity. If $K(z)=c(z)R(z)+d(z)Q(z)$, then

$$\begin{aligned} \frac{v(z)-v^*(\zeta)}{z-\bar\zeta} &= K^*(\zeta)^{-1}\,\frac{R(\zeta)^*Q(z)+Q(\zeta)^*R(z)}{i(\bar\zeta-z)}\,K(z)^{-1} \\ &\quad+\int_0^\ell \begin{bmatrix}I & iv^*(\zeta)\end{bmatrix}W^*(t,\zeta)H(t)W(t,z)\begin{bmatrix}I\\ -iv(z)\end{bmatrix}dt \end{aligned} \tag{2.3.2}$$

for $z,\zeta\in\Omega_{R,Q}$ such that $z\neq\bar\zeta$ and $K(z)$ and $K(\zeta)$ are invertible.

Proof. By (2.1.2) and (2.1.4),

$$\int_0^\ell W^*(t,\zeta)H(t)W(t,z)\,dt = \frac{\mathfrak{A}(\bar\zeta)J\mathfrak{A}^*(\bar z)-J}{i(z-\bar\zeta)}.$$

Hence

$$\begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} \frac{\mathfrak{A}(\bar\zeta) J \mathfrak{A}^*(\bar z) - J}{i(z-\bar\zeta)} \begin{bmatrix} I \\ -iv(z) \end{bmatrix} = \int_0^\ell \begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} W^*(t,\zeta) H(t) W(t,z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix} dt.$$

Breaking the left side into two parts and rearranging terms, we obtain

$$\begin{aligned} \frac{v(z)-v^*(\zeta)}{z-\bar\zeta} &= -\frac{\begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} J \begin{bmatrix} I \\ -iv(z) \end{bmatrix}}{i(z-\bar\zeta)} \\ &= \frac{\begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} \mathfrak{A}(\bar\zeta) J \mathfrak{A}^*(\bar z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix}}{i(\bar\zeta - z)} \\ &\quad + \int_0^\ell \begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} W^*(t,\zeta) H(t) W(t,z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix} dt\,. \end{aligned} \tag{2.3.3}$$

Recall that $K(z) = c(z)R(z) + d(z)Q(z)$. In addition, set $H(z) = a(z)R(z) + b(z)Q(z)$. Then by (2.3.1), $v(z) = iH(z)K(z)^{-1}$. By (2.1.4),

$$\begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \begin{bmatrix} a(z)R(z) + b(z)Q(z) \\ c(z)R(z) + d(z)Q(z) \end{bmatrix} = \mathfrak{A}(z) \begin{bmatrix} R(z) \\ Q(z) \end{bmatrix},$$

and therefore

$$\mathfrak{A}(z) \begin{bmatrix} R(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} H(z)K(z)^{-1} \\ I \end{bmatrix} K(z) = \begin{bmatrix} -iv(z) \\ I \end{bmatrix} K(z).$$

By (2.1.1), $\mathfrak{A}(z) J \mathfrak{A}^*(\bar z) = J$. Hence $\mathfrak{A}(z)^{-1} = J\mathfrak{A}^*(\bar z) J$ and

$$\begin{bmatrix} R(z) \\ Q(z) \end{bmatrix} K(z)^{-1} = \mathfrak{A}(z)^{-1} \begin{bmatrix} -iv(z) \\ I \end{bmatrix} = J\mathfrak{A}^*(\bar z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix}.$$

Thus by (2.3.3),

$$\begin{aligned} \frac{v(z)-v^*(\zeta)}{z-\bar\zeta} &= \frac{\begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} \mathfrak{A}(\bar\zeta) JJJ \mathfrak{A}^*(\bar z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix}}{i(\bar\zeta - z)} \\ &\quad + \int_0^\ell \begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} W^*(t,\zeta) H(t) W(t,z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix} dt \\ &= \frac{K^*(\zeta)^{-1} \begin{bmatrix} R^*(\zeta) & Q^*(\zeta) \end{bmatrix} J \begin{bmatrix} R(z) \\ Q(z) \end{bmatrix} K(z)^{-1}}{i(\bar\zeta - z)} \\ &\quad + \int_0^\ell \begin{bmatrix} I & iv^*(\zeta) \end{bmatrix} W^*(t,\zeta) H(t) W(t,z) \begin{bmatrix} I \\ -iv(z) \end{bmatrix} dt \end{aligned}$$

$$= K^*(\zeta)^{-1}\, \frac{R(\zeta)^*Q(z) + Q(\zeta)^*R(z)}{i(\bar\zeta - z)}\, K(z)^{-1} + \int_0^\ell \begin{bmatrix} I & iv^*(\zeta)\end{bmatrix} W^*(t,\zeta)H(t)W(t,z)\begin{bmatrix} I \\ -iv(z)\end{bmatrix}\, dt\,,$$

which is (2.3.2). The identity $v(z) = v^*(\bar z)$ is easily deduced from (2.3.2). □

3. Systems with boundary conditions

Let $R(z), Q(z)$ be a Nevanlinna pair of functions analytic on $\Omega_{R,Q}$ such that $c(z)R(z) + d(z)Q(z)$ is invertible except at isolated points. We study the eigenfunctions and resolvent operators for the system

$$\begin{gathered} \frac{dY}{dx} = izJH(x)Y, \qquad 0 \le x \le \ell, \\ Y_1(0,z) = 0, \qquad R^*(\bar z)Y_1(\ell,z) + Q^*(\bar z)Y_2(\ell,z) = 0, \end{gathered} \tag{3.0.1}$$

where $z \in \Omega_{R,Q}$ and the Hamiltonian $H(x)$ satisfies the conditions in §2.

3.1. Eigenfunctions and resolvents

Consider a system (3.0.1) with Hamiltonian $H(x) = H^*(x)$.

Definition 3.1.1. For every $\zeta \in \Omega_{R,Q}$, let $\mathfrak{L}_\zeta$ be the linear subspace of $L^2(Hdx)$ consisting of all solutions of (3.0.1) with $z = \zeta$.

We call a point $\zeta \in \Omega_{R,Q}$ an **eigenvalue** for (3.0.1) if $\mathfrak{L}_\zeta$ contains a function $Y(x,\zeta)$ such that $Y \ne 0$ as an element of $L^2(Hdx)$. In this case, we call any such Y an **eigenfunction** and $\mathfrak{L}_\zeta$ the **eigenspace** for the eigenvalue ζ.

Proposition 3.1.2. *For any $\zeta \in \Omega_{R,Q}$, $\mathfrak{L}_\zeta$ is the set of functions of the form*

$$\begin{gathered} Y(x,\zeta) = W(x,\zeta)\begin{bmatrix} 0 \\ g\end{bmatrix}, \qquad g \in \mathbb{C}^m, \\ [R^*(\bar\zeta)c^*(\bar\zeta) + Q^*(\bar\zeta)d^*(\bar\zeta)]g = 0\,. \end{gathered} \tag{3.1.1}$$

Hence $\mathfrak{L}_\zeta = \{0\}$ except at isolated points of $\Omega_{R,Q}$.

Proof. Let $Y = Y(x,\zeta) \in \mathfrak{L}_\zeta$, and set

$$Y(0,\zeta) = \begin{bmatrix} Y_1(0,\zeta) \\ Y_2(0,\zeta)\end{bmatrix} = \begin{bmatrix} \tilde g \\ g\end{bmatrix}.$$

Since $dY/dx = i\zeta JH(x)Y$ on $[0,\ell]$ and $W(0,\zeta) = I_{2m}$,

$$Y(x,\zeta) = W(x,\zeta)\begin{bmatrix} \tilde g \\ g\end{bmatrix}.$$

The condition $Y_1(0,\zeta)=0$ says that $\tilde g = 0$, so

$$\begin{bmatrix} Y_1(\ell,\zeta) \\ Y_2(\ell,\zeta) \end{bmatrix} = \begin{bmatrix} a^*(\bar\zeta) & c^*(\bar\zeta) \\ b^*(\bar\zeta) & d^*(\bar\zeta) \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} c^*(\bar\zeta)g \\ d^*(\bar\zeta)g \end{bmatrix}.$$

From $R^*(\bar\zeta)Y_1(\ell,\zeta) + Q^*(\bar\zeta)Y_2(\ell,\zeta) = 0$, we get $[R^*(\bar\zeta)c^*(\bar\zeta) + Q^*(\bar\zeta)d^*(\bar\zeta)]g = 0$. Thus Y has the form (3.1.1). These steps are reversible. □

Lemma 3.1.3. *Let $\zeta \in \Omega_{R,Q}$. Assume that $R^*(\zeta)R(\zeta)+Q^*(\zeta)Q(\zeta)$ and $R^*(\bar\zeta)R(\bar\zeta)+Q^*(\bar\zeta)Q(\bar\zeta)$ are invertible. Set*

$$M_\zeta = \operatorname{ran} \begin{bmatrix} R(\zeta) \\ Q(\zeta) \end{bmatrix} \qquad \text{and} \qquad M_{\bar\zeta} = \operatorname{ran} \begin{bmatrix} R(\bar\zeta) \\ Q(\bar\zeta) \end{bmatrix}. \tag{3.1.2}$$

Then $\dim M_\zeta = \dim M_{\bar\zeta} = m$, $M_\zeta^\perp = JM_{\bar\zeta}$, and $M_{\bar\zeta}^\perp = JM_\zeta$.

Proof. Our assumptions imply that

$$\ker \begin{bmatrix} R(\zeta) \\ Q(\zeta) \end{bmatrix} = \ker \begin{bmatrix} R(\bar\zeta) \\ Q(\bar\zeta) \end{bmatrix} = \{0\},$$

and this implies $\dim M_\zeta = \dim M_{\bar\zeta} = m$. By the definition of a Nevanlinna pair, $R^*(\bar\zeta)Q(\zeta) + Q^*(\bar\zeta)R(\zeta) = 0$, and hence $JM_{\bar\zeta} \subseteq M_\zeta^\perp$. Equality holds because M_ζ and $M_{\bar\zeta}$ have dimension m. Similarly, $M_{\bar\zeta}^\perp = JM_\zeta$. □

Lemma 3.1.4. *Let $\zeta \in \Omega_{R,Q}$. Assume that $R^*(\zeta)R(\zeta)+Q^*(\zeta)Q(\zeta)$ and $R^*(\bar\zeta)R(\bar\zeta)+Q^*(\bar\zeta)Q(\bar\zeta)$ are invertible. Then $c(\zeta)R(\zeta) + d(\zeta)Q(\zeta)$ is invertible if and only if $c(\bar\zeta)R(\bar\zeta) + d(\bar\zeta)Q(\bar\zeta)$ is invertible.*

Proof. Suppose that $c(\bar\zeta)R(\bar\zeta)+d(\bar\zeta)Q(\bar\zeta)$ is not invertible. Then there is a nonzero vector g in $\mathbb{C}^m$ such that $[R^*(\bar\zeta)c^*(\bar\zeta) + Q^*(\bar\zeta)d^*(\bar\zeta)]g = 0$. Therefore

$$\begin{bmatrix} R^*(\bar\zeta) & Q^*(\bar\zeta) \end{bmatrix} \begin{bmatrix} c^*(\bar\zeta)g \\ d^*(\bar\zeta)g \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} c^*(\bar\zeta)g \\ d^*(\bar\zeta)g \end{bmatrix} \in M_{\bar\zeta}^\perp = JM_\zeta,$$

where M_ζ and $M_{\bar\zeta}$ are as in Lemma 3.1.3. Hence there is g_1 in $\mathbb{C}^m$ such that

$$\begin{bmatrix} c^*(\bar\zeta)g \\ d^*(\bar\zeta)g \end{bmatrix} = J \begin{bmatrix} R(\zeta)g_1 \\ Q(\zeta)g_1 \end{bmatrix} = \begin{bmatrix} Q(\zeta)g_1 \\ R(\zeta)g_1 \end{bmatrix}.$$

Thus by (2.1.5),

$$[c(\zeta)R(\zeta) + d(\zeta)Q(\zeta)]g_1 = c(\zeta)d^*(\bar\zeta)g + d(\zeta)c^*(\bar\zeta)g = 0.$$

We show that $g_1 \neq 0$. In fact, if $g_1 = 0$, then $d^*(\bar\zeta)g = c^*(\bar\zeta)g = 0$, which by (2.1.5) implies that $g = [a(\zeta)d^*(\bar\zeta) + b(\zeta)c^*(\bar\zeta)]g = 0$, a contradiction. Since $c(\zeta)R(\zeta) + d(\zeta)Q(\zeta)$ has a nontrivial kernel, it is not invertible. The result follows on interchanging the roles of ζ and $\bar\zeta$. □

The next result prepares the way for the definition of a resolvent operator in Definition 3.1.6.

Proposition 3.1.5. *Suppose that $z \in \Omega_{R,Q}$ and $c(z)R(z)+d(z)Q(z)$ and $c(\bar z)R(\bar z)+d(\bar z)Q(\bar z)$ are invertible. Then for any $f \in L^2(Hdx)$, the system*

$$\begin{aligned} \frac{dV}{dx} &= izJH(x)V + JH(x)f(x), \qquad 0 \le x \le \ell, \\ V_1(0,z) = 0, &\qquad R^*(\bar z)V_1(\ell,z) + Q^*(\bar z)V_2(\ell,z) = 0, \end{aligned} \tag{3.1.3}$$

has a unique solution given by

$$\begin{aligned} V(x,z) = W(x,z)\Bigg\{ &\begin{bmatrix} 0 & I_m \\ 0 & -iv(z) \end{bmatrix} \int_0^x W^*(t,\bar z)H(t)f(t)\,dt \\ &+ \begin{bmatrix} 0 & 0 \\ -I_m & -iv(z) \end{bmatrix} \int_x^\ell W^*(t,\bar z)H(t)f(t)\,dt \Bigg\}, \end{aligned} \tag{3.1.4}$$

where $v(z)$ is defined by (2.3.1).

Proof. If a solution $V(x,z)$ exists and $V(x,z) = W(x,z)U(x,z)$, then by (1.0.3) and (2.1.1),

$$\frac{dU}{dx} = W(x,z)^{-1}JH(x)f(x) = JW^*(x,\bar z)JH(x)f(x),$$

and so

$$U(x,z) = U(0,z) + \int_0^x JW^*(t,\bar z)H(t)f(t)\,dt. \tag{3.1.5}$$

The boundary condition $V_1(0,z) = 0$ and relation $V(0,z) = W(0,z)U(0,z) = U(0,z)$ imply that $U_1(0,z) = 0$, and hence

$$U(0,z) = \begin{bmatrix} 0 \\ U_2(0,z) \end{bmatrix}. \tag{3.1.6}$$

The boundary condition $R^*(\bar z)V_1(\ell,z) + Q^*(\bar z)V_2(\ell,z) = 0$ can be written in the form $\begin{bmatrix} R^*(\bar z) & Q^*(\bar z) \end{bmatrix} V(\ell,z) = 0$. Here $V(\ell,z) = W(\ell,z)U(\ell,z)$, and so by (2.1.4), (3.1.5), and (3.1.6),

$$\begin{aligned} 0 &= \begin{bmatrix} R^*(\bar z) & Q^*(\bar z) \end{bmatrix} \begin{bmatrix} a^*(\bar z) & c^*(\bar z) \\ b^*(\bar z) & d^*(\bar z) \end{bmatrix} U(\ell,z) \\ &= \begin{bmatrix} R^*(\bar z)a^*(\bar z) + Q^*(\bar z)b^*(\bar z) & R^*(\bar z)c^*(\bar z) + Q^*(\bar z)d^*(\bar z) \end{bmatrix} \cdot \\ &\qquad \cdot \left\{ \begin{bmatrix} 0 \\ U_2(0,z) \end{bmatrix} + \int_0^\ell W(t,z)^{-1}JH(t)f(t)\,dt \right\}. \end{aligned}$$

By (2.1.1),

$$\begin{aligned} 0 = \begin{bmatrix} R^*(\bar z)a^*(\bar z) + Q^*(\bar z)b^*(\bar z) & R^*(\bar z)c^*(\bar z) + Q^*(\bar z)d^*(\bar z) \end{bmatrix} \cdot \\ \cdot \left\{ \begin{bmatrix} 0 \\ U_2(0,z) \end{bmatrix} + \int_0^\ell JW^*(t,\bar z)H(t)f(t)\,dt \right\} \end{aligned}$$

$$= [R^*(\bar z)c^*(\bar z) + Q^*(\bar z)d^*(\bar z)]U_2(0,z)$$
$$+ \begin{bmatrix} R^*(\bar z)a^*(\bar z) + Q^*(\bar z)b^*(\bar z) & R^*(\bar z)c^*(\bar z) + Q^*(\bar z)d^*(\bar z)\end{bmatrix}\cdot$$
$$\cdot \int_0^\ell W^*(t,\bar z)H(t)f(t)\,dt.$$

Solving for $U_2(0,z)$, we get

$$U_2(0,z) = -\begin{bmatrix} I_m & \varphi(z)\end{bmatrix}\int_0^\ell W^*(t,\bar z)H(t)f(t)\,dt,$$

where $\varphi(z) = i(-i)[R^*(\bar z)c^*(\bar z)+Q^*(\bar z)d^*(\bar z)]^{-1}[R^*(\bar z)a^*(\bar z)+Q^*(\bar z)b^*(\bar z)] = iv^*(\bar z)$, that is, $\varphi(z) = iv(z)$. Thus

$$U(0,z) = \begin{bmatrix} 0 & 0 \\ -iv(z) & -I_m\end{bmatrix} J\int_0^\ell W^*(t,\bar z)H(t)f(t)\,dt. \tag{3.1.7}$$

Then by (3.1.5) and (3.1.7),

$$\begin{aligned} V(x,z) &= W(x,z)\Bigg\{\begin{bmatrix} 0 & 0 \\ -iv(z) & -I_m\end{bmatrix} J\int_0^\ell W^*(t,\bar z)H(t)f(t)\,dt \\ &\qquad + \begin{bmatrix} I_m & 0 \\ 0 & I_m\end{bmatrix} J\int_0^x W^*(t,\bar z)H(t)f(t)\,dt\Bigg\} \\ &= W(x,z)\Bigg\{\begin{bmatrix} I_m & 0 \\ -iv(z) & 0\end{bmatrix} J\int_0^x W^*(t,\bar z)H(t)f(t)\,dt \\ &\qquad + \begin{bmatrix} 0 & 0 \\ -iv(z) & -I_m\end{bmatrix} J\int_x^\ell W^*(t,\bar z)H(t)f(t)\,dt\Bigg\} \\ &= W(x,z)\Bigg\{\begin{bmatrix} 0 & I_m \\ 0 & -iv(z)\end{bmatrix} \int_0^x W^*(t,\bar z)H(t)f(t)\,dt \\ &\qquad + \begin{bmatrix} 0 & 0 \\ -I_m & -iv(z)\end{bmatrix} \int_x^\ell W^*(t,\bar z)H(t)f(t)\,dt\Bigg\}, \end{aligned}$$

which is one direction of the theorem. The other direction follows on reversing the steps. □

Definition 3.1.6. Let Ω_v be the maximum domain of analyticity of the function $v(z)$ defined by (2.3.1). For each z in Ω_v, define a **resolvent operator** $B(z)$ on $L^2(H dx)$ by

$$B(z)f = V(x,z), \qquad f \in L^2(H dx), \tag{3.1.8}$$

where $V(x,z)$ is given by (3.1.4).

The domain Ω_v contains all removable singularities of (2.3.1) as well as real intervals across which this function has an analytic extension.

Proposition 3.1.7. *The resolvent $B(z)$ is analytic as a function of z, has compact values, and satisfies $B^*(\bar z) = -B(z)$ on Ω_v.*

Proof of Proposition 3.1.7. The fact that $B(z)$ is analytic and has compact values follows from (3.1.4). We show that $B^*(\bar z) = -B(z)$. First use (3.1.8) and (3.1.4) to write $B(z) = B_1(z) + B_2(z)$, where

$$B_1(z)f = W(x,z)\begin{bmatrix} 0 & I_m \\ 0 & -iv(z) \end{bmatrix} \int_0^x W^*(t,\bar z)H(t)f(t)\,dt, \tag{3.1.9}$$

$$B_2(z)g = W(x,z)\begin{bmatrix} 0 & 0 \\ -I_m & -iv(z) \end{bmatrix} \int_x^\ell W^*(t,\bar z)H(t)g(t)\,dt, \tag{3.1.10}$$

for any $f, g \in L^2(Hdx)$. By (3.1.9),

$$\begin{aligned} \langle B_1(z)f, g\rangle_H &= \int_0^\ell g^*(x)H(x)W(x,z)\begin{bmatrix} 0 & I_m \\ 0 & -iv(z) \end{bmatrix} \int_0^x W^*(t,\bar z)H(t)f(t)\,dt\,dx \\ &= \int_0^\ell h^*(t)H(t)f(t)\,dt, \end{aligned}$$

where

$$h(x) = W(x,\bar z)\begin{bmatrix} 0 & 0 \\ I_m & iv^*(z) \end{bmatrix} \int_x^\ell W^*(u,z)H(u)g(u)\,du.$$

By (3.1.10),

$$h(x) = -W(x,\bar z)\begin{bmatrix} 0 & 0 \\ -I_m & -iv(\bar z) \end{bmatrix} \int_x^\ell W^*(u,z)H(u)g(u)\,du = -B_2(\bar z)g.$$

Therefore $B_1^*(z) = -B_2(\bar z)$, and the assertion follows. □

Proposition 3.1.8. *For each $f \in L^2(Hdx)$ and z in Ω_v,*

$$i\langle B(z)f, f\rangle_H = F^*(\bar z)v(z)F(z) + i\Gamma_f(z), \tag{3.1.11}$$

where $F(z)$ is given by (2.2.1), *and*

$$\Gamma_f(z) = -\overline{\Gamma_f(\bar z)} = \int_0^\ell \int_0^\ell f^*(x)M(x,t,z)H(t)f(t)\,dt\,dx,$$

$$M(x,t,z) = \begin{cases} W(x,z)\begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix} W^*(t,\bar z), & x > t, \\ W(x,z)\begin{bmatrix} 0 & 0 \\ -I_m & 0 \end{bmatrix} W^*(t,\bar z), & x < t. \end{cases}$$

Proof. By (3.1.8) and (3.1.4),

$$\langle B(z)f,f\rangle_H = \int_0^\ell f^*(x)H(x)W(x,z)\Bigg\{\begin{bmatrix}0 & I_m\\ 0 & -iv(z)\end{bmatrix}\int_0^x W^*(t,\bar z)H(t)f(t)\,dt + \begin{bmatrix}0 & 0\\ -I_m & -iv(z)\end{bmatrix}\int_x^\ell W^*(t,\bar z)H(t)f(t)\,dt\Bigg\}\,dx.$$

The parts of the two integrals on the right containing $-iv(z)$ combine to give

$$\begin{aligned}\langle B(z)f,f\rangle_H &= \int_0^\ell f^*(x)H(x)W(x,z)\Bigg\{\begin{bmatrix}0 & 0\\ 0 & -iv(z)\end{bmatrix}\int_0^\ell W^*(t,\bar z)H(t)f(t)\,dt\\ &\qquad + \begin{bmatrix}0 & I_m\\ 0 & 0\end{bmatrix}\int_0^x W^*(t,\bar z)H(t)f(t)\,dt\\ &\qquad + \begin{bmatrix}0 & 0\\ -I_m & 0\end{bmatrix}\int_x^\ell W^*(t,\bar z)H(t)f(t)\,dt\Bigg\}\,dx\\ &= -iF^*(\bar z)v(z)F(z) + \int_0^\ell\int_0^\ell f^*(x)H(x)M(x,t,z)H(t)f(t)\,dt\,dx.\end{aligned}$$

This yields the formula for $\Gamma_f(z)$ in (3.1.11). The equality $\Gamma_f(z) = -\overline{\Gamma_f(\bar z)}$ follows from the identities $B^*(\bar z) = -B(z)$ and $v^*(\bar z) = v(z)$. □

3.2. Definite case: V as a contraction operator

We again assume given a system (3.0.1), but now in addition we assume that $H(x) \ge 0$ a.e. Then $L^2(Hdx)$ is a Hilbert space. There are no nonreal eigenvalues in this case (Proposition 3.2.1). We derive a Cauchy representation for the resolvent and show its consequence for the transform V (Theorem 3.2.4).

Proposition 3.2.1. *If $R^*(z)R(z)+Q^*(z)Q(z)$ is invertible for every nonreal z, then* (3.0.1) *has no nonreal eigenvalues.*

Proof. Fix $\zeta \ne \bar\zeta$, and suppose that $Y(t,\zeta) \in \mathfrak{L}_\zeta$. We show that $Y = 0$ in $L^2(Hdx)$. We borrow a formula from the proof of Proposition 4.1.1, which is valid under the present assumptions as well:

$$i(\zeta-\bar\zeta)\int_0^\ell Y^*(t,\zeta)H(t)Y(t,\zeta)\,dt = Y^*(\ell,\zeta)JY(\ell,\zeta). \tag{3.2.1}$$

Define M_ζ and $M_{\bar\zeta}$ by (3.1.2). The boundary condition at ℓ in (3.0.1) implies that $Y(\ell,\zeta)$ is orthogonal to $M_{\bar\zeta}$. Since we assume that $R^*(\zeta)R(\zeta)+Q^*(\zeta)Q(\zeta)$ and $R^*(\bar\zeta)R(\bar\zeta)+Q^*(\bar\zeta)Q(\bar\zeta)$ are invertible, it follows from Lemma 3.1.3 that $Y(\ell,\zeta) \in M_{\bar\zeta}^\perp = JM_\zeta$. Therefore

$$Y(\ell,\zeta) = J\begin{bmatrix}R(\zeta)\\ Q(\zeta)\end{bmatrix}g$$

for some $g \in \mathbb{C}^m$. Then by (3.2.1) and the definition of a Nevanlinna pair,

$$\begin{aligned}\int_0^\ell Y^*(t,\zeta)H(t)Y(t,\zeta)\,dt &= g^*\,\frac{R^*(\zeta)Q(\zeta)+Q^*(\zeta)R(\zeta)}{i(\zeta-\bar\zeta)}\,g \\ &= -g^*\,i\,\frac{R^*(\zeta)Q(\zeta)+Q^*(\zeta)R(\zeta)}{\zeta-\bar\zeta}\,g \le 0.\end{aligned}$$

Since $H(x)\ge 0$ a.e., the left side is nonnegative and hence equal to zero. □

In the definite case, it follows from (2.3.2) that

$$v(z) = i\,[a(z)R(z)+b(z)Q(z)]\,[c(z)R(z)+d(z)Q(z)]^{-1} \tag{3.2.2}$$

is a Nevanlinna function. This means that all nonreal singularities of $v(z)$ are removable, $v(z) = v^*(\bar z)$ for all nonreal z, and $v(z)$ has nonnegative imaginary part on $\mathbb{C}_+$. Hence $v(z)$ has a Nevanlinna representation

$$v(z) = \alpha + \beta z + \int_{-\infty}^{\infty}\left[\frac{1}{t-z}-\frac{t}{1+t^2}\right]\,d\tau(t). \tag{3.2.3}$$

Here α and β are $m\times m$ matrices such that $\alpha=\alpha^*$, $\beta\ge 0$, and $\tau(t)$ is a nondecreasing $m\times m$ matrix-valued function such that $\int_{-\infty}^{\infty} d\tau(t)/(1+t^2)$ is convergent. In particular, the resolvent $B(z)$ is analytic on $\Omega_v \supseteq \mathbb{C}_+\cup\mathbb{C}_-$.

Lemma 3.2.2. *For each $z\in\mathbb{C}_+$ and $f\in L^2(Hdx)$,*

$$\operatorname{Re}\langle B(z)f,f\rangle_H \ge \operatorname{Im} z\,\langle B(z)f, B(z)f\rangle_H \ge 0. \tag{3.2.4}$$

Proof. Without loss of generality, we can assume that $c(z)R(z)+d(z)Q(z)$ and $c(\bar z)R(\bar z)+d(\bar z)Q(\bar z)$ are invertible. Then $B(z)f = V$ is the unique solution of (3.1.3), and thus

$$\begin{aligned}\langle JV',V\rangle_{L^2_{2m}(0,\ell)} &= iz\langle H(x)V,V\rangle_{L^2_{2m}(0,\ell)} + \langle H(x)f,V\rangle_{L^2_{2m}(0,\ell)} \\ &= iz\langle V,V\rangle_H + \langle f,V\rangle_H.\end{aligned}$$

Hence

$$\begin{aligned}2\operatorname{Re}\langle f,V\rangle_H &= 2\operatorname{Im} z\,\langle V,V\rangle_H + \int_0^\ell \left[V^*(t,z)JV'(t,z)+V'^*(t,z)JV(t,z)\right]\,dt \\ &= 2\operatorname{Im} z\,\langle V,V\rangle_H + V^*(\ell,z)JV(\ell,z) - V^*(0,z)JV(0,z) \\ &= 2\operatorname{Im} z\,\langle V,V\rangle_H + V^*(\ell,z)JV(\ell,z),\end{aligned}$$

since $V^*(0,z)JV(0,z)=0$ by the boundary condition at 0 in (3.1.3). Let M_z and $M_{\bar z}$ be as in Lemma 3.1.3, so $JM_z = M_{\bar z}^{\perp}$. By the boundary condition at ℓ,

$$\left\langle V(\ell,z), \begin{bmatrix} R(\bar z)\\ Q(\bar z)\end{bmatrix} g\right\rangle_{\mathbb{C}^{2m}} = 0, \qquad g\in\mathbb{C}^m.$$

Hence $V(\ell,z)\in M_{\bar z}^{\perp} = JM_z$, so $V(\ell,z) = \begin{bmatrix} Q(z)\\ R(z)\end{bmatrix} g_z$ for some $g_z\in\mathbb{C}^m$. Thus

$$V^*(\ell,z)JV(\ell,z) = 2\operatorname{Im} z\, g_z^*\,\frac{Q^*(z)R(z)+R^*(z)Q(z)}{i(\bar z - z)}\,g_z \ge 0$$

for $\operatorname{Im} z \geq 0$ by the definition of a Nevanlinna pair (see §2.3). Recalling that $V = B(z)f$, we deduce (3.2.4). □

Lemma 3.2.3. *The resolvent operators have a representation*

$$iB(z) = \int_{-\infty}^{\infty} \frac{dG(t)}{t-z}, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.2.5}$$

where $G(x)$ is a nondecreasing function of real x whose values are operators on $L^2(Hdx)$ such that $G(x) = \frac{1}{2}\big[G(x+0) - G(x-0)\big]$ for all real x and $\int_{-\infty}^{\infty} dG(t) \leq I$.

Proof. By Lemma 3.2.2 and the identity $B^*(\bar{z}) = -B(z)$, $iB(z)$ is a Nevanlinna function and hence has a representation

$$iB(z) = C_1 + C_2 z + \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] dG(t), \tag{3.2.6}$$

where $C_1 = C_1^*$, $C_2 \geq 0$, and $G(x)$ is a nondecreasing function satisfying $G(x) = \frac{1}{2}\big[G(x+0) - G(x-0)\big]$ for all real x such that the integral $\int_{-\infty}^{\infty} dG(t)/(1+t^2)$ is weakly convergent. By Lemma 3.2.2, if $f \in L^2(Hdx)$,

$$\|iB(iy)f\|_H^2 \leq \frac{1}{y}\, \|iB(iy)f\|_H \|f\|_H,$$

and hence $y\|iB(iy)\| \leq 1$ for $y > 0$. It follows that $C_2 = 0$. Therefore

$$\begin{aligned} iB(iy) &= C_1 + \int_{-\infty}^{\infty} \left[\frac{1}{t-iy} - \frac{t}{1+t^2}\right] dG(t) \\ &= C_1 + \int_{-\infty}^{\infty} \left[\frac{t(1-y^2)}{(t^2+y^2)(1+t^2)} + i\, \frac{y}{t^2+y^2}\right] dG(t), \end{aligned}$$

and so

$$\int_{-\infty}^{\infty} \frac{y^2}{t^2+y^2}\, dG(t) = y \operatorname{Im}\,[iB(iy)].$$

Since $y\|iB(iy)\| \leq 1$ for $y > 0$, $\int_{-\infty}^{\infty} dG(t) \leq I$. The representation (3.2.6) can thus be written in the form

$$iB(z) = C_1 + \int_{-\infty}^{\infty} \frac{dG(t)}{t-z} - \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, dG(t). \tag{3.2.7}$$

Since $y\|iB(iy)\|$ is bounded for $y > 0$, $C_1 = \int_{-\infty}^{\infty} t(1+t^2)^{-1}\, dG(t)$ and so (3.2.7) reduces to (3.2.5). □

Theorem 3.2.4. *For any $f \in L^2(Hdx)$,*

$$\langle iB(z)f, f\rangle_H = \int_{-\infty}^{\infty} \frac{F^*(t)d\tau(t)F(t)}{t-z}, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.2.8}$$

where $\tau(x)$ is as in (3.2.3) and $F(z)$ is given by (2.2.1). Moreover,

$$\int_{-\infty}^{\infty} F^*(t)d\tau(t)F(t) \leq \int_0^{\ell} f^*(t)H(t)f(t)\, dt. \tag{3.2.9}$$

That is, the transform V acts as a contraction from $L^2(H)$ into $L^2(d\tau)$.

Proof. Apply Lemma 3.2.3 and Proposition 3.1.8 to write

$$\langle iB(z)f, f\rangle_H = \int_{-\infty}^{\infty} \frac{d\langle G(t)f, f\rangle_H}{t-z} = F^*(\bar{z})v(z)F(z) + i\Gamma_f(z).$$

By the Livšic-Stieltjes inversion formula, for any a, b, $-\infty < a < b < \infty$,

$$\langle [G(b) - G(a)]f, f\rangle_H = \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^b \mathrm{Im}\left[F^*(t-iy)v(t+iy)F(t+iy)\right] dt + \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^b i\Gamma_f(t+iy)\, dt = \int_a^b F^*(t)d\tau(t)F(t).$$

In the last equality the term involving $\Gamma_f(z)$ makes no contribution because $\Gamma_f(z)$ is continuous and real on the real axis. This proves (3.2.8). We deduce (3.2.9) from the inequality $\int_{-\infty}^{\infty} dG(t) \le I$ in Lemma 3.2.3. □

4. Constant boundary conditions and main results

Our main results construct pseudospectral data and pseudospectral functions for a system (1.0.1) by considering boundary conditions as in §3. For this purpose it is necessary to assume that the Nevanlinna pair $R(z) \equiv R$ and $Q(z) \equiv Q$ in (3.0.1) is constant. In this case, the domain of analyticity of the pair is $\Omega_{R,Q} = \mathbb{C}$. Thus throughout this section we assume given a system

$$\begin{gathered} \frac{dY}{dx} = izJH(x)Y, \qquad 0 \le x \le \ell, \\ Y_1(0,z) = 0, \qquad R^*Y_1(\ell, z) + Q^*Y_2(\ell, z) = 0, \end{gathered} \tag{4.0.1}$$

where R, Q are $m \times m$ matrices such that $R^*Q + Q^*R = 0$, the entire function $c(z)R + d(z)Q$ is invertible except at isolated points, and z is any complex number. As usual, we assume that the Hamiltonian $H(x)$ satisfies the conditions in §2. Notice that $R^*R + Q^*Q$ is invertible, since otherwise $c(z)R + d(z)Q$ cannot be invertible at any point.

4.1. Construction of pseudospectral data

Assume given a system (4.0.1) with Hamiltonian $H(x) = H^*(x)$. The goal of this section is Theorem 4.1.11, which is the basis for the notion of pseudospectral data. We also derive additional properties of eigenfunctions and resolvents that will be important for later results.

By Proposition 3.1.2, for each $\zeta \in \mathbb{C}$, $\mathfrak{L}_\zeta$ is the set of functions

$$\begin{gathered} Y(x,\zeta) = W(x,\zeta)\begin{bmatrix} 0 \\ g \end{bmatrix}, \qquad g \in \mathbb{C}^m, \\ [R^*c^*(\bar{\zeta}) + Q^*d^*(\bar{\zeta})]g = 0\,. \end{gathered} \tag{4.1.1}$$

Proposition 4.1.1. *For any* $\zeta_1, \zeta_2 \in \mathbb{C}$ *and* $Y(x,\zeta_1) \in \mathfrak{L}_{\zeta_1}$ *and* $Y(x,\zeta_2) \in \mathfrak{L}_{\zeta_2}$*,*

$$i(\zeta_1 - \bar{\zeta}_2) \int_0^\ell Y^*(t,\zeta_2)H(t)Y(t,\zeta_1)\,dt = 0. \tag{4.1.2}$$

Hence $\mathfrak{L}_{\zeta_1} \perp \mathfrak{L}_{\zeta_2}$ *if* $\zeta_1 \neq \bar{\zeta}_2$*, and* $\mathfrak{L}_\zeta$ *is a neutral subspace of* $L^2(Hdx)$ *if* $\zeta \neq \bar{\zeta}$*.*

Proof. By the differential equation in (4.0.1),

$$\begin{aligned} i(\zeta_1 - \bar{\zeta}_2) &\int_0^\ell Y^*(t,\zeta_2)H(t)Y(t,\zeta_1)\,dt \\ &= \int_0^\ell Y^*(t,\zeta_2)J[i\zeta_1 JH(t)Y(t,\zeta_1)]\,dt + \int_0^\ell [i\zeta_2 JH(t)Y(t,\zeta_2)]^* JY(t,\zeta_1)\,dt \\ &= \int_0^\ell \left[Y^*(t,\zeta_2)JY'(t,\zeta_1) + Y^{*\prime}(t,\zeta_2)JY(t,\zeta_1)\right]dt \\ &= Y^*(\ell,\zeta_2)JY(\ell,\zeta_1) - Y^*(0,\zeta_2)JY(0,\zeta_1) \\ &= Y^*(\ell,\zeta_2)JY(\ell,\zeta_1), \end{aligned}$$

where at the last stage we used the initial conditions $Y_1(0,\zeta_1) = Y_1(0,\zeta_2) = 0$. Applying Lemma 3.1.3 to the subspace

$$M = \operatorname{ran} \begin{bmatrix} R \\ Q \end{bmatrix} \subseteq \mathbb{C}^{2m}, \tag{4.1.3}$$

we see that $\dim M = m$ and $M^\perp = JM$. Since $R^*Y_1(\ell,\zeta_1) + Q^*Y_2(\ell,\zeta_1) = 0$ and $R^*Y_1(\ell,\zeta_2) + Q^*Y_2(\ell,\zeta_2) = 0$, $Y(\ell,\zeta_1)$ and $Y(\ell,\zeta_2)$ are orthogonal to M. Therefore $Y(\ell,\zeta_1) \in M^\perp = JM$ and $JY(\ell,\zeta_1) \in M$. Since $Y(\ell,\zeta_2) \in M^\perp$, $Y^*(\ell,\zeta_2)JY(\ell,\zeta_1) = 0$. This proves (4.1.2).

The last part of the lemma follows in a straightforward way from (4.1.2), provided that whenever (4.1.2) is applied with $\zeta_1 = \zeta_2 = \zeta$ then $Y(x,\zeta_1)$ and $Y(x,\zeta_2)$ are understood to be possibly different elements of $\mathfrak{L}_\zeta$. □

Given a linear operator T on some linear space and an eigenvalue γ for T, let $\mathfrak{R}_0(T,\gamma) = \ker(T - \gamma I)$. If $\mathfrak{R}_j(T,\gamma)$ has been defined for $j = 0,\dots,k$, let $\mathfrak{R}_{k+1}(T,\gamma)$ be the set of all vectors f such that $(T-\gamma I)f \in \mathfrak{R}_k(T,\gamma)$. We call $\mathfrak{R}_0(T,\gamma), \mathfrak{R}_1(T,\gamma), \dots$ the **root subspaces** for T for the eigenvalue γ.

Lemma 4.1.2. *Given a linear operator* T *with eigenvalue* γ*,*

(i) *the subspaces* $\mathfrak{R}_0(T,\gamma), \mathfrak{R}_1(T,\gamma), \dots$ *are invariant for* T*;*
(ii) $\{0\} \subseteq \mathfrak{R}_0(T,\gamma) \subseteq \mathfrak{R}_1(T,\gamma) \subseteq \cdots$*, and if equality holds at one stage, it holds at all later stages;*
(iii) *if* $\mathfrak{R}_0(T,\gamma)$ *is finite dimensional, so is* $\mathfrak{R}_k(T,\gamma)$ *for every* $k = 0,1,2,\dots$*;*
(iv) *if* $\mathfrak{R}_0(T,\gamma)$ *is finite dimensional and* $\gamma \neq 0$*, then* T *is a one-to-one mapping from* $\mathfrak{R}_k(T,\gamma)$ *onto itself for each* $k = 0,1,2,\dots$*.*

The details are elementary and omitted. We introduce an analogous notion for canonical systems.

Definition 4.1.3. Assume given a system (4.0.1) with eigenvalue ζ. Let $\mathfrak{L}_\zeta^{(0)} = \mathfrak{L}_\zeta$ be the corresponding eigenspace. If the subspaces $\mathfrak{L}_\zeta^{(0)}, \dots, \mathfrak{L}_\zeta^{(k)}$ have been defined, let $\mathfrak{L}_\zeta^{(k+1)}$ be the set of all functions Y which satisfy

$$\begin{aligned} \frac{dY}{dx} &= i\zeta JH(x)Y + JH(x)Y^{(k)}, \\ \begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) = 0, & \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0, \end{aligned} \tag{4.1.4}$$

for some $Y^{(k)} \in \mathfrak{L}_\zeta^{(k)}$. We call $\mathfrak{L}_\zeta^{(0)}, \mathfrak{L}_\zeta^{(1)}, \dots$ the **root subspaces** for (4.0.1) for the eigenvalue ζ.

It is easy to see that for any eigenvalue ζ of (4.0.1), $\mathfrak{L}_\zeta^{(0)} \subseteq \mathfrak{L}_\zeta^{(1)} \subseteq \cdots$, and if equality holds at one stage, it holds at all subsequent stages.

Theorem 4.1.4. *Let $z_0 \in \mathbb{C}$, and assume that $c(z_0)R + d(z_0)Q$ is invertible.*

(i) *The nonzero eigenvalues of the resolvent operator $B(z_0)$ coincide with the set of numbers $i/(z_0 - \zeta)$ where ζ is an eigenvalue of* (4.0.1).
(ii) *For each eigenvalue ζ of* (4.0.1) *and every $k = 0, 1, 2, \dots$,*

$$\mathfrak{L}_\zeta^{(k)} = \mathfrak{R}_k(B(z_0), i/(z_0 - \zeta)).$$

Proof. Let ζ be an eigenvalue for (4.0.1). Thus $\mathfrak{L}_\zeta^{(0)} = \mathfrak{L}_\zeta \neq \{0\}$. If $Y \in \mathfrak{L}_\zeta^{(0)}$, then

$$\frac{dY}{dx} = i\zeta JH(x)Y = iz_0 JH(x)Y + JH(x)[i(\zeta - z_0)Y],$$

$\begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) = 0$, and $\begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0$. Therefore $B(z_0)[i(\zeta - z_0)Y] = Y$. It follows that $i/(z_0 - \zeta)$ is a nonzero eigenvalue of $B(z_0)$, and

$$\mathfrak{L}_\zeta^{(0)} \subseteq \mathfrak{R}_0(B(z_0), i/(z_0 - \zeta)).$$

On the other hand, if $Y \in \mathfrak{R}_0(B(z_0), i/(z_0 - \zeta))$, we can reverse these steps to show that $Y \in \mathfrak{L}_\zeta^{(0)}$. Thus $i/(z_0 - \zeta)$ is a nonzero eigenvalue of $B(z_0)$, and

$$\mathfrak{L}_\zeta^{(0)} = \mathfrak{R}_0(B(z_0), i/(z_0 - \zeta)).$$

Conversely, suppose that γ is a nonzero eigenvalue of $B(z_0)$. Consider any eigenvector Y. Then $B(z_0)[\gamma^{-1}Y] = Y$. This means that

$$\begin{aligned} \frac{dY}{dx} &= iz_0 JH(x)Y + JH(x)[\gamma^{-1}Y], \\ \begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) = 0, & \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0. \end{aligned}$$

Define ζ by $\gamma = i/(z_0 - \zeta)$. Then

$$\frac{dY}{dx} = iz_0 JH(x)Y + JH(x)[i(\zeta - z_0)Y] = i\zeta JH(x)Y,$$

$\begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) = 0$, and $\begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0$. Hence Y is an eigenvector for (4.0.1).

Thus far we have proved (i). We have also proved (ii) for $k = 0$. In particular, for any eigenvalue ζ of (4.0.1),

$$\dim \mathfrak{R}_0(B(z_0), i/(z_0 - \zeta)) = \dim \mathfrak{L}_\zeta^{(0)} < \infty$$

by the representation (4.1.1) of the elements of an eigenspace. It remains to complete the proof of (ii). Suppose we know that

$$\mathfrak{R}_k(B(z_0), i/(z_0 - \zeta)) = \mathfrak{L}_\zeta^{(k)}$$

for some $k \geq 0$. Let $Y \in \mathfrak{L}_\zeta^{(k+1)}$. Choose $Y^{(k)} \in \mathfrak{L}_\zeta^{(k)}$ satisfying (4.1.4). Then

$$\frac{dY}{dx} = iz_0 JH(x)Y + JH(x)[i(\zeta - z_0)Y + Y^{(k)}],$$

$$\begin{bmatrix} I_m & 0 \end{bmatrix} Y(0) = 0, \qquad \begin{bmatrix} R^* & Q^* \end{bmatrix} Y(\ell) = 0,$$

which means that $B(z_0)[i(\zeta - z_0)Y + Y^{(k)}] = Y$, or

$$\Big[B(z_0) - \frac{i}{z_0 - \zeta}\, I \Big] Y = -\frac{i}{z_0 - \zeta} B(z_0) Y^{(k)}. \tag{4.1.5}$$

Since $Y^{(k)} \in \mathfrak{L}_\zeta^{(k)} = \mathfrak{R}_k(B(z_0), i/(z_0 - \zeta))$, by (4.1.5) and Lemma 4.1.2(iv),

$$\Big[B(z_0) - \frac{i}{z_0 - \zeta}\, I \Big] Y \in \mathfrak{R}_k(B(z_0), i/(z_0 - \zeta)), \tag{4.1.6}$$

and so $Y \in \mathfrak{R}_{k+1}(B(z_0), i/(z_0 - \zeta))$. Thus

$$\mathfrak{L}_\zeta^{(k+1)} \subseteq \mathfrak{R}_{k+1}(B(z_0), i/(z_0 - \zeta)). \tag{4.1.7}$$

To prove the reverse inclusion, consider any $Y \in \mathfrak{R}_{k+1}(B(z_0), i/(z_0 - \zeta))$. Then Y satisfies (4.1.6). Using Lemma 4.1.2(iv), we deduce (4.1.5) for some

$$Y^{(k)} \in \mathfrak{R}_k(B(z_0), i/(z_0 - \zeta)) = \mathfrak{L}_\zeta^{(k)}.$$

Now we can reverse the steps and conclude that $Y \in \mathfrak{L}_\zeta^{(k+1)}$. Therefore equality holds in (4.1.7), and the proof of (ii) is complete. □

We say that a real interval (a, b) is ***H*-indivisible** if

$$H(x) = \eta h(x) \eta^* \quad \text{a.e. on } (a.b), \quad \eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{4.1.8}$$

where $h(x)$ is a measurable function with selfadjoint $m \times m$ matrix values, and α and β are $m \times m$ matrices such that $\eta^* J \eta = \alpha^* \beta + \beta^* \alpha = 0$. The notion of an H-indivisible interval is due to Kac [7]. See also Hassi, de Snoo, and Winkler [6] and Kaltenbäck and Woracek [8, Part IV]. It should be noted that some authors use a trace-normed Hamiltonian, and their formulas have a different appearance.

Definition 4.1.5. Let $\widehat{L}^2(Hdx)$ be the subspace of $L^2(Hdx)$ consisting of all f such that for every H-indivisible interval (a, b), there is a $c \in \mathbb{C}^{2m}$ satisfying

$$H(x) f(x) = H(x) c \quad \text{a.e. on } (a.b).$$

When $H(x) \geq 0$ a.e., $L^2(Hdx)$ is a Hilbert space, and an argument in Kac [7] can be used to show that the subspace $\widehat{L}^2(Hdx)$ is closed. In the general case $H(x) = H^*(x)$, the inner product of $L^2(Hdx)$ is indefinite, and we make no similar assertion.

Proposition 4.1.6. (1) *Let $F = Vf$ be given by* (2.2.1) *for some f in $L^2(Hdx)$. If f is orthogonal to the subspace $\widehat{L}^2(Hdx)$, then $F(z) \equiv 0$.*
(2) *The root subspaces $\mathfrak{L}_\zeta^{(0)}, \mathfrak{L}_\zeta^{(1)}, \dots$ of* (4.0.1) *are contained in $\widehat{L}^2(Hdx)$.*

Proof. (1) Let $g \in \mathbb{C}^m$ and $z \in \mathbb{C}$. By (2.2.1),

$$g^*F(z) = \left\langle f(x), W(x,\bar{z}) \begin{bmatrix} 0 \\ g \end{bmatrix} \right\rangle_H. \tag{4.1.9}$$

We show that the function

$$Y(x) = W(x,\bar{z}) \begin{bmatrix} 0 \\ g \end{bmatrix}$$

belongs to $\widehat{L}^2(Hdx)$. Consider an H-indivisible interval (a,b), and suppose that $H(x)$ is represented as in (4.1.8) on (a,b). By (1.0.3), $dY/dx = i\bar{z}JH(x)Y$ a.e. By (4.1.8) and the identity $\eta^*J\eta = 0$,

$$\frac{d}{dx}\left(\eta^*Y(x)\right) = i\bar{z}\eta^*J\eta h(x)\eta^*Y(x) = 0$$

a.e. on (a,b). Therefore $\eta^*Y(x) \equiv$ const. on (a,b). The constant belongs to the range of η^*, and so $\eta^*Y(x) = \eta^*c$ on (a,b) for some $c \in \mathbb{C}^{2m}$. Then

$$H(x)Y(x) = \eta h(x)\eta^*Y(x) = \eta h(x)\eta^*c = H(x)c$$

a.e. on (a,b). Hence $Y \in \widehat{L}^2(Hdx)$. Since f is orthogonal to $\widehat{L}^2(Hdx)$, $g^*F(z) = 0$ by (4.1.9). By the arbitrariness of g, $F(z) \equiv 0$.

(2) Eigenfunctions have the form (4.1.1) and hence belong to $\widehat{L}^2(Hdx)$ by the proof of (1). Thus $\mathfrak{L}_\zeta^{(0)} \subseteq \widehat{L}^2(Hdx)$. Let $Y = Y(x) \in \mathfrak{L}_\zeta^{(k+1)}$, $k \geq 0$. Then Y satisfies an equation (4.1.4). Let (a,b) be an H-indivisible interval with $H(x)$ represented in the form (4.1.8) on (a,b). By (4.1.4) and (4.1.8),

$$\frac{d}{dx}(\eta^*Y(x)) = i\zeta\eta^*J\eta h(x)\eta^*Y(x) + \eta^*J\eta h(x)\eta^*Y^{(k)}(x) = 0$$

a.e. on (a,b) because $\eta^*J\eta = 0$. Therefore $\eta^*Y(x) \equiv$ const. on (a,b). As in the proof of (1), we deduce that $Y \in \widehat{L}^2(Hdx)$. □

Proposition 4.1.7. *The identity $B(z) - B(w) = i(z-w)B(z)B(w)$ holds at all points w, z such that $B(z)$ and $B(w)$ are defined. Hence the subspace $\mathfrak{K} = \ker B(z)$ is independent of z.*

Proposition 4.1.7 is a statement about resolvent operators for systems (4.0.1) with constant boundary conditions. The resolvent identity does not hold in general for systems (3.0.1) with variable boundary conditions.

Proof. Fix $f \in L^2(Hdx)$. Without loss of generality assume that $c(z)R + d(z)Q$ and $c(w)R + d(w)Q$ are invertible. Then according to Definition 3.1.6, $B(z)f$ and $B(w)f$ are determined by Proposition 3.1.5. Set

$$B(w)f = h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

By Proposition 3.1.5,

$$\frac{dh}{dx} = iwJH(x)h(x) + JH(x)f(x), \qquad 0 \le x \le \ell,$$
$$h_1(0) = 0, \qquad R^* h_1(\ell) + Q^* h_2(\ell) = 0,$$

Hence

$$\frac{dh}{dx} = izJH(x)h(x) + JH(x)\big[f(x) - i(z - w)h(x)\big], \qquad 0 \le x \le \ell,$$
$$h_1(0) = 0, \qquad R^* h_1(\ell) + Q^* h_2(\ell) = 0,$$

Therefore $B(z)\big[f - i(z - w)h\big] = h$, and the result follows. □

Proposition 4.1.8. *For any complex number* ζ*, the following are equivalent:*

(i) ζ *is an eigenvalue for* (4.0.1)*;*
(ii) $c(\zeta)R + d(\zeta)Q$ *is not invertible.*

Proof. (ii) ⇒ (i) If $c(\zeta)R + d(\zeta)Q$ is not invertible, neither is $c(\bar\zeta)R + d(\bar\zeta)Q$ by Lemma 3.1.4. Hence we can choose $g \neq 0$ in $\mathbb{C}^m$ such that $[R^* c^*(\bar\zeta) + Q^* d^*(\bar\zeta)]g = 0$. Thus (see (4.1.1))

$$Y(x, \zeta) = W(x, \zeta) \begin{bmatrix} 0 \\ g \end{bmatrix} \in \mathfrak{L}_\zeta.$$

We show that $Y \neq 0$ as an element of $L^2(Hdx)$. Argue by contradiction. If Y is equivalent to zero in $L^2(Hdx)$, then

$$\int_0^\ell f^*(t)H(t)Y(t, \zeta)\, dt = 0$$

for all $f \in L^2(Hdx)$. This implies that $H(x)Y(x, \zeta) = 0$ a.e., and hence $dY/dx = i\zeta JH(x)Y = 0$ a.e. on $[0, \ell]$. Therefore Y is constant, and so

$$Y(x, \zeta) = Y(0, \zeta) = \begin{bmatrix} 0 \\ g \end{bmatrix} \qquad \text{and} \qquad H(x)\begin{bmatrix} 0 \\ g \end{bmatrix} = 0 \quad \text{a.e.}$$

By the nondegeneracy condition (iii) in our assumptions on $H(x)$ in §2, $g = 0$, a contradiction. Therefore $Y \neq 0$ in $L^2(Hdx)$ and ζ is an eigenvalue for (4.0.1).

(i) ⇒ (ii) If ζ is an eigenvalue for (4.0.1), there is a function (4.1.1) which is not equivalent to zero in $L^2(Hdx)$. The vector g in (4.1.1) can therefore not be zero, and so $c(\bar\zeta)R + d(\bar\zeta)Q$ is not invertible. Then $c(\zeta)R + d(\zeta)Q$ is not invertible by Lemma 3.1.4. □

Since the functions $R(z) \equiv R$ and $Q(z) \equiv Q$ in (2.3.1) are constant,

$$v(z) = i\ [a(z)R + b(z)Q]\,[c(z)R + d(z)Q]^{-1} \tag{4.1.10}$$

is meromorphic in the complex plane. The remaining results in this section add the hypothesis that $v(z)$ has only simple poles.

Proposition 4.1.9. *Assume that $v(z)$ has only simple poles. Define $\gamma(\zeta)$ for every by $\zeta \in \mathbb{C}$ by*

$$v(z) = \frac{\gamma(\zeta)}{\zeta - z} + v_1(z), \tag{4.1.11}$$

where $v_1(z)$ is holomorphic at $z = \zeta$. Then $\gamma(\zeta) = 0$ except at the poles of $v(z)$, and $\gamma(\bar\zeta) = \gamma(\zeta)^*$. For each $\zeta \in \mathbb{C}$,

$$[R^* c^*(\bar\zeta) + Q^* d^*(\bar\zeta)]\gamma(\zeta) = 0, \tag{4.1.12}$$

and hence for every $g \in \mathbb{C}^m$,

$$Y(x, \zeta) = W(x, \zeta)\begin{bmatrix} 0 \\ \gamma(\zeta)g \end{bmatrix} \in \mathfrak{L}_\zeta. \tag{4.1.13}$$

Proof. Clearly $\gamma(\zeta) = 0$ except at the poles of $v(z)$. Since $v(z) = v^*(\bar z)$,

$$v(z) = \frac{\gamma(\zeta)}{\zeta - z} + v_1(z) = \frac{\gamma(\zeta)^*}{\bar\zeta - z} + v_1^*(\bar z), \tag{4.1.14}$$

and hence $\gamma(\bar\zeta) = \gamma(\zeta)^*$. By (4.1.10) and (4.1.14),

$$i(\bar\zeta - z)[a(z)R + b(z)Q] = [\gamma(\zeta)^* + (\bar\zeta - z)v_1^*(\bar z)][c(z)R + d(z)Q].$$

Letting $z \to \bar\zeta$, we deduce (4.1.12). Then (4.1.13) follows from (4.1.1). □

The eigenvalues of (4.0.1) are isolated in the complex plane and occur in conjugate pairs by Proposition 4.1.8 and Lemma 3.1.4. Assuming again that $v(z)$ has only simple poles, we fix the following notation for these points.

- Let $\{\lambda_j\}_{j=1}^r$ be the real eigenvalues of (4.0.1) ($0 \le r \le \infty$). For each $j = 1, \dots, r$, write
$$v(z) = \frac{\tau_j}{\lambda_j - z} + v_j(z), \qquad \tau_j = \tau_j^* = \gamma(\lambda_j).$$
- Let $\{\mu_k, \bar\mu_k\}_{j=1}^s$ be the nonreal pairs of eigenvalues of (4.0.1) ($0 \le s \le \infty$). For each $k = 1, \dots, s$, write
$$v(z) = \frac{\beta_k}{\mu_k - z} + v_k(z) = \frac{\beta_k^*}{\bar\mu_k - z} + v_k^*(\bar z), \qquad \beta_k = \gamma(\mu_k).$$
- Let $\boldsymbol{\tau} = \boldsymbol{\tau}_{R,Q}$ be the collection of all eigenvalues $\lambda_j, \mu_k, \bar\mu_k$ together with the matrices τ_j, β_k, $j = 1, \dots, r$ and $k = 1, \dots, s$.

By Proposition 4.1.1, $\mathfrak{L}_{\lambda_j} \perp \mathfrak{L}_{\lambda_k}$ if $j \neq k$, $\mathfrak{L}_{\lambda_j} \perp (\mathfrak{L}_{\mu_k} + \mathfrak{L}_{\bar\mu_k})$, and $\mathfrak{L}_{\mu_k}$ is a neutral subspace of $L^2(Hdx)$ for all $j = 1, \dots, r$ and $k = 1, \dots, s$.

Let $L^2_0(\boldsymbol{\tau})$ be the space of all $\mathbb{C}^m$-valued functions defined on the points $\lambda_j, \mu_k, \bar\mu_k$ having only finitely many nonzero values, in the inner product

$$\langle F, G\rangle_{L^2_0(\boldsymbol{\tau})} = \sum_{j=1}^r G(\lambda_j)^* \tau_j F(\lambda_j) + \sum_{k=1}^s \big[G(\bar\mu_k)^* \beta_k F(\mu_k) + G(\mu_k)^* \beta_k^* F(\bar\mu_k)\big].$$

Equivalently, we can consider the elements of $L^2_0(\boldsymbol{\tau})$ as $\mathbb{C}^m$-valued functions on the complex plane in the inner product

$$\langle F, G\rangle_{L^2_0(\boldsymbol{\tau})} = \sum_{\zeta\in\mathbb{C}} G^*(\bar\zeta)\gamma(\zeta)F(\zeta). \tag{4.1.15}$$

Two functions F_1 and F_2 are identified if

$$\gamma(\zeta)[F_1(\zeta) - F_2(\zeta)] = 0, \qquad \zeta \in \mathbb{C}. \tag{4.1.16}$$

The inner product in $L^2_0(\boldsymbol{\tau})$ is nondegenerate and in general indefinite.

We investigate the transform $F = Vf$ defined by (2.2.1) as a mapping from $L^2(Hdx)$ into $L^2_0(\boldsymbol{\tau})$.

Lemma 4.1.10. *Assume that $v(z)$ has only simple poles. Let $Y(x,\zeta)$ belong to $\mathfrak{L}_\zeta$ and have the form (4.1.1). Then VY belongs to $L^2_0(\boldsymbol{\tau})$ and is equivalent to the function F defined by*

$$F(z) = \begin{cases} \Delta_\zeta\, g, & z = \zeta, \\ 0, & z \neq \zeta, \end{cases} \tag{4.1.17}$$

where

$$\Delta_\zeta = \Delta^*_{\bar\zeta} = i[c'(\zeta)d^*(\bar\zeta) + d'(\zeta)c^*(\bar\zeta)]. \tag{4.1.18}$$

Proof. The identity $\Delta_\zeta = \Delta^*_{\bar\zeta}$ follows from (2.1.6). Let $G = VY$. For $z \neq \zeta$,

$$\begin{aligned} G(z) &= \begin{bmatrix} 0 & I_m \end{bmatrix} \int_0^\ell W(t,\bar z)^*\, H(t) W(t,\zeta)\, dt \begin{bmatrix} 0 \\ g \end{bmatrix} \\ &= \begin{bmatrix} 0 & I_m \end{bmatrix} \frac{W^*(\ell,\bar z) J W(\ell,\zeta) - J}{i(\zeta - z)} \begin{bmatrix} 0 \\ g \end{bmatrix} \\ &= \frac{1}{i(\zeta - z)} \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} J \begin{bmatrix} a^*(\bar\zeta) & c^*(\bar\zeta) \\ b^*(\bar\zeta) & d^*(\bar\zeta) \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \\ &= \frac{c(z)d^*(\bar\zeta) + d(z)c^*(\bar\zeta)}{i(\zeta - z)}\, g\,. \end{aligned}$$

By (2.1.5), $c(\zeta)d^*(\bar\zeta) + d(\zeta)c^*(\bar\zeta) = 0$, and so

$$\begin{aligned} G(\zeta) &= \lim_{z\to\zeta} \frac{[c(z) - c(\zeta)]d^*(\bar\zeta) + [d(z) - d(\zeta)]c^*(\bar\zeta)}{i(\zeta - z)}\, g \\ &= i[c'(\zeta)d^*(\bar\zeta) + d'(\zeta)c^*(\bar\zeta)]g = \Delta_\zeta\, g. \end{aligned}$$

Thus $F(\zeta) = G(\zeta)$. To show that G is equivalent to F in $L^2_0(\boldsymbol{\tau})$, by (4.1.17) we must show that for all $w \neq \zeta$, $\gamma(w)[F(w) - G(w)] = 0$, that is, $\gamma(w)G(w) = 0$, or

$$\gamma(w)\, \frac{c(w)d^*(\bar{\zeta}) + d(w)c^*(\bar{\zeta})}{i(\zeta - w)}\, g = 0\,. \tag{4.1.19}$$

Fix $w \neq \zeta$ and consider any $\tilde{g} \in \mathbb{C}^m$. Write

$$\tilde{g}^*\gamma(w)[c(w)d^*(\bar{\zeta}) + d(w)c^*(\bar{\zeta})]\, g = \left\langle \begin{bmatrix} d^*(\bar{\zeta})g \\ c^*(\bar{\zeta})g \end{bmatrix}, \begin{bmatrix} c^*(w)\gamma(w)^*\tilde{g} \\ d^*(w)\gamma(w)^*\tilde{g} \end{bmatrix} \right\rangle_{\mathbb{C}^{2m}}. \tag{4.1.20}$$

Define M as in (4.1.3), so $M^\perp = JM$. By (4.1.12), $[R^*c^*(w) + Q^*d^*(w)]\gamma(w)^* = 0$, and hence

$$\begin{bmatrix} c^*(w)\gamma(w)^*\tilde{g} \\ d^*(w)\gamma(w)^*\tilde{g} \end{bmatrix} \in M^\perp. \tag{4.1.21}$$

By (4.1.1), $[R^*c^*(\bar{\zeta}) + Q^*d^*(\bar{\zeta})]g = 0$, that is, $\begin{bmatrix} c^*(\bar{\zeta})g \\ d^*(\bar{\zeta})g \end{bmatrix} \in M^\perp = JM$. Hence

$$\begin{bmatrix} d^*(\bar{\zeta})g \\ c^*(\bar{\zeta})g \end{bmatrix} = J \begin{bmatrix} c^*(\bar{\zeta})g \\ d^*(\bar{\zeta})g \end{bmatrix} \in M \tag{4.1.22}$$

By (4.1.20), (4.1.21), and (4.1.22), $\tilde{g}^*\gamma(w)[c(w)d^*(\bar{\zeta}) + d(w)c^*(\bar{\zeta})]\, g = 0$. Since $\tilde{g}$ is arbitrary, this proves (4.1.19), and the result follows. □

Theorem 4.1.11. *Assume that $v(z)$ has only simple poles.*
(1) *If f_1 and f_2 are finite linear combinations of eigenfunctions of* (4.0.1), *then*

$$\int_0^\ell f_2^*(t)H(t)f_1(t)\, dt = \langle Vf_1, Vf_2 \rangle_{L^2_0(\boldsymbol{\tau})}.$$

(2) *If $f \in L^2(Hdx)$ and f is orthogonal to every eigenfunction of* (4.0.1), *then $Vf = 0$ as an element of $L^2_0(\boldsymbol{\tau})$.*

Definition 4.1.12. By **pseudospectral data** for (1.0.1) we mean a collection $\boldsymbol{\tau}$ of the type considered above satisfying the properties (1) and (2) in Theorem 4.1.11.

Proof. (1) By linearity, we may assume that $f_j(x) = Y(\zeta_j, x)$, where

$$Y(x, \zeta_j) = W(x, \zeta_j) \begin{bmatrix} 0 \\ g_j \end{bmatrix} \in \mathfrak{L}_{\zeta_j}, \qquad j = 1, 2, \tag{4.1.23}$$

as in (4.1.1) for some $\zeta_1, \zeta_2 \in \mathbb{C}$. By Lemma 4.1.10, $VY(x, \zeta_j)$ is equivalent to

$$F_j(z) = \begin{cases} \Delta_{\zeta_j}\, g_j, & z = \zeta_j, \\ 0, & z \neq \zeta_j, \end{cases} \tag{4.1.24}$$

$j = 1, 2$, where Δ_ζ is given by (4.1.18). To prove (1), we must show that

$$\int_0^\ell Y(t, \zeta_2)^* H(t) Y(t, \zeta_1)\, dt = \langle F_1, F_2 \rangle_{L^2_0(\boldsymbol{\tau})}\,. \tag{4.1.25}$$

Case 1: $\zeta_2 \neq \bar{\zeta}_1$. Then $\int_0^\ell Y(t,\zeta_2)^* H(t) Y(t,\zeta_1)\; dt = 0$ by Proposition 4.1.1. By (4.1.15),

$$\langle F_1, F_2\rangle_{L^2_0(\boldsymbol{\tau})} = \sum_{\zeta\in\mathbb{C}} F_2^*(\bar{\zeta})\gamma(\zeta)F_1(\zeta) = F_2^*(\bar{\zeta}_1)\gamma(\zeta)F_1(\zeta_1),$$

since $F_1(\zeta) = 0$ for $\zeta \neq \zeta_1$ by (4.1.24). In the same way, $F_2(\zeta) = 0$ for $\zeta \neq \zeta_2$, and hence $F_2(\bar{\zeta}_1) = 0$ because $\bar{\zeta}_1 \neq \zeta_2$. Thus $F_1 \perp F_2$ in $L^2_0(\boldsymbol{\tau})$, and (4.1.25) follows.

Case 2: $\zeta_2 = \bar{\zeta}_1$. Write the two points as $\zeta_1 = \zeta$ and $\zeta_2 = \bar{\zeta}$. By (2.1.6),

$$\begin{aligned}\int_0^\ell Y(t,\bar{\zeta})^* H(t) Y(t,\zeta)\; dt &= \begin{bmatrix} 0 & g_2^* \end{bmatrix} \int_0^\ell W(t,\bar{\zeta})^* H(t) W(t,\zeta)\; dt \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & g_2^* \end{bmatrix} \begin{bmatrix} * & * \\ * & i[c'(\zeta)d^*(\bar{\zeta}) + d'(\zeta)c^*(\bar{\zeta})] \end{bmatrix} \begin{bmatrix} 0 \\ g_1 \end{bmatrix} \\ &= g_2^* \Delta_\zeta g_1. \end{aligned} \tag{4.1.26}$$

Since

$$\langle F_1, F_2\rangle_{L^2_0(\boldsymbol{\tau})} = \sum_{z\in\mathbb{C}} F_2^*(\bar{z})\gamma(z)F_1(z) = F_2^*(\bar{\zeta})\gamma(\zeta)F_1(\zeta) = g_2^*\Delta_{\bar{\zeta}}^*\gamma(\zeta)\Delta_\zeta g_1\,,$$

in order to verify (4.1.25), we must show that

$$g_2^*\Delta_{\bar{\zeta}}^*\gamma(\zeta)\Delta_\zeta g_1 = g_2^*\Delta_\zeta g_1. \tag{4.1.27}$$

As in the proof of Lemma 4.1.10, $\begin{bmatrix} d^*(\bar{\zeta})g_1 \\ c^*(\bar{\zeta})g_1 \end{bmatrix} \in \operatorname{ran} \begin{bmatrix} R \\ Q \end{bmatrix}$. Hence

$$R\tilde{g}_1 = d^*(\bar{\zeta})g_1 \qquad \text{and} \qquad Q\tilde{g}_1 = c^*(\bar{\zeta})g_1 \tag{4.1.28}$$

for some $\tilde{g}_1 \in \mathbb{C}^m$. Thus

$$\Delta_\zeta g_1 = i\,[c'(\zeta)d^*(\bar{\zeta}) + d'(\zeta)c^*(\bar{\zeta})]g_1 = i\,[c'(\zeta)R + d'(\zeta)Q]\tilde{g}_1. \tag{4.1.29}$$

Next use (4.1.10) and (4.1.11) to write

$$i(\zeta - z)[a(z)R + b(z)Q] = [\gamma(\zeta) + (\zeta - z)v_1(z)][c(z)R + d(z)Q],$$

where $v_1(z)$ is holomorphic at $z = \zeta$. On differentiating this relation with respect to z and then setting $z = \zeta$, we obtain

$$-i[a(\zeta)R + b(\zeta)Q] = -v_1(\zeta)[c(\zeta)R + d(\zeta)Q] + \gamma(\zeta)[c'(\zeta)R + d'(\zeta)Q]. \tag{4.1.30}$$

Therefore

$$\begin{aligned} g_2^*\Delta_{\bar{\zeta}}^*\gamma(\zeta)\Delta_\zeta g_1 &\overset{(4.1.29)}{=} g_2^*\Delta_{\bar{\zeta}}^*\gamma(\zeta) i\,[c'(\zeta)R + d'(\zeta)Q]\tilde{g}_1 \\ &\overset{(4.1.30)}{=} i\,g_2^*\Delta_{\bar{\zeta}}^*\Big\{ -i[a(\zeta)R + b(\zeta)Q] + v_1(\zeta)[c(\zeta)R + d(\zeta)Q]\Big\}\tilde{g}_1 \\ &\overset{(4.1.28)}{=} g_2^*\Delta_{\bar{\zeta}}^*\Big\{[a(\zeta)d^*(\bar{\zeta}) + b(\zeta)c^*(\bar{\zeta})]g_1 \\ &\qquad\qquad + iv_1(\zeta)[c(\zeta)d^*(\bar{\zeta}) + d(\zeta)c^*(\bar{\zeta})]g_1\Big\}. \end{aligned}$$

Thus by (2.1.5) and (4.1.18), $g_2^*\Delta_{\bar\zeta}^*\gamma(\zeta)\Delta_\zeta g_1 = g_2^*\Delta_{\bar\zeta}^* g_1 = g_2^*\Delta_\zeta g_1$. This proves (4.1.27) and verifies (4.1.25) in Case 2.

(2) Let $F = Vf$, where $f \in L^2(Hdx)$ and $f \perp \mathfrak{L}_\zeta$ for all $\zeta \in \mathbb{C}$. To show that $F = 0$ as an element of $L_0^2(\boldsymbol{\tau})$, by (4.1.16) we must show that $\gamma(\zeta)F(\zeta) = 0$ for every $\zeta \in \mathbb{C}$. In fact, for every $g \in \mathbb{C}^m$,

$$W(x,\bar\zeta)\begin{bmatrix} 0 \\ \gamma(\zeta)^* g \end{bmatrix} \in \mathfrak{L}_{\bar\zeta} \tag{4.1.31}$$

by Proposition 4.1.9. By assumption, f is orthogonal to (4.1.31), and so by (2.2.1),

$$g^*\gamma(\zeta)F(\zeta) = g^*\gamma(\zeta)\int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(t,\bar\zeta)^* H(t) f(t)\, dt = 0.$$

Since g is arbitrary, $\gamma(\zeta)F(\zeta) = 0$. □

4.2. Definite case: pseudospectral functions

Let a system (4.0.1) be given as before, and in addition assume that $H(x) \geq 0$ a.e. Then the function $v(z)$ defined by (4.1.10) is a Nevanlinna function by (2.3.2). The main results of this section appear in Theorems 4.2.2, 4.2.4, and 4.2.5.

Proposition 4.2.1. *The eigenvalues of* (4.0.1) *are real. For any complex number* ζ*, the following are equivalent:*

(i) ζ *is an eigenvalue for* (4.0.1)*;*
(ii) $c(\zeta)R + d(\zeta)Q$ *is not invertible;*
(iii) ζ *is a pole of* $v(z)$*.*

Proof. Since $H(x) \geq 0$, the eigenvalues of (4.0.1) are real by Proposition 4.1.1 (or Proposition 3.2.1). The equivalence of (i) and (ii) is shown in Proposition 4.1.8.

(iii) $\Longrightarrow$ (ii) This is obvious from the definition of $v(z)$ in (4.1.10).

(i) $\Longrightarrow$ (iii) If ζ is an eigenvalue of (4.0.1), then there is a $Y \neq 0$ in $L^2(Hdx)$ of the form (4.1.1). By Lemma 4.1.10, VY is equivalent to the function $F(x)$ given by (4.1.17). Since $H(x) \geq 0$, by Theorem 4.1.11(1) and (4.1.15),

$$0 < \int_0^\ell Y^*(t,\zeta)H(t)Y(t,\zeta)\, dt = \langle F, F\rangle_{L_0^2(\boldsymbol{\tau})} = F^*(\bar\zeta)\gamma(\zeta)F(\zeta).$$

So $\gamma(\zeta) \neq 0$, and hence ζ is a pole of $v(z)$ by (4.1.11). □

Since $v(z)$ is meromorphic in $\mathbb{C}$ and a Nevanlinna function, its poles are real and simple, and hence the pseudospectral data constructed in Theorem 4.1.11 take a simpler form. By Proposition 4.2.1, the poles of $v(z)$ coincide with the eigenvalues $\{\lambda_j\}_{j=1}^r$ of (4.0.1). Thus we have

$$v(z) = \frac{\tau_j}{\lambda_j - z} + v_j(z),$$

where $\tau_j \geq 0$ and $v_j(z)$ is holomorphic at λ_j, $j = 1, \ldots, r$, and

$$v(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \frac{t}{1 + t^2} \right] d\tau(t), \tag{4.2.1}$$

where $\tau(t)$ is a nondecreasing $m \times m$ matrix-valued step function with jumps τ_j at the points λ_j, $j = 1, \ldots, r$. The inner product space $L_0^2(\boldsymbol{\tau})$ is positive and has a Hilbert space completion to $L^2(d\tau)$.

Theorem 4.2.2. *The transform $F = Vf$,*

$$F(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W^*(x, \bar{z})\, H(x) f(x)\, dx,$$

acts as a partial isometry from $L^2(Hdx)$ into $L^2(d\tau)$ with initial space $\mathfrak{N}$ equal to the closed span $\mathfrak{N} = \bigvee_{j=0}^{r} \mathfrak{L}_{\lambda_j}$ of all eigenfunctions for the system (4.0.1).

It will be shown in Theorem 4.2.4 that the mapping in Theorem 4.2.2 is always onto. Theorem 4.2.2 constructs a family of pseudospectral functions for the system (1.0.1) in the sense of the following definition.

Definition 4.2.3. A **pseudospectral function** for (1.0.1) is a nondecreasing function $\tau(t)$ of real t such that the transform

$$(Vf)(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W^*(x, \bar{z})\, H(x) f(x)\, dx$$

acts as a partial isometry from $L^2(Hdx)$ into $L^2(d\tau)$. If the partial isometry is an isometry, we call $\tau(t)$ a **spectral function** for (1.0.1). We say that a pseudospectral function $\tau(t)$ is **orthogonal** if the range of the partial isometry is all of $L^2(d\tau)$.

Proof of Theorem 4.2.2. By Theorem 4.1.11(1), V acts isometrically from the linear span of all eigenfunctions into $L^2(d\tau)$. Hence V acts isometrically from $\mathfrak{N}$ into $L^2(d\tau)$. Theorem 4.1.11(2) asserts that every function in $L^2(Hdx)$ which is orthogonal to all eigenfunctions is mapped by V to the zero element of $L^2(d\tau)$. □

Alternate proof of part of Theorem 4.2.2. We give another proof that V is isometric on $\mathfrak{N}$, using resolvent operators and Theorems 4.1.4 and 3.2.4. This argument avoids any use of Lemma 4.1.10 and Theorem 4.1.11.

By (3.2.9), V is a contraction from $L^2(Hdx)$ into $L^2(d\tau)$. It is therefore sufficient to show that for any eigenvalues λ_j and λ_k,

$$\langle Y(x, \lambda_j), Y(x, \lambda_k) \rangle_H = \langle F_j(t), F_k(t) \rangle_{L^2(d\tau)}, \tag{4.2.2}$$

where $Y(x, \lambda_j)$ and $Y(x, \lambda_k)$ are corresponding eigenfunctions and $F_j(z)$ and $F_k(z)$ are their transforms under V. By Theorem 4.1.4,

$$B(z) Y(x, \lambda_j) = \frac{i}{z - \lambda_j} Y(x, \lambda_j)$$

for every z such that $c(z)R + d(z)Q$ is invertible. For such z,

$$\langle B(z)Y(x,\lambda_j), Y(x,\lambda_k)\rangle_H = \frac{i}{z-\lambda_j}\langle Y(x,\lambda_j), Y(x,\lambda_k)\rangle_H.$$

We deduce that

$$\lim_{y\to\infty} y\,\langle B(iy)Y(x,\lambda_j), Y(x,\lambda_k)\rangle_H = \langle Y(x,\lambda_j), Y(x,\lambda_k)\rangle_H\,, \tag{4.2.3}$$

where the limit is through points such that $c(iy)R + d(iy)Q$ is invertible. By the identity (3.2.8) in Theorem 3.2.4,

$$\begin{aligned}\lim_{y\to\infty} y\,\langle B(iy)Y(x,\lambda_j), Y(x,\lambda_k)\rangle_H &= \lim_{y\to\infty}\int_{-\infty}^{\infty}\frac{-iy}{t-iy}\,F_j^*(t)\,d\tau(t)\,F_k(t)\\ &= \langle F_j(t), F_k(t)\rangle_{L^2(d\tau)}.\end{aligned} \tag{4.2.4}$$

We obtain (4.2.2) from (4.2.3) and (4.2.4). □

Theorem 4.2.4. *The pseudospectral function $\tau(t)$ constructed in Theorem* 4.2.2 *is orthogonal.*

Proof. It is sufficient to show that for each $j = 1,\dots,r$, $V\mathfrak{L}_{\lambda_j} = \mathfrak{M}_j$, where $\mathfrak{M}_j$ is the subspace of functions in $L^2(d\tau)$ which are supported at λ_j.

By Lemma 4.1.10, $V\mathfrak{L}_{\lambda_j} \subseteq \mathfrak{M}_j$. Since $V|\mathfrak{L}_{\lambda_j}$ is one-to-one and $\dim\mathfrak{M}_j = \operatorname{rank}\tau_j$, we only need to show that $\dim\mathfrak{L}_{\lambda_j} \geq \operatorname{rank}\tau_j$. For any $g \in \mathbb{C}^m$,

$$Y(x,\lambda_j) = W(x,\lambda_j)\begin{bmatrix}0\\ \tau_j g\end{bmatrix} \in \mathfrak{L}_{\lambda_j} \tag{4.2.5}$$

by Proposition 4.1.9. If $Y = 0$ as an element of $L^2(H dx)$, then

$$\int_0^\ell Y^*(t,\lambda_j)H(t)Y(t,\lambda_j)\,dt = 0. \tag{4.2.6}$$

Since $H(x) \geq 0$ on $[0,\ell]$ we conclude that $H(x)^{1/2}Y(x,\lambda_j) = 0$, and hence

$$\frac{dY}{dx} = izJH(x)Y = 0$$

a.e. on $[0,\ell]$. Thus Y is constant, and so $Y(x,\lambda_j) \equiv \begin{bmatrix}0\\ \tau_j g\end{bmatrix}$. Then by (4.2.6),

$$\begin{bmatrix}0 & g^*\tau_j\end{bmatrix}\int_0^\ell H(t)\,dt\begin{bmatrix}0\\ \tau_j g\end{bmatrix} = 0,$$

and so $\tau_j g = 0$ by the condition (iii′) at the beginning of §2. Therefore we can find a linearly independent set of elements of $\mathfrak{L}_{\lambda_j}$ of the form (4.2.5) containing $\operatorname{rank}\tau_j$ elements. Hence $\dim\mathfrak{L}_{\lambda_j} \geq \operatorname{rank}\tau_j$, and the result follows. □

Theorem 4.2.5. *The following are equivalent.*

(i) *The function $\tau(t)$ in Theorem* 4.2.2 *is an orthogonal spectral function for the system* (1.0.1).

(ii) *The eigenfunctions for* (4.0.1) *are complete in* $L^2(Hdx)$.
(iii) *For some and hence any* z *in the domain of the resolvent,* $\ker B(z) = \{0\}$.

Proof. (i) ⇔ (ii) This follows from Theorems 4.2.2 and 4.2.4.

(ii) ⇔ (iii) By Proposition 4.1.7, $\mathfrak{K} = \ker B(z)$ is independent of z in the domain of the resolvent. Therefore in (iii) it is sufficient to consider some $z = z_0$ such that $z_0 = \bar{z}_0$ and $c(z_0)R + d(z_0)Q$ is invertible. Then $iB(z_0)$ is a compact self-adjoint operator by Proposition 3.1.7. By the spectral theorem, the eigenfunctions for $iB(z_0)$ are complete in $L^2(Hdx)$. By Theorem 4.1.4, the eigenfunctions for $iB(z_0)$ for its nonzero eigenvalues have the same closed span as the eigenfunctions for (4.0.1).

Now assume (ii). Then $L^2(Hdx)$ is the closed span of the eigenfunctions for $iB(z_0)$ for its nonzero eigenvalues. So the origin is not an eigenvalue of $iB(z_0)$. Thus $\ker B(z_0) = \{0\}$, and (iii) follows. The proof that (iii) implies (ii) follows on reversing these steps. □

Corollary 4.2.6. *Conditions* (i)–(iii) *in Theorem* 4.2.5 *hold if* $H(x)$ *has invertible values a.e.*

Proof. We verify condition (iii) in Theorem 4.2.5. Suppose $f \in \ker B(z_0)$ for some real number z_0 such that $c(z_0)R + d(z_0)Q$ is invertible. If $H(x)$ has invertible values, then a function in $L^2(Hdx)$ is equivalent to the zero element of the space if and only if it is equal to zero a.e. Hence by Definition 3.1.6 and (3.1.4),

$$W(x,z_0)\left\{\begin{bmatrix} I_m & 0 \\ -iv(z_0) & 0 \end{bmatrix} J \int_0^x W^*(t,z_0)H(t)f(t)\,dt + \begin{bmatrix} 0 & 0 \\ -iv(z_0) & -I_m \end{bmatrix} J \int_x^\ell W^*(t,z_0)H(t)f(t)\,dt\right\} \equiv 0.$$

Multiply by $W(x,z_0)^{-1}$, then differentiate to get $W^*(x,z_0)H(x)f(x) = 0$ a.e. Again since $H(x)$ has invertible values a.e., it follows that $f(x) = 0$ a.e. This verifies the condition (iii) in Theorem 4.2.5, and so the corollary follows. □

Recall that by Proposition 4.1.7, the subspace $\mathfrak{K} = \ker B(z)$ is independent of z in the domain of the resolvent. Let $\widehat{L}^2(Hdx)$ be as in Definition 4.1.5.

Proposition 4.2.7. (1) *The subspace* $\mathfrak{K} = \ker B(z)$ *contains* $L^2(Hdx) \ominus \widehat{L}^2(Hdx)$.
(2) *The eigenfunctions for* (4.0.1) *are complete in* $L^2(Hdx) \ominus \mathfrak{K}$.

Proof. (1) Let $f \in L^2(Hdx) \ominus \widehat{L}^2(Hdx)$. We must show that $B(z)f = 0$ for z in the domain of the resolvent. We may suppose that $z \in \mathbb{C}_+ \cup \mathbb{C}_-$, in which case we can use the representation (3.2.8). It follows from (3.2.8) that for any $g \in L^2(Hdx)$,

$$\langle iB(z)f, g\rangle_H = \int_{-\infty}^{\infty} \frac{G^*(t)d\tau(t)F(t)}{t-z},$$

where F and G are the transforms of f and g as in (2.2.1). Since we assume that f is orthogonal to $\widehat{L}^2(Hdx)$, $F \equiv 0$ by Proposition 4.1.6(1). Thus $iB(z)f \perp L^2(Hdx)$, and hence $B(z)f = 0$.

(2) Write $\mathfrak{K} = \ker B(z_0)$, where z_0 is a real number such that $c(z_0)R + d(z_0)Q$ is invertible. As in the proof of Theorem 4.2.5, the eigenfunctions for (4.0.1) have the same closed span as the eigenfunctions for $iB(z_0)$ for its nonzero eigenvalues. This closed span is $L^2(Hdx) \ominus \mathfrak{K}$ because $\mathfrak{K} = \ker B(z_0)$. □

In A. L. Sakhnovich [13], the term "pseudospectral function" is used in a little different way from our definition. What is called a "pseudospectral function" in [13] will be called a "strong pseudospectral function" here.

Definition 4.2.8. By a **strong pseudospectral function** for a system (1.0.1) we mean a pseudospectral function $\tau(t)$ such that the kernel of V as an operator from $L^2(Hdx)$ into $L^2(d\tau)$ coincides with the set of all f in $L^2(Hdx)$ such that $(Vf)(z) \equiv 0$ for all z. A **strong spectral function** for (1.0.1) is a spectral function $\tau(t)$ such that, whenever f belongs to $L^2(Hdx)$ and its transform $F = Vf$ is zero in $L^2(d\tau)$, then $(Vf)(z) \equiv 0$ for all z. The term **orthogonal** applied to these notions has the same meaning as in Definition 4.2.3.

Thus if $\tau(t)$ is a pseudospectral function, it may occur that the subspaces

$$\begin{aligned} \mathfrak{K}_+ &= \{f\colon f \in L^2(Hdx) \text{ and } Vf = 0 \text{ in } L^2(d\tau)\}, \\ \mathfrak{K}_- &= \{f\colon f \in L^2(Hdx) \text{ and } (Vf)(z) = 0 \text{ for all } z \in \mathbb{C}\}, \end{aligned} \tag{4.2.7}$$

do not coincide, although in every case $\mathfrak{K}_- \subseteq \mathfrak{K}_+$. The condition for $\tau(t)$ to be a strong pseudospectral function is that $\mathfrak{K}_+ = \mathfrak{K}_-$.

The next result follows easily from Theorem 4 of A. L. Sakhnovich [13]. Set

$$v_0(z) = i[a(z) + b(z)][c(z) + d(z)]^{-1}. \tag{4.2.8}$$

Since $c(z) + d(z)$ is entire and has value I_m for $z = 0$, it is invertible except at isolated points. It is easy to see that $\operatorname{Im} v_0(z) \geq 0$ on $\mathbb{C}_+$ (but $v_0(z) \neq v_0^*(\bar{z})$).

Proposition 4.2.9. *The pseudospectral function $\tau(t)$ constructed in Theorem 4.2.2 is strongly pseudospectral if*

$$\lim_{y\to\infty} \frac{1}{y} \, [c^*(-iy) - d^*(-iy)][v(iy) - v_0(iy)][c(iy) - d(iy)] = 0. \tag{4.2.9}$$

In this case, the closed span $\mathfrak{N} = \bigvee_{j=0}^{r} \mathfrak{L}_{\lambda_j}$ of the eigenfunctions of (4.0.1) is equal to the closed span of all functions

$$Y(x,z) = W(x,z)\begin{bmatrix} 0 \\ g \end{bmatrix}, \qquad z \in \mathbb{C},\ g \in \mathbb{C}^m. \tag{4.2.10}$$

Proof. Define $\mathfrak{K}_+$ and $\mathfrak{K}_-$ by (4.2.7). By Theorem 4.2.2, V acts as a partial isometry from $L^2(Hdx)$ into $L^2(d\tau)$ with initial space $\mathfrak{K}_+^{\perp} = \bigvee_{j=0}^{r} \mathfrak{L}_{\lambda_j}$. By [13, Theorem 4(a)], the condition (4.2.9) implies that the isometric set of V coincides with

$\mathfrak{K}_-^\perp$. Hence $\mathfrak{K}_-^\perp = \mathfrak{K}_+^\perp = \bigvee_{j=0}^r \mathfrak{L}_{\lambda_j}$. In particular, $\mathfrak{K}_- = \mathfrak{K}_+$, and therefore $\tau(t)$ is a strong pseudospectral function. The last statement follows from the equality $(\bigvee_{j=0}^r \mathfrak{L}_{\lambda_j})^\perp = \mathfrak{K}_-$. together with the observation that a function f in $L^2(Hdx)$ belongs to $\mathfrak{K}_-$ if and only if f is orthogonal to all functions of the form (4.2.10). □

Example 4.2.10. A simple example, adapted from Orcutt [9], illustrates some of our results. Take $m = 1$ and fix a number $0 < x_0 < \ell$. Consider a system (1.0.1) with

$$H(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & 0 \le t \le x_0, \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, & x_0 < t \le \ell. \end{cases} \tag{4.2.11}$$

Then $L^2(Hdx)$ is an infinite-dimensional Hilbert space. The intervals $(0, x_0)$ and (x_0, ℓ) are H-indivisible, and therefore the subspace $\widehat{L}^2(Hdx)$ of Definition 4.1.5 is two-dimensional. Straightforward calculations show that

$$W(t,z) = \begin{cases} \begin{bmatrix} 1 & 0 \\ izt & 1 \end{bmatrix}, & 0 \le t \le x_0, \\ \begin{bmatrix} 1 - x_0 z^2 (t - x_0) & iz(t - x_0) \\ i x_0 z & 1 \end{bmatrix}, & x_0 < t \le \ell, \end{cases} \tag{4.2.12}$$

and

$$(Vf)(z) = \int_{x_0}^{\ell} f_2(t)\, dt, \qquad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \in L^2(Hdx). \tag{4.2.13}$$

The functions (2.1.4) are given by

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = \begin{bmatrix} 1 - x_0 z^2 (\ell - x_0) & -i x_0 z \\ -iz(\ell - x_0) & 1 \end{bmatrix}. \tag{4.2.14}$$

As an illustration of Theorem 4.2.2, consider a system (4.0.1), where R and Q are numbers not both zero such that $\bar{R}Q + \bar{Q}R = 0$. Set

$$v(z) = i\,\frac{a(z)R + b(z)Q}{c(z)R + d(z)Q} = i\,\frac{[1 - x_0 z^2 (\ell - x_0)]R - i x_0 z Q}{-iz(\ell - x_0)R + Q}.$$

When $R \neq 0$,

$$v(z) = \frac{\tau_1}{\lambda_1 - z} + x_0 z\,,$$

where $\tau_1 = 1/(\ell - x_0)$, $\lambda_1 = \rho/(\ell - x_0)$, and $\rho = -iQ/R$. Thus $\tau(x)$ is a step function with a single jump at $x = \lambda_1$. The eigenspace $\mathfrak{L}_{\lambda_1}$ is one-dimensional and spanned by

$$Y(t,\lambda_1) = \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & 0 \le t \le x_0, \\ \begin{bmatrix} i\lambda_1(t-x_0) \\ 1 \end{bmatrix}, & x_0 \le t \le \ell. \end{cases} \tag{4.2.15}$$

We easily check that V is a partial isometry from $L^2(Hdx)$ onto $L^2(d\tau)$ with initial space $\mathfrak{L}_{\lambda_1}$. In particular, a pseudospectral function need not be a spectral function. When $R = 0$, $v(z) = x_0 z$ and $\tau(x)$ is constant. There are no poles and no eigenvalues, and we interpret the span of the eigenfunctions to be the zero subspace of $L^2(Hdx)$. Trivially V is the zero operator on $L^2(Hdx)$ to $L^2(d\tau) = \{0\}$.

The question arises if Theorem 4.2.2 generalizes to systems (3.0.1) with non-constant functions $R(z)$ and $Q(z)$. That is, is the function $\tau(x)$ in (3.2.3) is always a pseudospectral function? An example shows that this is not necessarily the case. Choose the Nevanlinna pair

$$R(z) = 1, \qquad Q(z) = -iqz, \qquad z \in \mathbb{C},$$

where $q > 0$. By (4.2.14),

$$v(z) = i\,\frac{a(z)R(z) + b(z)Q(z)}{c(z)R(z) + d(z)Q(z)} = x_0 z - \frac{1}{\ell - x_0 + q}\,\frac{1}{z},$$

and so

$$\tau(x) = \begin{cases} \dfrac{1}{\ell - x_0 + q}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

By (4.2.13), the tranform $F = Vf$ of any f in $L^2(Hdx)$ is constant. The orthogonal complement of $\ker V$ in $L^2(Hdx)$ is spanned by the element

$$f_0(x) = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & 0 < x < x_0, \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & x_0 < x < \ell, \end{cases}$$

whose transform $F_0 = Vf_0$ is given by $F_0(z) = \ell - x_0$. Thus

$$\|F_0\|^2_{L^2(d\tau)} = \frac{(\ell - x_0)^2}{\ell - x_0 + q} < \ell - x_0 = \|f_0\|^2_H,$$

so V is not isometric on the orthogonal complement of its kernel. Hence $\tau(x)$ is not a pseudospectral function. We remark that the inequality $\|F_0\|^2_{L^2(d\tau)} \le \|f_0\|^2_H$ is a special case of (3.2.9).

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