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INTERPOLATION PROBLEMS FOR MATRIX INTEGRO-DIFFERENTIAL OPERATORS WITH DIFFERENCE KERNELS AND WITH A FINITE NUMBER OF NEGATIVE SQUARES

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To the memory of Tiberiu Constantinescu.

ABSTRACT. The method of operator identities is used to investigate a continual analog of the Pick-Nevanlinna interpolation problem for matrix-valued generalized Nevanlinna functions. It is shown that solutions always exist in the nondegenerate case. A large class of solutions is parametrized in a linear fractional representation.

KEYWORDS: *Interpolation, operator identity, integro-differential operator, difference kernel, generalized Nevanlinna function.*

MSC (2000): Primary 47A57; Secondary 47B50.

1. INTRODUCTION

Let $\mathfrak{H}$ and $\mathfrak{G}$ be Hilbert spaces with $\dim \mathfrak{G} < \infty$. We are concerned with operator identities of the form

$$(1.1)\qquad \begin{cases} AS - SA^* = \mathrm{i}\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right], \\ S = S^*, \quad A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}). \end{cases}$$

Identities of this type provide a framework for interpolation theory and the spectral theory of canonical differential equations. Systematic accounts in the definite case, $S \geq 0$, are given in [10, 11], and indefinite generalizations appear, for example, in [4, 5]. Every operator identity (1.1) gives rise to an interpolation problem (1.2). The problem reduces to classical Pick-Nevanlinna interpolation when A, S, Φ_1, Φ_2 have the forms (1.3) and (1.4). In this paper, we study the interpolation problem for operators A, S, Φ_1, Φ_2 of the form (1.5)–(1.8). Such operators are important in a number of applied and theoretical problems [9]. In this case, solutions of the interpolation problem (1.2) give solutions of corresponding extension problems and spectral problems for canonical differential equations [4, 7, 11].

General criteria for the existence of solutions of an interpolation problem (1.2) are derived in [5]. The criteria involve checking a number of technical conditions. Our main result verifies that these conditions are always met for operators A, S, Φ_1, Φ_2 of the form (1.5)–(1.8) which have these two properties: (1) $\varkappa_S < \infty$, that is, the negative spectrum of S consists of a finite number of eigenvalues having total multiplicity $\varkappa_S$, and (2) S is invertible (nondegenerate case). We deduce not only the existence of solutions but also an explicit linear fractional representation for a large class of solutions. An application from [3], a representation theorem for Hermitian difference kernels, is stated without proof.

By the **generalized Nevanlinna class** $\mathbf{N}_\varkappa$ we mean the set of $m \times m$ matrix-valued meromorphic functions $v(z)$ on the union $\mathbf{C}_+ \cup \mathbf{C}_-$ of the upper and lower half-planes such that $v(z) = v(\bar{z})^*$ and the kernel $[v(z) - v(\zeta)^*]/(z - \bar{\zeta})$ has $\varkappa$ negative squares.

A class of interpolation problems is defined by specifying operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$. Within the class, the interpolation problem for a given operator identity (1.1) is to find all generalized Nevanlinna functions $v(z)$ such that

$$S = S_v \qquad \text{and} \qquad \Phi_1 = \Phi_{1,v}. \tag{1.2}$$

Here $S_v \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_{1,v} \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ are operators which are constructed from the given operators A, Φ_2 and a generalized Nevanlinna function $v(z)$, assuming that certain technical conditions are met. Any such operators satisfy

$$AS_v - S_v A^* = \mathrm{i}\left[\Phi_{1,v}\Phi_2^* + \Phi_2\Phi_{1,v}^*\right].$$

The definitions of S_v and $\Phi_{1,v}$ will be given in Section 2 below.

The interpolation problem (1.2) reduces to Pick-Nevanlinna interpolation when

$$A = \begin{bmatrix} z_1 I_m & 0 & \cdots & 0 \\ 0 & z_2 I_m & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & z_n I_m \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}, \tag{1.3}$$

where $\mathfrak{H} = \mathbf{C^m} \oplus \cdots \oplus \mathbf{C^m}$, $\mathfrak{G} = \mathbf{C^m}$, and $z_1, \ldots, z_n$ are distinct points in $\mathbf{C}_+$. For a given generalized Nevanlinna function $v(z)$ (see [5, Theorem 3.7]),

$$S_v = \left[\frac{w(z_j) - w(z_k)^*}{z_j - \bar{z}_k}\right]_{j,k=1}^n, \qquad \Phi_{1,v} = -\mathrm{i}\begin{bmatrix} w(z_1) \\ w(z_2) \\ \vdots \\ w(z_n) \end{bmatrix},$$

where $w(z) = -v(1/\bar{z})^*$. The definitions of S_v and $\Phi_{1,v}$ in [5] require that the poles of $w(z)$ are disjoint from $z_1, \ldots, z_n$. On the other hand, the identity (1.1) is satisfied by arbitrary data

$$S = \left[\frac{w_j - w_k^*}{z_j - \bar{z}_k} \right]_{j,k=1}^{n}, \qquad \Phi_1 = -\mathrm{i} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \tag{1.4}$$

where $w_1, \dots, w_n$ are $m \times m$ matrices. Thus when the operators A, S, Φ_1, Φ_2 are chosen as in (1.3) and (1.4), the interpolation problem (1.2) is equivalent to the classical Pick-Nevanlinna interpolation problem $w(z_j) = w_j$, $j = 1, \dots, n$.

We study a continual counterpart to the Pick-Nevanlinna problem in which (1.3) and (1.4) are replaced by operators of the form (1.5)–(1.8) below. Let $\mathfrak{H} = L^2_m(0,\ell)$ for a fixed positive number ℓ, and let $\mathfrak{G} = \mathbf{C^m}$. For any $f \in \mathfrak{H}$, $g \in \mathfrak{G}$, set

$$(Af)(x) = \mathrm{i} \int_0^x f(t)\, \mathrm{d}t, \tag{1.5}$$

$$(\Phi_2 g)(x) = \varphi_2(x) g, \qquad \varphi_2(x) \equiv I_m. \tag{1.6}$$

Define $S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ by requiring that

$$(Sf)(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\ell s(x-t) f(t)\, \mathrm{d}t, \tag{1.7}$$

$$(\Phi_1 g)(x) = \varphi_1(x) g, \qquad \varphi_1(x) = s(x), \quad 0 < x < \ell, \tag{1.8}$$

for all $f \in \mathfrak{H}$ and $g \in \mathfrak{G}$. Here we assume that $s(x)$ is an $m \times m$ matrix-valued function on $(-\ell, \ell)$ satisfying $s(x) = -s(-x)^*$ a.e. such that

(1) for every g in $\mathfrak{G}$, $s(x)g$ belongs to $L^2_m(0,\ell)$;
(2) for every f in $L^2_m(0,\ell)$, the function $H(x) = \int_0^\ell s(x-t) f(t)\, \mathrm{d}t$ is the indefinite integral of a function h in $L^2_m(0,\ell)$.

By the closed graph theorem, S is an everywhere defined and bounded operator on $\mathfrak{H}$. Any operators A, S, Φ_1, Φ_2 of the form (1.5)–(1.8) satisfy (1.1). Conversely, for any operator identity (1.1) with A and Φ_2 given by (1.5) and (1.6), S and Φ_1 have the form (1.7) and (1.8). In the scalar case, these assertions are proved in [9] (see Theorem 1.2 on p. 10 and Theorem 2.2 on p. 19). The matrix case is handled similarly (see [8, Lemma 2]).

The class of integro-differential operators (1.7) includes common forms of operators, such as

$$(Sf)(x) = f(x) + \int_0^\ell k(x-t) f(t)\, \mathrm{d}t, \tag{1.9}$$

where $k(x) = k(-x)^*$ is a bounded continuous $m \times m$ matrix-valued function on $(-\ell, \ell)$. In fact, (1.9) has the form (1.7) with

$$s(x) = \begin{cases} \frac{1}{2} I_m + \int_0^x k(u)\, \mathrm{d}u, & 0 < x < \ell, \\ -\frac{1}{2} I_m + \int_0^x k(u)\, \mathrm{d}u, & -\ell < x < 0. \end{cases} \tag{1.10}$$

Section 2 reviews background on operator identities for any Hilbert space $\mathfrak{H}$. In Section 3, we specialize to operators of the form (1.5)–(1.8) on the Hilbert space $L^2_m(0, \ell)$. Our main result, Theorem 3.1, establishes the existence of a large class of solutions of the interpolation problem in the nondegenerate case. The scalar case of Theorem 3.1, is due to A. L. Sakhnovich [7, Example 21, pp. 198–199]. An example of Theorem 3.1 is stated at the end of Section 3.

Notation. Throughout we take $\mathfrak{G} = \mathbf{C^m}$ in the Euclidean metric. Elements of $\mathfrak{G}$ are written as column vectors, and operators on $\mathfrak{G}$ are written as $m \times m$ matrices. We use $|\cdot|$ for the norm of an m-dimensional vector or $m \times m$ matrix. We write $g_1^* g_2 = \langle g_1, g_2 \rangle$ for any $g_1, g_2 \in \mathfrak{G}$.

2. PRELIMINARIES

Given a Hilbert space $\mathfrak{H}$ and operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$, we shall define operators S_v and $\Phi_{1,v}$, depending on an $m \times m$ matrix-valued generalized Nevanlinna function $v(z)$, such that the operator identity (1.1) is satisfied with $S = S_v$ and $\Phi_1 = \Phi_{1,v}$. The definitions use the Kreĭn-Langer integral representation of $v(z)$. The functions which occur in our applications satisfy

$$\lim_{|y| \to \infty} \frac{v(\mathrm{i}y)}{y} = 0. \tag{2.1}$$

In this case, the Kreĭn-Langer representation of $v(z)$ takes the form [1, 2, 6]

$$v(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z) \right] \mathrm{d}\tau(t) + R(z), \tag{2.2}$$

where $\Delta_1, \dots, \Delta_r$ are bounded open intervals having disjoint closures, Δ_0 is the complement of their union in the real line, and

(1°) there are points $\alpha_1, \dots, \alpha_r$ and positive integers $\rho_1, \dots, \rho_r$ such that $\alpha_j \in \Delta_j$, $j = 1, \dots, r$, and

$$\frac{1}{t-z} - S_j(t,z) = \frac{1}{t-z} \left(\frac{t-\alpha_j}{z-\alpha_j} \right)^{2\rho_j} \qquad \text{on } \Delta_j, \quad j = 1, \dots, r,$$

$$\frac{1}{t-z} - S_0(t,z) = \frac{1+tz}{t-z} \, \frac{1}{1+t^2} \qquad \text{on } \Delta_0\,;$$

(2°) $\tau(t)$ is an $m \times m$ matrix-valued function which is nondecreasing on each of the $r+1$ open intervals determined by $\alpha_1, \ldots, \alpha_r$ such that the integral

$$\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\cdots+\rho_r}} \, \frac{\mathrm{d}\tau(t)}{1+t^2}$$

is convergent;

(3°) $R(z)$ is an $m \times m$ matrix-valued rational function which is analytic at infinity and satisfies $R(z) = R(\bar{z})^*$; equivalently,

$$R(z) = C_0 - \sum_{k=1}^{s} \left[R_k\Big(\frac{1}{z-\lambda_k}\Big) + R_k\Big(\frac{1}{\bar{z}-\lambda_k}\Big)^* \right], \tag{2.3}$$

where $\lambda_1, \ldots, \lambda_s$ are distinct points in the closed upper half-plane, the functions $R_1(z), \ldots, R_s(z)$ are polynomials of the form

$$R_k(z) = R_{k1} z + \cdots + R_{k,\sigma_k} z^{\sigma_k}, \qquad k = 1, \ldots, s,$$

and $C_0 = R(\infty)$ is a selfadjoint $m \times m$ matrix.

A Stieltjes inversion formula recovers the increments of $\tau(t)$ from $v(z)$ in the intervals determined by the points $\alpha_1, \ldots, \alpha_r$ (see [6, Corollary 3.3]). However, the representation (2.2) itself is not unique. Conversely, every function of the form (2.2) satisfying the conditions (1°)–(3°) belongs to some class $\mathbf{N}_\varkappa$ and satisfies (2.1).

The definitions of S_v and $\Phi_{1,v}$ require that certain assumptions are met. In our applications $\sigma(A) = \{0\}$, and we restrict attention to this case. See [5] for the variations needed when $\sigma(A)$ contains nonzero points.

ASSUMPTIONS 2.1. *Let $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ be given operators, and let $v(z)$ be a generalized Nevanlinna function. We assume:*

(1) $v(\mathrm{i}y)/y \to 0$ *as* $|y| \to \infty$;

(2) $\sigma(A) = \{0\}$;

(3) *if $v(z)$ is represented in the form* (2.2)*, then*

$$\int_{\Delta_0} \Big\langle (I - At)^{-1} \Phi_2 \, [\mathrm{d}\tau(t)] \, \Phi_2^* (I - A^* t)^{-1} f, f \Big\rangle < \infty$$

for each f in $\mathfrak{H}$.

When the Assumptions 2.1 are met, we set

$$S_v = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ (I - At)^{-1} \Phi_2 [\mathrm{d}\tau(t)] \Phi_2^* \, (I - A^* t)^{-1} - \mathrm{d}\tau_j(t; A, \Phi_2) \Big\} - \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} (I - \lambda A)^{-1} \Phi_2 R(\lambda) \Phi_2^* (I - \lambda A^*)^{-1} \, \mathrm{d}\lambda \,, \tag{2.4}$$

(2.5) $$\mathrm{i}\,\Phi_{1,v} = \sum_{j=0}^{r} \int_{\Delta_j} \left\{ A(I-At)^{-1} - \mathfrak{S}_j(t;A) \right\} \Phi_2\,[\mathrm{d}\tau(t)] - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} A(I-\lambda A)^{-1}\Phi_2 R(\lambda)\,\mathrm{d}\lambda \;-\; \Phi_2 R(\infty)\,,$$

where Γ is any closed contour that winds once in the positive direction about each pole of $R(\lambda)$. For $j=0$, we define

$$\mathrm{d}\tau_0(t;A,\Phi_2) = 0, \qquad \mathfrak{S}_0(t;A) = -\frac{tI}{1+t^2}\,.$$

For $j = 1,\dots,r$, $\mathrm{d}\tau_j(t;A,\Phi_2)$ and $\mathfrak{S}_j(t;A)$ are defined to be the Taylor polynomials about $t=\alpha_j$ of order $2\rho_j - 1$ of the main terms in (2.4) and (2.5),

$$F(t) = (I-At)^{-1}\Phi_2[\mathrm{d}\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} \qquad \text{and} \qquad G(t) = A(I-At)^{-1}\,.$$

The integrals in (2.4) and (2.5) converge weakly. By Theorems 3.4 and 3.5 of [5], S_v and $\Phi_{1,v}$ are independent of the choice of Kreĭn-Langer representation (2.2) for $v(z)$, S_v is selfadjoint, $\varkappa_{S_v} < \infty$, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ satisfy (1.1).

Now consider the interpolation problem (1.2) for a given operator identity (1.1) such that $\sigma(A) = \{0\}$, S is invertible, and $\varkappa_S < \infty$. Following [5], we seek solutions $v(z) \in \mathbf{N}_\varkappa$ of (1.2) in the form

(2.6) $$v(z) = \mathrm{i}\,[a(z)P(z) + b(z)Q(z)]\,[c(z)P(z)+d(z)Q(z)]^{-1}\,.$$

Here $a(z), b(z), c(z), d(z)$ are $m \times m$ matrix-valued entire functions defined by

(2.7) $$\mathfrak{A}(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix},$$

where

(2.8) $$\mathfrak{A}(z) = I_{2m} - \mathrm{i}z \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \end{bmatrix} (I - zA^*)^{-1} S^{-1} \begin{bmatrix} \Phi_2 & \Phi_1 \end{bmatrix},$$

and $P(z)$ and $Q(z)$ are $m \times m$ matrix-valued functions which are analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$ except at isolated points, such that

(i) $P(\bar{z})^*Q(z) + Q(\bar{z})^*P(z) \equiv 0$;
(ii) $c(z)P(z) + d(z)Q(z)$ is invertible except at isolated points;
(iii) the kernel

(2.9) $$D_{P,Q}(z,\zeta) = \mathrm{i}\,\frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{z - \bar{\zeta}}$$

has a finite number $\varkappa_{P,Q}$ of negative squares.

DEFINITION 2.2. *Assume given an operator identity* (1.1) *such that* $\sigma(A) = \{0\}$, *S is invertible, and* $\varkappa_S < \infty$.

(1) *Let* $\mathbf{N}(\mathfrak{A})$ *be the set of functions* (2.6) *such that* $P(z)$ *and* $Q(z)$ *satisfy the conditions* (i)–(iii).

(2) *By* $\mathbf{N}_+(\mathfrak{A})$ *we mean the subclass of functions* (2.6) *in* $\mathbf{N}(\mathfrak{A})$ *such that*

$$D_{P,Q}(\mathrm{i}y, \mathrm{i}y) \geq 0$$

for all real y *with* $|y|$ *sufficiently large.*

Every $v(z) \in \mathbf{N}(\mathfrak{A})$ is a generalized Nevanlinna function by [5, Theorem 4.4]. Example 2.3 below shows that in general $\mathbf{N}_+(\mathfrak{A}) \neq \mathbf{N}(\mathfrak{A})$. Clearly $\mathbf{N}_+(\mathfrak{A})$ includes all functions $v(z)$ in $\mathbf{N}(\mathfrak{A})$ such that the kernel (2.9) is nonnegative, that is, $\varkappa_{P,Q} = 0$. Example 2.4 shows that there are functions $v(z)$ in $\mathbf{N}_+(\mathfrak{A})$ with $\varkappa_{P,Q} > 0$.

EXAMPLE 2.3. *We use the simplest case of operators* A, S, Φ_1, Φ_2 *of the form* (1.5)–(1.8). *Namely, we assume the scalar case* $m = 1$, *choose* $S = I$, *and* $\Phi_1 1 = \frac{1}{2}$. *By the identity*

$$(I - zA^*)^{-1} f = f(x) - \mathrm{i}z \int_x^\ell \mathrm{e}^{\mathrm{i}z(x-t)} f(t)\, \mathrm{d}t, \tag{2.10}$$

the functions (2.7) *are given by*

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\ell z} + 1) & \frac{1}{4}(\mathrm{e}^{-\mathrm{i}\ell z} - 1) \\ \mathrm{e}^{-\mathrm{i}\ell z} - 1 & \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\ell z} + 1) \end{bmatrix}.$$

Set

$$P(z) = 1, \qquad \mathrm{i}\, Q(z) = \frac{\alpha}{z}, \qquad z \in \mathbf{C}_+ \cup \mathbf{C}_-,$$

where $\alpha > 0$. *The conditions* (i)–(iii) *are satisfied with* $\varkappa_{P,Q} = 1$, *and thus* (2.6) *defines a function* $v(z)$ *in* $\mathbf{N}(\mathfrak{A})$. *But* $D_{P,Q}(\mathrm{i}y, \mathrm{i}y) = -\alpha/y^2 < 0$ *for all real* y, *and therefore* $v(z)$ *does not belong to* $\mathbf{N}_+(\mathfrak{A})$. *Hence in general* $\mathbf{N}_+(\mathfrak{A}) \neq \mathbf{N}(\mathfrak{A})$.

EXAMPLE 2.4. *In the same setting as Example* 2.3, *let* $P(z) = 1$ *for all* z *in* $\mathbf{C}_+ \cup \mathbf{C}_-$, *and set*

$$\mathrm{i}\, Q(z) = \begin{cases} \dfrac{1}{2}\mathrm{i} + \dfrac{\alpha}{z}, & z \in \mathbf{C}_+, \\[2ex] -\dfrac{1}{2}\mathrm{i} + \dfrac{\alpha}{z}, & z \in \mathbf{C}_-, \end{cases}$$

where $\alpha > 0$. *As in Example* 2.3, *the conditions* (i)–(iii) *are satisfied with* $\varkappa_{P,Q} = 1$, *and hence* (2.6) *defines a function* $v(z)$ *in* $\mathbf{N}(\mathfrak{A})$. *For all real* y,

$$D_{P,Q}(\mathrm{i}y, \mathrm{i}y) = \frac{1}{|y|}\left(\frac{1}{2} - \frac{\alpha}{|y|}\right) > 0$$

for all sufficiently large $|y|$, *and therefore* $v(z) \in \mathbf{N}_+(\mathfrak{A})$. *Since* $\varkappa_{P,Q} = 1$, *the kernel* $D_{P,Q}(z, \zeta)$ *is not nonnegative. Thus the condition that the kernel* $D_{P,Q}(z, \zeta)$ *is nonnegative is sufficient but not necessary that* $v(z) \in \mathbf{N}_+(\mathfrak{A})$.

Given $v(z) \in \mathbf{N}(\mathfrak{A})$, we set

$$B_v(z) = (I - zA)^{-1}\big[\Phi_1 - \mathrm{i}\, \Phi_2 v(z)\big], \tag{2.11}$$

$$B_{v,T}(z) = \Big[SA^* + \mathrm{i}\, B_v(z)\Phi_2^*\Big](I - zA^*)^{-1} \tag{2.12}$$
$$= (I - zA)^{-1}\Big[AS - \mathrm{i}\,\Phi_2 B_v(\bar{z})^*\Big].$$

These functions are defined on $\mathbf{C}_+ \cup \mathbf{C}_-$ except at the poles of $v(z)$.

THEOREM 2.5 ([5], Theorem 5.3). *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that*

- *S is invertible, and $\varkappa_S < \infty$;*
- *$\sigma(A) = \{0\}$, and $\|(I - \mathrm{i}yA)^{-1}f\| \neq \mathcal{O}(1)$ as $|y| \to \infty$ for every $f \neq 0$ in $\mathfrak{H}$.*

(1) *Let $v(z)$ belong to $\mathbf{N}(\mathfrak{A})$, and suppose that*

(i) *$v(\mathrm{i}y)/y \to 0$ as $|y| \to \infty$;*
(ii) *for all h in $\mathfrak{H}$ and g in $\mathfrak{G}$, $\langle B_v(\mathrm{i}y)g, h\rangle = \mathcal{O}(1)$ as $|y| \to \infty$;*
(iii) *for all h and k in $\mathfrak{H}$, $\langle B_{v,T}(\mathrm{i}y)h, k\rangle = \mathcal{O}(1/|y|)$ as $|y| \to \infty$.*

Then the Assumptions 2.1 *are met, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$.*
(2) *Conversely, if $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for some generalized Nevanlinna function $v(z)$ such that the Assumptions* 2.1 *are met, then $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$ and satisfies conditions* (i)–(iii) *in* (1).

3. THE CONTINUAL INTERPOLATION PROBLEM

We now solve the interpolation problem (1.2) for operators A, S, Φ_1, Φ_2 of the form (1.5)–(1.8) when S is invertible and $\varkappa_S < \infty$.

THEOREM 3.1. *Let $\mathfrak{H} = L^2_m(0, \ell)$, and let A, S, Φ_1, Φ_2 be operators of the form* (1.5)–(1.8) *such that S is invertible and $\varkappa_S < \infty$. Then the class $\mathbf{N}_+(\mathfrak{A})$ associated with the interpolation problem* (1.2) *as in §2 is not empty. Every function $v(z)$ in $\mathbf{N}_+(\mathfrak{A})$ satisfies the Assumptions* 2.1 *and provides a solution to the interpolation problem* (1.2) *for the given operators, that is, $S = S_v$ and $\Phi_1 = \Phi_{1,v}$.*

We first note some properties of the operators (1.5) and (1.6). The identities

$$(I - zA)^{-1}f = f(x) + \mathrm{i}z \int_0^x \mathrm{e}^{\mathrm{i}z(x-t)} f(t)\, \mathrm{d}t, \tag{3.1}$$

$$(I - zA)^{-1}\Phi_2 g = \mathrm{e}^{\mathrm{i}zx} g, \tag{3.2}$$

$$(I - zA^*)^{-1}\Phi_2 g = \mathrm{e}^{\mathrm{i}z(x-\ell)} g \tag{3.3}$$

$f \in \mathfrak{H}$, $g \in \mathfrak{G}$, $z \in \mathbf{C}$, are verified in a straightforward way.

LEMMA 3.2. *For every δ in $(0, \frac{1}{2}\pi)$, there is a constant $M_\delta > 0$ such that*

$$\|(I - zA)^{-1}\| \leq \begin{cases} M_\delta, & \delta \leq \arg z \leq \pi - \delta, \\ M_\delta\, \mathrm{e}^{\ell|\mathrm{Im}\, z|}, & -\pi + \delta \leq \arg z \leq -\delta. \end{cases} \tag{3.4}$$

This is known, but for the reader's convenience we sketch a proof.

Proof. By (3.1),

$$(I - zA)^{-1} = I + T_z,$$

where $T_z f = \int_0^\ell k_z(x-t) f(t)\,\mathrm{d}t$ and

$$k_z(t) = \begin{cases} \mathrm{i}z\,\mathrm{e}^{\mathrm{i}zt}, & 0 < t < \ell, \\ 0, & -\ell < t < 0. \end{cases}$$

Write $z = \sigma + \mathrm{i}\tau$. By [9, p. 24 (4.8)],

$$\|T_z\| \le \int_{-\ell}^{\ell} |k_z(t)|\,\mathrm{d}t = |z|\,\frac{\mathrm{e}^{-\tau\ell} - 1}{-\tau}.$$

If $\delta \le \arg z \le \pi - \delta$, then $|\sigma| \le \tau \cot\delta$, and $\|T_z\| \le |z|/\tau \le 1 + \cot\delta$. We obtain (3.4) with $M_\delta = 2 + \cot\delta$. The case $-\pi + \delta \le \arg z \le -\delta$ is similar, and again we can choose $M_\delta = 2 + \cot\delta$. ∎

LEMMA 3.3. *For every $h \in \mathfrak{H}$ and $\varepsilon > 0$, there is a $\rho > 0$ such that*

$$\frac{|\langle \mathrm{e}^{\mathrm{i}zx} g, h\rangle|}{\|\mathrm{e}^{\mathrm{i}zx} g\|} < \varepsilon, \quad 0 \ne g \in \mathfrak{G},$$

for every complex number z such that $|z| > \rho$.

Proof. Suppose h is continuously differentiable on $[0,\ell]$ and $h(0) = h(\ell) = 0$. Let $0 \ne g \in \mathfrak{G}$ and $z \in \mathbf{C}$. Integration by parts yields

$$\text{(3.5)} \qquad \left|\left\langle \mathrm{e}^{\mathrm{i}zx} g, h\right\rangle\right| = \left|\frac{1}{\mathrm{i}z}\int_0^\ell \mathrm{e}^{-\mathrm{i}\bar{z}x} g^* h'(x)\,\mathrm{d}x\right|$$
$$\le \frac{1}{|z|}\left(\int_0^\ell \mathrm{e}^{\mathrm{i}(z-\bar{z})x}\,\mathrm{d}x\right)^{\frac12} \|g\| \left(\int_0^\ell |h'(x)|^2\,\mathrm{d}x\right)^{\frac12} = \frac{C_h}{|z|}\left\|\mathrm{e}^{\mathrm{i}zx} g\right\|,$$

where $C_h = \left(\int_0^\ell |h'(x)|^2\,\mathrm{d}x\right)^{\frac12}$. The conclusion is clear in this case.

Now let $h \in \mathfrak{H}$ be arbitrary. Given $\varepsilon > 0$, choose h_1 as in the special case above such that $\|h - h_1\| < \varepsilon/2$. Let $0 \ne g \in \mathfrak{G}$. By (3.5) applied to h_1,

$$\frac{|\langle \mathrm{e}^{\mathrm{i}zx} g, h\rangle|}{\|\mathrm{e}^{\mathrm{i}zx} g\|} = \frac{|\langle \mathrm{e}^{\mathrm{i}zx} g, h_1\rangle + \langle \mathrm{e}^{\mathrm{i}zx} g, h - h_1\rangle|}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \le \frac{C_{h_1}}{|z|} + \|h - h_1\| < \frac{C_{h_1}}{|z|} + \frac{\varepsilon}{2}.$$

The result follows on choosing $\rho = 2C_{h_1}/\varepsilon$. ∎

LEMMA 3.4. *Assume that $S \in \mathfrak{L}(\mathfrak{H})$ is selfadjoint, invertible, and $\varkappa_S < \infty$. Then there exist numbers $\varepsilon > 0$ and $\rho > 0$ such that*

$$\text{(3.6)} \qquad \left\langle S^{-1}\mathrm{e}^{\mathrm{i}zx} g, \mathrm{e}^{\mathrm{i}zx} g\right\rangle \ge \varepsilon\,\|\mathrm{e}^{\mathrm{i}zx} g\|^2, \qquad g \in \mathfrak{G},$$

for all complex numbers z such that $|z| > \rho$.

Proof. Observe first that we can write $S^{-1} = X + F$, where X and F are selfadjoint, $X \geq 2\varepsilon I$ for some $\varepsilon > 0$, and F has finite rank. In fact, suppose that P_+ and P_- are the spectral projections for S for the sets $(0, \infty)$ and $(-\infty, 0)$. Choose $\varepsilon > 0$ such that the restriction of S^{-1} to $P_+\mathfrak{H}$ is bounded below by 2ε. Then the operators $X = S^{-1}P_+ + 2\varepsilon P_-$ and $F = S^{-1}P_- - 2\varepsilon P_-$ have the required properties.

For any nonzero $g \in \mathfrak{G}$ and $z \in \mathbf{C}$,

$$\left\langle S^{-1}\, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \right\rangle \geq 2\varepsilon + \left\langle F\, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \right\rangle \geq 2\varepsilon - \left| \left\langle F\, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \right\rangle \right|. \tag{3.7}$$

Write $F = \sum_{k=1}^{r} \mu_k \langle \cdot, h_k \rangle h_k$, where $\mu_k \neq 0$ and $h_k \in \mathfrak{H}$, $k = 1, \dots, r$. Then

$$\left| \left\langle F\, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \right\rangle \right| \leq \sum_{k=1}^{r} |\mu_k| \left| \left\langle \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, h_k \right\rangle \right|^2.$$

By Lemma 3.3, we can choose $\rho > 0$ such that

$$\left| \left\langle \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, h_k \right\rangle \right|^2 < \frac{\varepsilon}{r|\mu_k|}, \qquad 0 \neq g \in \mathfrak{G}, \qquad k = 1, \dots, r,$$

whenever $|z| > \rho$. Hence for $|z| > \rho$,

$$\left\langle F\, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|}, \frac{\mathrm{e}^{\mathrm{i}zx} g}{\|\mathrm{e}^{\mathrm{i}zx} g\|} \right\rangle \leq \sum_{k=1}^{r} |\mu_k|\, \frac{\varepsilon}{r|\mu_k|} = \varepsilon, \qquad 0 \neq g \in \mathfrak{G}.$$

The inequality (3.6) then follows from (3.7). ∎

Proof of Theorem 3.1. To see that $\mathbf{N}_+(\mathfrak{A}) \neq \emptyset$, choose $P \equiv 0$ and $Q \equiv I_m$ in (2.6). The matrix-valued entire function $d(z)$ has value I_m at the origin, and therefore $d(z)$ has invertible values except at isolated points in the plane. The other conditions required of P and Q are clearly met, and so $v(z) = \mathrm{i}\, b(z) d(z)^{-1}$ belongs to $\mathbf{N}_+(\mathfrak{A})$.

Let $v(z)$ be any function in $\mathbf{N}_+(\mathfrak{A})$ represented in the form (2.6). We prove the theorem by showing that the conditions (i)–(iii) in Theorem 2.5(1) are met. As a preliminary, note that by [5, Theorem 4.4],

$$\frac{v(z) - v(\zeta)^*}{z - \bar{\zeta}} = K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} + B_v(\zeta)^* S^{-1} B_v(z)\,,$$

$$\frac{B_{v,T}(z) - B_{v,T}(\zeta)^*}{z - \bar{\zeta}} = L_2(\bar{\zeta}) K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} L_2(\bar{z})^* + B_{v,T}(\zeta)^* S^{-1} B_{v,T}(z)\,,$$

where $K(z) = [c(z)P(z) + d(z)Q(z)]^{-1}$ and $L_2(z) = (I - zA)^{-1}\Phi_2$. Since $v(z) \in \mathbf{N}_+(\mathfrak{A})$, we deduce that

$$\frac{v(\mathrm{i}y) - v(\mathrm{i}y)^*}{2\mathrm{i}y} \geq B_v(\mathrm{i}y)^* S^{-1} B_v(\mathrm{i}y)\,, \tag{3.8}$$

$$\frac{B_{v,T}(\mathrm{i}y) - B_{v,T}(\mathrm{i}y)^*}{2\mathrm{i}y} \geq B_{v,T}(\mathrm{i}y)^* S^{-1} B_{v,T}(\mathrm{i}y)\,, \tag{3.9}$$

for all real y with $|y|$ sufficiently large.

Proof of (i). We shall show more, namely, for any $p > \frac{1}{2}$,

$$\lim_{|y|\to\infty} \frac{v(\mathrm{i}y)}{|y|^p} = 0. \tag{3.10}$$

We assume that (3.10) is false and deduce a contradiction. Since $\mathfrak{G}$ is finite dimensional, the uniform, strong operator, and weak operator topologies on $\mathfrak{L}(\mathfrak{G})$ coincide, and any one of these can be used to interpret convergence in (3.10). Since $v(z) = v(\bar{z})^*$, if (3.10) fails, we can find a vector g in $\mathfrak{G}$ and a number $c > 0$ such that

$$|v(\mathrm{i}y_n)g| \geq c\,|y_n|^p \tag{3.11}$$

for some sequence $y_n \to -\infty$. It can be assumed that (3.8) holds for every term $y = y_n$ of the sequence. We fix the vector g and sequence $y_1, y_2, \ldots$.

Claim. *If* (3.11) *is satisfied, then there is an* $\eta > 0$ *such that*

$$g^* B_v(\mathrm{i}y_n)^* S^{-1} B_v(\mathrm{i}y_n) g \geq \eta\, \|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|^2 \tag{3.12}$$

for all sufficiently large n.

Granting the claim, we obtain the desired contradiction. For by (3.8) and (3.12),

$$g^* \frac{v(\mathrm{i}y_n) - v(\mathrm{i}y_n)^*}{2\mathrm{i}y_n}\, g \geq \eta\, \|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|^2 = \eta\, \frac{\mathrm{e}^{-2y_n \ell} - 1}{-2y_n}\, |v(\mathrm{i}y_n) g|^2.$$

Hence

$$\frac{|v(\mathrm{i}y_n)g||g|}{|y_n|} \geq \eta\, \frac{\mathrm{e}^{-2y_n \ell} - 1}{-2y_n}\, |v(\mathrm{i}y_n) g|^2,$$

and so

$$\frac{|g|}{|v(\mathrm{i}y_n)g|} \geq \eta\, \frac{\mathrm{e}^{-2y_n \ell} - 1}{2}.$$

This is impossible since the right side tends to $+\infty$ as $y_n \to -\infty$, whereas the left side has limit zero by (3.11).

We prove the claim. By (2.11) and (3.2), for any $y < 0$,

$$g^* B_v(\mathrm{i}y)^* S^{-1} B_v(\mathrm{i}y) g$$

$$= \left\langle S^{-1}(I - \mathrm{i}yA)^{-1}\Phi_1 g, (I - \mathrm{i}yA)^{-1}\Phi_1 g \right\rangle + \mathrm{i} \left\langle S^{-1}(I - \mathrm{i}yA)^{-1}\Phi_1 g, \mathrm{e}^{-yx} v(\mathrm{i}y) g \right\rangle$$

$$- \mathrm{i} \left\langle S^{-1} \mathrm{e}^{-yx} v(\mathrm{i}y) g, (I - \mathrm{i}yA)^{-1} \Phi_1 g \right\rangle + \left\langle S^{-1} \mathrm{e}^{-yx} v(\mathrm{i}y) g, \mathrm{e}^{-yx} v(\mathrm{i}y) g \right\rangle$$

$$\stackrel{\text{def}}{=} T_1(y) + T_2(y) + T_3(y) + T_4(y).$$

By Lemma 3.4, there is an $\varepsilon > 0$ such that

$$T_4(y_n) \geq \varepsilon \, \|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|^2 \tag{3.13}$$

for all sufficiently large n. Observe next that

$$\lim_{n \to \infty} \frac{\|(I - \mathrm{i}y_n A)^{-1} \Phi_1 g\|}{\|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|} = 0. \tag{3.14}$$

In fact by (3.4),

$$\frac{\|(I - \mathrm{i}y_n A)^{-1} \Phi_1 g\|^2}{\|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|^2} \leq \frac{\mathrm{e}^{2\ell |y_n|} \|\Phi_1 g\|^2}{(2|y_n|)^{-1} \left[\mathrm{e}^{2\ell |y_n|} - 1\right] |v(\mathrm{i}y_n) g|^2} \leq M \frac{|y_n|}{|v(\mathrm{i}y_n) g|^2}$$

for all sufficiently large n and some constant M. Thus (3.14) follows from (3.11). By (3.14),

$$\lim_{n \to \infty} \frac{T_k(y_n)}{\|\mathrm{e}^{-y_n x} v(\mathrm{i}y_n) g\|^2} = 0, \qquad k = 1, 2, 3. \tag{3.15}$$

In view of (3.13), the claim follows with $\eta = \varepsilon/2$. This completes the proof of (i).

Proof of (ii). Argue by contradiction, and assume that (ii) is false. By the principle of uniform boundedness, (ii) is equivalent to the assertion that for every vector $g \in \mathfrak{G}$, $\|B_v(\mathrm{i}y) g\| = \mathcal{O}(1)$ as $|y| \to \infty$. Hence if (ii) fails, there is a vector $g \in \mathfrak{G}$ and a sequence of real numbers y_n such that $|y_n| \to \infty$ and $\|B_v(\mathrm{i}y_n) g\| \to \infty$. By (2.10), (3.1), and (3.2), we can write

$$B_v(z) = \mathrm{e}^{\mathrm{i}zx} w(z) + F(z), \tag{3.16}$$

where

$$w(z) = \begin{cases} -\mathrm{i}\, v(z), & z \in \mathbf{C}_+ \\ -\mathrm{i}\, v(z) + \mathrm{i}z \int_0^\ell \mathrm{e}^{-\mathrm{i}zt} \varphi_1(t) \, \mathrm{d}t, & z \in \mathbf{C}_-, \end{cases} \tag{3.17}$$

and

$$F(z) = \begin{cases} (I - zA)^{-1} \Phi_1, & z \in \mathbf{C}_+, \\ (I - zA^*)^{-1} \Phi_1, & z \in \mathbf{C}_-. \end{cases} \tag{3.18}$$

Moreover, $\|F(\mathrm{i}y) g\| = \mathcal{O}(1)$ as $|y| \to \infty$ by (3.4). Therefore

$$\lim_{n \to \infty} \|\mathrm{e}^{-y_n x} w(\mathrm{i}y_n) g\| = +\infty. \tag{3.19}$$

By (3.8)

$$g^* \, \frac{v(\mathrm{i}y_n) - v(\mathrm{i}y_n)^*}{2\mathrm{i}y_n} \, g \geq g^* B_v(\mathrm{i}y_n)^* S^{-1} B_v(\mathrm{i}y_n) g$$

for all sufficiently large n. Write

$$\begin{aligned} g^* B_v(\mathrm{i}y)^* S^{-1} B_v(\mathrm{i}y) g &= \left\langle S^{-1}\mathrm{e}^{-yx} w(\mathrm{i}y)g, \mathrm{e}^{-yx} w(\mathrm{i}y)g \right\rangle + \left\langle S^{-1}\mathrm{e}^{-yx} w(\mathrm{i}y)g, F(\mathrm{i}y)g \right\rangle \\ &\quad + \left\langle S^{-1} F(\mathrm{i}y)g, \mathrm{e}^{-yx} w(\mathrm{i}y)g \right\rangle + \left\langle S^{-1} F(\mathrm{i}y)g, F(\mathrm{i}y)g \right\rangle \\ &\stackrel{\mathrm{def}}{=} T_1(y) + T_2(y) + T_3(y) + T_4(y). \end{aligned}$$

By Lemma 3.4, there is an $\varepsilon > 0$ such that

$$T_1(y_n) \ge \varepsilon \, \|\mathrm{e}^{-y_n x} w(\mathrm{i}y_n) g\|^2$$

for all sufficiently large n. Because of (3.19) and the boundedness of $F(\mathrm{i}y_n)g$,

$$\lim_{n\to\infty} \frac{T_k(y_n)}{\|\mathrm{e}^{-y_n x} w(\mathrm{i}y_n) g\|^2} = 0, \qquad k = 2, 3, 4\,.$$

Hence on setting $\eta = \varepsilon/2$, we obtain

$$g^* \, \frac{v(\mathrm{i}y_n) - v(\mathrm{i}y_n)^*}{2\mathrm{i}y_n} \, g \ge g^* B_v(\mathrm{i}y_n)^* S^{-1} B_v(\mathrm{i}y_n) g \ge \eta \, \|\mathrm{e}^{-y_n x} w(\mathrm{i}y_n) g\|^2$$

for all sufficiently large n. This is impossible, since by condition (i), which has already been proved, the term on the left of the last relation has limit zero as $n \to \infty$, whereas the term on the right is unbounded. This proves (ii).

Proof of (iii). Since $B_{v,T}(z) = B_{v,T}(\bar{z})^*$, it is enough to show that $\langle B_{v,T}(\mathrm{i}y)h, k\rangle = \mathcal{O}(1/|y|)$ as $y \to -\infty$ for all $h, k \in \mathfrak{H}$. By the uniform bounded principle, it is the same thing to show that $\|B_v(\mathrm{i}y)h\| = \mathcal{O}(1/|y|)$ as $y \to -\infty$ for all $h \in \mathfrak{H}$. We assume that this is false and derive a contradiction. If the assertion is false, we can choose an $h \in \mathfrak{H}$ and a sequence $y_n \to -\infty$ such that

$$\lim_{n\to\infty} |y_n| \, \|B_{v,T}(\mathrm{i}y_n)h\| = +\infty. \tag{3.20}$$

Suppose that z belongs to a sector $-\pi + \delta < \arg z < -\delta$, $0 < \delta < \frac{1}{2}\pi$. By (3.4),

$$A^*(I - zA^*)^{-1} = \frac{1}{z}\left[(I - zA^*)^{-1} - I\right] = \mathcal{O}(1/|z|), \qquad |z| \to \infty\,.$$

Hence

$$B_{v,T}(z) = [SA^* + \mathrm{i}\, B_v(z)\Phi_2^*](I - zA^*)^{-1} = \mathrm{i}\, B_v(z)\Phi_2^*(I - zA^*)^{-1} + \mathcal{O}(1/|z|)$$

as $|z| \to \infty$. Writing

$$B_v(z) = \mathrm{e}^{\mathrm{i}zx} w(z) + (I - zA^*)^{-1}\Phi_1$$

as in (3.16)–(3.18), we obtain

$$\begin{aligned} B_{v,T}(z) &= \mathrm{i}\, \mathrm{e}^{\mathrm{i}zx} w(z)\Phi_2^*(I - zA^*)^{-1} \\ &\quad + \mathrm{i}\,(I - zA^*)^{-1}\Phi_1\Phi_2^*(I - zA^*)^{-1} + \mathcal{O}(1/|z|). \end{aligned} \tag{3.21}$$

By (1.1) and the identity $A - A^* = \mathrm{i}\,\Phi_2\Phi_2^*$,

(3.22)

$$\begin{aligned}
&(I - zA^*)^{-1}\Phi_1\Phi_2^*(I - zA^*)^{-1} \\
&\quad = -\mathrm{i}\,(I - zA^*)^{-1}(AS - SA^*)(I - zA^*)^{-1} \\
&\qquad - (I - zA^*)^{-1}\Phi_2\Phi_1^*(I - zA^*)^{-1} \\
&\quad = -\mathrm{i}\,(I - zA^*)^{-1}(A^* + \mathrm{i}\,\Phi_2\Phi_2^*)S(I - zA^*)^{-1} \\
&\qquad + \mathrm{i}\,(I - zA^*)^{-1}SA^*(I - zA^*)^{-1} - (I - zA^*)^{-1}\Phi_2\Phi_1^*(I - zA^*)^{-1} \\
&\quad = (I - zA^*)^{-1}\Phi_2\Phi_2^*S(I - zA^*)^{-1} \\
&\qquad - (I - zA^*)^{-1}\Phi_2\Phi_1^*(I - zA^*)^{-1} + \mathcal{O}(1/|z|)\,,
\end{aligned}$$

$|z| \to \infty$. Hence by (3.3), (3.21), and (3.22),

$$B_{v,T}(z) = \mathrm{e}^{\mathrm{i}zx}c(z) + G(z), \tag{3.23}$$

where $c(z)$ has values in $\mathfrak{L}(\mathfrak{H}, \mathfrak{G})$ and $G(z) = \mathcal{O}(1/|z|)$ as $|z| \to \infty$. By (3.20) and (3.23),

$$\lim_{n\to\infty} \|y_n \mathrm{e}^{-y_n x} c(\mathrm{i}y_n)h\| = +\infty. \tag{3.24}$$

By (3.9), for all sufficiently large n,

$$\begin{aligned}
\|yB_{v,T}(\mathrm{i}y)h\|\,\|h\| &\geq y^2\, \frac{\langle B_{v,T}(\mathrm{i}y)h, h\rangle - \langle h, B_{v,T}(\mathrm{i}y)h\rangle}{2\mathrm{i}y} \\
&\geq \left\langle S^{-1}yB_{v,T}(\mathrm{i}y)h, yB_{v,T}(\mathrm{i}y)h\right\rangle.
\end{aligned} \tag{3.25}$$

Write

$$\begin{aligned}
&\left\langle S^{-1}yB_{v,T}(\mathrm{i}y)h, yB_{v,T}(\mathrm{i}y)h\right\rangle \\
&\qquad = \left\langle S^{-1}y\mathrm{e}^{-yx}c(\mathrm{i}y)h, y\mathrm{e}^{-yx}c(\mathrm{i}y)h\right\rangle + \left\langle S^{-1}y\mathrm{e}^{-yx}c(\mathrm{i}y)h, yG(\mathrm{i}y)h\right\rangle \\
&\qquad\quad + \left\langle S^{-1}yG(\mathrm{i}y)h, y\mathrm{e}^{-yx}c(\mathrm{i}y)h\right\rangle + \left\langle S^{-1}yG(\mathrm{i}y)h, yG(\mathrm{i}y)h\right\rangle \\
&\qquad \stackrel{\text{def}}{=} T_1(y) + T_2(y) + T_3(y) + T_4(y).
\end{aligned}$$

By Lemma 3.4, there is an $\varepsilon > 0$ such that

$$T_1(y_n) \geq \varepsilon\,\|y_n\mathrm{e}^{-y_n x}c(\mathrm{i}y_n)h\|^2$$

for all sufficiently large n. Since $y_nG(\mathrm{i}y_n)h = \mathcal{O}(1)$, (3.24) implies that

$$\lim_{n\to\infty} \frac{T_k(y_n)}{\|y_n\mathrm{e}^{-y_n x}c(\mathrm{i}y_n)h\|^2} = 0\,, \qquad k = 2, 3, 4\,.$$

Setting $\eta = \varepsilon/2$, we obtain

$$\left\langle S^{-1} y_n B_{v,T}(\mathrm{i}y_n)h, y_n B_{v,T}(\mathrm{i}y_n)h \right\rangle \geq \eta \, \|y_n \mathrm{e}^{-y_n x} c(\mathrm{i}y_n)h\|^2$$

for all sufficiently large n. Hence by (3.25), for all sufficiently large n,

$$\begin{aligned} \|y_n \mathrm{e}^{-y_n x} c(\mathrm{i}y_n)h + y_n G(\mathrm{i}y_n)h\| \, \|h\| &= \|y_n B_{v,T}(\mathrm{i}y_n)h\| \, \|h\| \\ &\geq \eta \, \|y_n \mathrm{e}^{-y_n x} c(\mathrm{i}y_n)h\|^2. \end{aligned}$$

This is impossible by (3.24). Thus (iii) is proved, and the result follows. ∎

In work in progress [3], we apply Theorem 3.1 to derive a representation theorem for Hermitian difference kernels. To conclude, we state this result without proof and refer the reader to [3] for details.

Given a nonnegative integer $\varkappa$ and positive number ℓ, define $\mathfrak{S}_{\ell,\varkappa}$ as the set of all measurable $m \times m$ matrix-valued functions $s(x)$ on $(-\ell,\ell)$ such that $s(x) = -s(-x)^*$ a.e. and such that (1.7) defines a bounded operator S with $\varkappa_S = \varkappa$. Set

$$\begin{aligned} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}t} &= \sum_{\nu=0}^{\infty} P_\nu(\lambda, x)(t-\lambda)^\nu \,, \\ \frac{1 + \mathrm{i}tx - \mathrm{e}^{\mathrm{i}tx}}{t^2} &= \sum_{\nu=0}^{\infty} Q_\nu(\lambda, x)(t-\lambda)^\nu \,, \end{aligned}$$

for any complex number λ and real numbers t, x.

EXAMPLE 3.5. *Let $s(x) \in \mathfrak{S}_{\ell,\varkappa}$, and assume that the associated operator S defined by* (1.7) *is invertible. Then there is a generalized Nevanlinna function $v(z)$ represented as in* (2.2) *and* (1°)–(3°) *such that*

$$s(x) = \sum_{j=0}^{r} s_j(x) + s_d(x) \,,$$

where

$$\begin{aligned} s_0(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_{\Delta_0} \left[1 + \frac{\mathrm{i}tx}{1+t^2} - \mathrm{e}^{\mathrm{i}tx} \right] \frac{\mathrm{d}\tau(t)}{t^2} \,, \\ s_j(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_{\Delta_j} \left[\frac{1 + \mathrm{i}tx - \mathrm{e}^{\mathrm{i}tx}}{t^2} - \sum_{\nu=0}^{2\rho_j - 1} Q_\nu(\alpha_j, x)(t-\alpha_j)^\nu \right] \mathrm{d}\tau(t) \,, \end{aligned}$$

$j = 1, \ldots, r$, and

$$s_d(x) = \sum_{k=1}^{s} \sum_{\nu=1}^{\sigma_k} \left[P_{\nu-1}(\lambda_k, x) R_{k\nu} + P_{\nu-1}(\bar{\lambda}_k, x) R_{k\nu}^* \right] \; + \; \mathrm{i}\, C_0,$$

a.e. on $(-\ell,\ell)$.

This generalizes the formula

$$s(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} \left[1 + \frac{\mathrm{i}tx}{1+t^2} - \mathrm{e}^{\mathrm{i}tx}\right] \frac{\mathrm{d}\tau(t)}{t^2} + \mathrm{i}\, C_0$$

given in [10, p. 22] and [12, p. 503] in the definite case $S \geq 0$.

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