Integral representations for generalized difference kernels having a finite number of negative squares

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Abstract. An integral representation is derived for matrix-valued generalized difference kernels which have a finite number of negative squares. The representation is used to extend such kernels to the real line with a bound on the number of negative squares. The main results are obtained by means of an operator interpolation theorem. The nondegenerate case is assumed.

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1. Introduction

If $s(x)$ is a measurable $m \times m$ matrix-valued function on a finite interval $(-\ell, \ell)$, then the formula

$$(Sf)(x) = \frac{d}{dx} \int_0^\ell s(x-t) f(t)\, dt \tag{1.1}$$

defines a bounded operator on the Lebesgue space $L^2_m(0,\ell)$ of m-dimensional vector-valued functions on $(0,\ell)$ if

(1) for every g in $\mathbb{C}^m$, $s(x)g$ belongs to $L^2_m(-\ell,\ell)$, and
(2) for every f in $L^2_m(0,\ell)$, the function $F(x) = \int_0^\ell s(x-t)f(t)\, dt$ is absolutely continuous and has derivative in $L^2_m(0,\ell)$.

In this case we call $s(x-t)$ a generalized difference kernel. The operator S is selfadjoint if $s(x) = -s(-x)^*$ a.e. on $(-\ell,\ell)$. The class of bounded selfadjoint operators of the form (1.1) coincides with the set of solutions of an operator identity involving the classical Volterra operator (see Theorem 2.1). The class includes many integral operators with ordinary difference kernels. For example, if C is a selfadjoint $m \times m$ matrix and $k(x) = k(-x)^*$ is an integrable $m \times m$ matrix-valued function on $(-\ell, \ell)$, the operator

$$(Sf)(x) = Cf(x) + \int_0^\ell k(x-t)f(t)\,dt \tag{1.2}$$

on $L^2_m(0,\ell)$ has the form (1.1) with $s(x) = \frac{1}{2}\,\mathrm{sgn}(x)\,C + \int_0^x k(u)\,du$ on $(-\ell,\ell)$.

Bounded selfadjoint operators of the form (1.1) which satisfy the additional condition $S \geq 0$ play an important role in many areas including interpolation problems and the spectral theory of canonical differential systems [17, 18, 19]. The condition $S \geq 0$ in (1.1) implies that $s(x)$ has a representation

$$s(x) = \frac{d}{dx}\int_{-\infty}^{\infty}\left[1 + \frac{itx}{1+t^2} - e^{itx}\right]\frac{d\tau(t)}{t^2} + iC_0, \tag{1.3}$$

where $\tau(t)$ is a nondecreasing $m\times m$ matrix-valued function such that $d\tau(t)/(1+t^2)$ is integrable over the real line and C_0 is a constant selfadjoint $m \times m$ matrix (see [18, p. 22] and [20, p. 503]). The formula (1.3) is derived from the Nevanlinna representation of a certain Nevanlinna function which solves a related abstract interpolation problem. The condition $S \geq 0$ is referred to as the definite case.

In this paper we adapt similar ideas to the indefinite theory. The condition $S \geq 0$ is replaced by the requirement that

$$\varkappa_S < \infty, \tag{1.4}$$

that is, the negative spectrum of S consists of finitely many eigenvalues having finite total multiplicity $\varkappa_S$. When (1.4) is satisfied, we say that the generalized difference kernel $s(x-t)$ has a finite number of negative squares. In the indefinite case, we use generalized Nevanlinna functions and recent extensions of the theory of operator identities which appear in [13, 14]. A substitute for the Nevanlinna representation is available in the Kreĭn-Langer integral representation [2, 9, 12] of a generalized Nevanlinna function. Our main result, Theorem 3.1, is a generalization of (1.3) to generalized difference kernels $s(x-t)$ having a finite number of negative squares. For technical reasons, we also assume that S is invertible (nondegenerate case). An immediate consequence of the representation is that $s(x-t)$ can be extended to arbitrarily large intervals with a bound on the number of negative squares (see Theorem 3.3). Our results extend those of A. L. Sakhnovich [15, Section 3].

The study of difference kernels on finite intervals for both scalar- and matrix-valued functions has a long history, which we only partially give. In the case of positive kernels, the study was initiated by Kreĭn [7], who showed that if $k(x)$ is a continuous scalar-valued function on a finite interval $[-\ell,\ell]$ such that the kernel $k(x-t)$ is positive definite, then

$$k(x) = \int_{-\infty}^{\infty} e^{ixt}\,d\tau(t)$$

for some bounded nondecreasing function $\tau(t)$. Analogous problems for operators of the form (1.2) are said to be of accelerant type. See Arov and Dym [1] for a comprehensive account and literature references in the positive case. In the indefinite case, an extension theorem and integral representation for continuous Hermitian kernels was given by Kreĭn [8]. Grossmann and Langer [3, p. 314] and Kaltenbäck, Winkler, and Woracek [5, p. 270] obtain more precise extension theorems for such kernels. Connections with generalized Nevanlinna functions, interpolation theory, and canonical differential systems are detailed in Kreĭn and Langer [9, 10]. Related questions in the indefinite theory are investigated in Kaltenbäck and Woracek [6] and Langer, Langer, and Sasvári [11].

Section 2 is devoted to preliminaries on generalized difference kernels, operator identities, and generalized Nevanlinna functions. Our main results are presented in Section 3. A technical detail is deferred to an appendix.

2. Operator identities and generalized Nevanlinna functions

We use the theory of operator identities

$$AS - SA^* = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right], \tag{2.1}$$

where $A, S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ for some Hilbert spaces $\mathfrak{H}$ and $\mathfrak{G}$ with $\dim \mathfrak{G} < \infty$. In our applications, $\mathfrak{H} = L^2_m(0,\ell)$, $\mathfrak{G} = \mathbb{C}^m$, where ℓ is a positive number and m is a positive integer (which are fixed throughout), and

$$(Af)(x) = i\int_0^x f(t)\, dt, \tag{2.2}$$

$$\Phi_2 g = g, \tag{2.3}$$

for all f in $\mathfrak{H}$ and g in $\mathfrak{G}$.

When A is given by (2.2), an operator S which satisfies an identity of the form (2.1) is necessarily selfadjoint. In fact, by (2.1) the operator $X = S^* - S$ satisfies $AX - XA^* = 0$, and hence $X = 0$, that is, $S = S^*$. Our first result identifies the class of selfadjoint operators satisfying an operator identity (2.1) as the class of selfadjoint operators of the form (1.1).

Theorem 2.1. *Let $\mathfrak{H} = L^2_m(0,\ell)$, $\mathfrak{G} = \mathbb{C}^m$, and define $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ by* (2.2) *and* (2.3)*.*

(i) *If $S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ satisfy* (2.1)*, there exists a measurable $m \times m$ matrix-valued function $s(x)$ on $(-\ell, \ell)$ such that $s(x) = -s(-x)^*$ a.e. and*

$$(Sf)(x) = \frac{d}{dx}\int_0^\ell s(x-t) f(t)\, dt, \tag{2.4}$$

$$(\Phi_1 g)(x) = \varphi_1(x) g, \qquad \varphi_1(x) = s(x), \quad 0 < x < \ell, \tag{2.5}$$

for each f in $\mathfrak{H}$ and g in $\mathfrak{G}$.

(ii) *Conversely, if $S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ have the form* (2.4) *and* (2.5) *and $s(x) = -s(-x)^*$ a.e. on $(-\ell, \ell)$, then S and Φ_1 satisfy* (2.1)*.*

In the scalar case, this result is given in [17, pp. 10,19] The matrix case can be proved similarly using [16, Lemma 2]. For completeness, we outline a direct proof.

Proof. (i) Since $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$, there is a measurable $m \times m$ matrix-valued function $\varphi_1(x)$ on $(0, \ell)$ such that $(\Phi_1 g)(x) = \varphi_1(x) g$ for all g in $\mathfrak{G}$. Define $s(x) = \varphi_1(x)$ on $(0, \ell)$, and extend the definition so that $s(x) = -s(-x)^*$ a.e. on $(-\ell, \ell)$. Then (2.5) holds by construction. To prove (2.4), define an operator $S_1 \in \mathfrak{L}(\mathfrak{H})$ by

$$S_1 f = \int_0^\ell \int_0^t \big[s(x-u) - s(-u)\big]\, du\, f(t)\, dt.$$

A straightforward calculation shows that

$$AS_1 - S_1 A^* = i\, A\big[\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*\big] A^*.$$

The same identity is satisfied with S_1 replaced by ASA^*, and therefore the operator $X = S_1 - ASA^*$ satisfies $AX - XA^* = 0$. It follows that $X = 0$, that is, $S_1 = ASA^*$. Hence for any h in $\mathfrak{H}$,

$$\begin{aligned} ASA^* h &= \int_0^\ell \int_0^t \big[s(x-u) - s(-u)\big]\, du\, h(t)\, dt \\ &= i \int_0^\ell \big[s(x-u) - s(-u)\big](-i) \int_u^\ell h(t)\, dt\, du \\ &= i \int_0^\ell \big[s(x-u) - s(-u)\big](A^* h)(u)\, du. \end{aligned}$$

Therefore for a dense set of functions f in $\mathfrak{H}$,

$$ASf = i \int_0^\ell \big[s(x-t) - s(-t)\big]\, f(t)\, dt. \tag{2.6}$$

By approximation, the same equation holds for all f in $\mathfrak{H}$. In other words, for each f in $\mathfrak{H}$ and x in $(0, \ell)$,

$$\int_0^x (Sf)(t)\, dt = \int_0^\ell s(x-t) f(t)\, dt - \int_0^\ell s(-t) f(t)\, dt. \tag{2.7}$$

This proves (2.4), and (i) follows.

(ii) Let S and Φ_1 have the forms (2.4) and (2.5). From the definition of S we deduce (2.7). Hence AS is given by (2.6). The condition $s(x) = -s(-x)^*$ implies that $S = S^*$, and therefore a formula for $SA^* = (AS)^*$ can obtained from (2.6). Then (2.1) follows by a routine calculation. □

By the **generalized Nevanlinna class $\mathbf{N}_\varkappa$** we mean the set of $m \times m$ matrix-valued meromorphic functions $v(z)$ on the union $\mathbb{C}_+ \cup \mathbb{C}_-$ of the upper and lower half-planes such that $v(z) = v(\bar{z})^*$ and the kernel $[v(z) - v(\zeta)^*]/(z - \bar{\zeta})$ has $\varkappa$ negative squares. The generalized Nevanlinna functions which occur in our applications satisfy

$$\lim_{|y|\to\infty} \frac{v(iy)}{y} = 0. \tag{2.8}$$

Every generalized Nevanlinna function $v(z)$ which satisfies (2.8) has a Kreĭn-Langer integral representation

$$v(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z) \right] d\tau(t) + R(z), \tag{2.9}$$

where $\Delta_1, \dots, \Delta_r$ are bounded open intervals having disjoint closures, Δ_0 is the complement of their union in the real line, and

(1°) there are points $\alpha_j \in \Delta_j$, $j = 1, \dots, r$, and positive integers $\rho_1, \dots, \rho_r$ such that

$$\frac{1}{t-z} - S_j(t,z) = \frac{1}{t-z} \left(\frac{t-\alpha_j}{z-\alpha_j} \right)^{2\rho_j} \qquad \text{on } \Delta_j, \quad j = 1, \dots, r,$$
$$\frac{1}{t-z} - S_0(t,z) = \frac{1+tz}{t-z}\, \frac{1}{1+t^2} \qquad \text{on } \Delta_0\,;$$

(2°) $\tau(t)$ is an $m \times m$ matrix-valued function which is nondecreasing on each of the $r+1$ open intervals determined by $\alpha_1, \dots, \alpha_r$ such that the integral

$$\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\cdots+\rho_r}}\, \frac{d\tau(t)}{1+t^2}$$

is convergent;

(3°) $R(z)$ is an $m \times m$ matrix-valued rational function which is analytic at infinity and satisfies $R(z) = R(\bar z)^*$; equivalently,

$$R(z) = C_0 - \sum_{k=1}^{s} \left[R_k\Big(\frac{1}{z-\lambda_k}\Big) + R_k\Big(\frac{1}{\bar z - \lambda_k}\Big)^* \right], \tag{2.10}$$

where $\lambda_1, \dots, \lambda_s$ are distinct points in the closed upper half-plane, the functions $R_1(z), \dots, R_s(z)$ are polynomials of the form

$$R_k(z) = R_{k1} z + \cdots + R_{k,\sigma_k} z^{\sigma_k}, \qquad k = 1, \dots, s,$$

and $C_0 = R(\infty)$ is a selfadjoint $m \times m$ matrix.

Conversely, every function of the form (2.9) is a generalized Nevanlinna function which satisfies (2.8). A Stieltjes inversion formula recovers increments of $\tau(t)$ in the open intervals of the real line determined by the points $\alpha_1, \dots, \alpha_r$. For details, see [2, 9, 12].

We recall some constructions from [14]. Consider arbitrary Hilbert spaces $\mathfrak{H}$ and $\mathfrak{G}$, $\dim \mathfrak{G} < \infty$, and operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$. Let $v(z)$ be a generalized Nevanlinna function satisfying (2.8) and having Kreĭn-Langer representation (2.9). Assume that the spectrum of A contains only the point at the origin and that

$$\int_{\Delta_0} \langle d\tau(t)\, \Phi_2^*(I-A^*t)^{-1}h, \Phi_2^*(I-A^*t)^{-1}h\rangle < \infty, \qquad h \in \mathfrak{H}. \tag{2.11}$$

Then the formulas

$$S_v = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} - d\tau_j(t;A,\Phi_2) \Big\} \tag{2.12}$$
$$- \frac{1}{2\pi i}\int_\Gamma (I-\lambda A)^{-1}\Phi_2 R(\lambda)\Phi_2^*(I-\lambda A^*)^{-1}\, d\lambda\,,$$

$$i\,\Phi_{1,v} = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ A(I-At)^{-1} - \mathfrak{S}_j(t;A) \Big\}\, \Phi_2\,[d\tau(t)] \tag{2.13}$$
$$- \frac{1}{2\pi i}\int_\Gamma A(I-\lambda A)^{-1}\Phi_2 R(\lambda)\, d\lambda \,-\, \Phi_2 R(\infty)\,.$$

define operators $S_v \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_{1,v} \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$. Here Γ is any closed contour that winds once counterclockwise about each pole of $R(\lambda)$. The convergence terms $d\tau_j(t;A,\Phi_2)$ and $\mathfrak{S}_j(t;A)$, $j=1,\dots,r$, are defined to be the Taylor polynomials about $t=\alpha_j$ of order $2\rho_j - 1$ of the main terms

$$F(t) = (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} \qquad \text{and} \qquad G(t) = A(I-At)^{-1}\,.$$

For $j=0$, $d\tau_0(t;A,\Phi_2)=0$ and $\mathfrak{S}_0(t;A) = -tI/(1+t^2)$. With these choices, the integrals in (2.12) and (2.13) converge weakly by the conditions (1°) and (2°) and the assumption (2.11).

By Theorems 3.4 and 3.5 of [14], the definitions of S_v and $\Phi_{1,v}$ are independent of the choice of Kreĭn-Langer representation, S_v is selfadjoint, and

$$AS_v - S_vA^* = i\,\big[\Phi_{1,v}\Phi_2^* + \Phi_2\Phi_{1,v}^*\big]. \tag{2.14}$$

In an appendix, we prove that

$$\varkappa_{S_v} \le m\Big(\sum_{j=1}^{r}\rho_j + \sum_{k=1}^{s}\sigma_k\Big). \tag{2.15}$$

We return to the case when $\mathfrak{H} = L^2_m(0,\ell)$, $\mathfrak{G} = \mathbb{C}^m$, and A and Φ_2 are given by (2.2) and (2.3). Then the preceding formulas take a more concrete form. In particular, the hypothesis (2.11) is expressed as an integrability property of certain Fourier integrals. Let $v(z)$ be a generalized Nevanlinna function satisfying (2.8) and having Kreĭn-Langer representation (2.9). Since

$$\Phi_2^*(I-A^*z)^{-1}f = \int_0^\ell e^{-izt}f(t)\,dt, \tag{2.16}$$

the condition (2.11) asserts that for all $f \in L^2_m(0,\ell)$,

$$\int_{\Delta_0} F(t)^*\,[d\tau(t)]\,F(t) < \infty \quad \text{where} \quad F(z) = \int_0^\ell e^{-izt}f(t)\,dt. \tag{2.17}$$

In view of the identity

$$\int_0^{2\ell} e^{-izt} f(t)\, dt = \int_0^{\ell} e^{-izt} f(t)\, dt + e^{-iz\ell} \int_0^{\ell} e^{-izt} f(t+\ell)\, dt,$$

this property is independent of the choice of ℓ.

Definition 2.2. A generalized Nevanlinna function $v(z)$ with representation (2.9) is called **admissible** if it satisfies (2.17) for some and hence every $\ell > 0$.

If $v(z)$ is admissible, then operators S_v and $\Phi_{1,v}$ are defined, and they have the forms (2.4) and (2.5) by Theorem 2.1.

Theorem 2.3. *Let $\mathfrak{H} = L^2_m(0,\ell)$ and $\mathfrak{G} = \mathbb{C}^m$, and let A and Φ_2 be given by* (2.2) *and* (2.3). *Let $S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1 \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$ satisfy* (2.1). *If $\varkappa_S < \infty$ and S is invertible, there is an admissible generalized Nevanlinna function $v(z)$ such that*

$$S = S_v \qquad \text{and} \qquad \Phi_1 = \Phi_{1,v}. \tag{2.18}$$

Theorem 2.3 is proved in [13] by an explicit construction of a family of admissible generalized Nevanlinna functions $v(z)$ satisfying (2.18).

3. Main results

Given a nonnegative integer $\varkappa$, let $\mathfrak{S}_{\ell,\varkappa}$ be the set of measurable $m \times m$ matrix-valued functions $s(x)$ on $(-\ell,\ell)$ such that $s(x) = -s(-x)^*$ a.e. and such that (1.1) defines a bounded operator S with $\varkappa_S = \varkappa$.

Theorem 3.1. *Let $s(x) \in \mathfrak{S}_{\ell,\varkappa}$, and assume that the associated operator S defined by* (1.1) *is invertible. Then there is an admissible generalized Nevanlinna function $v(z)$ such that $S = S_v$. If $v(z)$ is represented in the form* (2.9), *then*

$$s(\xi) = \sum_{j=0}^{r} s_j(\xi) + s_d(\xi)\,, \tag{3.1}$$

where

$$s_0(\xi) = \frac{d}{d\xi} \int_{\Delta_0} \left[1 + \frac{it\xi}{1+t^2} - e^{it\xi} \right] \frac{d\tau(t)}{t^2}\,, \tag{3.2}$$

$$s_j(\xi) = \frac{d}{d\xi} \int_{\Delta_j} \left[\frac{1 + it\xi - e^{it\xi}}{t^2} - \sum_{\nu=0}^{2\rho_j - 1} Q_\nu(\alpha_j,\xi)(t-\alpha_j)^\nu \right] d\tau(t)\,, \tag{3.3}$$

$j = 1,\dots,r$, and

$$s_d(\xi) = \sum_{k=1}^{s} \sum_{\nu=1}^{\sigma_k} \left[P_{\nu-1}(\lambda_k,\xi) R_{k\nu} + P_{\nu-1}(\bar\lambda_k,\xi) R^*_{k\nu} \right] \; + \; iC_0, \tag{3.4}$$

a.e. on $(-\ell,\ell)$.

The formulas (3.1)–(3.4) use integrated forms of the Fourier kernel, which we write as

$$\frac{e^{itx}-1}{it} = \sum_{\nu=0}^{\infty} P_\nu(\lambda,x)(t-\lambda)^\nu\,,$$
$$\frac{1+itx-e^{itx}}{t^2} = \sum_{\nu=0}^{\infty} Q_\nu(\lambda,x)(t-\lambda)^\nu\,.$$

For each $\nu = 0,1,2,\dots$,

$$\begin{aligned} P_\nu(\lambda,x) &= \frac{1}{\nu!}\Big(\frac{\partial}{\partial t}\Big)^\nu \frac{e^{itx}-1}{it}\Big|_{t=\lambda} \\ &= \frac{1}{i}\,\frac{(-1)^\nu}{\lambda^{\nu+1}}\left[e^{i\lambda x}\sum_{k=0}^{\nu}\frac{(-i\lambda x)^k}{k!} - 1\right], \\ Q_\nu(\lambda,x) &= \frac{1}{\nu!}\Big(\frac{\partial}{\partial t}\Big)^\nu \frac{1+itx-e^{itx}}{t^2}\Big|_{t=\lambda} \\ &= \frac{(-1)^{\nu+1}}{\lambda^{\nu+2}}\left[e^{i\lambda x}\sum_{k=0}^{\nu}(\nu-k+1)\,\frac{(-i\lambda x)^k}{k!} - \nu - 1 - i\lambda x\right]. \end{aligned}$$

When $\lambda = 0$, these expressions are interpreted by continuity:

$$P_\nu(0,x) = \frac{i^\nu x^{\nu+1}}{(\nu+1)!}\,, \qquad Q_\nu(0,x) = \frac{i^\nu x^{\nu+2}}{(\nu+2)!}\,.$$

The equations (3.2) and (3.3) will be proved in the forms

$$\int_0^\xi s_0(u)\,du = \int_{\Delta_0}\left[\frac{1+it\xi-e^{it\xi}}{t^2} - \frac{it\xi}{1+t^2}\right]d\tau(t), \tag{3.5}$$

$$\int_0^\xi s_j(u)\,du = \int_{\Delta_j}\left[\frac{1+it\xi-e^{it\xi}}{t^2} - \sum_{\nu=0}^{2\rho_j-1} Q_\nu(\alpha_j,\xi)(t-\alpha_j)^\nu\right]d\tau(t)\,, \tag{3.6}$$

where $-\ell < \xi < \ell$. The integrals on the right sides of (3.5) and (3.6) converge absolutely for all real ξ by the condition (2°) in the Kreĭn-Langer representation of $v(z)$.

Proof. Let $\mathfrak{H} = L^2_m(0,\ell)$, $\mathfrak{G} = \mathbb{C}^m$, and define $A, S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G},\mathfrak{H})$ by (2.2)–(2.5). By Theorem 2.1, these operators satisfy (2.1). Since $s(x) \in \mathfrak{S}_{\ell,\varkappa}$, $\varkappa_S < \infty$. Hence by Theorem 2.3, $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for some admissible generalized Nevanlinna function $v(z)$. Let $v(z)$ have the representation (2.9). Write (2.13) in the form $\Phi_{1,v} = \sum_{j=0}^r \Phi_{1,v}^{(j)} + \Phi_{1,v}^{(d)}$, where

$$\Phi_{1,v}^{(j)} = \frac{1}{i}\int_{\Delta_j}\Big\{A(I-At)^{-1} - \mathfrak{S}_j(t;A)\Big\}\,\Phi_2\,[d\tau(t)]\,, \qquad j = 0,\dots,r, \tag{3.7}$$

$$\Phi_{1,v}^{(d)} = -\frac{1}{i}\left[\frac{1}{2\pi i}\int_\Gamma A(I-\lambda A)^{-1}\Phi_2 R(\lambda)\,d\lambda \,+\, \Phi_2 R(\infty)\right]. \tag{3.8}$$

Then

$$\Phi_{1,v}^{(j)} g = s_j(x)g, \quad j=0,\dots,r, \qquad \text{and} \qquad \Phi_{1,v}^{(d)} g = s_d(x)g, \tag{3.9}$$

for some $m\times m$ matrix-valued functions $s_j(x)$, $j=0,\dots,r$, and $s_d(x)$ on $(0,\ell)$. Since $\Phi_1 = \Phi_{1,v}$, (3.1) holds on $(0,\ell)$. If we extend the functions in (3.9) so that $s_j(x) = -s_j(-x)^*$, $j=0,\dots,r$, and $s_d(x) = -s_d(-x)^*$, then (3.1) holds on $(-\ell,\ell)$.

It remains to prove (3.5), (3.6), and (3.4). It is sufficient to take $0<\xi<\ell$. In fact, the integrals on the right sides of (3.5) and (3.6) are unchanged when they are conjugated and ξ is replaced by $-\xi$, and from this property we easily see that the identities (3.5) and (3.6) hold on $(-\ell,\ell)$ if they hold on $(0,\ell)$. Similarly, (3.4) holds on $(-\ell,\ell)$ if it holds on $(0,\ell)$.

Proofs of (3.5) **and** (3.6)**.** Fix ξ in $(0,\ell)$. Then

$$\int_0^\xi g_2^* s_j(u) g_1\,du = \left\langle \Phi_{1,v}^{(j)} g_1, h_\xi \right\rangle, \qquad j=0,\dots,r, \tag{3.10}$$

where g_1, g_2 are arbitrary vectors in $\mathfrak{G}$ and

$$h_\xi(x) = \begin{cases} g_2, & 0<x<\xi,\\ 0, & \xi<x<\ell. \end{cases}$$

Case 1: $j=0$**.** As in the definite case [18, p. 2], we use the identity

$$A(I-At)^{-1} - \mathfrak{S}_0(t;A) = A(I-At)^{-1} + \frac{tI}{1+t^2} = \frac{(A+tI)(I-tA)^{-1}}{1+t^2}$$

and (3.7) to write

$$\begin{aligned}\left\langle \Phi_{1,v}^{(0)} g_1, h_\xi \right\rangle &= \frac{1}{i}\int_{\Delta_0} \left\langle (A+tI)(I-tA)^{-1}\Phi_2\,d\tau(t)\,\frac{g_1}{1+t^2}, h_\xi \right\rangle \\ &= \frac{1}{i}\int_{\Delta_0} \left\langle d\tau(t)\,\frac{g_1}{1+t^2}, \Phi_2^*(I-tA^*)^{-1}A^* h_\xi \right\rangle \\ &\qquad + \frac{1}{i}\int_{\Delta_0} \left\langle d\tau(t)\,\frac{tg_1}{1+t^2}, \Phi_2^*(I-tA^*)^{-1} h_\xi \right\rangle .\end{aligned}$$

Short calculations yield

$$\Phi_2^*(I-tA^*)^{-1}h_\xi = i\,\frac{e^{-it\xi}-1}{t}\,g_2\,,$$

$$\Phi_2^*(I-tA^*)^{-1}A^*h_\xi = i\,\frac{e^{-it\xi}-1+it\xi}{t^2}\,g_2\,.$$

Thus by (3.10),

$$\begin{aligned}\int_0^\xi g_2^* s_0(u) g_1\, du &= \left\langle \Phi_{1,v}^{(0)} g_1, h_\xi \right\rangle \\ &= \frac{1}{i}\int_{\Delta_0} \left\langle d\tau(t)\, \frac{g_1}{1+t^2}, i\, \frac{e^{-it\xi}-1+it\xi}{t^2}\, g_2 \right\rangle \\ &\quad + \frac{1}{i}\int_{\Delta_0} \left\langle d\tau(t)\, \frac{tg_1}{1+t^2}, i\, \frac{e^{-it\xi}-1}{t}\, g_2 \right\rangle \\ &= \frac{1}{i}\int_{\Delta_0} \left[-i\, \frac{e^{it\xi}-1-it\xi}{t^2(1+t^2)} - i\, \frac{e^{it\xi}-1}{1+t^2} \right] g_2^* d\tau(t) g_1 \\ &= \int_{\Delta_0} \left[\frac{1+it\xi-e^{it\xi}}{t^2} - \frac{it\xi}{1+t^2} \right] g_2^* d\tau(t) g_1 .\end{aligned}$$

This proves (3.5).

Case 2: $j = 1, \dots, r$**.** By (3.7),

$$\begin{aligned}&\left\langle \Phi_{1,v}^{(j)} g_1, h_\xi \right\rangle \\ &\quad = \frac{1}{i}\int_{\Delta_j} \left\langle d\tau(t)(t-\alpha_j)^{\rho_j} g_1\,, \frac{\Phi_2^*\big[A^*(I-tA^*)^{-1} - \mathfrak{S}_j(t;A)^*\big] h_\xi}{(t-\alpha_j)^{\rho_j}} \right\rangle .\end{aligned} \tag{3.11}$$

By [14, Theorem 3.2], the convergence term $\mathfrak{S}_j(t;A)$ is given by

$$\mathfrak{S}_j(t;A) = \sum_{\nu=0}^{2\rho_j-1} A^{\nu+1}(I-\alpha_j A)^{-\nu-1}(t-\alpha_j)^\nu .$$

By induction, for any $g \in \mathfrak{G}$,

$$A^{\nu+1}(I-\lambda A)^{-\nu-1}\Phi_2 g = iP_\nu(\lambda,x) g\,, \qquad \nu = 0,1,2,\dots . \tag{3.12}$$

Hence

$$\begin{aligned}&\big[A(I-tA)^{-1} - \mathfrak{S}_j(t;A)\big]\, \Phi_2 g \\ &\qquad = A(I-tA)^{-1}\Phi_2 g - \sum_{\nu=0}^{2\rho_j-1} A^{\nu+1}(I-\alpha_j A)^{-\nu-1}\Phi_2 g\, (t-\alpha_j)^\nu \\ &\qquad = \left[\frac{e^{itx}-1}{t} - \sum_{\nu=0}^{2\rho_j-1} iP_\nu(\alpha_j,x)(t-\alpha_j)^\nu \right] g\,,\end{aligned}$$

and so

$$\Phi_2^*\big[A^*(I-tA^*)^{-1} - \mathfrak{S}_j(t;A)^*\big] h_\xi$$

$$= \int_0^\xi \left[\frac{e^{-itx} - 1}{t} - \sum_{\nu=0}^{2\rho_j - 1} [iP_\nu(\alpha_j, x)]^- (t - \alpha_j)^\nu \right] dx\, g_2 \,.$$

Thus (3.11) yields

$$\left\langle \Phi_{1,v}^{(j)} g_1, h_\xi \right\rangle = \frac{1}{i} \int_{\Delta_j} \int_0^\xi \left[\frac{e^{itx} - 1}{t} - \sum_{\nu=0}^{2\rho_j - 1} iP_\nu(\alpha_j, x)(t - \alpha_j)^\nu \right] dx\, g_2^* \, d\tau(t)\, g_1$$

$$= \int_{\Delta_j} \left[\frac{1 + it\xi - e^{it\xi}}{t^2} - \sum_{\nu=0}^{2\rho_j - 1} Q_\nu(\alpha_j, \xi)(t - \alpha_j)^\nu \right] g_2^* \, d\tau(t)\, g_1 \,,$$

where in the last equality we used the identity

$$Q_\nu(\lambda, \xi) = \int_0^\xi P_\nu(\lambda, x)\, dx, \qquad \nu = 0, 1, 2, \dots \,.$$

By (3.10),

$$\int_0^\xi g_2^* s_j(u) g_1 \, du = \left\langle \Phi_{1,v}^{(j)} g_1, h_\xi \right\rangle$$

$$= \int_{\Delta_j} \left[\frac{1 + it\xi - e^{it\xi}}{t^2} - \sum_{\nu=0}^{2\rho_j - 1} Q_\nu(\alpha_j, \xi)(t - \alpha_j)^\nu \right] g_2^* \, d\tau(t)\, g_1,$$

and (3.6) follows.

Proof of (3.4)**.** The integral in (3.8) can be evaluated as a sum of residues:

$$\Phi_{1,v}^{(d)} = i \sum_{k=1}^{s} \operatorname*{Res}_{\lambda=\lambda_k} A(I - \lambda A)^{-1} \Phi_2 R_k \left(\frac{1}{\lambda - \lambda_k} \right)$$

$$+ i \sum_{k=1}^{s} \operatorname*{Res}_{\lambda=\bar\lambda_k} A(I - \lambda A)^{-1} \Phi_2 R_k \left(\frac{1}{\bar\lambda - \lambda_k} \right)^* \; + \; i\, \Phi_2 C_0 \,.$$

For each $k = 1, \dots, s$, by (3.12),

$$\operatorname*{Res}_{\lambda=\lambda_k} A(I - \lambda A)^{-1} \Phi_2 R_k \left(\frac{1}{\lambda - \lambda_k} \right) g = \sum_{\nu=1}^{\sigma_k} \operatorname*{Res}_{\lambda=\lambda_k} \frac{A(I - \lambda A)^{-1} \Phi_2 R_{k\nu}}{(\lambda - \lambda_k)^\nu} g$$

$$= \sum_{\nu=1}^{\sigma_k} A^\nu (I - \lambda_k A)^\nu \Phi_2 R_{k\nu} g = \sum_{\nu=1}^{\sigma_k} iP_{\nu-1}(\lambda_k, x) R_{k\nu} g.$$

The residue is calculated with the aid of the formula

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{(I - \lambda A)^{-1}}{(\lambda - \lambda_0)^\nu} = A^{\nu-1} (I - \lambda_0 A)^{-\nu}.$$

A similar equation holds with λ_k replaced by $\bar\lambda_k$ and $R_{k\nu}$ replaced by $R_{k\nu}^*$. This yields (3.4). □

We prove a converse result.

Theorem 3.2. *Let $v(z)$ be an admissible generalized Nevanlinna function having the representation* (2.9)*. Then* (3.1)–(3.4) *define a measurable $m \times m$ matrix-valued function $s(x)$ on $(-\infty, \infty)$ such that for every $\ell > 0$, the restriction of $s(x)$ to $(-\ell, \ell)$ belongs to a class $\mathfrak{S}_{\ell, \varkappa_\ell}$ with $\varkappa_\ell \le m\big(\sum_{j=1}^r \rho_j + \sum_{k=1}^s \sigma_k\big)$.*

Proof. Fix ℓ, and let $\mathfrak{H} = L^2_m(0, \ell)$ and $\mathfrak{G} = \mathbb{C}^m$. By the definition of an admissible function, operators $S_v \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_{1,v} \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ are defined for the given function $v(z)$. These operators satisfy (2.14) with A and Φ_2 given by (2.2) and (2.3). Hence by Theorem 2.1,

$$(S_v f)(x) = \frac{d}{dx} \int_0^\ell s_v(x-t) f(t)\, dt,$$

$$(\Phi_{1,v} g)(x) = \varphi_{1,v}(x) g, \qquad \varphi_{1,v}(x) = s_v(x), \quad 0 < x < \ell,$$

for some $m \times m$ matrix-valued function $s_v(x) = -s_v(-x)^*$ on $(-\ell, \ell)$. The function $s_v(x)$ is calculated from $v(z)$ as in proof of Theorem 3.1 (the invertibility hypothesis in Theorem 3.1 is not used in the calculation). The calculation shows that $s_v(x)$ agrees with the restriction of the function $s(x)$ defined by (3.1)–(3.4) to $(-\ell, \ell)$. Thus the restriction of $s(x)$ to $(-\ell, \ell)$ belongs to $\mathfrak{S}_{\ell, \varkappa_\ell}$, where $\varkappa_\ell = \varkappa_{S_v}$. By (2.15), $\varkappa_\ell \le m\big(\sum_{j=1}^r \rho_j + \sum_{k=1}^s \sigma_k\big)$. □

Theorems 3.1 and 3.2 together provide an extension theorem.

Theorem 3.3. *Let $s(x) \in \mathfrak{S}_{\ell, \varkappa}$, and assume that the corresponding operator* (1.1) *is invertible. Then $s(x)$ has an extension to the real line such that for every $\tilde{\ell} > \ell$, $s(x)$ belongs to a class $\mathfrak{S}_{\tilde{\ell}, \tilde{\varkappa}}$ with $\tilde{\varkappa} \le m\big(\sum_{j=1}^r \rho_j + \sum_{k=1}^s \sigma_k\big)$.*

Proof. Theorem 3.1 represents $s(x)$ in the form (3.1)–(3.4). Then the required extensions are provided by Theorem 3.2. □

4. Appendix: An estimate of negative squares

We sketch a proof of the inequality (2.15) for any generalized Nevanlinna function $v(z)$ represented in the form (2.9). As a preliminary, note the identities

$$\sum_{\nu=0}^{n} \sum_{\substack{p+q=\nu \\ p,q \ge 0}} X_p^* M_\nu X_q = X^* M X, \tag{4.1}$$

$$\sum_{\nu=0}^{n} \sum_{\substack{p+q=\nu \\ p,q \ge 0}} \Big[X_p^* M_\nu Y_q + Y_p^* M_\nu^* X_q \Big] = \begin{bmatrix} X \\ Y \end{bmatrix}^* \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \tag{4.2}$$

which hold for any block operator matrices

$$X = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{bmatrix}, \qquad M = \begin{bmatrix} M_0 & M_2 & \cdots & M_{n-1} & M_n \\ M_1 & M_3 & \cdots & M_n & 0 \\ & & \cdots & & \\ M_n & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Lemma 4.1. *Let M be a selfadjoint $2k \times 2k$ matrix of the form*

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},$$

where each block has size $k \times k$. Then $\varkappa_M \le k$.

Proof. Define M_ε by the same formula with A and B replaced by $A_\varepsilon = A + \varepsilon I$ and $B_\varepsilon = B + \varepsilon I$ for any $\varepsilon > 0$. Choose $\delta > 0$ such that A_ε and B_ε are invertible for $0 < \varepsilon < \delta$. Since

$$M_\varepsilon = \begin{bmatrix} I & 0 \\ B_\varepsilon^* A_\varepsilon^{-1} & I \end{bmatrix} \begin{bmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon^* A_\varepsilon^{-1} B_\varepsilon \end{bmatrix} \begin{bmatrix} I & A_\varepsilon^{-1} B_\varepsilon \\ 0 & I \end{bmatrix},$$

two applications of Sylvester's law of inertia [4, p. 223] show that $\varkappa_{M_\varepsilon} = k$ for $0 < \varepsilon < \delta$. In the limit, we get $\varkappa_M \le k$ by [4, p. 540]. □

Proof of (2.15). It is sufficient to prove (2.15) when $v(z)$ is a single term in (2.9).

Case 1: $v(z) = \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z) \right] d\tau(t)$ for some $j = 1, \dots, r$

We show that $\varkappa_{S_v} \le m\rho_j$ in this case. By definition,

$$S_v = \int_{\Delta_j} \Big\{ (I - At)^{-1} \Phi_2 [d\tau(t)] \Phi_2^* (I - A^* t)^{-1} - d\tau_j(t; A, \Phi_2) \Big\}.$$

By [14, Theorem 3.2],

$$d\tau_j(t; A, \Phi_2) = \sum_{\nu=0}^{2\rho_j - 1} \sum_{\substack{p+q=\nu \\ p,q \ge 0}} A_p(\alpha_j) \Phi_2 (t - \alpha_j)^\nu \, [d\tau(t)] \, \Phi_2^* A_q(\alpha_j)^* \Big\},$$

where $A_p(\lambda) = A^p (I - \lambda A)^{-p-1}$ for each $p \ge 0$. Thus

$$S_v = \int_{\Delta_j} \Big\{ (I - At)^{-1} \Phi_2 [d\tau(t)] \Phi_2^* (I - A^* t)^{-1} - \sum_{\nu=0}^{2\rho_j - 1} \sum_{\substack{p+q=\nu \\ p,q \ge 0}} A_p(\alpha_j) \Phi_2 (t - \alpha_j)^\nu \, [d\tau(t)] \, \Phi_2^* A_q(\alpha_j)^* \Big\}.$$

By approximation, we may assume that $\tau(t)$ is constant in small intervals $(\alpha_j - \varepsilon, \alpha_j)$ and $(\alpha_j, \alpha_j + \varepsilon)$. In this case, $S_v = T_1 + T_2$, where

$$T_1 = \int_{\Delta_j} (I - At)^{-1} \Phi_2 [d\tau(t)] \Phi_2^* (I - A^* t)^{-1},$$

$$T_2 = \sum_{\nu=0}^{2\rho_j-1} \sum_{\substack{p+q=\nu \\ p,q\geq 0}} A_p(\alpha_j)\Phi_2 H_\nu \Phi_2^* A_q(\alpha_j)^*,$$

and

$$H_\nu = -\int_{\Delta_j} (t-\alpha_j)^\nu \, d\tau(t), \qquad \nu = 0,1,2,\dots .$$

Clearly, $T_1 \geq 0$, so $\varkappa_{T_1} = 0$. By (4.1) and Lemma 4.1, $\varkappa_{T_2} \leq m\rho_j$, and hence $\varkappa_{S_v} \leq \varkappa_{T_1} + \varkappa_{T_2} \leq m\rho_j$.

Case 2: $v(z) = \displaystyle\int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z)\right] d\tau(t)$

In this case, the operator

$$S_v = \int_{\Delta_0} (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*(I-A^*t)^{-1}$$

is nonnegative, and so $\varkappa_{S_v} = 0$.

Case 3: $v(z)$ is one of the terms in (2.10)

The constant C_0 makes no contribution, so suppose that

$$v(z) = -R_k\Big(\frac{1}{z-\lambda_k}\Big) - R_k\Big(\frac{1}{\bar z - \lambda_k}\Big)^*,$$

where $R_k(z) = R_{k1}z + \cdots + R_{k,\sigma_k}z^{\sigma_k}$ for some $k = 1,\dots,s$. Then

$$\begin{aligned} S_v &= \frac{1}{2\pi i}\int_\Gamma (I-\lambda A)^{-1}\Phi_2 R_k\Big(\frac{1}{\lambda-\lambda_k}\Big)\Phi_2^*(I-\lambda A^*)^{-1}\,d\lambda \\ &\quad + \frac{1}{2\pi i}\int_\Gamma (I-\lambda A)^{-1}\Phi_2 R_k\Big(\frac{1}{\bar\lambda-\lambda_k}\Big)^*\Phi_2^*(I-\lambda A^*)^{-1}\,d\lambda \\ &= \sum_{\nu=1}^{\sigma_k} \operatorname*{Res}_{\lambda=\lambda_k} \frac{(I-\lambda A)^{-1}\Phi_2 R_{k\nu}\Phi_2^*(I-\lambda A^*)^{-1}}{(\lambda-\lambda_k)^\nu} \\ &\quad + \sum_{\nu=1}^{\sigma_k} \operatorname*{Res}_{\lambda=\bar\lambda_k} \frac{(I-\lambda A)^{-1}\Phi_2 R_{k\nu}^*\Phi_2^*(I-\lambda A^*)^{-1}}{(\lambda-\bar\lambda_k)^\nu} . \end{aligned}$$

To evaluate the residues, we use the formula

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{(I-\lambda A)^{-1}C(I-\lambda A^*)^{-1}}{(\lambda-\lambda_0)^\nu} = \sum_{\substack{p+q=\nu-1 \\ p,q\geq 0}} A_p(\lambda_0)CA_q(\bar\lambda_0)^*,$$

where $A_p(\lambda) = A^p(I-\lambda A)^{-p-1}$ for each $p \geq 0$. This yields

$$S_v = \sum_{\nu=1}^{\sigma_k} \sum_{\substack{p+q=\nu-1 \\ p,q\geq 0}} \Big[A_p(\lambda_k)\Phi_2 R_{k\nu}\Phi_2^* A_q(\bar\lambda_k)^* + A_p(\bar\lambda_k)\Phi_2 R_{k\nu}^*\Phi_2^* A_q(\lambda_k)^*\Big].$$

Then by (4.2) and Lemma 4.1, $\varkappa_{S_v} \leq m\sigma_k$.

All terms in the formula (2.12) have been considered, and the proof of (2.15) is complete. □

Added in proof. In Theorems 2.3 and 3.1, $v(z)$ can be chosen such that $\varkappa_v \leq \varkappa_S$, where $\varkappa_v$ is the number of negative squares of $[v(z) - v(\zeta)^*]/(z - \bar{\zeta})$. We omit the details but remark that this follows from the proof of [13, Theorem 3.1], where it can be seen that $v(z)$ may be chosen with this property.

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