ON GENERALIZED SCHWARZ-PICK ESTIMATES

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1. Introduction

By Pick's invariant form of Schwarz's lemma, an analytic function $B(z)$ which is bounded by one in the unit disk $D = \{z\colon |z| < 1\}$ satisfies the inequality

$$|B'(\alpha)| \leq \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2} \tag{1}$$

at each point α of D. Recently several authors [2, 10, 11] have obtained more general estimates for higher order derivatives. Best possible estimates are due to Ruscheweyh [12]. Below in §2 we use a Hilbert space method to derive Ruscheweyh's results. The operator method applies equally well to operator-valued functions, and this generalization is outlined in §3.

Theorem 1. *If $B(z)$ is analytic and bounded by one in the unit disk, then*

$$(1 - |\alpha|)^{n-1} \left| \frac{B^{(n)}(\alpha)}{n!} \right| \leq \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2} \tag{2}$$

for each α in D and $n = 1, 2, \ldots$. For fixed α and n,

$$\sup_B \, (1 - |\alpha|)^{n-1} \left| \frac{B^{(n)}(\alpha)}{n!} \right| \frac{1 - |\alpha|^2}{1 - |B(\alpha)|^2} = 1, \tag{3}$$

where the supremum is over all nonconstant analytic functions $B(z)$ which are bounded by one on D.

The case $\alpha = 0$ in (2) asserts that if $B(z) = B_0 + B_1 z + B_2 z^2 + \cdots$, then

$$|B_n| \leq 1 - |B_0|^2$$

for every $n \geq 1$. This result is classical and due to F. Wiener; see e.g. [2, 4, 9].

Results for functions of several variables have also been obtained in [2] and [11]. In particular, in [2] rather intricate estimates in several variables are obtained from the above inequality of Wiener.

Theorem 2. (i) *When $n = 1$, equality holds in (2) for some α in D if and only if it holds for all α in D, and this occurs if and only if*

$$B(z) = \gamma \, \frac{z - \beta}{1 - \bar{\beta} z} \tag{4}$$

for some numbers γ and β such that $|\gamma| = 1$ and $|\beta| < 1$.
(ii) *If $\alpha = 0$, then for any $n \geq 1$ equality holds in (2) if and only if*

$$B(z) = \gamma \, \frac{z^n - \beta}{1 - \bar{\beta} z^n}, \tag{5}$$

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where $|\gamma| = 1$ *and* $|\beta| < 1$.
(*iii*) *If* $\alpha \in D \setminus \{0\}$ *and* $n \geq 2$, *equality holds in* (2) *only for a constant of absolute value one.*

Theorem 2(ii) can also be proved from the fact that the extremal function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of modulus at most one which maximizes Re a_n subject to the constraint that $f(0) = a_0$ is unique. A proof of this may be deduced as a special case of [7, Theorem 1.5, p. 67]. See also [3] for related estimates of Taylor coefficients of functions in the unit ball of H^p, $1 \leq p \leq \infty$.

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2. Hilbert space method

Given a function $B(z)$ which is analytic and bounded by one in D, let $\mathcal{H}(B)$ be the Hilbert space of functions on D with reproducing kernel (see [6, 13])

$$K_B(w, z) = \frac{1 - B(z)\overline{B(w)}}{1 - z\bar{w}}, \qquad w, z \in D.$$

Every such space $\mathcal{H}(B)$ is a linear subspace of the Hardy class H^2, and the inclusion mapping from $\mathcal{H}(B)$ into H^2 is contractive. A space $\mathcal{H}(B)$ is contained isometrically in H^2 if and only if $B(z)$ is an inner function, and in this case $\mathcal{H}(B) = H^2 \ominus BH^2$. For any space $\mathcal{H}(B)$, the transformation $R(0)\colon f(z) \to [f(z) - f(0)]/z$ is a contraction from $\mathcal{H}(B)$ into itself and satisfies

$$\left\| \frac{f(z) - f(0)}{z} \right\|^2_{\mathcal{H}(B)} \leq \|f(z)\|^2_{\mathcal{H}(B)} - |f(0)|^2 \tag{6}$$

for every $f(z)$ in $\mathcal{H}(B)$ (see [6, Problem 47]). More generally, for each α in D, the transformation $R(\alpha)\colon f(z) \to [f(z) - f(\alpha)]/(z - \alpha)$ is everywhere defined on $\mathcal{H}(B)$ into itself, and

$$\|R(\alpha)\| \leq \frac{1}{1 - |\alpha|}\,. \tag{7}$$

This follows from [6, Problem 82, p. 45]. In fact,

$$R(\alpha) = [I - \alpha R(0)]^{-1} R(0) = \sum_{k=0}^{\infty} \alpha^k R(0)^{k+1}, \tag{8}$$

and therefore $\|R(\alpha)\| \leq \sum_{k=0}^{\infty} |\alpha|^k = 1/(1 - |\alpha|)$.

We remark that equality holds in (6) for every $f(z)$ in $\mathcal{H}(B)$ if and only if one of the following equivalent conditions holds (see [6, Theorem 16] and [13, Chapters IV and V]):

(i) $B(z) \notin \mathcal{H}(B)$;
(ii) $B(z)$ is an extreme point of the unit ball of H^∞;
(iii) $\int_0^{2\pi} \log\left[1 - |B(e^{i\theta})|^2\right] d\theta = -\infty$.

The extremal functions in Theorem 2 both satisfy (iii) above, and we raise the question if there is a better bound in (2) when $B(z)$ is subject to the constraint

$$\int_0^{2\pi} \log\left[1 - |B(e^{i\theta})|^2\right] d\theta = -K > -\infty.$$

Lemma 3. *If* $|\alpha| < 1$*, then* $[B(z) - B(\alpha)]/(z - \alpha)$ *belongs to* $\mathcal{H}(B)$ *and*

$$\left\| \frac{B(z) - B(\alpha)}{z - \alpha} \right\|^2_{\mathcal{H}(B)} \leq \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}.$$

Proof. Write $\widetilde{B}(z) = \overline{B(\bar{z})}$ for all z in D. It is known that $[B(z) - B(w)]/(z - w)$ belongs to $\mathcal{H}(B)$ for every w in D, and that there is a contractive transformation W from $\mathcal{H}(B)$ into $\mathcal{H}(\widetilde{B})$ such that $Wf = g$ is given by

$$g(w) = \left\langle f(z), \frac{B(z) - B(\bar{w})}{z - \bar{w}} \right\rangle_{\mathcal{H}(B)}, \qquad w \in D.$$

This follows from the scalar case of Theorem 5, p. 350, in the Appendix of [5]; for a different proof, see [1, p. 93]. If $f(z) = K_B(\alpha, z)$, then $g(z) = [\widetilde{B}(z) - \widetilde{B}(\bar{\alpha})]/(z - \bar{\alpha})$. Since W is a contraction,

$$\left\| \frac{\widetilde{B}(z) - \widetilde{B}(\bar{\alpha})}{z - \bar{\alpha}} \right\|^2_{\mathcal{H}(\widetilde{B})} \leq \|K_B(\alpha, z)\|^2_{\mathcal{H}(B)} = \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}.$$

The lemma follows from this inequality on replacing $B(z)$ by $\widetilde{B}(z)$ and α by $\bar{\alpha}$. □

Proof of Theorem 1. Fix $\alpha \in D$. We first prove (2). Set

$$f_1(z) = \frac{B(z) - B(\alpha)}{z - \alpha}$$

and $f_n = R(\alpha)^{n-1} f_1$, $n = 2, 3, \ldots$. For each $n \geq 1$,

$$f_n(z) = \sum_{k=n}^{\infty} \frac{B^{(k)}(\alpha)}{k!} (z - \alpha)^{k-n}$$

pointwise in a neighborhood of α, and so

$$f_n(\alpha) = \frac{B^{(n)}(\alpha)}{n!}.$$

Hence by the Schwarz inequality, Lemma 3, and the inequality (7),

$$(9) \qquad \begin{aligned} \left| \frac{B^{(n)}(\alpha)}{n!} \right| &= \left| \langle f_n(z), K_B(\alpha, z) \rangle_{\mathcal{H}(B)} \right| \\ &= \left| \left\langle R(\alpha)^{n-1} f_1(z), K_B(\alpha, z) \right\rangle_{\mathcal{H}(B)} \right| \\ &\leq \|R(\alpha)\|^{n-1} \|f_1(z)\|_{\mathcal{H}(B)} \|K_B(\alpha, z)\|_{\mathcal{H}(B)} \\ &\leq \frac{1}{(1 - |\alpha|)^{n-1}} \sqrt{\frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}} \sqrt{\frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}} \\ &= \frac{1}{(1 - |\alpha|)^{n-1}} \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}. \end{aligned}$$

This is equivalent to (2).

To prove (3) it is enough to suppose that $n \geq 2$ (when $n = 1$, equality in (2) is attained for any automorphism of D). Consider $B(z) = (z - c)/(1 - \bar{c}z)$ for a fixed c in D. A short calculation yields

$$B^{(n)}(z) = n!\,\bar{c}^{\,n-1}\,(1 - |c|^2)(1 - \bar{c}z)^{-n-1}.$$

Hence

$$\begin{aligned}
(1 - |\alpha|)^{n-1} &\left| \frac{B^{(n)}(\alpha)}{n!} \right| \frac{1 - |\alpha|^2}{1 - |B(\alpha)|^2} \\
&= \frac{(1 - |\alpha|)^{n-1}(1 - |\alpha|^2)}{n!} \, \frac{|n!\,\bar{c}^{\,n-1}\,(1 - |c|^2)(1 - \bar{c}\alpha)^{-n-1}|}{1 - |(\alpha - c)/(1 - \bar{c}\alpha)|^2} \\
&= \frac{(1 - |\alpha|)^{n-1}(1 - |\alpha|^2)|c|^{n-1}(1 - |c|^2)|1 - \bar{c}\alpha|^{-n+1}}{|1 - \bar{c}\alpha|^2 - |\alpha - c|^2} \\
&= \frac{(1 - |\alpha|)^{n-1}(1 - |\alpha|^2)|c|^{n-1}(1 - |c|^2)|1 - \bar{c}\alpha|^{-n+1}}{(1 - |\alpha|^2)(1 - |c|^2)} \\
&= |c|^{n-1} \left(\frac{1 - |\alpha|}{|1 - \bar{c}\alpha|} \right)^{n-1}.
\end{aligned}$$

The last expression tends to 1 as $c \to e^{i\theta}$ where θ is chosen such that $e^{-i\theta}\alpha = |\alpha|$, and (3) follows. □

Proof of Theorem 2. (i) This is well known.

(ii) The case $n = 1$ is included in (i), so we may suppose $n \geq 2$. If $B(z)$ has the form (5), then $B(z) = \gamma\big[-\beta + (1 - |\beta|^2)z^n + \cdots\big]$, and so

$$|B^{(n)}(0)| = n!\,(1 - |\beta|^2) = n!\,\big[1 - |B(0)|^2\big].$$

This is (2) with equality.

Conversely, assume that equality holds in (2) with $\alpha = 0$. If $B(z) = B_0 + B_1 z + B_2 z^2 + \cdots$, then in the notation of the proof of Theorem 1,

$$f_1(z) = \frac{B(z) - B(0)}{z} = B_1 + B_2 z + \cdots.$$

By (9) with $\alpha = 0$,

$$\begin{aligned}
\left| \frac{B^{(n)}(0)}{n!} \right| &= \left| \big\langle R(0)^{n-1} f_1(z), K_B(0, z) \big\rangle_{\mathcal{H}(B)} \right| \\
&\leq \|R(0)^{n-1} f_1(z)\|_{\mathcal{H}(B)} \|K_B(0, z)\|_{\mathcal{H}(B)} \\
&\leq \|f_1(z)\|_{\mathcal{H}(B)} \|K_B(0, z)\|_{\mathcal{H}(B)} \\
&\leq 1 - |B(0)|^2.
\end{aligned} \tag{10}$$

Thus equality holds throughout. In particular,

$$\|R(0)^{n-1} f_1(z)\|_{\mathcal{H}(B)} = \|f_1(z)\|_{\mathcal{H}(B)}.$$

By iterating the difference-quotient inequality (6), we get

$$\|f_1(z)\|^2_{\mathcal{H}(B)} = \|R(0)^{n-1} f_1(z)\|^2_{\mathcal{H}(B)} \leq \|f_1(z)\|^2_{\mathcal{H}(B)} - |B_1|^2 - \cdots - |B_{n-1}|^2,$$

and therefore $B_1 = \cdots = B_{n-1} = 0$. Hence $f_1(z) = B_n z^{n-1} + B_{n+1} z^n + \cdots$, so

$$R(0)^{n-1} f_1(z) = \frac{B(z) - B(0)}{z^n}.$$

For equality to hold in (2), equality must also hold in the Schwarz inequality in (10), which implies that

$$R(0)^{n-1} f_1(z) = \gamma K_B(0, z)$$

for some constant γ. Hence

$$\frac{B(z) - B(0)}{z^n} = \gamma[1 - B(z)\overline{B(0)}],$$

and so

$$B(z) = \frac{\gamma z^n + B(0)}{1 + \gamma \overline{B(0)} z^n} = B(0) + \gamma[1 - |B(0)|^2] z^n + \cdots .$$

In particular, $B^{(n)}(0)/n! = \gamma[1 - |B(0)|^2]$. Therefore since equality holds in (2), $|\gamma| = 1$. Thus $B(z)$ has the form (5) with $\beta = -\bar{\gamma} B(0)$.

(iii) Fix $\alpha \in D \setminus \{0\}$ and $n \geq 2$. Assume that $B(z)$ is analytic and bounded by one in D and

$$\left| \frac{B^{(n)}(\alpha)}{n!} \right| = \frac{1}{(1 - |\alpha|)^{n-1}} \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}. \tag{11}$$

By (8), $R(\alpha) = \sum_{k=0}^{\infty} \alpha^k R(0)^{k+1}$. Thus since $n \geq 2$, we can write the chain of relations (9) in the form

$$\begin{aligned}
\left| \frac{B^{(n)}(\alpha)}{n!} \right| &= \left| \left\langle R(\alpha)^{n-2} R(\alpha) f_1(z), K_B(\alpha, z) \right\rangle_{\mathcal{H}(B)} \right| \\
&\leq \frac{1}{(1 - |\alpha|)^{n-2}} \; \|R(\alpha) f_1\|_{\mathcal{H}(B)} \sqrt{\frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}} \\
&\leq \frac{1}{(1 - |\alpha|)^{n-2}} \sum_{k=0}^{\infty} |\alpha|^k \|R(0)^{k+1} f_1\|_{\mathcal{H}(B)} \sqrt{\frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}} \\
&\leq \frac{1}{(1 - |\alpha|)^{n-2}} \sum_{k=0}^{\infty} |\alpha|^k \|f_1\|_{\mathcal{H}(B)} \sqrt{\frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}} \\
&\leq \frac{1}{(1 - |\alpha|)^{n-1}} \frac{1 - |B(\alpha)|^2}{1 - |\alpha|^2}.
\end{aligned}$$

By (11), equality holds at every stage. Since $\alpha \neq 0$ and $R(0)$ is a contraction, we deduce that

$$\|R(0)^{k+1} f_1\|_{\mathcal{H}(B)} = \|f_1\|_{\mathcal{H}(B)}, \qquad k = 0, 1, 2, \ldots . \tag{12}$$

Let $f_1(z) = \sum_{k=0}^{\infty} a_k z^k$. Iteration of (6) yields

$$\|R(0)^{k+1} f_1\|^2_{\mathcal{H}(B)} \leq \|f_1(z)\|^2_{\mathcal{H}(B)} - |a_0|^2 - \cdots - |a_k|^2, \qquad k = 0, 1, 2, \ldots .$$

Hence by (12), $f_1(z) = [B(z) - B(\alpha]/(z - \alpha)$ is identically zero. But then $B(z)$ is constant, and this constant has absolute value one by (11). □

3. Operator-valued functions

The Hilbert space method applies with minor modifications to operator-valued functions.

Fix a Hilbert space $\mathcal{C}$, and write $\mathfrak{L}(\mathcal{C})$ for the space of bounded linear operators on $\mathcal{C}$. The norms on $\mathcal{C}$ and $\mathfrak{L}(\mathcal{C})$ are both written $|\cdot|$, and the adjoint of an operator $A \in \mathfrak{L}(\mathcal{C})$ is denoted A^*. We write $v^*u = \langle u, v\rangle_{\mathcal{C}}$ for the inner product of two vectors u and v in $\mathcal{C}$.

Let $B(z)$ be an analytic function on D with values in $\mathfrak{L}(\mathcal{C})$ such that $|B(z)| \le 1$ on D. A particular case of Ginzburg's inequality (see e.g. [1, p. 93]) or [8, p. 369]) states that

$$\left| v^* \, \frac{B(\beta) - B(\alpha)}{\beta - \alpha} \, u \right|^2 \le u^* \, \frac{I - B(\alpha)^* B(\alpha)}{1 - |\alpha|^2} \, u \cdot v^* \, \frac{I - B(\beta) B(\beta)^*}{1 - |\beta|^2} \, v$$

for any points α, β in D and vectors u, v in $\mathcal{C}$. Letting β approach α, we deduce that

$$|B'(\alpha)| \le \frac{|I - B(\alpha)^* B(\alpha)|^{\frac{1}{2}} |I - B(\alpha) B(\alpha)^*|^{\frac{1}{2}}}{1 - |\alpha|^2}$$

for every α in D. This is an operator counterpart to the Schwarz-Pick inequality (1).

Theorem 4. *Let $B(z)$ be an analytic function with values in $\mathfrak{L}(\mathcal{C})$ and satisfying $|B(z)| \le 1$ on D. Then*

$$(1 - |\alpha|)^{n-1} \left| \frac{B^{(n)}(\alpha)}{n!} \right| \le \frac{|I - B(\alpha)^* B(\alpha)|^{\frac{1}{2}} |I - B(\alpha) B(\alpha)^*|^{\frac{1}{2}}}{1 - |\alpha|^2}$$

for each α in D and $n = 1, 2, \ldots$.

The space $\mathcal{H}(B)$ is now a space of $\mathcal{C}$-valued analytic functions on D having reproducing kernel (see [1, 5])

$$K_B(w, z) = \frac{I - B(z) B(w)^*}{1 - z\bar{w}}.$$

Thus for every $w \in D$ and $u \in \mathcal{C}$, $K_B(w, z)u$ belongs to $\mathcal{H}(B)$ as a function of z, and the identity

$$\langle f(z), K_B(w, z) u \rangle_{\mathcal{H}(B)} = u^* f(w)$$

holds for every $f(z)$ in $\mathcal{H}(B)$. As in the scalar case, $[f(z) - f(0)]/z$ belongs to $\mathcal{H}(B)$ whenever $f(z)$ is in $\mathcal{H}(B)$, and the inequality (6) holds. If α is any point of D, $R(\alpha)\colon f(z) \to [f(z) - f(\alpha)]/(z - \alpha)$ maps $\mathcal{H}(B)$ into itself, and $\|R(\alpha)\| \le 1/(1 - |\alpha|)$.

Lemma 5. *If $|\alpha| < 1$ and $u \in \mathcal{C}$, then $[B(z) - B(\alpha)]u/(z - \alpha)$ belongs to $\mathcal{H}(B)$ and*

$$\left\| \frac{B(z) - B(\alpha)}{z - \alpha} \, u \right\|^2_{\mathcal{H}(B)} \le u^* \, \frac{1 - B(\alpha)^* B(\alpha)}{1 - |\alpha|^2} \, u.$$

Proof. This follows from [5, Theorem 5, p. 350] as in the proof of Lemma 3. □

Proof of Theorem 4. Fix $\alpha \in D$. Let u and v be arbitrary vectors in $\mathcal{C}$. Define $f_1(z) = [B(z) - B(\alpha)]u/(z - \alpha)$ and $f_n = R(\alpha)^{n-1} f_1$, $n = 2, 3, \ldots$. For any $n \ge 1$,

$f_n(\alpha) = B^{(n)}(\alpha)u/n!$. Thus

$$\begin{aligned}
\left| v^* \, \frac{B^{(n)}(\alpha)}{n!} \, u \right| &= \left| \langle f_n(z), K_B(\alpha, z)v \rangle_{\mathcal{H}(B)} \right| \\
&= \left| \left\langle R(\alpha)^{n-1} f_1(z), K_B(\alpha, z)v \right\rangle_{\mathcal{H}(B)} \right| \\
&\le \|R(\alpha)\|^{n-1} \|f_1(z)\|_{\mathcal{H}(B)} \|K_B(\alpha, z)\|_{\mathcal{H}(B)} \\
&\le \frac{1}{(1-|\alpha|)^{n-1}} \left(u^* \, \frac{1 - B(\alpha)^* B(\alpha)}{1 - |\alpha|^2} \, u \right)^{\frac{1}{2}} \left(v^* \, \frac{1 - B(\alpha) B(\alpha)^*}{1 - |\alpha|^2} \, v \right)^{\frac{1}{2}} .
\end{aligned}$$

The result follows on taking the supremum over all unit vectors u and v. □

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