Inverse Problems for Canonical Differential Equations with Singularities

J. Rovnyak and L. A. Sakhnovich

Abstract. The inverse problem for canonical differential equations is investigated for Hamiltonians with singularities. The usual notion of a spectral function is not adequate in this generality, and it is replaced by a more general notion of spectral data. The method of operator identities is used to describe a solution of the inverse problem in this setting. The solution is explicitly computable in many cases, and a number of examples are constructed.

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1. Introduction

By a canonical differential equation we understand a system of the form

$$\begin{aligned} \frac{dY}{dx} &= izJH(x)Y, \qquad 0 \le x < \ell, \\ D_2 Y_1(0,z) &+ D_1 Y_2(0,z) = 0, \end{aligned} \tag{1.1}$$

where $H(x) = H(x)^*$ has $2m \times 2m$ matrix values and satisfies

$$H(x) \ge 0 \tag{1.2}$$

on $[0, \ell)$. Here ℓ is a finite positive number, z is a complex parameter,

$$J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \qquad Y(x,z) = \begin{bmatrix} Y_1(x,z) \\ Y_2(x,z) \end{bmatrix}, \tag{1.3}$$

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where $Y_1(x,z)$, $Y_2(x,z)$ have $m\times 1$ matrix values, and D_1, D_2 are $m\times m$ matrices such that $D_1D_2^*+D_2D_1^*=0$ and $D_1D_1^*+D_2D_2^*=I_m$. Without loss of generality (see [21, p. 52]), we can take

$$D_1=0, \qquad D_2=I_m. \tag{1.4}$$

The **fundamental solution** is the $2m\times 2m$ matrix-valued function $W(x,z)$ such that

$$\frac{dW}{dx}=izJH(x)W, \qquad W(0,z)=I_{2m}. \tag{1.5}$$

With the aid of this function, we define a transform

$$Vf=F,$$

$$F(z)=\int_0^\ell \begin{bmatrix}0 & I_m\end{bmatrix} W(x,\bar z)^*\,H(x)f(x)\,dx,$$

where $f(x)$ is a $2m\times 1$ matrix-valued function on $[0,\ell)$ and $F(z)$ is an $m\times 1$ matrix-valued entire function. A nondecreasing $m\times m$ matrix-valued function $\tau(t)$ on the real line is called a **spectral function** for (1.1) if

$$\int_0^\ell f(x)^*H(x)f(x)\,dx=\int_{-\infty}^\infty F(t)^*\,[d\tau(t)]\,F(t) \tag{1.6}$$

for any transform pair $f(x),F(z)$. The direct problem of spectral theory is to find all spectral functions $\tau(t)$ for a given system (1.1). The inverse problem is find a system (1.1) having a given spectral function $\tau(t)$.

We recall how the inverse problem is solved in [18, 21] for systems (1.1) having locally integrable Hamiltonians $H(x)$. Let $v(z)$ be an $m\times m$ matrix-valued Nevanlinna function such that $v(iy)/y\to 0$ as $y\to\infty$. Then

$$v(z)=C_0+\int_{-\infty}^\infty\left[\frac{1}{t-z}-\frac{t}{1+t^2}\right]d\tau(t), \tag{1.7}$$

where $\tau(t)$ is a nondecreasing matrix-valued function such that $\int_{-\infty}^\infty d\tau(t)/(1+t^2)$ converges and C_0 is a constant selfadjoint $m\times m$ matrix. To construct a system (1.1) which has $\tau(t)$ as a spectral function, we choose a Hilbert space $\mathfrak{H}$, a Volterra operator $A\in\mathfrak{L}(\mathfrak{H})$, and an operator $\Phi_2\in\mathfrak{L}(\mathfrak{G},\mathfrak{H})$ where $\mathfrak{G}=\mathbf{C}^m$ in the Euclidean metric. Define operators $S=S_v$ in $\mathfrak{L}(\mathfrak{H})$ and $\Phi_1=\Phi_{1,v}$ in $\mathfrak{L}(\mathfrak{G},\mathfrak{H})$ by

$$S_v=\int_{-\infty}^\infty (I-At)^{-1}\Phi_2\,[d\tau(t)]\,\Phi_2^*(I-A^*t)^{-1}, \tag{1.8}$$

$$\Phi_{1,v}=-i\int_{-\infty}^\infty\left[A(I-At)^{-1}+\frac{tI}{t^2+1}\right]\Phi_2\,[d\tau(t)]+i\Phi_2C_0. \tag{1.9}$$

If the integral in (1.8) is weakly convergent, then so is the integral in (1.9). In this case,

$$AS-SA^*=i\left[\Phi_1\Phi_2^*+\Phi_2\Phi_1^*\right], \tag{1.10}$$

and $S\ge 0$. Let A^* have an eigenchain of projections P_x, $0\le x\le\ell$, with $P_0=0$ and $P_\ell=I$. Write $\mathfrak{H}_x=P_x\mathfrak{H}$, and assume that the operators $S_x=P_xSP_x|\mathfrak{H}_x$ are invertible. Then under conditions detailed in in [21, pp. 54–55, Theorems 2.1, 2.2], the function

$$W(x,z) = I_{2m} + izJ\,\Pi^* P_x S_x^{-1} P_x (I - zA)^{-1}\Pi, \quad \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \tag{1.11}$$

has a continual product representation

$$\begin{aligned} W(x,z) &= \lim\, \exp\left\{ \int_{t_{n-1}}^{t_n} izJH(t)\,dt \right\} \cdots \exp\left\{ \int_{t_0}^{t_1} izJH(t)\,dt \right\} \\ &= \overset{\curvearrowleft}{\int_0^x} \exp\left\{ izJH(t)\,dt \right\}, \end{aligned} \tag{1.12}$$

where $0 = t_0 < t_1 < \cdots < t_n = x$ is a partition of the interval $[0,x]$ and the limit is taken as the maximum length of the intervals in the partition tends to zero. The function $H(x)$ is extracted from this representation by the formula

$$H(x) = \frac{d}{dx}\,\Pi^* P_x S_x^{-1} P_x \Pi. \tag{1.13}$$

Moreover, the function $W(x,z)$ given by (1.11) is the fundamental solution of a canonical differential system (1.1) with Hamiltonian (1.13), and $\tau(t)$ is a spectral function for this system.

In this paper we generalize the preceding approach to the inverse problem. We retain the assumption of positivity but allow the Hamiltonian $H(x)$ to have singularities $0 < x_1 < x_2 < \cdots < \ell$ (that is, points where $H(x)$ is not locally integrable). Thus in place of (1.2) we have

$$H(x) \geq 0, \qquad x \neq x_1, x_2, \ldots\,. \tag{1.14}$$

Consider now a generalized Nevanlinna function $v(z)$ satisfying $v(iy)/y \to 0$ as $y \to \infty$. The representation (1.7) is replaced by the Kreĭn-Langer integral representation,

$$v(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z) \right] d\tau(t) + R(z). \tag{1.15}$$

This representation depends on certain quantities

$$\boldsymbol{\tau} = \{\tau(t); \Delta_0, \ldots, \Delta_r; \alpha_1, \ldots, \alpha_r; \rho_1, \ldots, \rho_r; R(z)\} \tag{1.16}$$

that we call Kreĭn-Langer data (see Theorem 2.1). A transform V is defined for systems (1.1) as before. We say that (1.1) admits $\boldsymbol{\tau}$ as spectral data if

$$\int_0^\ell f(x)^* H(x) f(x)\,dx = \langle F(z), F(z) \rangle_{\boldsymbol{\tau}} \tag{1.17}$$

for all transform pairs $f(x)$ and $F(z)$, where $\langle \cdot,\cdot \rangle_{\boldsymbol{\tau}}$ is an inner product that generalizes the right side of (1.6).

To solve the inverse problem for systems with singularities, we use formulas from [12] that generalize (1.8) and (1.9) to construct an operator identity (1.10). Now we assume only that the operators S_x in the previous scheme are invertible except at certain points $0 < x_1 < x_2 < \cdots$. Then (1.11) and (1.13) define the fundamental solution and Hamiltonian of a system (1.1) satisfying (1.2) but having singularities at the points $x_1, x_2, \ldots$. We emphasize that a system constructed in this way satisfies (1.14). Hence by the Parseval relation (1.17), the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{\tau}}$ is positive on the range of the transform V. In what follows, precise conditions will be given for the validity of the procedure just described.

It will be shown in examples that there are cases in which the calculations can be carried out explicitly. The singularities which occur are of pole type. The examples can be expanded to the complex domain, and in a number of cases it is possible to construct the global solutions to (1.5) in an explicit form.

The study of systems (1.1) has a long history, and we only mention a part of this development. Gohberg and Kreĭn [6] considered such systems on a finite interval, named them canonical differential equations, and introduced notions of eigenvalue and eigenfunction. L. de Branges [4] has obtained deep results on inverse problems by an analysis of families of Hilbert spaces of entire functions. The approach to inverse problems from the viewpoint of factorization problems and operator identities is given by L. A. Sakhnovich [18, 21]. The properties of spectral functions for canonical systems have also been investigated by A. L. Sakhnovich [16]. In a series of papers including [1] and [2], Arov and Dym have made thorough studies of inverse monodromy, inverse scattering, and inverse impedance problems for canonical systems with an emphasis on the strongly regular case. An indefinite theory is initiated in Kreĭn and Langer [9] and developed in an interesting paper by Langer and Winkler [11]. The indefinite case of canonical differential systems presents new technical difficulties. The theory of de Branges has a successful generalization to Pontryagin spaces, due to Kaltenbäck and Woracek [7]. Indefinite problems for canonical systems are studied in the simplest case of discrete systems by the authors [13], by the method of factorization and operator identities. Continuous systems are added to this theory in [15] under some simplifying assumptions. This list of references is not complete, and the sources cited here should be consulted for additional references.

The purpose of this paper is to describe classes of inverse problems in which the solution by means of operator identities produces examples of canonical differential systems (1.1) such that $H(x) \geq 0$ and $H(x)$ has singularities. We note how this paper differs from [15]. In [15] we considered systems (4.1) such that $B(x)$ has at most simple discontinuities at isolated points in $[0, \ell)$. Here we allow these points to be singularities, that is, points where $B(x)$ and $H(x) = B'(x)$ may fail to be locally integrable. Using the results of [12] and [14], we are also able to extend the theory to the full class of generalized Nevanlinna functions: in this paper we allow the points $\alpha_1, \ldots, \alpha_r$ in Theorem 2.1(1°), whereas such points are excluded in [15]. In [15] we also considered some problems with $\varkappa = \infty$, but such problems are not considered here.

In Sections 2 and 3 we formulate results from [12] and [14] that are needed for what follows. These concern the Kreĭn-Langer integral representation and operator identities associated with generalized Nevanlinna functions. In Section 4 we construct a system with a given operator identity. The main scheme to solve the inverse problem is described in Section 5. Section 6 gives additional results for integral operators. Concrete examples are constructed in Section 7.

Notation. Throughout $\mathfrak{H}$ is a separable Hilbert space, m is a positive integer, and $\mathfrak{G} = \mathbf{C}^m$ in the Euclidean metric. By $\mathfrak{L}(\mathfrak{H})$ and $\mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ we mean the usual spaces of bounded linear operators on $\mathfrak{H}$ into itself and on $\mathfrak{G}$ into $\mathfrak{H}$. Write $\mathbf{C}_\pm$ for the open upper and lower half-planes. The matrix J is as in (1.3). Let $\mathbf{N}_\varkappa$ be the **generalized Nevanlinna class** of $m \times m$ matrix-valued functions $v(z)$ which are meromorphic on $\mathbf{C}_+ \cup \mathbf{C}_-$ such that $v(z) = v(\bar{z})^*$ and the kernel

$$\frac{v(z) - v(\zeta)^*}{z - \bar{\zeta}}$$

has $\varkappa$ negative squares ($\varkappa$ a nonnegative integer). If $S \in \mathfrak{L}(\mathfrak{H})$ is a selfadjoint operator on a Hilbert space, $\varkappa_S$ is the dimension of the spectral subspace for the set $(-\infty, 0)$. Thus $\varkappa_S < \infty$ if and only if the negative spectrum consists of eigenvalues of finite total multiplicity.

2. The Kreĭn-Langer integral representation

The Kreĭn-Langer integral representation of a generalized Nevanlinna function $v(z)$ generalizes the integral formula (1.7) for classical Nevanlinna functions. The functions which occur in our applications satisfy the additional condition

$$\lim_{y \to \infty} \frac{v(iy)}{y} = 0, \tag{2.1}$$

and we state the result for this case. For the general case, see [3, 10, 14].

Theorem 2.1. *Every $m \times m$ matrix-valued function $v(z)$ which belongs to some class $\mathbf{N}_\varkappa$, $\varkappa \geq 0$, and satisfies* (2.1) *can be written*

$$v(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{1}{t - z} - S_j(t, z) \right] d\tau(t) + R(z), \tag{2.2}$$

where $\Delta_1, \ldots, \Delta_r$ are bounded open intervals having disjoint closures, Δ_0 is the complement of their union in the real line, and

(1°) *there are points $\alpha_1, \ldots, \alpha_r$ and positive integers $\rho_1, \ldots, \rho_r$ such that $\alpha_j \in \Delta_j$, $j = 1, \ldots, r$, and*

$$\frac{1}{t - z} - S_j(t, z) = \frac{1}{t - z} \left(\frac{t - \alpha_j}{z - \alpha_j} \right)^{2\rho_j} \quad \text{on } \Delta_j, \quad j = 1, \ldots, r\,,$$

$$\frac{1}{t - z} - S_0(t, z) = \frac{1 + tz}{t - z} \, \frac{1}{1 + t^2} \quad \text{on } \Delta_0\,;$$

(2°) *$\tau(t)$ is an $m \times m$ matrix-valued function which is nondecreasing on each of the $r+1$ open intervals determined by $\alpha_1, \dots, \alpha_r$ such that the integral*

$$\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\cdots+\rho_r}} \frac{d\tau(t)}{1+t^2}$$

is convergent;

(3°) *$R(z)$ is an $m \times m$ matrix-valued rational function which is analytic at infinity and satisfies $R(z) = R(\bar{z})^*$.*

Conversely, every function of the form (2.2) *belongs to some class $\mathbf{N}_\varkappa$ and satisfies* (2.1).

Proof. This follows from Theorems 2.1 and 4.1 in [14]. □

The function $\tau(t)$ in (2.2) is essentially unique and can be recovered from $v(z)$ by a Stieltjes inversion formula [14, Corollary 3.3]. However, the other quantities in (2.2) are not unique.

We note that any function $R(z)$ satisfying (3°) can be written

$$R(z) = C_0 - \sum_{k=1}^{s} \left[R_k\Big(\frac{1}{z-\lambda_k}\Big) + R_k\Big(\frac{1}{\bar{z}-\lambda_k}\Big)^* \right], \tag{2.3}$$

where C_0 is a constant selfadjoint $m \times m$ matrix, $\lambda_1, \dots, \lambda_s$ are distinct points in the closed upper half-plane, and $R_1(z), \dots, R_s(z)$ are polynomials such that $R_1(0) = \cdots = R_s(0) = 0$.

Definition 2.2. *The quantities*

$$\boldsymbol{\tau} = \{\tau(t); \Delta_0, \Delta_1, \dots, \Delta_r; \alpha_1, \dots, \alpha_r; \rho_1, \dots, \rho_r; R(z)\} \tag{2.4}$$

appearing in a representation (2.2) *are called* **Kreĭn-Langer data** *for $v(z)$.*

3. Operator identities

Generalized Nevanlinna functions $v(z)$ and their Kreĭn-Langer integral representations (2.2) are used to construct operator identities

$$\begin{gathered} AS - SA^* = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right], \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}), \end{gathered} \tag{3.1}$$

where $\mathfrak{H}$ is a Hilbert space, $\mathfrak{G} = \mathbf{C}^m$, and $S = S^*$. The method that we use here follows [12] and generalizes the formulas (1.8) and (1.9) from the definite case [20, 21]. In place of the inequality $S \geq 0$ which is used in [20, 21], it is assumed here and in [12] that $\varkappa_S < \infty$. We do not require the full generality of [12], since in the present applications A is a Volterra operator and $v(z)$ satisfies (2.1). In this section, we review background from [12] in the form needed in this paper.

By a Volterra operator A we mean a compact operator on a Hilbert space such that $\sigma(A) = \{0\}$.

Assumptions 3.1. *Let* $A \in \mathfrak{L}(\mathfrak{H})$ *and* $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ *be given operators, and let* $v(z)$ *be an* $m \times m$ *matrix-valued generalized Nevanlinna function satisfying* (2.1) *which is represented in the form* (2.2) *for associated Kreĭn-Langer data* (2.4). *Assume*

(i) A *is a Volterra operator, and*
(ii) *the integral* $\int_{\Delta_0} (I - At)^{-1}\Phi_2\,[d\tau(t)]\,\Phi_2^*(I - A^*t)^{-1}$ *converges weakly.*

When these conditions are met, then following [12] we define operators $S_v \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_{1,v} \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ by

$$\begin{aligned} S_v = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ (I - At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I - A^*t)^{-1} - d\tau_j(t; A, \Phi_2) \Big\} \\ - \frac{1}{2\pi i} \int_\Gamma (I - \lambda A)^{-1}\Phi_2 R(\lambda)\Phi_2^*(I - \lambda A^*)^{-1}\, d\lambda\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} i\,\Phi_{1,v} = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ A(I - At)^{-1} - \mathfrak{S}_j(t; A) \Big\}\,\Phi_2\,[d\tau(t)] \\ - \frac{1}{2\pi i} \int_\Gamma A(I - \lambda A)^{-1}\Phi_2 R(\lambda)\, d\lambda \;-\; \Phi_2 C_0. \end{aligned} \tag{3.3}$$

In (3.2) and (3.3), Γ is any closed contour that winds once counterclockwise about each of the poles of $R(\lambda)$, that is, about each of the points $\lambda_1, \dots, \lambda_s$ in a representation (2.3). Explicit formulas for these contour integrals are given in [12, Section 3]. The constant selfadjoint matrix C_0 plays no role and can be chosen arbitrarily. If C_0 is chosen is as in (2.3), that is, $C_0 = R(\infty)$, then (3.2) and (3.3) reduce to (1.8) and (1.9) when $\varkappa = 0$.

The convergence terms $d\tau_j(t; A, \Phi_2)$ and $\mathfrak{S}_j(t; A)$ in (3.2) and (3.3) are defined in this way. For $j = 0$, define

$$d\tau_0(t; A, \Phi_2) = 0, \qquad \mathfrak{S}_0(t; A) = -\frac{tI}{1 + t^2}\,.$$

For $j = 1, \dots, r$, use the Taylor expansion of $(I - tA)^{-1}$ about α_j to write

$$\begin{aligned} &(I - tA)^{-1}\Phi_2[d\tau(t)]\Phi_2^*(I - tA^*)^{-1} \\ &\qquad = \sum_{\ell=0}^{\infty} (t - \alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q\ge 0}} A_p(\alpha_j)\,\Phi_2[d\tau(t)]\Phi_2^*\,A_q(\alpha_j)^*, \\ &A(I - tA)^{-1} = \sum_{p=0}^{\infty} (t - \alpha_j)^p A_p(\alpha_j) A, \end{aligned}$$

where $A_p(\alpha_j) = A^p(I - \alpha_j A)^{-p-1}$ for all $p \ge 0$. Then take

$$d\tau_j(t; A, \Phi_2) = \sum_{\ell=0}^{2\rho_j - 1} (t - \alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q\ge 0}} A_p(\alpha_j)\,\Phi_2[d\tau(t)]\Phi_2^*\,A_q(\alpha_j)^*,$$

$$\mathfrak{S}_j(t;A) = \sum_{p=0}^{2\rho_j-1} (t-\alpha_j)^p A_p(\alpha_j) A.$$

With these definitions, the integrals in (3.2) and (3.3) converge weakly.

The formulas (3.2) and (3.3) that define S_v and $\Phi_{1,v}$ agree with the corresponding formulas in [12]. From Theorems 3.4 and 3.5 in [12], we obtain:

Theorem 3.2. *Let A, Φ_2, and $v(z)$ satisfy Assumptions* 3.1. *Then*

(i) *the definitions of the operators S_v and $\Phi_{1,v}$ are independent of the choice of Kreĭn-Langer representation* (2.2) *for $v(z)$;*

(ii) *the operator S_v is selfadjoint, and $\varkappa_{S_v} < \infty$;*

(iii) *the operators A, Φ_2, $S = S_v$, and $\Phi_1 = \Phi_{1,v}$ satisfy*

$$AS - SA^* = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right].$$

4. Systems associated with operator identities

The main result of this section, Theorem 4.1, shows how to construct a canonical differential equation from an operator identity. We first pass to an integral form of a system (1.1):

$$\begin{gathered} Y(x,z) = Y(0,z) + izJ\int_0^x [dB(t)]\,Y(t,z), \\ D_2Y_1(0,z) + D_1Y_2(0,z) = 0, \end{gathered} \tag{4.1}$$

$0 \le x < \ell$. Here $Y(x,z)$ and J are as in (1.3). As before, we take

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix}.$$

In (4.1), we allow singularities at points $0 < x_1 < x_2 < \cdots$ which have no limit point in $[0,\ell)$. Thus we assume that $B(x)$ has selfadjoint $2m \times 2m$ matrix values and is continuous and nondecreasing on the intervals

$$[0,x_1), (x_1,x_2), (x_2,x_3), \ldots\ . \tag{4.2}$$

For an interval (x_n, x_{n+1}) with $n \ge 1$, we interpret (4.1) to mean that

$$Y(b,z) - Y(a,z) = izJ\int_a^b [dB(t)]\,Y(t,z) \tag{4.3}$$

whenever $[a,b] \subseteq (x_n,x_{n+1})$. A similar meaning is attached to the equation

$$W(x,z) = I_{2m} + izJ\int_0^x [dB(t)]\,W(t,z), \tag{4.4}$$

where $W(x,z)$ is a $2m \times 2m$ matrix-valued function. On (x_n,x_{n+1}) with $n \ge 1$, we interpret (4.4) to mean that

$$W(b,z) - W(a,z) = izJ\int_a^b [dB(t)]\,W(t,z) \tag{4.5}$$

whenever $[a, b] \subseteq (x_n, x_{n+1})$. In particular, (4.3) and (4.5) hold in each of the intervals (4.2). In the usual way, (4.1) reduces to (1.1) when $B(x)$ is absolutely continuous and $H(x) = B'(x)$.

We call any solution $W(x, z)$ of (4.4) a **fundamental solution** for the system (4.1). The fundamental solution is not unique when singularities are present due to the way in which we interpret (4.4) in the intervals between the points $x_1, x_2, \ldots$. A fundamental solution is continuous in x on the intervals (4.2) for fixed z, and entire in z for each fixed x. In Theorem 5.3 we show that for systems associated with operator identities, there is a distinguished choice of fundamental solution.

Theorem 4.1. *Let $A, S \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ satisfy* (3.1) *with A Volterra, S selfadjoint, and $\varkappa_S < \infty$. Let A^* have a strongly continuous eigenchain of projections P_x, $0 \le x \le \ell$, satisfying an inequality*

$$\|(P_{x+\Delta x} - P_x)A(P_{x+\Delta x} - P_x)\| \le M\,\Delta x$$

whenever $0 \le x < x + \Delta x \le \ell$ for some $M > 0$. Assume:

(i) *there are points $0 < x_1 < x_2 < \cdots$ having no limit point in $[0, \ell)$ such that the operator $S_x = P_x S P_x|_{\mathfrak{H}_x}$ is invertible on $\mathfrak{H}_x = P_x \mathfrak{H}$ for each x in $[0, \ell) \setminus \{x_1, x_2, \ldots\}$;*
(ii) *$S_x^{-1} P_x$ is a strongly continuous function of x on the intervals* (4.2)*.*

Then the $2m \times 2m$ matrix-valued function

$$B(x) = \Pi^* P_x S_x^{-1} P_x \Pi, \qquad \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \tag{4.6}$$

is continuous and nondecreasing in each of the intervals (4.2)*, and*

$$W(x, z) = I_{2m} + izJ\,\Pi^* P_x S_x^{-1} P_x (I - zA)^{-1} \Pi \tag{4.7}$$

is a fundamental solution for the system (4.1) *with $B(x)$ defined by* (4.6)*.*

In Theorem 4.1, we assume that the eigenchain is indexed so that $P_0 = 0$ and $P_\ell = I$.

Lemma 4.2. *Under the assumptions in Theorem* 5.3*,*

$$P_\xi S_\xi^{-1} P_\xi S P_\eta S_\eta^{-1} P_\eta = P_\zeta S_\zeta^{-1} P_\zeta,$$

where $\zeta = \min\{\xi, \eta\}$ and ξ, η are any points in $[0, \ell]$ such that the inverses exist.

Proof of Lemma 4.2. If $\xi < \eta$, then $P_\xi S P_\eta|_{\mathfrak{H}_\eta} = P_\xi P_\eta S P_\eta|_{\mathfrak{H}_\eta} = P_\xi S_\eta$, and

$$P_\xi S_\xi^{-1} P_\xi S P_\eta S_\eta^{-1} P_\eta = P_\xi S_\xi^{-1} P_\xi S_\eta S_\eta^{-1} P_\eta = P_\xi S_\xi^{-1} P_\xi.$$

If $\xi \ge \eta$, then $P_\xi S P_\eta = P_\xi S P_\xi P_\eta = S_\xi P_\eta$, and

$$P_\xi S_\xi^{-1} P_\xi S P_\eta S_\eta^{-1} P_\eta = P_\xi S_\xi^{-1} S_\xi P_\eta S_\eta^{-1} P_\eta = P_\eta S_\eta^{-1} P_\eta,$$

as was to be shown. □

Proof of Theorem 4.1. By (ii), $B(x)$ is continuous in each of the intervals (4.2). To show that it is nondecreasing in these intervals, it is sufficient to show that for each $u \in \mathfrak{G} \times \mathfrak{G}$, the function

$$\beta(x) = u^* B(x) u$$

is nondecreasing in the intervals. We assume that $\beta(a) > \beta(b)$ for some compact subinterval $[a, b]$ of one of the intervals (4.2) and derive a contradiction. Since $\beta(x)$ is continuous on $[a, b]$, by the intermediate value theorem, for any positive integer $r > \varkappa_S$ we can find points $a_1 > b_1 > a_2 > b_2 > \cdots > a_r > b_r$ in $[a, b]$ such that

$$\beta(a_1) > \beta(b_1) > \beta(a_2) > \beta(b_2) > \cdots > \beta(a_r) > \beta(b_r).$$

For each $j = 1, \dots, r$, set

$$\begin{aligned} \delta_j &= \beta(b_j) - \beta(a_j), \\ f_j &= P_{b_j} S_{b_j}^{-1} P_{b_j} \Pi u - P_{a_j} S_{a_j}^{-1} P_{a_j} \Pi u. \end{aligned}$$

By Lemma 4.2, if $\xi, \eta \in [a, b]$,

$$\left\langle S P_\eta S_\eta^{-1} P_\eta \Pi u, P_\xi S_\xi^{-1} P_\xi \Pi u \right\rangle = u^* \Pi^* P_\zeta S_\zeta^{-1} P_\zeta \Pi u = \beta(\zeta),$$

where $\zeta = \min\{\xi, \eta\}$. It follows that

$$\langle S f_j, f_k \rangle = \begin{cases} \delta_j, & j = k, \\ 0, & j \neq k. \end{cases}$$

For when $j = k$,

$$\langle S f_j, f_j \rangle = \beta(b_j) - \beta(a_j) - \beta(a_j) + \beta(a_j) = \delta_j.$$

If $j < k$,

$$\langle S f_j, f_k \rangle = \beta(b_j) - \beta(b_j) - \beta(a_j) + \beta(a_j) = 0,$$

and similarly if $j > k$. Since $\delta_j < 0$ for each j, $\mathfrak{H}$ contains an r-dimensional subspace $\mathfrak{N}$ which is the antispace of a Hilbert space in the inner product

$$\langle S f, g \rangle, \qquad f, g \in \mathfrak{N}.$$

This is impossible since $r > \varkappa_S$ (because the projection of $\mathfrak{N}$ into the spectral subspace of S for the negative axis is one-to-one). It follows that $B(x)$ is nondecreasing on each of the intervals (4.2). [In the case $S \geq 0$, a different argument to show that $B(x)$ is nondecreasing is given in [21, p. 42].]

The proof that $W(x, z)$ is a fundamental solution for the resulting system is essentially identical to the first part of the argument in [12, Theorem 3.3]. An extra condition is used in [15, Theorem 3.3], namely, that $\|S_x^{-1}\|$ is bounded on $[0, \ell]$. In our case, $\|S_x^{-1}\|$ is locally bounded by (ii) and the uniform boundedness principle, and this is all that is needed in the argument. □

5. Spectral data and the inverse problem

Consider a system (4.1) with fundamental solution $W(x,z)$. Define a transform

$$Vf = F,$$
$$F(z) = \int_0^\ell \begin{bmatrix} 0 & I_m \end{bmatrix} W(x,\bar{z})^* \, [dB(t)] \, f(x), \tag{5.1}$$

where $f(x)$ is a $2m \times 1$ matrix-valued function on $[0,\ell)$. We assume that $f(x)$ is compactly supported, vanishes in an open interval about each point $x_1, x_2, \dots$, and is continuous except for a finite number of simple discontinuities. For each such $f(x)$, the corresponding $F(z)$ is an $m \times 1$ matrix-valued entire function.

Let $v(z)$ be an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$ satisfying (2.1) with Kreĭn-Langer data $\boldsymbol{\tau}$ given by (2.4). If $F(z)$ and $G(z)$ are $m \times 1$ matrix-valued entire functions such that the integrals

$$\int_{\Delta_0} F(t)^* \, [d\tau(t)] \, F(t), \qquad \int_{\Delta_0} G(t)^* \, [d\tau(t)] \, G(t) \tag{5.2}$$

converge, we define

$$\langle F(z), G(z) \rangle_{\boldsymbol{\tau}} = \sum_{j=0}^{r} \int_{\Delta_j} \Big\{ G(t)^* \, [d\tau(t)] \, F(t) - d\sigma_j(t; F, G) \Big\}$$
$$- \frac{1}{2\pi i} \int_\Gamma G(\bar{\lambda})^* R(\lambda) F(\lambda) \, d\lambda. \tag{5.3}$$

In the last term, Γ is any closed contour that winds once counterclockwise about each of the poles of $R(\lambda)$. In the first integral term, we take

$$d\sigma_0(t; F, G) = 0.$$

Then the integral $\int_{\Delta_0}$ in (5.3) converges since the two integrals in (5.2) converge. For $j = 1, \dots, r$, use the Taylor series $F(t) = \sum_{p=0}^\infty F_p(\alpha_j)(t-\alpha_j)^p$ and $G(t) = \sum_{q=0}^\infty G_q(\alpha_j)(t-\alpha_j)^q$ to formally write

$$G(t)^* \, [d\tau(t)] \, F(t) = \sum_{\ell=0}^{\infty} (t-\alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q \geq 0}} G_q(\alpha_j)^* \, [d\tau(t)] \, F_p(\alpha_j).$$

Then choose

$$d\sigma_j(t; F, G) = \sum_{\ell=0}^{2\rho_j - 1} (t-\alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q \geq 0}} G_q(\alpha_j)^* \, [d\tau(t)] \, F_p(\alpha_j).$$

With this choice, the integral $\int_{\Delta_j}$ in (5.3) converges by the condition (2°) in Theorem 2.1.

Lemma 5.1. *Let $\boldsymbol{\tau}$ be Kreĭn-Langer data for a function $v(z) \in \mathbf{N}_\varkappa$ which satisfies* (2.1). *Suppose $F(z) = \Phi_2^*(I - zA^*)^{-1} f$ and $G(z) = \Phi_2^*(I - zA^*)^{-1} g$, $f, g \in \mathfrak{H}$, where $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ satisfy the conditions in Assumptions* 3.1 *and S_v is defined by* (3.2). *Then*

$$\langle F(z), G(z) \rangle_{\boldsymbol{\tau}} = \langle S_v f, g \rangle .$$

Proof. If the integrals in (3.2) are interpreted in the weak sense, the inner product $\langle S_v f, g \rangle$ reduces to (5.3) by the definition of S_v in Section 3. □

Definition 5.2. *Consider a system* (4.1) *with fundamental solution $W(x, z)$ and transform V defined by* (5.1). *Let*

$$\boldsymbol{\tau} = \{\tau(t); \Delta_0, \Delta_1, \dots, \Delta_r; \alpha_1, \dots, \alpha_r; \rho_1, \dots, \rho_r; R(z)\}$$

be Kreĭn-Langer data for an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$ satisfying (2.1). *We call $\boldsymbol{\tau}$* **spectral data** *for the system* (4.1) *if the Parseval identity*

$$\int_0^\ell g(t)^* \, [dB(t)] \, f(t) = \langle F(z), G(z) \rangle_{\boldsymbol{\tau}} \tag{5.4}$$

holds for all transform pairs $f(t), F(z)$ and $g(t), G(z)$.

We now describe a solution to the inverse problem for systems (4.1).

Theorem 5.3. *Let $v(z) \in \mathbf{N}_\varkappa$ be an $m \times m$ matrix-valued function satisfying* (2.1) *which has Kreĭn-Langer data $\boldsymbol{\tau}$. Choose operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ satisfying Assumptions* 3.1, *and define $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ by* (3.2) *and* (3.3). *Then S is selfadjoint, $\varkappa_S < \infty$, and A, S, Φ_1, Φ_2 satisfy* (3.1). *Any system* (4.1) *constructed from these operators by means of the formulas* (4.6) *and* (4.7) *in Theorem* 4.1 *has spectral data $\boldsymbol{\tau}$.*

Proof. The stated properties of A, S, Φ_1, Φ_2 follow from Theorem 3.2. Let γ, δ of $[0, \ell)$ be subintervals of $[0, \ell)$ whose closures do not contain any of the points $x_1, x_2, \dots$. We first prove the identity

$$\int_0^\ell g_\delta(t)^* \, [dB(t)] \, f_\gamma(t) = \langle F_\gamma(z), G_\delta(z) \rangle_{\boldsymbol{\tau}} \tag{5.5}$$

for any transform pairs $f_\gamma(x), F_\gamma(z)$ and $g_\delta(x), G_\delta(z)$ such that

$$f_\gamma(x) = \chi_\gamma(x) u, \qquad g_\delta(x) = \chi_\delta(x) v, \tag{5.6}$$

where $u, v \in \mathfrak{G}$. We allow the possibility that γ and δ are contained different intervals in the list (4.2). Clearly,

$$\int_0^\ell g_\delta(t)^* \, [dB(t)] \, f_\gamma(t) = \int_{\gamma \cap \delta} v^* \, [dB(t)] \, u. \tag{5.7}$$

If $\gamma = [a, b]$, then by (4.5),

$$F_\gamma(z) = \int_a^b \begin{bmatrix} 0 & I_m \end{bmatrix} W(t, \bar{z})^* \, [dB(t)] \, u$$

$$= \begin{bmatrix} 0 & I_m \end{bmatrix} \frac{W(b,\bar{z})^*J - W(a,\bar{z})^*J}{-iz}\, u.$$

Hence by (4.7),

$$\begin{aligned} F_\gamma(z) &= \begin{bmatrix} 0 & I_m \end{bmatrix} \Big\{ \Pi^*(I - zA^*)^{-1} P_b S_b^{-1} P_b \Pi u \\ &\qquad\qquad - \Pi^*(I - zA^*)^{-1} P_a S_a^{-1} P_a \Pi u \Big\} \\ &= \Phi_2^*(I - zA^*)^{-1}(h_b - h_a), \end{aligned}$$

where $h_a = P_a S_a^{-1} P_a \Pi u$ and $h_b = P_b S_b^{-1} P_b \Pi u$. Similarly, if $\delta = [c,d]$, then

$$G_\delta(z) = \Phi_2^*(I - zA^*)^{-1}(k_d - k_c),$$

where $k_c = P_c S_c^{-1} P_c \Pi v$ and $k_d = P_d S_d^{-1} P_d \Pi v$. Now set $S = S_v$, and apply Lemma 5.1 to get

$$\begin{aligned} &\langle F_\gamma(z), G_\delta(z) \rangle_{\boldsymbol{\tau}} = \langle S(h_b - h_a), k_d - k_c \rangle \\ &\quad = \left\langle SP_b S_b^{-1} P_b \Pi u, P_d S_d^{-1} P_d \Pi v \right\rangle - \left\langle SP_b S_b^{-1} P_b \Pi u, P_c S_c^{-1} P_c \Pi v \right\rangle \\ &\qquad - \left\langle SP_a S_a^{-1} P_a \Pi u, P_d S_d^{-1} P_d \Pi v \right\rangle + \left\langle SP_a S_a^{-1} P_a \Pi u, P_c S_c^{-1} P_c \Pi v \right\rangle. \end{aligned} \tag{5.8}$$

Case 1: $\gamma \cap \delta = \emptyset$. If $a < b < c < d$, then by (5.7), (5.8), and Lemma 4.2,

$$\begin{aligned} &\langle F_\gamma(z), G_\delta(z) \rangle_{\boldsymbol{\tau}} = v^*\Pi^* P_b S_b^{-1} P_b \Pi u - v^*\Pi^* P_b S_b^{-1} P_b \Pi u \\ &\quad - v^*\Pi^* P_a S_a^{-1} P_a \Pi u + v^*\Pi^* P_a S_a^{-1} P_a \Pi u = 0 = \int_0^\ell g_\delta(t)^* \, [dB(t)] \, f_\gamma(t). \end{aligned}$$

Case 2: $\gamma \cap \delta \neq \emptyset$. Here we can assume that $a \le c \le b \le d$. As above,

$$\begin{aligned} \langle F_\gamma(z), G_\delta(z) \rangle_{\boldsymbol{\tau}} &= v^*\Pi^* P_b S_b^{-1} P_b \Pi u - v^*\Pi^* P_c S_c^{-1} P_c \Pi u \\ &\qquad - v^*\Pi^* P_a S_a^{-1} P_a \Pi u + v^*\Pi^* P_a S_a^{-1} P_a \Pi u \\ &= \int_c^b g_\delta(t)^* \, [dB(t)] \, f_\gamma(t) = \int_0^\ell g_\delta(t)^* \, [dB(t)] \, f_\gamma(t). \end{aligned}$$

The general case follows by linearity and approximation. □

Corollary 5.4. *In the situation of Theorem* 5.3, $\langle F(z), G(z) \rangle_{\boldsymbol{\tau}}$ *is a strictly positive inner product on the range of the transform* (5.1).

Proof. The inner product is nonnegative by the Parseval formula (5.4) and the fact, established in Theorem 4.1, that $B(x)$ is nondecreasing in the intervals (4.2). If $\langle F(z), F(z) \rangle_{\boldsymbol{\tau}} = 0$ for some transform pair $f(x), F(z)$, the same identity implies that $\int_0^\ell f(t)^* \, [dB(t)] \, f(t) = 0$. Then $F(z) \equiv 0$ by (5.1) and the Cauchy-Schwarz inequality. □

In many examples of Theorem 5.3, (4.1) is equivalent to a system

$$\begin{gathered} \frac{dY}{dx} = izJH(x)Y, \qquad 0 \le x < \ell, \\ Y_1(0,z) = 0, \end{gathered} \tag{5.9}$$

where $H(x)$ has the form (see Theorem 6.2)

$$H(x) = \begin{bmatrix} h_1(x)^* \\ h_2(x)^* \end{bmatrix} \begin{bmatrix} h_1(x) & h_2(x) \end{bmatrix}. \tag{5.10}$$

Then it is natural to consider an alternative form for the transform (5.1). In (5.1) write

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \qquad W(x,z) = \begin{bmatrix} W_{11}(x,z) & W_{12}(x,z) \\ W_{21}(x,z) & W_{22}(x,z) \end{bmatrix}$$

and

$$\begin{gathered} g(x) = h_1(x)f_1(x) + h_2(x)f_2(x), \\ \psi(x,z) = W_{12}(x,\bar z)^*\, h_1(x)^* + W_{22}(x,\bar z)^*\, h_2(x)^* \end{gathered}$$

Then (5.1) assumes the form

$$\begin{gathered} \widetilde{V} g = G, \\ G(z) = \int_0^\ell \psi(x,z)g(x)\,dx. \end{gathered} \tag{5.11}$$

The Parseval relation (5.4) becomes

$$\int_0^\ell g(t)^* g(t)\,dt = \langle G(z), G(z)\rangle_{\boldsymbol{\tau}} \tag{5.12}$$

in this case.

Paley-Wiener example. The simplest example of Theorem 5.3 yields the Paley-Wiener transform. Consider the spectral data $\tau(t) = t/(2\pi)$ on $\Delta_0 = (-\infty,\infty)$ and associated Nevanlinna function $v(z) = i/2$, $\operatorname{Im} z > 0$. We apply Theorem 5.3 with $\mathfrak{H} = L^2(0,\ell)$, $\mathfrak{G} = \mathbf{C}$, and

$$(Af)(x) = i\int_0^x f(t)\,dt \qquad \text{and} \qquad (\Phi_2 c)(x) = c$$

for all $f \in L^2(0,\ell)$ and $c \in \mathbf{C}$. We find that $S_v = I$ and $(\Phi_{1,v}c)(x) = \frac{1}{2}c$ for all $c \in \mathbf{C}$. If P_ξ is the projection onto $L^2(0,\xi)$, short calculations of the quantities (4.6) and (4.7) yield

$$B(\xi) = \begin{bmatrix} \frac14 & \frac12 \\ \frac12 & 1 \end{bmatrix}\xi, \qquad H(\xi) = \begin{bmatrix} \frac14 & \frac12 \\ \frac12 & 1 \end{bmatrix},$$

$$W(\xi,z) = \begin{bmatrix} \frac12 & -1 \\ -\frac14 & \frac12 \end{bmatrix} + e^{iz\xi}\begin{bmatrix} \frac12 & 1 \\ \frac14 & \frac12 \end{bmatrix}.$$

The transform (5.11) and Parseval relation (5.12) are given by

$$G(z) = \int_0^\ell e^{-izx} g(x)\, dx \tag{5.13}$$

and

$$\int_0^\ell |g(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(x)|^2\, dx. \tag{5.14}$$

The associated canonical differential equation, of course, has no singularities. Examples with singularities are given in Section 7.

6. Integral operators

In Theorem 6.1 we identify a large class of operator identities (3.1) for which the hypotheses of Theorem 4.1 are satisfied. Theorem 6.2 shows that in many cases a $2m \times 2m$ matrix-valued Hamiltonian $H(x) = B'(x)$ obtained from (4.6) satisfies $\operatorname{rank} H(x) \equiv m$ except at the points of singularity. For difference-kernel operators, the Hamiltonian has a special form, which is given in Theorem 6.3.

Let $\mathfrak{H} = L^2_m(0,\ell)$ and $\mathfrak{G} = \mathbf{C}^m$ for some positive integer m. Let P_ξ be the projection of $\mathfrak{H}$ onto $\mathfrak{H}_\xi = L^2_m(0,\xi)$, $0 \le \xi \le \ell$. Assume that

$$(Af)(x) = i \int_0^x f(t)\, dt, \qquad f \in L^2_m(0,\ell), \tag{6.1}$$

and that Φ_1 and Φ_2 are operators on $\mathfrak{G}$ into $\mathfrak{H}$ given by

$$(\Phi_1 g)(x) = \varphi_1(x) g, \qquad (\Phi_2 g)(x) = \varphi_2(x) g, \qquad g \in \mathbf{C}^m, \tag{6.2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are continuous $m \times m$ matrix-valued functions. Let

$$\begin{aligned} (Sf)(x) &= f(x) + \int_0^\ell K(x,t) f(t)\, dt, \\ K(x,t) &= K(t,x)^*, \quad x,t \in (0,\ell), \end{aligned} \tag{6.3}$$

where $K(x,t)$ is a bounded continuous $m \times m$ matrix-valued function. We assume that the identity

$$AS - SA^* = i\left[\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*\right] \tag{6.4}$$

is satisfied.

When the operators A, S, Φ_1, Φ_2 are as in (6.1)–(6.4), the formula (4.6) for the Hamiltonian takes the form

$$H(\xi) = B'(\xi) = \frac{d}{d\xi} \begin{bmatrix} \left\langle S_\xi^{-1}\varphi_1, \varphi_1 \right\rangle_\xi & \left\langle S_\xi^{-1}\varphi_2, \varphi_1 \right\rangle_\xi \\ \left\langle S_\xi^{-1}\varphi_1, \varphi_2 \right\rangle_\xi & \left\langle S_\xi^{-1}\varphi_2, \varphi_2 \right\rangle_\xi \end{bmatrix}, \tag{6.5}$$

where $\langle \cdot, \cdot \rangle_\xi$ denotes the inner product in $L^2_m(0,\xi)$. In (6.5) and below, we understand that operations on matrix-valued functions are performed as required on the columns of the functions.

The next result gives a sufficient condition for the technical hypotheses of Theorem 4.1 to be met.

Theorem 6.1. *The hypotheses of Theorem* 4.1 *are satisfied if* A, S, Φ_1, Φ_2 *are as in* (6.1)–(6.4)*, and if* $K(x,t)$ *has an extension to a function* $K(z,\bar{w})$ *which is bounded and analytic as functions of* z *and* w *in a region* G *such that* G *contains the interval* $(0,\ell)$ *and* $zt, \bar{z}t \in G$ *whenever* $z \in G$ *and* $0 < t \leq 1$.

Proof. It is clear from (6.3) that S is selfadjoint, and $\varkappa_S < \infty$ since S is a compact perturbation of the identity operator. The assumptions on A are verified in a routine way. The main problem is to check the conditions (i) and (ii) in Theorem 4.1.

(i) For small ξ, the operator

$$(S_\xi f)(x) = f(x) + \int_0^\xi K(x,t) f(t)\, dt$$

on $L^2_m(0,\xi)$ differs from the identity operator by an operator of norm less than one. Therefore S_ξ is invertible for $0 \leq \xi < \varepsilon$ for some $\varepsilon > 0$; for $\xi = 0$, S_ξ is the identity operator on the zero space and hence invertible.

For each ξ in $(0,\ell)$, define U_ξ from $L^2_m(0,\ell)$ to $L^2_m(0,\xi)$ by

$$(U_\xi f)(x) = \sqrt{\frac{\ell}{\xi}}\, f\left(\frac{\ell x}{\xi}\right), \qquad 0 < x < \xi.$$

Then U_ξ maps $L^2_m(0,\ell)$ isometrically onto $L^2_m(0,\xi)$, and

$$(U_\xi^{-1} g)(x) = \sqrt{\frac{\xi}{\ell}}\, g\left(\frac{\xi x}{\ell}\right), \qquad 0 < x < \ell.$$

Hence $U_\xi^{-1} S_\xi U_\xi$ is a bounded operator on $L^2_m(0,\ell)$ given by

$$(U_\xi^{-1} S_\xi U_\xi f)(x) = f(x) + \frac{\xi}{\ell} \int_0^\ell K\left(\frac{\xi x}{\ell}, \frac{\xi t}{\ell}\right) f(t)\, dt.$$

Clearly S_ξ is invertible if and only if $U_\xi^{-1} S_\xi U_\xi$ is invertible. Write

$$U_\xi^{-1} S_\xi U_\xi = I + T(\xi). \tag{6.6}$$

The assumptions on G allow us to define an operator $T(z)$ on $L^2_m(0,\ell)$ by

$$(T(z)f)(x) = \frac{z}{\ell} \int_0^\ell K\left(\frac{zx}{\ell}, \frac{\bar{z}t}{\ell}\right) f(t)\, dt, \qquad z \in G.$$

The operator $T(z)$ is compact and depends holomorphically on z, and $T(z)$ agrees with the operator $T(\xi)$ defined by (6.6) when $z = \xi$ is a point of $(0,\ell)$. Since $I + T(\xi)$ is invertible for small positive ξ, $I + T(z)$ is invertible except at isolated points of G (see Kato [8], Theorem 1.9 on p. 370). In particular, (i) follows.

(ii) In this condition, we interpret S_ξ^{-1} as acting from $L^2_m(0,\ell)$ into itself. Hence, for $0 < \xi < \ell$,

$$S_\xi^{-1} P_\xi = E_\xi U_\xi F(\xi) U_\xi^{-1} P_\xi, \tag{6.7}$$

where $F(\xi) = [I + T(\xi)]^{-1}$ and E_ξ is the natural embedding of $L^2_m(0,\xi)$ into $L^2_m(0,\ell)$. In any interval that does not include singularities, all of the operators on the right side of (6.7) are locally bounded. The function $F(\xi)$ is continuous in the operator norm, and one checks easily that $U_\xi^{-1}P_\xi$ and $E_\xi U_\xi$ are strongly continuous. It follows that $S_\xi^{-1}P_\xi$ is strongly continuous at any point ξ in $(0,\ell)$ which is not one of the singularities $x_1, x_2, \dots$. The strong continuity of $S_\xi^{-1}P_\xi$ at the point $\xi = 0$ is clear because $\|S_\xi^{-1}\|$ is bounded for small ξ by (6.7), and $\|P_\xi f\| \to 0$ as $\xi \to 0$ for every f in $L^2_m(0,\ell)$. □

Theorem 6.2. *Let $B(x)$ be constructed by (4.6) for operators A, S, Φ_1, Φ_2 as in (6.1)–(6.4). Then $B(x)$ is continuously differentiable in the intervals between singularities, and in these intervals*

$$H(\xi) = B'(\xi) = \begin{bmatrix} h_1(\xi)^* \\ h_2(\xi)^* \end{bmatrix} \begin{bmatrix} h_1(\xi) & h_2(\xi) \end{bmatrix}, \tag{6.8}$$

where $h_1(\xi)$ and $h_2(\xi)$ are continuous $m \times m$ matrix-valued functions.

Proof. We use results from [6, Chapter IV, §7], which should be consulted for additional details. For each ξ,

$$(S_\xi f)(x) = f(x) + \int_0^\xi K(x,t) f(t)\, dt, \qquad f \in L^2_m(0,\xi).$$

Suppose that S_ξ is invertible for $x_1 < \xi < x_2$. Then

$$(S_\xi^{-1} f)(x) = f(x) + \int_0^\xi \Gamma_\xi(x,t) f(t)\, dt, \qquad f \in L^2_m(0,\xi),$$

where $\Gamma_\xi(x,t)$ is continuous in x and t and differentiable in ξ, and

$$\frac{\partial}{\partial \xi}\, \Gamma_\xi(x,t) = \Gamma_\xi(x,\xi)\Gamma_\xi(\xi,t). \tag{6.9}$$

The last formula is (7–10) in [6, Chapter IV, §7]. We shall compute $H(\xi)$ using (6.5). For any continuous functions f and g in $L^2_m(0,\ell)$,

$$\Big\langle S_\xi^{-1} P_\xi f, P_\xi g \Big\rangle_\xi = \int_0^\xi g(x)^* \left[f(x) + \int_0^\xi \Gamma_\xi(x,t) f(t)\, dt \right] dx.$$

Differentiation yields

$$\begin{aligned} \frac{d}{d\xi} \Big\langle S_\xi^{-1} P_\xi f, P_\xi g \Big\rangle_\xi &= g(\xi)^* f(\xi) + \int_0^\xi g(\xi)^* \Gamma_\xi(\xi,t) f(t)\, dt \\ &\quad + \int_0^\xi g(x)^* \Gamma_\xi(x,\xi) f(\xi)\, dx \\ &\quad + \int_0^\xi \int_0^\xi g(x)^* \frac{\partial}{\partial \xi}\, \Gamma_\xi(x,t)\, f(t)\, dt\, dx\,. \end{aligned}$$

By (6.9),

$$\begin{aligned}\frac{d}{d\xi}\Big\langle S_\xi^{-1}P_\xi f, P_\xi g\Big\rangle_\xi &= g(\xi)^* f(\xi) + \int_0^\xi g(\xi)^*\Gamma_\xi(\xi,t)f(t)\,dt \\ &\quad + \int_0^\xi g(x)^*\Gamma_\xi(x,\xi)f(\xi)\,dx \\ &\quad + \int_0^\xi\int_0^\xi g(x)^*\Gamma_\xi(x,\xi)\Gamma_\xi(\xi,t)f(t)\,dt\,dx \\ &= \left[g(\xi)^* + \int_0^\xi g(x)^*\Gamma_\xi(x,\xi)\,dx\right]\cdot \\ &\qquad \cdot\left[f(\xi) + \int_0^\xi \Gamma_\xi(\xi,t)f(t)\,dt\right].\end{aligned}$$

By (6.5), on choosing $f = \varphi_j$ and $g = \varphi_k$, $j,k = 1,2$, we obtain (6.8) with

$$\begin{aligned}h_1(\xi) &= \varphi_1(\xi) + \int_0^\xi \Gamma_\xi(\xi,t)\varphi_1(t)\,dt, \\ h_2(\xi) &= \varphi_2(\xi) + \int_0^\xi \Gamma_\xi(\xi,t)\varphi_2(t)\,dt,\end{aligned}$$

which yields the result. □

We suppose next that the operator (6.2) has a difference kernel: $K(x,t) = k(x-t)$. That is, we assume that S is defined on $L^2_m(0,\ell)$ by

$$\begin{aligned}(Sf)(x) &= f(x) + \int_0^\ell k(x-t)f(t)\,dt, \\ k(x) &= k(-x)^*, \quad x \in (-\ell,\ell),\end{aligned} \tag{6.10}$$

where $k(x)$ is a bounded continuous $m\times m$ matrix-valued function on $(-\ell,\ell)$. By writing

$$s(x) = \begin{cases} \frac{1}{2}I_m + \int_0^x k(u)\,du, & 0 < x < \ell, \\ -\frac{1}{2}I_m + \int_0^x k(u)\,du, & -\ell < x < 0, \end{cases} \tag{6.11}$$

we can bring (6.10) to the form (see [19]):

$$\begin{aligned}(Sf)(x) &= \frac{d}{dx}\int_0^\ell s(x-t)f(t)\,dt, \\ s(x) &= -s(-x)^*, \quad x \in (-\ell,\ell).\end{aligned} \tag{6.12}$$

Define A, Φ_1, and Φ_2 by (6.1) and (6.2), with

$$\varphi_1(x) = s(x), \qquad \varphi_2(x) = I_m, \qquad 0 < x < \ell. \tag{6.13}$$

The condition (6.4) is easily checked by direct calculation.

Theorem 6.3. *Let $B(x)$ be constructed by* (4.6) *for operators A, S, Φ_1, Φ_2 as in* (6.10)–(6.13). *Assume also that $k(x)$ has selfadjoint values. Then in the intervals between singularities,*

$$H(x) = \frac{1}{2} \begin{bmatrix} Q(x) & I_m \\ I_m & Q(x)^{-1} \end{bmatrix}, \tag{6.14}$$

where $Q(x)$ is a continuous $m \times m$ matrix-valued function whose values are nonnegative and invertible.

A similar result is obtained in [22, p. 507] under different assumptions, namely, $S \geq 0$ and S is factorable.

Lemma 6.4. *Define an involution U on $L^2_m(0, \ell)$ by*

$$(Uf)(x) = f(\ell - x), \qquad f \in L^2_m(0, \ell).$$

Let S have the form (6.10)*, and assume also that $k(x) = k(x)^*$ on $(-\ell, \ell)$. Then $USU = S$.*

Proof of Lemma 6.4. Write S in the form (6.12) with $s(x)$ given by (6.11). The assumptions on $k(x)$ imply that $s(x)^* = s(x) = -s(-x)^*$ on $(-\ell, \ell)$. It is sufficient to show that $USUf = Sf$ whenever f is continuously differentiable on $[0, \ell]$ and $f(0) = f(\ell) = 0$. For such f, integration by parts yields

$$(Sf)(x) = \int_0^\ell s(x - t) f'(t)\, dt.$$

Therefore

$$(USUf)(x) = -\int_0^\ell s(-x + t) f'(t)\, dt = \int_0^\ell s(x - t) f'(t)\, dt = (Sf)(x),$$

as was to be shown. □

Proof of Theorem 6.3. By Theorem 6.2, $H(x)$ has the form (6.8). To deduce (6.14), it is sufficient to show that $h_2^*(\xi) h_1(\xi) = \frac{1}{2} I_m$, that is,

$$\frac{d}{d\xi} \left\langle S_\xi^{-1} \varphi_1, \varphi_2 \right\rangle_\xi = \tfrac{1}{2} I_m \tag{6.15}$$

for ξ in any interval between singularities. For such ξ,

$$(S_\xi f)(x) = \frac{d}{dx} \int_0^\xi s(x - t) f(t)\, dt, \qquad f \in L^2_m(0, \xi). \tag{6.16}$$

Since $s(x)^* = s(x) = -s(-x)^*$ on $(-\ell, \ell)$,

$$S_\xi I_m = \frac{d}{dx} \int_0^\xi s(x - t)\, dt = \frac{d}{dx} \int_{x-\xi}^x s(u)\, du$$
$$= s(x) - s(x - \xi) = s(x) + s(\xi - x),$$

$0 < x < \xi$. Therefore by (6.13),

$$S_\xi I_m = \varphi_1 + U_\xi \varphi_1,$$

where $(U_\xi f)(x) = f(\xi - x)$ for all $f \in L^2_m(0,\xi)$. By Lemma 6.4, $U_\xi S_\xi U_\xi = S_\xi$, and so

$$I_m = S_\xi^{-1}\varphi_1 + S_\xi^{-1} U_\xi \varphi_1 = I_m = S_\xi^{-1}\varphi_1 + U_\xi S_\xi^{-1}\varphi_1.$$

Writing $S_\xi^{-1}\varphi_1 = f$ and integrating, we get

$$2\left\langle S_\xi^{-1}\varphi_1, \varphi_2 \right\rangle_\xi = 2\int_0^\xi f(x)\,dx = \int_0^\xi [f(x) + f(\xi - x)]\,dx = \xi I_m.$$

This yields (6.15) and hence the result. □

7. Examples

The examples in this section illustrate Theorems 4.1 and 5.3 in a number of ways. Each example features operators S, A, Φ_1, Φ_2 satisfying (3.1). We exhibit a corresponding canonical differential system (1.1) and spectral data. The systems which are constructed in the examples have Hamiltonians which are analytic except for poles. The calculations are straightforward but sometimes lengthy, and we only give the final results.

Let us first fix notation for the examples. In all cases, the underlying spaces are $\mathfrak{H} = L^2_m(0,\ell)$ and $\mathfrak{G} = \mathbf{C}^m$ for some positive integer m. The operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ are the same in all of the examples:

$$(Af)(x) = i\int_0^x f(t)\,dt \quad \text{and} \quad (\Phi_2 c)(x) = \varphi_2(x)c, \quad \varphi_2(x) \equiv I_m. \tag{7.1}$$

The operators S and Φ_1 are special to each example. Since $\Phi_1 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$, we always have

$$(\Phi_1 c)(x) = \varphi_1(x)c,$$

where $\varphi_1(x)$ is an $m \times m$ matrix-valued function. Let P_ξ be the projection of $\mathfrak{H}$ onto $\mathfrak{H}_\xi = L^2_m(0,\xi)$, $0 \le \xi \le \ell$, and let $S_\xi = P_\xi S P_\xi|_{\mathfrak{H}_\xi}$. Then according to Theorem 4.1, the system (4.1) associated with the operator identity (3.1) is obtained with

$$B(\xi) = \Pi^* P_\xi S_\xi^{-1} P_\xi \Pi = \begin{bmatrix} \left\langle S_\xi^{-1}\varphi_1, \varphi_1 \right\rangle_\xi & \left\langle S_\xi^{-1}\varphi_2, \varphi_1 \right\rangle_\xi \\ \left\langle S_\xi^{-1}\varphi_1, \varphi_2 \right\rangle_\xi & \left\langle S_\xi^{-1}\varphi_2, \varphi_2 \right\rangle_\xi \end{bmatrix}. \tag{7.2}$$

Here $\langle \cdot,\cdot \rangle_\xi$ denotes an inner product in $L^2_m(0,\xi)$. In (7.2), we understand that φ_1 and φ_2 are first restricted to $(0,\xi)$, and we interpret $\left\langle S_\xi^{-1}\varphi_j, \varphi_k \right\rangle_\xi$ as an $m \times m$ matrix by viewing S_ξ^{-1} as acting on the columns of the matrix-valued functions φ_j and φ_k, $j,k = 1,2$.

In the examples we are mainly concerned with the scalar case, $m = 1$. In this case the underlying spaces are $\mathfrak{H} = L^2(0,\ell)$ and $\mathfrak{G} = \mathbf{C}$, and we use standard scalar notation.

Example 1. Assume the scalar case: $\mathfrak{H} = L^2(0,\ell)$ and $\mathfrak{G} = \mathbf{C}$. Define A and Φ_2 by (7.1), and let

$$(Sf)(x) = f(x) + \beta \int_0^\ell f(t)\, dt,$$
$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = \tfrac{1}{2} + \beta x,$$

where β is real and $\beta < 0$.

The operator identity (3.1) is satisfied, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where $v(z) = \frac{1}{2}i - \beta/z$ for $\operatorname{Im} z > 0$. The function $v(z)$ belongs to $\mathbf{N}_1$ and has the representation

$$v(z) = \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] \frac{dt}{2\pi} - \frac{\beta}{z}.$$

Thus $v(z)$ has Kreĭn-Langer data $\boldsymbol{\tau} = \{\tau(t); \Delta_0; R(z)\}$, where $\tau(t) = t/(2\pi)$ on $\Delta_0 = (-\infty,\infty)$ and $R(z) = -\beta/z$. We obtain

$$(S_\xi f)(x) = f(x) + \beta \int_0^\xi f(t)\, dt,$$
$$(S_\xi^{-1} f)(x) = f(x) - \beta(1+\beta\xi)^{-1} \int_0^\xi f(t)\, dt,$$

on $L^2(0,\xi)$. The function (4.6) in Theorem 4.1 is given by

$$B(\xi) = \begin{bmatrix} (3\xi + 3\beta\xi^2 + \beta^2\xi^2)/12 & \frac{1}{2}\xi \\ \frac{1}{2}\xi & \xi/(1+\beta\xi) \end{bmatrix}.$$

The solution to the inverse problem in Theorem 5.3 is the system

$$\frac{dY}{dx} = izJH(x)Y, \qquad Y_1(0,z) = 0,$$

$$H(x) = B'(x) = \frac{1}{2} \begin{bmatrix} (1+\beta x)^2/2 & 1 \\ 1 & 2/(1+\beta x)^2 \end{bmatrix},$$

$0 \le x < \ell$. The Hamiltonian has a singularity at $x_1 = -1/\beta$ if $-1/\beta < \ell$. The transform (5.11) and Parseval relation (5.12) are given by

$$G(z) = \int_0^\ell \left(e^{-izx} - \frac{\beta}{1+\beta x}\, \frac{e^{-izx} - 1}{-iz} \right) g(x)\, dx,$$
$$\int_0^\ell |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(x)|^2\, dx + \beta\, |G(0)|^2.$$

Thus we obtain a perturbation of the Paley-Wiener example (5.13)–(5.14).

Example 2. Assume the scalar case as in Example 1. The operator identity (3.1) is satisfied with A and Φ_2 given by (7.1), and

$$(Sf)(x) = f(x) + \beta \int_0^\ell \left[e^{i\lambda(x-t)} + e^{-i\lambda(x-t)} \right] f(t)\, dt,$$
$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = \tfrac{1}{2} + \beta\, \frac{e^{i\lambda x} - e^{-i\lambda x}}{i\lambda},$$

where β and λ are real numbers such that $\beta < 0$ and $\lambda > 0$.

We have $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where

$$v(z) = \tfrac{1}{2}\, i - \frac{\beta}{z-\lambda} - \frac{\beta}{z+\lambda}, \qquad \operatorname{Im} z > 0.$$

This function belongs to $\mathbf{N}_2$ and has Kreĭn-Langer data $\boldsymbol{\tau} = \{\tau(t); \Delta_0; R(z)\}$, where $\tau(t) = t/(2\pi)$ on $\Delta_0 = (-\infty, \infty)$ and $R(z) = -\beta/(z-\lambda) - \beta/(z+\lambda)$. We find

$$(S_\xi f)(x) = f(x) + \beta \int_0^\xi \left[e^{i\lambda(x-t)} + e^{-i\lambda(x-t)} \right] f(t)\, dt,$$
$$(S_\xi^{-1} f)(x) = f(x) - K(x)T(\xi)^{-1} \int_0^\xi K(t)^* f(t)\, dt,$$

where $K(x) = \begin{bmatrix} e^{i\lambda x} & e^{-i\lambda x} \end{bmatrix}$, and

$$T(\xi) = \begin{bmatrix} \xi + \beta^{-1} & \gamma(\xi) \\ \overline{\gamma(\xi)} & \xi + \beta^{-1} \end{bmatrix}, \qquad \gamma(\xi) = \frac{e^{-2i\lambda\xi} - 1}{-2i\lambda}.$$

Using Theorem 5.3, we obtain a solution to the inverse problem given by

$$\frac{dY}{dx} = izJH(x)Y, \qquad Y_1(0,z) = 0,$$
$$H(x) = \begin{bmatrix} \overline{h_1(x)} \\ \overline{h_2(x)} \end{bmatrix} \begin{bmatrix} h_1(x) & h_2(x) \end{bmatrix},$$

where

$$h_1(x) = \tfrac{1}{2}\, \frac{x + \beta^{-1} + \lambda^{-1}\sin(\lambda x)}{x + \beta^{-1} - \lambda^{-1}\sin(\lambda x)},$$
$$h_2(x) = \frac{x + \beta^{-1} - \lambda^{-1}\sin(\lambda x)}{x + \beta^{-1} + \lambda^{-1}\sin(\lambda x)}.$$

Singularities occur when $\det T(x) = (x + \beta^{-1})^2 - \lambda^{-2}\sin^2(\lambda x) = 0$.

Example 3. Let $\mathfrak{H} = L^2_m(0,\ell)$ and $\mathfrak{G} = \mathbf{C}^m$. Define A and Φ_2 by (7.1), and let

$$(Sf)(x) = f(x) + \int_0^\ell \sum_{j=1}^r \beta_j e^{i\lambda_j(x-t)}\, f(t)\, dt,$$

$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = \tfrac{1}{2} + \sum_{j=1}^{r} \beta_j \, \frac{e^{i\lambda_j x} - 1}{i\lambda_j},$$

where $\beta_1, \dots, \beta_r$ are invertible selfadjoint $m \times m$ matrices and $\lambda_1, \dots, \lambda_r$ are distinct real numbers; in the formula for $\varphi_1(x)$, if $\lambda_j = 0$ for some j, the expression $[e^{i\lambda_j x} - 1]/(i\lambda_j)$ is interpreted as

$$\left.\frac{e^{i\lambda_j x} - 1}{i\lambda_j}\right|_{\lambda_j = 0} = x.$$

The operator identity (3.1) is satisfied, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where

$$v(z) = \tfrac{1}{2}\, i\, I_m - \sum_{j=1}^{r} \frac{\beta_j}{z - \lambda_j}, \qquad \operatorname{Im} z > 0.$$

This function has Kreĭn-Langer data $\boldsymbol{\tau} = \{\tau(t); \Delta_0; R(z)\}$, where $\tau(t) = t/(2\pi)$ on $\Delta_0 = (-\infty, \infty)$ and $R(z) = -\sum_{j=1}^{r} \beta_j/(z - \lambda_j)$. We find

$$(S_\xi f)(x) = f(x) + \int_0^\xi K(x) C K(t)^* f(t)\, dt, \tag{7.3}$$

$$(S_\xi^{-1} f)(x) = f(x) - \int_0^\xi K(x) \rho(\xi)^{-1} K(t)^* f(t)\, dt, \tag{7.4}$$

where $K(x) = \begin{bmatrix} e^{i\lambda_1 x} & \cdots & e^{i\lambda_r x} \end{bmatrix}$, $C = \operatorname{diag}\{\beta_1, \dots, \beta_r\}$, and

$$\rho(\xi) = C^{-1} + \int_0^\xi K(t)^* K(t)\, dt. \tag{7.5}$$

The inverse operator S_ξ^{-1} exists when $\det \rho(\xi) \neq 0$. Theorem 5.3 yields a solution to the inverse problem given by

$$\frac{dY}{dx} = izJH(x)Y, \qquad Y_1(0, z) = 0, \tag{7.6}$$

$$H(x) = \begin{bmatrix} h_1(x)^* \\ h_2(x)^* \end{bmatrix} \begin{bmatrix} h_1(x) & h_2(x) \end{bmatrix}, \tag{7.7}$$

where

$$h_1(x) = \varphi_1(x) - K(x)\rho(x)^{-1} \int_0^x K(t)^* \varphi_1(t)\, dt, \tag{7.8}$$

$$h_2(x) = 1 - K(x)\rho(x)^{-1} \int_0^x K(t)^*\, dt. \tag{7.9}$$

These functions are computable in closed form, but the expressions are not simple except in particular cases. We note that Examples 1 and 2 are special cases of this example with $m = 1$.

Example 4. In Examples 1–3, $v(z)$ is analytic for nonreal z. In this example, $v(z)$ has nonreal poles. Fix complex numbers $\lambda \neq \bar{\lambda}$ and $\beta \neq 0$. Again with $m = 1$, define A and Φ_2 by (7.1), and let

$$(Sf)(x) = f(x) + \int_0^\ell \left[\beta\, e^{i\lambda(x-t)} + \bar{\beta}\, e^{i\bar{\lambda}(x-t)}\right] f(t)\, dt,$$

$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = \tfrac{1}{2} + \beta\,\frac{e^{i\lambda x} - 1}{i\lambda} + \bar{\beta}\,\frac{e^{i\bar{\lambda}x} - 1}{i\bar{\lambda}}\,.$$

The identity (3.1) is satisfied, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where

$$v(z) = \tfrac{1}{2}\, i - \frac{\beta}{z - \lambda} - \frac{\bar{\beta}}{z - \bar{\lambda}}\,, \qquad \operatorname{Im} z > 0.$$

This function belongs to $\mathbf{N}_1$ and has Kreĭn-Langer data $\boldsymbol{\tau} = \{\tau(t); \Delta_0; R(z)\}$, where $\tau(t) = t/(2\pi)$ on $\Delta_0 = (-\infty, \infty)$ and $R(z) = -\beta/(z-\lambda) - \bar{\beta}/(z - \bar{\lambda})$. The inverse problem is solved by Theorem 5.3 using the identical formulas (7.3)–(7.9) from Example 3, but now taken with

$$K(x) = \begin{bmatrix} e^{i\lambda x} & e^{i\bar{\lambda}x} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{bmatrix},$$

$$\varphi_1(x) = \tfrac{1}{2} + \beta\,\frac{e^{i\lambda x} - 1}{i\lambda} + \bar{\beta}\,\frac{e^{i\bar{\lambda}x} - 1}{i\bar{\lambda}}\,.$$

Example 5. Let $\mathfrak{H} = L^2(0,\ell)$ and $\mathfrak{G} = \mathbf{C}$. Define A and Φ_2 by (7.1), and let

$$(Sf)(x) = f(x) + ia \int_0^x f(t)\, dt - ia \int_x^\ell f(t)\, dt\,,$$

$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = \tfrac{1}{2} + iax,$$

where $a \neq 0$ is a real number. The identity (3.1) is satisfied.

A priori we do not know a generalized Nevanlinna function $v(z)$ such that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, but we shall determine such a function later. Nevertheless, we may apply Theorem 4.1 since S is a compact perturbation of the identity operator and hence $\varkappa_S < \infty$.

We find

$$(S_\xi f)(x) = f(x) + ia \int_0^x f(t)\, dt - ia \int_x^\xi f(t)\, dt\,,$$

$$(S_\xi^{-1} f)(x) = f(x) - 2ia \int_0^x e^{2ia(t-x)} f(t)\, dt + \frac{2ia}{e^{2ia\xi} + 1} \int_0^\xi e^{2ia(t-x)} f(t)\, dt,$$

for all ξ such that $e^{2ia\xi} + 1 \neq 0$, that is, for all points $\xi = \pi n/a$, $n = 1, 2, \dots$, that lie in $[0, \ell)$. The system constructed in Theorem 4.1 for the operator identity (3.1) is

$$\frac{dY}{dx} = izJH(x)Y, \qquad Y_1(0,z) = 0,$$

$$H(x) = \begin{bmatrix} \overline{h_1(x)} \\ \overline{h_2(x)} \end{bmatrix} \begin{bmatrix} h_1(x) & h_2(x) \end{bmatrix},$$

where

$$h_1(x) = \tfrac{1}{2}\, e^{iax} + \tfrac{1}{2}\, i\, \frac{ax}{\cos(ax)}, \qquad h_2(x) = \frac{1}{\cos(ax)}.$$

Next we determine $v(z)$ belonging to some class $\mathbf{N}_\varkappa$ such that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$. This is an interpolation problem of a type first solved by A. L. Sakhnovich [17]. Alternatively, we may use [12, Theorem 5.3]. Thus we may choose

$$v(z) = ia(z)/c(z), \tag{7.10}$$

where

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = I_2 - iz\Pi^*(I - zA^*)^{-1}S^{-1}\Pi J, \tag{7.11}$$

$\Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$. It can be shown that all of the conditions required in [12, Theorem 5.3] are met. We obtain $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where

$$v(z) = -\left(\frac{1}{2} + \frac{a}{z}\right) \cot \frac{(z + 2a)\ell}{2}.$$

We remark that this provides a nontrivial example of the interpolation result in [12, Theorem 5.3].

Example 6. Let $\mathfrak{H} = L^2(0, \ell)$ and $\mathfrak{G} = \mathbf{C}$, and let α, β be real numbers, $\alpha\beta \neq 0$. The operator identity (3.1) is satisfied with A and Φ_2 defined by (7.1), and

$$(Sf)(x) = 2f(x) + \beta \int_0^\ell e^{-\alpha|x-t|} f(t)\, dt,$$

$$(\Phi_1 c)(x) = \varphi_1(x)c, \qquad \varphi_1(x) = 1 + \beta\, \frac{e^{-\alpha x} - 1}{-\alpha}.$$

The hypotheses of Theorem 4.1 are satisfied. By Theorem 6.3, the Hamiltonian $H(\xi) = B'(\xi)$ constructed from (4.6) has the form

$$H(\xi) = \frac{1}{2} \begin{bmatrix} Q(\xi) & 1 \\ 1 & Q(\xi)^{-1} \end{bmatrix},$$

where $Q(\xi)$ is a positive continuous function in the intervals between singularities. The location of singularities and form of $Q(\xi)$ depend on cases.

Case 1: $\alpha^2 + \alpha\beta > 0$. Put $\omega^2 = \alpha^2 + \alpha\beta$, $\omega > 0$. We obtain

$$Q(\xi) = \left[\frac{\omega}{\alpha}\, \frac{(\omega + \alpha)e^{\frac{1}{2}\omega\xi} - (\omega - \alpha)e^{-\frac{1}{2}\omega\xi}}{(\omega + \alpha)e^{\frac{1}{2}\omega\xi} + (\omega - \alpha)e^{-\frac{1}{2}\omega\xi}} \right]^2. \tag{7.12}$$

There are no singularities if $\alpha > 0$. If $\alpha < 0$, there is one singularity at

$$\xi_1 = \frac{1}{\omega} \log \left| \frac{\omega - \alpha}{\omega + \alpha} \right|$$

if this point is less than ℓ. We have $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ with

$$v(z) = i - \frac{\beta}{z + i\alpha}, \qquad \operatorname{Im} z > 0.$$

For $\alpha > 0$, $v(z)$ belongs to $\mathbf{N}_0$, that is, it is a classical Nevanlinna function; its Kreĭn-Langer (Nevanlinna) representation is

$$v(z) = \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \frac{t}{1 + t^2} \right] d\tau(t),$$

where

$$d\tau(t) = \left[\frac{1}{\pi} + \frac{\beta}{\pi} \frac{\alpha}{t^2 + \alpha^2} \right] dt. \tag{7.13}$$

For $\alpha < 0$, $v(z)$ belongs to $\mathbf{N}_1$ and has Kreĭn-Langer representation

$$v(z) = \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \frac{t}{1 + t^2} \right] d\tau(t) - \frac{\beta}{z + i\alpha} - \frac{\beta}{z - i\alpha},$$

where $d\tau(t)$ has the same form (7.13).

Case 2: $\alpha^2 + \alpha\beta = 0$. In this case, $\beta = -\alpha$. We find that

$$Q(\xi) = (1 + \tfrac{1}{2}\alpha\xi)^{-2}.$$

There are no singularities if $\alpha > 0$. If $\alpha < 0$, there is one singularity at

$$\xi_1 = -\frac{2}{\alpha}$$

if this point is less than ℓ. We have $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for the same $v(z)$ as in Case 1 taken with $\beta = -\alpha$.

Case 3: $\alpha^2 + \alpha\beta < 0$. In this case,

$$Q(\xi) = \frac{\omega^2}{\alpha^2} \cot^2\left(\tfrac{1}{2}\,\omega\xi + \rho\right),$$

where $\omega^2 = |\alpha^2 + \alpha\beta|$, $\omega > 0$, and $\tan\rho = \omega/\alpha$. There are singularities at all of the points

$$\xi_k = \frac{1}{\omega}\,(k\pi - 2\rho), \qquad k = 0, \pm 1, \pm 2, \dots$$

which lie in $(0, \ell)$. An explicit choice of generalized Nevanlinna function $v(z)$ such that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ can be constructed as in Example 5 using the formulas (7.10) and (7.11). We obtain

$$v(z) = i - \frac{\beta}{z + i\alpha} + \frac{i}{c(z)} \left[1 + \frac{\beta\alpha^{-1} z}{z + i\alpha}\, c(-i\alpha) \right],$$

where

$$c(z) = -iz \int_0^{\ell} e^{-izt} \left[-\frac{\alpha^2}{2\omega^2} + \sigma^{-1} \cos\left(\omega(t - \tfrac{1}{2}\,\ell)\right) \right] dt$$

and $\sigma = 2\omega^2[\alpha\cos(\frac{1}{2}\omega\ell) - \omega\sin(\frac{1}{2}\omega\ell)]/[\alpha(\alpha^2 + \omega^2)]$. The integral in the formula for $c(z)$ is easily computed in terms of elementary functions.

Example 7. This example uses a generalized Nevanlinna function $v(z)$ whose Kreĭn-Langer representation (2.2) involves a nontrivial term with $\Delta_1 = (-1,1)$. Let $\mathfrak{H} = L^2(0,\ell)$ and $\mathfrak{G} = \mathbf{C}$, and let α, β be real numbers with $\alpha \neq 0$. The operator identity (3.1) is satisfied with A and Φ_2 defined by (7.1), and

$$\begin{aligned} Sf &= f(x) + \int_0^\ell \big[\alpha|x-t| + \beta\big]\, f(t)\, dt \\ &= \frac{d}{dx}\int_0^\ell s(x-t) f(t)\, dt, \\ \Phi_1 g &= \varphi_1(x) g, \end{aligned}$$

where

$$s(x) = \begin{cases} \tfrac{1}{2} + \beta x + \tfrac{1}{2}\,\alpha x^2, & 0 < x < \ell, \\ -\tfrac{1}{2} + \beta x - \tfrac{1}{2}\,\alpha x^2, & -\ell < x < 0, \end{cases}$$

and $\varphi_1(x) = s(x)$ for $0 < x < \ell$. Theorems 4.1 and 6.3 produce a system with Hamiltonian $H(\xi) = B'(\xi)$ of the form

$$H(\xi) = \frac{1}{2}\begin{bmatrix} Q(\xi) & 1 \\ 1 & Q(\xi)^{-1} \end{bmatrix},$$

where $Q(\xi)$ is positive and continuous in the intervals between singularities.

Case 1: $\alpha < 0$**.** For sufficiently large ℓ there is one singularity, and

$$\begin{aligned} Q(\xi) &= \frac{1}{2}\left[1 + \frac{\alpha\xi + 2\beta}{\sqrt{2|\alpha|}}\; \frac{e^{\xi\sqrt{|\alpha|/2}} - e^{-\xi\sqrt{|\alpha|/2}}}{e^{\xi\sqrt{|\alpha|/2}} + e^{-\xi\sqrt{|\alpha|/2}}}\right]^2 \\ &= \frac{1}{2}\left[1 + \frac{\alpha\xi + 2\beta}{\sqrt{2|\alpha|}}\, \tanh\left(\xi\sqrt{|\alpha|/2}\right)\right]^2. \end{aligned}$$

In this case $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where $v(z) \in \mathbf{N}_1$ is given by

$$v(z) = \tfrac{1}{2}\, i - \frac{i\alpha}{z^2} - \frac{\beta}{z}, \qquad y > 0.$$

This function has the Kreĭn-Langer representation

$$\begin{aligned} v(z) = &\int_{-1}^{1}\left[\frac{1}{t-z} - S_1(t,z)\right] d\tau(t) \\ &+ \left(\int_{-\infty}^{-1} + \int_1^\infty\right)\left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\tau(t) - \frac{C}{z}, \end{aligned}$$

where $C = (\pi\beta - 2|\alpha| - 1)/\pi$ and

$$\tau(t) = \frac{1}{2\pi}\, t - \frac{|\alpha|}{\pi}\,\frac{1}{t}, \qquad t \neq 0.$$

In the definition of $S_1(t,z)$, we take $\alpha_1 = 0$ and $\rho_1 = 1$:

$$\frac{1}{t-z} - S_1(t,z) = \frac{1}{t-z}\,\frac{t^2}{z^2}\,.$$

Case 2: $\alpha > 0$**.** In this case,

$$Q(\xi) = \frac{1}{2}\left[1 + \frac{\alpha\xi + 2\beta}{\sqrt{2\alpha}}\,\tan\left(\xi\sqrt{\alpha/2}\right)\right]^2 .$$

For large ℓ there can be an arbitrarily large number of singularities. These occur at the points in $(0,\ell)$ where $Q(\xi)$ is zero or undefined. To find $v(z)$ in some class $\mathbf{N}_\varkappa$ such that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, we again use the formulas (7.10) and (7.11) as in Example 5. Setting $\alpha = \frac{1}{2}\omega^2$, we get

$$\begin{aligned} v(z) = \tfrac{1}{2}\,i - \frac{\beta}{z} - \frac{i\alpha}{z^2} \\ - \left(\frac{A}{z} + \frac{B}{z^2}\right) e^{\frac{1}{2}iz\ell}\,\frac{z^2-\omega^2}{z\sin(\frac{1}{2}z\ell)\cos(\frac{1}{2}\omega\ell) - \omega\cos(\frac{1}{2}z\ell)\sin(\frac{1}{2}\omega\ell)}, \end{aligned}$$

where

$$\begin{aligned} A &= \tfrac{1}{2}\cos(\tfrac{1}{2}\omega\ell) + \left(\tfrac{1}{2}\omega\ell + \frac{2\beta}{\omega}\right)\sin(\tfrac{1}{2}\omega\ell), \\ B &= -\frac{i\alpha}{\omega}\sin(\tfrac{1}{2}\omega\ell). \end{aligned}$$

Example 8. We return to Example 1 and show a connection with Bessel's equation. Write the system constructed in Example 1 as

$$\begin{aligned} \frac{dY}{dx} = izJH(x)Y, \qquad Y_1(0,z) = 0, \\ H(x) = \tfrac{1}{2}\begin{bmatrix} Q(x) & 1 \\ 1 & Q(x)^{-1} \end{bmatrix}, \\ Q(x) = \tfrac{1}{2}(1+\beta x)^2. \end{aligned} \tag{7.14}$$

Here $\beta < 0$. In the regular case, the form of Hamiltonian in (7.14) occurs in the theory of dual systems [22]. We apply similar constructions and set

$$U(x,z) = \begin{bmatrix} U_1(x,z) \\ U_2(x,z) \end{bmatrix} = Y(2x,z)e^{-ixz}.$$

This leads to the system

$$\begin{aligned} \frac{dU}{dx} = izJ\begin{bmatrix} P(x) & 0 \\ 0 & P(x)^{-1} \end{bmatrix} U, \qquad U_1(0,z) = 0, \\ P(x) = Q(2x), \qquad 0 \le x < \tfrac{1}{2}\,\ell. \end{aligned} \tag{7.15}$$

We refer to [5, 22] for the notion of dual equations. In our case, the dual equations derived from (7.15) have the form

$$U_1'' + \frac{P'(x)}{P(x)}\,U_1' + z^2 U_1 = 0,$$
$$U_2'' - \frac{P'(x)}{P(x)}\,U_2' + z^2 U_2 = 0,$$

with appropriate boundary conditions that play no role here. Writing

$$x_1 = -\frac{1}{\beta},$$

we obtain $Q(x) = \frac{1}{2}\,\beta^2(x - x_1)^2$ and $P(x) = 2\beta^2(x - \frac{1}{2}\,x_1)^2$. The dual equations become

$$U_1'' + \frac{2}{x - \frac{1}{2}\,x_1}\,U_1' + z^2 U_1 = 0,$$
$$U_2'' - \frac{2}{x - \frac{1}{2}\,x_1}\,U_2' + z^2 U_2 = 0.$$

On setting $U_1 = (x - \frac{1}{2}\,x_1)^{-1} y_1$ and $U_2 = (x - \frac{1}{2}\,x_1) y_2$, we obtain

$$y_1'' + z^2 y_1 = 0,$$
$$y_2'' + \left(z^2 - \frac{2}{(x - \frac{1}{2}\,x_1)^2} \right) y_2 = 0,$$

which are forms of Bessel's equation for the orders $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$ (see Watson [23, §4.3, p. 95]).

8. Open problems

A number of open problems are suggested by our results, among them:

(1) Investigate the direct problem.

(2) Hamiltonians of the form

$$H(x) = \frac{1}{2} \begin{bmatrix} Q(x) & 1 \\ 1 & Q(x)^{-1} \end{bmatrix}$$

arise in Theorem 6.3. In the regular case such Hamiltonians give rise to a pair of dual equations [5, 22]. An example of a dual pair for a system with singularities is given in Example 8. The theory of dual equations should be generalized to systems with singularities. This requires introducing a new notion of spectral data for selfadjoint second order equations, and relating such a notion to spectral data for the associated canonical differential system.

(3) According to Theorem 6.1, the hypotheses of Theorem 4.1 hold when an analyticity condition is met. The examples in Section 7 show that the hypotheses of Theorem 4.1 also hold in situations which are not covered by Theorem 6.1. It is likely that there are general criteria that cover such examples.

Errata. *In* [15]*, Section* 3*, citations to theorems and definitions are shifted by one beginning with Theorem* 3.2*. For example, on p.* 130*, line* 1*, replace "conclusions of Theorem* 3.2*" by "conclusions of Theorem* 3.3*".*

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