Operator Theory:
Advances and Applications, Vol. 1, 1–43

On indefinite cases of operator identities which arise in interpolation theory

J. Rovnyak and L. A. Sakhnovich

Abstract. Operator identities involving nonnegative selfadjoint operators play a fundamental role in interpolation theory and its applications. The theory is generalized here to selfadjoint operators whose negative spectra consist of a finite number of eigenvalues of finite total multiplicity. It is shown that such identities are closely associated with generalized Nevanlinna functions by means of the Kreĭn-Langer integral representation. The Potapov fundamental matrix inequality is generalized to this situation, and it is used to formulate and solve an operator interpolation problem analogous to the definite case.

Mathematics Subject Classification (2000). Primary 47A57; Secondary 47A56, 30E05, 47B50, 46C20.

1. Introduction

One approach to interpolation theory and spectral problems for canonical differential equations is based on operator identities of the form

$$\begin{cases} AS - SA^* = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right], \\ A, S \in \mathfrak{L}(\mathfrak{H}), \quad \Phi_1, \Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H}), \end{cases} \tag{1.1}$$

where $\mathfrak{H}$ and $\mathfrak{G}$ are Hilbert spaces, $\dim \mathfrak{G} < \infty$, and $S = S^*$. A well-known matrix example is $A = \operatorname{diag}\{z_1, z_2, \dots, z_n\}$,

$$S = \left[\frac{w_\mu - \bar{w}_\nu}{z_\mu - \bar{z}_\nu}\right]_{\mu,\nu=1}^{n}, \quad \Phi_1 = -i\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

J. Rovnyak was supported by the National Science Foundation Grant DMS-0100437.

where $z_1, \dots, z_n$ are points in the upper half-plane and $w_1, \dots, w_n$ are any complex numbers. In the definite case, that is, when

$$S \geq 0, \tag{1.2}$$

a systematic treatment of identities (1.1) and their applications is given in [12, 13]. A large class of operator identities (1.1) satisfying (1.2) is obtained by first choosing operators $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ for some Hilbert space $\mathfrak{H}$ and $\mathfrak{G} = \mathbf{C}^m$, and an $m \times m$ matrix-valued Nevanlinna function $v(z)$. The Nevanlinna representation of $v(z)$ has the form

$$v(z) = \alpha + \beta\, z + \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\tau(t), \tag{1.3}$$

depending on data

$$\boldsymbol{\tau} = \{\tau(t), \alpha, \beta\}, \tag{1.4}$$

where α and β are selfadjoint matrices, $\beta \geq 0$, and $\tau(t)$ is a nondecreasing matrix-valued function such that the integral $\int_{-\infty}^{\infty} d\tau(t)/(1+t^2)$ is convergent. We define operators

$$S_v = \int_{-\infty}^{\infty} (I - At)^{-1} \Phi_2 \, [d\tau(t)] \, \Phi_2^* (I - A^* t)^{-1} + FF^*, \tag{1.5}$$

$$\Phi_{1,v} = -i \int_{-\infty}^{\infty} \left[A(I - At)^{-1} + \frac{tI}{t^2+1} \right] \Phi_2 \, [d\tau(t)] + i\big(\Phi_2 \alpha + F\beta^{1/2}\big), \tag{1.6}$$

where $F = A^{-1}\Phi_2\, \beta^{1/2}$ if A is invertible and $F = 0$ otherwise. Conditions for convergence of the integrals are given in [12, p. 2], and whenever the integrals converge the operators $S = S_v$, $\Phi_1 = \Phi_{1,v}$, A, and Φ_2 satisfy (1.1) and (1.2). Conversely, given an operator identity (1.1) satisfying (1.2), the abstract interpolation problem is to determine all representations of S and Φ_1 in the form $S = S_v$ and $\Phi_1 = \Phi_{1,v}$. Solutions are obtained in [12] with the aid of an operator analog of Potapov's fundamental matrix inequality. These results are applied in [12, 13] to concrete interpolation problems and spectral problems for canonical differential systems. The theory is simplest in the nondegenerate case, that is, when S is invertible.

In this paper, in place of (1.2) we assume that the negative spectrum of S consists of eigenvalues of finite total multiplicity. We replace (1.3) and (1.4) by the Kreĭn-Langer representation (2.1) of a generalized Nevanlinna function and corresponding data (2.2) and extend the definite theory. Such a generalization was initiated by A. L. Sakhnovich [10] in the scalar case. Other special cases are treated by the authors [9, 7]. We now take up the general case.

We state the Kreĭn-Langer representation of a generalized Nevanlinna function in Section 2. Our main results are formulated in Sections 3–5, with proofs deferred to Section 6. Section 3 is devoted to the construction of operators S_v and $\Phi_{1,v}$ generalizing (1.5) and (1.6). The constructions of S_v and $\Phi_{1,v}$ differ depending whether $0 \notin \sigma(A)$ or $0 \in \sigma(A)$, and these are referred to as Case 1 and

Case 2 throughout the paper. The abstract interpolation problem is formulated in Definition 3.6.

Section 4 derives an indefinite generalization of Potapov's fundamental matrix inequality. The generalization takes the form of a condition on the number of negative squares of a two-variable kernel. The results of Section 4 are used in Section 5 to characterize solutions of the abstract interpolation problem in the nondegenerate case.

Notation and preliminaries:

Let $\mathbf{C}, \mathbf{C}_\pm$ be the complex plane and open upper and lower half-planes. Throughout, we use a finite-dimensional Hilbert space $\mathfrak{G}$, which we take to be $\mathfrak{G} = \mathbf{C}^m$ for a fixed positive integer m. Operators on $\mathfrak{G}$ are represented as $m \times m$ matrices. The **generalized Nevanlinna class $\mathbf{N}_\varkappa$**, $\varkappa = 0, 1, 2, \ldots$, is the set of $m \times m$ matrix-valued functions $v(z)$ which are meromorphic on $\mathbf{C}_+ \cup \mathbf{C}_-$ such that $v(z) = v(\bar{z})^*$ and the kernel $[v(z) - v(\zeta)^*]/(z - \bar{\zeta})$ has $\varkappa = \varkappa_v$ negative squares (for example, see [1, 2, 5]). We also write $\varkappa_K$ for the number of negative squares of any Hermitian kernel $K(z, \zeta)$. If S is a selfadjoint operator on a Hilbert space, $\varkappa_S$ denotes the dimension of the eigenspace for $(0, \infty)$.

We assume familiarity with Stieltjes integrals $\int_\Delta f(t)\,[d\tau(t)]\,g(t)$, where $\tau(t)$ is an $m \times m$ matrix-valued function and $f(t)$ and $g(t)$ are matrix-valued functions of orders $p \times m$ and $m \times q$. In our applications, Δ is either an interval or a finite union of intervals and $\tau(t)$ is nondecreasing on each interval in Δ. By $L^2(d\tau)$ we mean a completion of the space of continuous $\mathfrak{G}$-valued functions $g(t)$ on Δ such that $\|g\|^2 = \int_\Delta g(t)^*\,[d\tau(t)]\,g(t) < \infty$. We also use integrals of the form

$$\int_\Delta G(t)^*\,[d\tau(t)]\,F(t), \tag{1.7}$$

where $F(t)$ and $G(t)$ are continuous functions with values in $\mathfrak{L}(\mathfrak{H}, \mathfrak{G})$ and $\mathfrak{L}(\mathfrak{K}, \mathfrak{G})$ for some Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. If $F(t)h$ and $G(t)k$ belong to $L^2(d\tau)$ for all vectors h in $\mathfrak{H}$ and k in $\mathfrak{K}$, we define (1.7) as the unique operator in $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that

$$\left\langle \left(\int_\Delta G(t)^*\,[d\tau(t)]\,F(t) \right) h, k \right\rangle_{L^2(d\tau)} = \int_\Delta [G(t)k]^*[d\tau(t)]F(t)h\,,$$

for all $h \in \mathfrak{H}$ and $k \in \mathfrak{K}$. Integrals of the type (1.7) also appear in the form

$$\int_\Delta \sum_{j=1}^r G_j(t)^*[d\tau(t)]F_j(t) = \int_\Delta \begin{bmatrix} G_1(t)^* & \cdots & G_r(t)^* \end{bmatrix} [d\tau(t)] \begin{bmatrix} F_1(t) \\ \vdots \\ F_r(t) \end{bmatrix}.$$

In practice, to prove convergence of such an integral we show that

$$\sum_{j=1}^r G_j(t)^*[d\tau(t)]F_j(t) = \sum_{k=1}^s \widetilde{G}_k(t)^*[d\tau(t)]\widetilde{F}_k(t),$$

where the integrals $\int_\Delta \widetilde{G}_j(t)^*\,[d\tau(t)]\,\widetilde{F}_j(t)$, $j = 1, \ldots, s$, exist separately.

2. Kreĭn-Langer integral representation

The Kreĭn-Langer integral representation of a generalized Nevanlinna function is given in [5] in the scalar case. The matrix case of the representation is due to Daho and Langer [2], and we use this case in a form given in [8].

Theorem 2.1. *Let $v(z)$ be an $m \times m$ matrix-valued meromorphic function such that $v(\bar{z})^* = v(z)$ on $\mathbf{C}_+ \cup \mathbf{C}_-$. A necessary and sufficient condition that $v(z)$ belong to some class $\mathbf{N}_\varkappa$, $\varkappa \geq 0$, is that it can be written in the form*

$$\begin{aligned} v(z) = & \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \sum_{j=0}^{r} S_j(t,z) \right] d\tau(t) \qquad (2.1) \\ & + R_0(z) - \sum_{j=1}^{r} R_j\left(\frac{1}{z-\alpha_j}\right) \\ & - \sum_{k=1}^{s} \left[M_k\left(\frac{1}{z-\beta_k}\right) + M_k\left(\frac{1}{\bar{z}-\beta_k}\right)^* \right], \end{aligned}$$

where $\alpha_1, \ldots, \alpha_r \in (-\infty, \infty)$ and $\beta_1, \ldots, \beta_s \in \mathbf{C}_+$ are distinct numbers, and

(1°) *the real line is a union of sets $\Delta_0, \Delta_1, \ldots, \Delta_r$ such that $\Delta_1, \ldots, \Delta_r$ are bounded open intervals containing $\alpha_1, \ldots, \alpha_r$ and having disjoint closures, Δ_0 is their complement, and*

$$\begin{aligned} \frac{1}{t-z} - S_j(t,z) &= \frac{1}{t-z}\left(\frac{t-\alpha_j}{z-\alpha_j}\right)^{2\rho_j} \quad \text{on } \Delta_j, \quad j = 1, \ldots, r, \\ \frac{1}{t-z} - S_0(t,z) &= \frac{1+tz}{t-z}\,\frac{(1+z^2)^{\rho_0}}{(1+t^2)^{\rho_0+1}} \quad \text{on } \Delta_0, \end{aligned}$$

for some positive integers $\rho_1, \ldots, \rho_r$ and some nonnegative integer ρ_0, and $S_j(t,z) = 0$ off Δ_j for each $j = 0, 1, \ldots, r$;

(2°) *$\tau(t)$ is an $m \times m$ matrix-valued function which is nondecreasing on each of the $r+1$ open intervals of the real line determined by the points $\alpha_1, \ldots, \alpha_r$ such that the integral*

$$\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\cdots+\rho_r}} \; \frac{d\tau(t)}{(1+t^2)^{\rho_0+1}}$$

converges;

(3°) *for each $j = 0, 1, \ldots, r$, $R_j(z)$ is a polynomial of degree at most $2\rho_j + 1$, having selfadjoint $m \times m$ matrix coefficients, such that if a term of maximum degree $C_j z^{2\rho_j+1}$ is present then $C_j \geq 0$, and $R_1(0) = \cdots = R_r(0) = 0$;*

(4°) *for each $k = 1, \ldots, s$, $M_k(z)$ is a polynomial $\not\equiv 0$ with $m \times m$ matrix coefficients such that $M_k(0) = 0$.*

The sets $\Delta_0, \Delta_1, \ldots, \Delta_r$ in (2.1) can be chosen arbitrarily subject to the conditions in (1°).

Definition 2.2. By **Kreĭn-Langer data** we mean a collection of quantities

$$\boldsymbol{\tau} = \{\tau(t);\ \alpha_1, \dots, \alpha_r;\ \beta_1, \dots, \beta_s;\ \rho_0, \dots, \rho_r;\ \Delta_0, \dots, \Delta_r; \\ R_0(z), \dots, R_r(z);\ M_1(z), \dots, M_s(z)\} \tag{2.2}$$

having the properties listed in Theorem 2.1. Given data $\boldsymbol{\tau}$, we write $v_{\boldsymbol{\tau}}(z)$ for the associated function (2.1).

The identities

$$\frac{1}{t-z}\left(\frac{t-\alpha}{z-\alpha}\right)^{2p} = \frac{1}{t-z} + \sum_{j=0}^{2p-1} \frac{(t-\alpha)^j}{(z-\alpha)^{j+1}} \tag{2.3}$$

$$\frac{(1+z^2)^p}{(1+t^2)^{p+1}}\,\frac{1+tz}{t-z} = \frac{1}{t-z} - (t+z)\sum_{j=0}^{p-1} \frac{(1+z^2)^j}{(1+t^2)^{j+1}} \\ - t\,\frac{(1+z^2)^p}{(1+t^2)^{p+1}} \tag{2.4}$$

show that the convergence terms in (2.1) are given by

$$S_j(t,z) = -\sum_{p=0}^{2\rho_j-1} \frac{(t-\alpha_j)^p}{(z-\alpha_j)^{p+1}}\,\chi_{\Delta_j}(t), \qquad j = 1, \dots, r, \tag{2.5}$$

$$S_0(t,z) = \left\{(t+z)\sum_{p=0}^{\rho_0-1} \frac{(1+z^2)^p}{(1+t^2)^{p+1}} + t\,\frac{(1+z^2)^{\rho_0}}{(1+t^2)^{\rho_0+1}}\right\}\chi_{\Delta_0}(t). \tag{2.6}$$

3. Interpolation problem for operator identities

Throughout this section we understand that $\mathfrak{H}$ is some Hilbert space, and, as usual, $\mathfrak{G} = \mathbf{C}^m$. Our first task is to construct operators S_v and $\Phi_{1,v}$ corresponding to a given generalized Nevanlinna function $v(z)$. These operators appear in the statement of the abstract interpolation problem in Definition 3.6.

Assumptions 3.1. *Let $A \in \mathfrak{L}(\mathfrak{H})$ and $\Phi_2 \in \mathfrak{L}(\mathfrak{G}, \mathfrak{H})$ be given operators, and let $v(z)$ be a generalized Nevanlinna function which is represented in the form (2.1) for some Kreĭn-Langer data (2.2). Assume that $\sigma(A)$ is a finite set that contains no point $1/\beta_k, 1/\bar{\beta}_k$, $k = 1, \dots, s$, and no real point except perhaps 0.*

Case 1: $0 \notin \sigma(A)$**.** *There are no additional assumptions in this case.*

Case 2: $0 \in \sigma(A)$**.** *Here we assume further that (2.1) can be chosen such that $\rho_0 = 0$, $R_0(z)$ is constant, and*

$$\int_{\Delta_0} \langle d\tau(t)\,\Phi_2^*(I - A^*t)^{-1}h, \Phi_2^*(I - A^*t)^{-1}h\rangle < \infty, \qquad h \in \mathfrak{H}. \tag{3.1}$$

By [8, Theorem 4.1], the condition that $v(iy)/y \to 0$ as $y \to \infty$ is necessary and sufficient that a representation (2.1) can be chosen such that $\rho_0 = 0$ and $R_0(z)$ is constant.

Under the Assumptions 3.1, in both Case 1 and Case 2, we shall define operators

$$S_v = \int_{-\infty}^{\infty} \left\{ (I - At)^{-1} \Phi_2[d\tau(t)] \Phi_2^* \, (I - A^* t)^{-1} - \sum_{j=0}^{r} d\tau_j(t; A, \Phi_2) \right\} \tag{3.2}$$
$$+ \sum_{j=0}^{r} \mathfrak{R}_j + \sum_{k=1}^{s} \big[\mathfrak{M}_{1k} + \mathfrak{M}_{2k} \big],$$

$$\Phi_{1,v} = -i \int_{-\infty}^{\infty} \left\{ A(I - At)^{-1} - \sum_{j=0}^{r} \mathfrak{S}_j(t; A) \right\} \Phi_2 \, [d\tau(t)] \tag{3.3}$$
$$- i \left(\sum_{j=0}^{r} \widehat{\mathfrak{R}}_j + \sum_{k=1}^{s} \big[\widehat{\mathfrak{M}}_{1k} + \widehat{\mathfrak{M}}_{2k} \big] \right),$$

which generalize (1.5) and (1.6) to the indefinite case. The terms in (3.2) and (3.3) are associated with the parts in the Kreĭn-Langer representation (2.1) of $v(z)$. The definitions differ slightly in Case 1 and Case 2 of the Assumptions 3.1, that is, according as $0 \notin \sigma(A)$ or $0 \in \sigma(A)$.

Definition of S_v and $\Phi_{1,v}$ in Case 1. In Case 1, $0 \notin \sigma(A)$ and so $\sigma(A)$ contains no real point. We first define the convergence terms $d\tau_j(t; A, \Phi_2)$ and $\mathfrak{S}_j(t; A)$ in the integral parts of (3.2) and (3.3), $j = 0, \dots, r$. These terms are defined to be zero off Δ_j, $j = 0, \dots, r$. For $j = 1, \dots, r$, we expand

$$(I - tA)^{-1} \Phi_2[d\tau(t)] \Phi_2^* (I - tA^*)^{-1} \qquad \text{and} \qquad A(I - tA)^{-1} \tag{3.4}$$

in powers of $t - \alpha_j$ using the series

$$(I - tA)^{-1} = \sum_{p=0}^{\infty} A_p(\alpha_j)(t - \alpha_j)^p, \quad A_p(\alpha_j) = A^p (I - \alpha_j A)^{-p-1}, \tag{3.5}$$

and we define $d\tau_j(t; A, \Phi_2)$ and $\mathfrak{S}_j(t; A)$ on Δ_j to be the expressions that remain after discarding all terms that are $\mathcal{O}\big((t - \alpha_j)^{2\rho_j}\big)$ as $t \to \alpha_j$.

To define $d\tau_0(t; A, \Phi_2)$ and $\mathfrak{S}_0(t; A)$ on Δ_0, we expand (3.4) in a neighborhood of infinity using

$$\begin{aligned} (I - tA)^{-1} &= (I + tA)\Big(I + A^2 - (1 + t^2)A^2\Big)^{-1} \\ &= -\frac{(A^{-2} + tA^{-1})}{1 + t^2} \left(I - \frac{I + A^{-2}}{1 + t^2} \right)^{-1} \\ &= -\sum_{p=0}^{\infty} \frac{(A^{-2} + tA^{-1})(I + A^{-2})^p}{(1 + t^2)^{p+1}}, \end{aligned} \tag{3.6}$$

and collect into terms $1/(1+t^2)^\ell$ and $t/(1+t^2)^\ell$, $\ell \geq 1$. After discarding all terms that are $\mathcal{O}\big(1/(1+t^2)^{\rho_0+1}\big)$ as $|t| \to \infty$, the expressions that remain are defined to be $d\tau_0(t; A, \Phi_2)$ and $\mathfrak{S}_0(t; A)$ on Δ_0. These definitions assure that the integrals in (3.2) and (3.3) converge weakly.

In the discrete parts of (3.2) and (3.3), we define

$$\begin{cases} \mathfrak{R}_0 = \operatorname*{Res}_{\lambda=0} \Big[(A-\lambda I)^{-1}\Phi_2 R_0(\lambda^{-1})\Phi_2^*(A^*-\lambda I)^{-1}\Big]\,, \\ \widehat{\mathfrak{R}}_0 = -\operatorname*{Res}_{\lambda=0} \Big[A(A-\lambda I)^{-1}\Phi_2 R_0(\lambda^{-1})\lambda^{-1}\Big]\,. \end{cases} \tag{3.7}$$

For $j = 1, \dots, r$, set

$$\begin{cases} \mathfrak{R}_j = \operatorname*{Res}_{\lambda=\alpha_j} \Big[(I-\lambda A)^{-1}\Phi_2 R_j\Big(\dfrac{1}{\lambda-\alpha_j}\Big)\Phi_2^*(I-\lambda A^*)^{-1}\Big]\,, \\ \widehat{\mathfrak{R}}_j = \operatorname*{Res}_{\lambda=\alpha_j} \Big[A(I-\lambda A)^{-1}\Phi_2 R_j\Big(\dfrac{1}{\lambda-\alpha_j}\Big)\Big]\,. \end{cases} \tag{3.8}$$

For $k = 1, \dots, s$, set

$$\begin{cases} \mathfrak{M}_{1k} = \operatorname*{Res}_{\lambda=\beta_k} \Big[(I-\lambda A)^{-1}\Phi_2 M_k\Big(\dfrac{1}{\lambda-\beta_k}\Big)\Phi_2^*(I-\lambda A^*)^{-1}\Big]\,, \\ \widehat{\mathfrak{M}}_{1k} = \operatorname*{Res}_{\lambda=\beta_k} \Big[A(I-\lambda A)^{-1}\Phi_2 M_k\Big(\dfrac{1}{\lambda-\beta_k}\Big)\Big]\,, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \mathfrak{M}_{2k} = \operatorname*{Res}_{\lambda=\bar\beta_k} \Big[(I-\lambda A)^{-1}\Phi_2 M_k\Big(\dfrac{1}{\bar\lambda-\beta_k}\Big)^*\Phi_2^*(I-\lambda A^*)^{-1}\Big]\,, \\ \widehat{\mathfrak{M}}_{2k} = \operatorname*{Res}_{\lambda=\bar\beta_k} \Big[A(I-\lambda A)^{-1}\Phi_2 M_k\Big(\dfrac{1}{\bar\lambda-\beta_k}\Big)^*\Big]\,. \end{cases} \tag{3.10}$$

Definition of S_v and $\Phi_{1,v}$ in Case 2. Now $0 \in \sigma(A)$, $\rho_0 = 0$, $R_0(z) = C_0$ is constant, and (3.1) holds. In this case we define

$$\begin{cases} d\tau_0(t; A, \Phi_2) = 0, \qquad \mathfrak{S}_0(t; A) = -\dfrac{t\,I}{1+t^2}\,, \\ \qquad \mathfrak{R}_0 = 0, \qquad \widehat{\mathfrak{R}}_0 = -\Phi_2 C_0. \end{cases} \tag{3.11}$$

All other terms are defined as in Case 1. The integral in (3.2) converges weakly by construction. The proof that the integral in (3.3) converges weakly in Case 2 is similar to an argument in [12, p. 2].

Theorem 3.2. *Under the Assumptions* 3.1*, on* Δ_j*,* $j = 1, \dots, r$*,*

$$\begin{aligned} d\tau_j(t; A, \Phi_2) &= \sum_{\ell=0}^{2\rho_j-1} (t-\alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q\geq 0}} A_p(\alpha_j)\,\Phi_2[d\tau(t)]\Phi_2^*\,A_q(\alpha_j)^* \\ &= -\operatorname*{Res}_{\lambda=\alpha_j} \Big[(I-\lambda A)^{-1} S_j(t,\lambda)\,\Phi_2[d\tau(t)]\,\Phi_2^*(I-\lambda A^*)^{-1}\Big]\,, \end{aligned}$$

$$\begin{aligned}\mathfrak{S}_j(t;A) &= \sum_{p=0}^{2\rho_j-1} (t-\alpha_j)^p A_p(\alpha_j) A \\ &= -\operatorname*{Res}_{\lambda=\alpha_j} \Big[A\,(I-\lambda A)^{-1}\, S_j(t,\lambda) \Big] .\end{aligned}$$

If $0 \notin \sigma(A)$, *then also on* Δ_0,

$$\begin{aligned} d\tau_0(t;A,\Phi_2) &= \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} A^{-j}\Phi_2[d\tau(t)]\,\Phi_2^* A^{*-k} \\ &\quad + \sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\geq 1}} A^{-j}\Phi_2[d\tau(t)]\,\Phi_2^* A^{*-k} \\ &= \operatorname*{Res}_{\lambda=0} \Big[(A-\lambda I)^{-1} S_0(t,\lambda^{-1})\,\Phi_2[d\tau(t)]\,\Phi_2^*(A^*-\lambda I)^{-1} \Big] , \\ \mathfrak{S}_0(t;A) &= -S_0(t,A^{-1}) \\ &= -\operatorname*{Res}_{\lambda=0} \Big[\lambda^{-1}\, A\,(A-\lambda I)^{-1} S_0(t,\lambda^{-1}) \Big] .\end{aligned}$$

In the case $0 \notin \sigma(A)$, the identity

$$\mathfrak{S}_j(t;A) = -S_j(t,A^{-1}) \tag{3.12}$$

holds for all $j = 0,\ldots,r$.

Theorem 3.3. *Under the Assumptions* 3.1, *if*

$$R_0(z) = \sum_{p=0}^{2\rho_0+1} R_{0p}\, z^p\,, \quad R_j(z) = \sum_{p=1}^{2\rho_j+1} R_{jp}\, z^p, \quad M_k(z) = \sum_{p=1}^{\sigma_k} M_{kp}\, z^p,$$

are the polynomials in the representation (2.1) *of* $v(z)$, *then*

$$\begin{aligned} \mathfrak{R}_j &= \sum_{p=1}^{2\rho_j+1} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\geq 1}} A_{\mu-1}(\alpha_j)\Phi_2 R_{jp}\Phi_2^* A_{\nu-1}(\alpha_j)^*\,, \\ \widehat{\mathfrak{R}}_j &= \sum_{p=1}^{2\rho_j+1} A\,A_{p-1}(\alpha_j)\Phi_2 R_{jp}\,, \\ \mathfrak{M}_{1k} &= \sum_{p=1}^{\sigma_k} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\geq 1}} A_{\mu-1}(\beta_k)\Phi_2 M_{kp}\Phi_2^* A_{\nu-1}(\bar\beta_k)^*\,, \\ \widehat{\mathfrak{M}}_{1k} &= \sum_{p=1}^{\sigma_k} A\,A_{p-1}(\beta_k)\Phi_2 M_{kp}\,, \end{aligned}$$

$$\mathfrak{M}_{2k} = \sum_{p=1}^{\sigma_k} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\geq 1}} A_{\mu-1}(\bar{\beta}_k)\Phi_2 M_{kp}^* \Phi_2^* A_{\nu-1}(\beta_k)^* \,,$$

$$\widehat{\mathfrak{M}}_{2k} = \sum_{p=1}^{\sigma_k} A\, A_{p-1}(\bar{\beta}_k)\Phi_2 M_{kp}^* \,,$$

$j = 1, \dots, r$ and $k = 1, \dots, s$. If $0 \notin \sigma(A)$, then also

$$\mathfrak{R}_0 = \sum_{p=1}^{2\rho_0+1} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\geq 1}} A^{-\mu}\Phi_2 R_{0p}\Phi_2^* A^{*-\nu} \,,$$

$$\widehat{\mathfrak{R}}_0 = -\sum_{p=0}^{2\rho_0+1} A^{-p}\Phi_2 R_{0p} \,.$$

When $\varkappa = 0$, the Kreĭn-Langer representation (2.1) reduces to the Nevanlinna representation (1.3); the Nevanlinna representation is unique, and when $\varkappa = 0$ the definitions of S_v and $\Phi_{1,v}$ given above reduce to the known forms (1.5) and (1.6). In contrast, the Kreĭn-Langer representation (2.1) in general is not unique. We show that the definitions of S_v and $\Phi_{1,v}$ do not depend on the choice of representation (2.1) for $v(z)$.

Theorem 3.4. *So long as the Assumptions* 3.1 *are met, the definitions of S_v and $\Phi_{1,v}$ do not depend on the choice of Kreĭn-Langer data* (2.2) *in the representation* (2.1).

We obtain a large class of examples of the operator identity (1.1).

Theorem 3.5. *Under the Assumptions* 3.1*, in both Case* 1 *and Case* 2*, the operator S_v is selfadjoint, $\varkappa_{S_v} < \infty$, and the operators $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ together with the given operators A and Φ_2 satisfy* (1.1).

Definition 3.6. The **abstract interpolation problem** for a given operator identity (1.1) is to find all generalized Nevanlinna functions $v(z)$ such that

$$S = S_v \qquad \text{and} \qquad \Phi_1 = \Phi_{1,v} \,. \tag{3.13}$$

The motivating example is classical Pick-Nevanlinna interpolation, where we choose $\mathfrak{H} = \mathbf{C}^m \oplus \cdots \oplus \mathbf{C}^m$ with n summands.

Theorem 3.7. *Let $z_1, \dots, z_n$ be distinct points in $\mathbf{C}_+$, and set*

$$A = \begin{bmatrix} z_1 I_m & 0 & \cdots & 0 \\ 0 & z_2 I_m & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & z_n I_m \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}. \tag{3.14}$$

Let $v(z)$ be a generalized Nevanlinna function. Set $w(z) = -v(1/\bar{z})^*$, and assume that the poles of $w(z)$ are disjoint from $z_1, \dots, z_n$. Then the operators (3.2) *and* (3.3) *are given by*

$$S_v = \left[\frac{w(z_\mu) - w(z_\nu)^*}{z_\mu - \bar{z}_\nu} \right]_{\mu,\nu=1}^n, \qquad \Phi_{1,v} = -i \begin{bmatrix} w(z_1) \\ w(z_2) \\ \vdots \\ w(z_n) \end{bmatrix}. \tag{3.15}$$

For the same A and Φ_2, the operator identity (1.1) is satisfied with

$$S = \left[\frac{w_\mu - w_\nu^*}{z_\mu - \bar{z}_\nu} \right]_{\mu,\nu=1}^n, \qquad \Phi_1 = -i \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

where $w_1, \dots, w_n$ are given matrices. Solutions of the abstract interpolation problem in this case correspond to solutions of the classical Pick-Nevanlinna interpolation problem $w(z_\mu) = w_\mu$, $\mu = 1, \dots, n$.

4. Generalization of the fundamental matrix inequality

In this section we introduce and study two kernels, $L_v(z,\zeta)$ and $L_{v,T}(z,\zeta)$ defined by (4.2) and (4.4)–(4.5) below, which are associated with any given operator identity (1.1) and generalized Nevanlinna function $v(z)$. These kernels are related to linear fractional transformations and the fundamental matrix inequality. The fundamental matrix inequality is used to solve classical interpolation problems by Kovalishina and Potapov [4] and Katsnelson [3], for example. In the definite case, the fundamental matrix inequality is adapted to the abstract interpolation problem in [12, Theorem 1.2.1]; it asserts that the kernel $L_v(z,\zeta)$ defined by (4.2) is nonnegative on the diagonal $z = \zeta$ for any solution $v(z)$. Our generalization, Theorem 4.5, asserts that in the indefinite case the two-variable kernel $L_v(z,\zeta)$ has a finite number of negative squares for any solution $v(z)$ of an abstract interpolation problem.

If A, S, Φ_1, Φ_2 are operators which satisfy (1.1), we shall also write

$$AS - SA^* = i\,\Pi J \Pi^*, \quad \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{4.1}$$

For any generalized Nevanlinna function $v(z)$, we define a kernel

$$L_v(z,\zeta) = \begin{bmatrix} S & B_v(z) \\ B_v(\zeta)^* & C_v(z,\zeta) \end{bmatrix}, \tag{4.2}$$

where

$$\begin{cases} B_v(z) = (I - zA)^{-1}\big[\Phi_1 - i\Phi_2 v(z)\big]\,, \\ C_v(z,\zeta) = \dfrac{v(z) - v(\zeta)^*}{z - \bar{\zeta}}\,. \end{cases} \tag{4.3}$$

We also use the transformed kernel

$$L_{v,T}(z,\zeta) = \begin{bmatrix} S & -iB_{v,T}(z) \\ iB_{v,T}(\zeta)^* & C_{v,T}(z,\zeta) \end{bmatrix} \tag{4.4}$$

defined by

$$L_{v,T}(z,\zeta) = \begin{bmatrix} I & 0 \\ L_0(\bar{\zeta}) & L_2(\bar{\zeta}) \end{bmatrix} \begin{bmatrix} S & B_v(z) \\ B_v(\zeta)^* & C_v(z,\zeta) \end{bmatrix} \begin{bmatrix} I & L_0(\bar{z})^* \\ 0 & L_2(\bar{z})^* \end{bmatrix}, \tag{4.5}$$

where

$$L_0(z) = iA(I - zA)^{-1} \qquad \text{and} \qquad L_2(z) = (I - zA)^{-1}\Phi_2, \tag{4.6}$$

as in [12, p. 6]. By direct calculation,

$$B_{v,T}(z) - B_{v,T}(\zeta)^* = (z - \bar{\zeta})C_{v,T}(z,\zeta),$$

and hence

$$B_{v,T}(z) = B_{v,T}(\bar{z})^*, \qquad C_{v,T}(z,\zeta) = \frac{B_{v,T}(z) - B_{v,T}(\zeta)^*}{z - \bar{\zeta}}, \tag{4.7}$$

at all points z and ζ in $\mathbf{C}_+ \cup \mathbf{C}_-$ where the functions are defined.

The nondegenerate case (S invertible) is assumed in Theorems 4.1, 4.2, 4.4 and Definition 4.3.

Theorem 4.1. *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that S is invertible. Define*

$$\mathfrak{A}(z) = I - iz\Pi^*(I - zA^*)^{-1}S^{-1}\Pi J \tag{4.8}$$

on the set $\Omega_{\mathfrak{A}}$ of all $z \in \mathbf{C}$ such that the inverse exists. For all $\bar{z}, \bar{\zeta} \in \Omega_{\mathfrak{A}}$,

$$\frac{J - \mathfrak{A}(\bar{\zeta})J\mathfrak{A}(\bar{z})^*}{i(\bar{\zeta} - z)} = \Pi^*(I - \bar{\zeta}A^*)^{-1}S^{-1}(I - zA)^{-1}\Pi. \tag{4.9}$$

If $z, \bar{z} \in \Omega_{\mathfrak{A}}$, $\mathfrak{A}(z)$ is invertible and $\mathfrak{A}(z)^{-1} = J\mathfrak{A}(\bar{z})^*J$.

In particular, $\mathfrak{A}(z)$ has invertible values except at isolated points of its domain.

Theorem 4.2. *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that S is invertible. Given any generalized Nevanlinna function $v(z)$, set*

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \mathfrak{A}(z)^{-1} \begin{bmatrix} -iv(z) \\ I \end{bmatrix}. \tag{4.10}$$

Then $P(\bar{z})^*Q(z) + Q(\bar{z})^*P(z) = 0$ and

$$L_v(z,\zeta) = \begin{bmatrix} I & 0 \\ B_v(\zeta)^*S^{-1} & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D_v(z,\zeta) \end{bmatrix} \begin{bmatrix} I & S^{-1}B_v(z) \\ 0 & I \end{bmatrix}, \tag{4.11}$$

where

$$D_v(z,\zeta) = \frac{v(z)-v(\zeta)^*}{z-\bar\zeta} - B_v(\zeta)^* S^{-1} B_v(z) \tag{4.12}$$
$$= i\,\frac{P(\zeta)^*Q(z)+Q(\zeta)^*P(z)}{z-\bar\zeta}$$

at all points where the functions are defined.

The block entries $a(z), b(z), c(z), d(z)$ of $\mathfrak{A}(z)$ are used as coefficients of a class of linear fractional transformations.

Definition 4.3. Let A, S, Φ_1, Φ_2 be operators which satisfy (1.1) such that $\sigma(A)$ is a finite set, S is invertible, and $\varkappa_S < \infty$. Write

$$\mathfrak{A}(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \tag{4.13}$$

where $a(z), b(z), c(z), d(z)$ are $m \times m$ matrix-valued functions. By $\mathbf{N}(\mathfrak{A})$ we mean the set of functions

$$v(z) = i\,[a(z)P(z)+b(z)Q(z)]\,[c(z)P(z)+d(z)Q(z)]^{-1}, \tag{4.14}$$

where $P(z)$ and $Q(z)$ are $m \times m$ matrix-valued functions which are analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$ except at isolated points, such that

(i) $P(\bar z)^*Q(z)+Q(\bar z)^*P(z) \equiv 0$;
(ii) $c(z)P(z)+d(z)Q(z)$ is invertible except at isolated points;
(iii) the kernel

$$D_{P,Q}(z,\zeta) = i\,\frac{P(\zeta)^*Q(z)+Q(\zeta)^*P(z)}{z-\bar\zeta}$$

has a finite number $\varkappa_{P,Q}$ of negative squares.

Theorem 4.4. *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that $\sigma(A)$ is a finite set, S is invertible, and $\varkappa_S < \infty$. If $v(z) \in \mathbf{N}(\mathfrak{A})$ and has the representation* (4.14)*, then $v(z) = v(\bar z)^*$ at all points where the functions are defined, and $v(z) \in \mathbf{N}_{\varkappa_v}$ where*

$$\varkappa_v \le \varkappa_{P,Q} + \varkappa_S = \varkappa_{L_v}. \tag{4.15}$$

In particular, $\varkappa_{L_v} = \varkappa_{P,Q} + \varkappa_S < \infty$. Moreover,

$$\frac{v(z)-v(\zeta)^*}{z-\bar\zeta} = K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} + B_v(\zeta)^* S^{-1} B_v(z), \tag{4.16}$$

and

$$\frac{B_{v,T}(z)-B_{v,T}(\zeta)^*}{z-\bar\zeta} = L_2(\bar\zeta) K(\zeta)^{*-1} D_{P,Q}(z,\zeta) K(z)^{-1} L_2(\bar z)^* + B_{v,T}(\zeta)^* S^{-1} B_{v,T}(z), \tag{4.17}$$

where $K(z) = [c(z)P(z)+d(z)Q(z)]^{-1}$ except at isolated points.

A consequence of Theorem 4.4 is that the kernel $L_v(z,\zeta)$ has a finite number of negative squares in a particular case in which S is invertible. Theorems 4.5, 4.6, and 4.7 below do not presume that S is invertible. Theorem 4.5 describes another case in which $L_v(z,\zeta)$ has a finite number of negative squares, but its proof does not give a simple description of the exact value of $\varkappa_{L_v}$ as in Theorem 4.4. Theorem 4.5 can be viewed as a generalization of the fundamental matrix inequality to the indefinite setting.

Theorem 4.5. *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1)*, where $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ are defined by* (3.2) *and* (3.3) *for some generalized Schur function $v(z)$. Then in both Case* 1 *and Case* 2 *of the Assumptions* 3.1*, the kernel $L_v(z,\zeta)$ defined by* (4.2) *has a finite number of negative squares.*

The next two results establish companions to Theorem 4.5 that provide additional necessary conditions on solutions of the abstract interpolation problem particular to Case 1 and Case 2.

Theorem 4.6. *In Theorem* 4.5*, Case* 1*, the function $B_v(z)$ in* (4.2) *is analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$ except perhaps for poles at the poles $\beta_k, \bar{\beta}_k$, $k = 1, \dots, s$, of $v(z)$. Hence $B_v(z)$ is analytic at every point λ such that $1/\lambda \in \sigma(A)$.*

Theorem 4.7. *In Theorem* 4.5*, Case* 2*, the functions $B_v(z)$ and $B_{v,T}(z)$ in* (4.2) *and* (4.4) *satisfy*

$$\|B_v(z)\| = \mathcal{O}(1) \tag{4.18}$$

and

$$\|B_{v,T}(z)\| = \mathcal{O}\left(\frac{1}{|z|}\right) \tag{4.19}$$

as $|z| \to \infty$ in any set $D_\delta = \{z\colon 0 < |\arg z| < \pi - \delta\}$ where $0 < \delta < \pi$.

5. Interpolation theorems

We now begin with an operator identity $AS - SA^* = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right]$ such that $\sigma(A)$ is a finite set, S is invertible, and $\varkappa_S < \infty$. The abstract interpolation problem (3.13) is to characterize all generalized Nevanlinna functions $v(z)$ such that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$. We shall see that such a function $v(z)$ belongs to the class $\mathbf{N}(\mathfrak{A})$ introduced in Definition 4.3. We give necessary and sufficient conditions on a function $v(z)$ in $\mathbf{N}(\mathfrak{A})$ that it is a solution of the abstract interpolation problem. They assert, roughly, that the necessary conditions on the functions

$$B_v(z) = (I - zA)^{-1}\left[\Phi_1 - i\Phi_2 v(z)\right]$$

and

$$\begin{aligned} B_{v,T}(z) &= \Big[SA^* + iB_v(z)\Phi_2^*\Big](I - zA^*)^{-1} \\ &= (I - zA)^{-1}\Big[AS - i\Phi_2 B_v(\bar{z})^*\Big], \end{aligned}$$

defined as in Section 4 are sufficient in the nondegenerate case, that is, when S is invertible.

Theorem 5.1 (Interpolation in Case 1). *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that*

- *S is invertible, and $\varkappa_S < \infty$;*
- *$\sigma(A)$ is a finite set and $\sigma(A) \cap \sigma(A^*) = \emptyset$.*

(1) *Suppose that $v(z) \in \mathbf{N}(\mathfrak{A})$, and that*

(i) *$v(z)$ has at most a removable singularity at every nonreal number λ such that $1/\lambda \in \sigma(A)$;*
(ii) *$B_v(z)$ has at most a removable singularity at every nonreal number λ such that $1/\lambda \in \sigma(A)$.*

Then $v(z)$ is a generalized Nevanlinna function, the conditions of Assumptions 3.1, *Case* 1, *are met, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$.*

(2) *Conversely, if $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for some generalized Nevanlinna function $v(z)$ having a representation* (2.1) *which satisfies Assumptions* 3.1, *Case* 1, *then $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$ and satisfies conditions* (i) *and* (ii) *in* (1).

We note a sufficient condition that the technical conditions (i) and (ii) in Theorem 5.1(1) are satisfied.

Theorem 5.2. *Conditions* (i) *and* (ii) *in Theorem* 5.1(1) *hold if $v(z)$ has a representation* (4.14) *such that every point λ satisfying $1/\lambda \in \sigma(A)$ belongs to the domain of holomorphy of $P(z)$ and $Q(z)$ and $c(\lambda)P(\lambda) + d(\lambda)Q(\lambda)$ is invertible.*

Theorem 5.3 (Interpolation in Case 2). *Let A, S, Φ_1, Φ_2 be operators which satisfy* (1.1) *such that*

- *S is invertible, and $\varkappa_S < \infty$;*
- *$\sigma(A) = \{0\}$, and $\|(I - iyA)^{-1} f\| \neq \mathcal{O}(1)$ as $|y| \to \infty$ for every $f \neq 0$ in $\mathfrak{H}$.*

(1) *Let $v(z)$ belong to $\mathbf{N}(\mathfrak{A})$, and suppose that*

(i) *$v(iy)/y \to 0$ as $|y| \to \infty$;*
(ii) *for all h in $\mathfrak{H}$ and g in $\mathbf{C}^m$, $\langle B_v(iy)g, h\rangle = \mathcal{O}(1)$ as $|y| \to \infty$;*
(iii) *for all h and k in $\mathfrak{H}$, $\langle B_{v,T}(iy)h, k\rangle = \mathcal{O}(1/|y|)$ as $|y| \to \infty$.*

Then $v(z)$ is a generalized Nevanlinna function, the conditions of Assumptions 3.1, *Case* 2, *are met, and $S = S_v$ and $\Phi_1 = \Phi_{1,v}$.*

(2) *Conversely, if $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for some generalized Nevanlinna function $v(z)$ having a representation* (2.1) *which satisfies Assumptions* 3.1, *Case* 2, *then $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$ and satisfies conditions* (i)–(iii) *in* (1).

It can be shown that the conditions on A required in Theorem 5.3 are met, for example, for

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad \text{on} \qquad \mathbf{C}^n,$$

and for

$$(Af)(x) = i \int_0^x f(t)\, dt \qquad \text{on} \qquad L^2_m(0,\ell), \tag{5.1}$$

for any positive integers m and n.

The hypotheses in Theorem 5.3 can be weakened when $\ker A = \{0\}$.

Theorem 5.4. *Theorem* 5.3 *remains true if the hypothesis on A is changed to read:*

- $\sigma(A) = \{0\}$, $\ker A = \{0\}$, *and* $y\,\|(I - iyA)^{-1} f\| \neq \mathcal{O}(1)$ *as* $|y| \to \infty$ *for every* $f \neq 0$ *in* $\mathfrak{H}$.

Example 5.5. In Theorem 5.3(1), the conditions (i)–(iii) are satisfied in an important concrete situation. Let $\mathfrak{H} = L^2_m(0,\ell)$ and $\mathfrak{G} = \mathbf{C}^m$ for some positive integer m and positive number ℓ. Let A be given by (5.1), and assume that S has the form (see [11])

$$(Sf)(x) = \frac{d}{dx} \int_0^\ell s(x-t) f(t)\, dt,$$

where $s(x)$ is a matrix-valued function such that $s(x) = -s(-x)^*$ on $(-\ell,\ell)$ and $s(x)g \in L^2_m(-\ell,\ell)$ for every $g \in \mathfrak{G}$. A particular case of such an operator is an integral operator of the form

$$(Sf)(x) = f(x) + \int_0^\ell k(x-t) f(t)\, dt,$$

where $k(x) = k(-x)^*$ is a bounded continuous matrix-valued function on $(-\ell,\ell)$. The operator identity (1.1) is satisfied with natural choices of operators Φ_1 and Φ_2. If the integro-differential operator S is bounded, invertible, and $\varkappa_S < \infty$, then the conditions (i)–(iii) in part (1) of Theorem 5.3 are satisfied for every function $v(z)$ in $\mathbf{N}(\mathfrak{A})$ given by (4.14) such that the kernel $D_{P,Q}(z,\zeta)$ is nonnegative. This generalizes a result of A. L. Sakhnovich [10]. The method of proof is interesting and applicable in other examples. Details will appear elsewhere.

6. Proofs of the theorems

We state some elementary lemmas that will be used in what follows. The proofs of the lemmas are straightforward, and details are omitted.

Lemma 6.1. *Define* $\binom{p}{k}$ *for* $p, k \geq 0$ *by* $(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$. *If* $p, k \geq 1$,

$$\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}. \tag{6.1}$$

For $q, s, u \geq 0$,

$$\sum_{\substack{p+r=q\\ p\geq 0,\ r\geq 0}} \binom{p}{s}\binom{r}{u} = \binom{q+1}{s+u+1}. \tag{6.2}$$

The residue formulas in Lemmas 6.2, 6.3, and 6.4 are deduced from elementary expansions, such as

$$(A-\lambda I)^{-1} = \sum_{\mu=0}^{\infty} A^{-\mu-1}\lambda^{\mu}, \quad (I-\lambda A)^{-1} = \sum_{p=0}^{\infty} A_p(\lambda_0)(\lambda-\lambda_0)^p,$$

where $A_p(\lambda) = A^p(I-\lambda A)^{-p-1}$. We assume here that A and C are bounded operators on appropriate spaces for which the expressions are meaningful, λ_0 and z are complex numbers, and p is a nonnegative integer.

Lemma 6.2. *If* $0 \notin \sigma(A)$, *then*

$$\operatorname*{Res}_{\lambda=0} \frac{(A-\lambda I)^{-1}C(A^*-\lambda I)^{-1}}{\lambda^p} = \begin{cases} \displaystyle\sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\geq 1}} A^{-\mu}CA^{*-\nu}, & p\geq 1,\\ 0, & p=0,\end{cases}$$

$$\operatorname*{Res}_{\lambda=0} \frac{\lambda^{-1}A(A-\lambda I)^{-1}C}{\lambda^p} = A^{-p}C\,,$$

$$\operatorname*{Res}_{\lambda=0} \frac{(A-\lambda I)^{-1}}{1-\lambda z}\,\frac{C}{\lambda^p} = \begin{cases} \displaystyle\sum_{\substack{j+k=p+1\\ j,k\geq 1}} A^{-j}C\,z^{k-1}, & p\geq 1,\\ 0, & p=0.\end{cases}$$

Lemma 6.3. (1) *If* $I-\lambda_0 A$ *and* $I-\lambda_0 A^*$ *are invertible, then*

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{(I-\lambda A)^{-1}C(I-\lambda A^*)^{-1}}{(\lambda-\lambda_0)^{p+1}} = \sum_{\substack{\mu+\nu=p\\ \mu,\nu\geq 0}} A_\mu(\lambda_0)CA_\nu(\bar\lambda_0)^*\,.$$

(2) *If* $I-\lambda_0 A$ *is invertible and* $z \neq \lambda_0$, *then*

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{(I-\lambda A)^{-1}C}{(\lambda-\lambda_0)^{p+1}} = A_p(\lambda_0)C\,,$$

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{(I-\lambda A)^{-1}}{z-\lambda}\,\frac{C}{(\lambda-\lambda_0)^{p+1}} = \sum_{\mu+\nu=p} \frac{A_\nu(\lambda_0)\,C}{(z-\lambda_0)^{\mu+1}}.$$

Lemma 6.4. (1) *If $P(z)$ is a polynomial with $P(0)=0$ and $z \neq \lambda_0$, then*

$$\operatorname*{Res}_{\lambda=\lambda_0} \frac{1}{z-\lambda}\, P\left(\frac{1}{\lambda-\lambda_0}\right) = P\left(\frac{1}{z-\lambda_0}\right).$$

(2) *If $P(z)$ is a polynomial, then*

$$\operatorname*{Res}_{\lambda=0} \frac{1}{1-\lambda z}\,\frac{1}{\lambda}\, P\left(\frac{1}{\lambda}\right) = P(z).$$

Proof of Theorem 3.2. In each case we prove the first of the two formulas; the residue versions then follow from Lemmas 6.2 and 6.3. Writing $dT = \Phi_2[d\tau(t)]\Phi_2^*$ and using (3.5), we obtain

$$\begin{aligned}(I-tA)^{-1}dT(I-tA^*)^{-1} &= \sum_{p=0}^{\infty}(t-\alpha_j)^p A_p(\alpha_j)\; dT\; \sum_{q=0}^{\infty}(t-\alpha_j)^q A_q(\alpha_j)^* \\ &\sim \sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^{\ell} \sum_{\substack{p+q=\ell\\ p,q\geq 0}} A_p(\alpha_j)\; dT\; A_q(\alpha_j)^*\,,\end{aligned}$$

where "$\sim$" indicates that we have dropped terms that are $\mathcal{O}\big((t-\alpha_j)^{2\rho_j}\big)$ as $t \to \alpha_j$. This yields the formula for $d\tau_j(t;A,\Phi_2)$ on Δ_j, $j = 1,\dots,r$. The formula for $\mathfrak{S}_j(t;A)$, $j=1,\dots,r$, is immediate from the definition.

Now assume that $0 \notin \sigma(A)$. To derive the formula for $d\tau_0(t;A,\Phi_2)$, we use (3.6) in the form

$$(I-tA)^{-1} = -\sum_{p=0}^{\infty}\frac{A^{-2}B^p + tA^{-1}B^p}{(1+t^2)^{p+1}}\,, \qquad B = I + A^{-2}.$$

Let "$\sim$" now indicate that we are dropping terms that are $\mathcal{O}\big(1/(1+t^2)^{\rho_0+1}\big)$ as $|t| \to \infty$. Then on Δ_0,

$$\begin{aligned}(I-tA)^{-1}dT(I-tA^*)^{-1} &\sim \sum_{p=0}^{\rho_0-1}\frac{A^{-2}B^p + tA^{-1}B^p}{(1+t^2)^{p+1}}\; dT \sum_{q=0}^{\rho_0-1}\frac{A^{-2}B^q + tA^{-1}B^q}{(1+t^2)^{q+1}} \\ &\sim \sum_{\ell=0}^{\rho_0-1}\frac{t}{(1+t^2)^{\ell+2}}\, P_\ell + \sum_{\ell=0}^{\rho_0-1}\frac{1}{(1+t^2)^{\ell+1}}\, Q_\ell\,.\end{aligned}$$

For $\ell \geq 0$, by Lemma 6.1,

$$\begin{aligned}P_\ell &= \sum_{p+q=\ell}\Big(A^{-2}B^p dT B^{*q} A^{*-1} + A^{-1}B^p dT B^{*q} A^{*-2}\Big) \\ &= \sum_{p+q=\ell}\sum_{\mu,\nu=0}^{\ell}\binom{p}{\mu}\binom{q}{\nu} A^{-2\mu-2}dT A^{*-2\nu-1}\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{p+q=\ell} \sum_{\mu,\nu=0}^{\ell} \binom{p}{\mu}\binom{q}{\nu} A^{-2\mu-1} dT A^{*-2\nu-2} \\
&= \sum_{p+q=\ell} \binom{\ell+1}{\mu+\nu+1} A^{-2\mu-2} dT A^{*-2\nu-1} \\
&\quad + \sum_{p+q=\ell} \binom{\ell+1}{\mu+\nu+1} A^{-2\mu-1} dT A^{*-2\nu-2} \\
&= \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1 \\ j,k\geq 1}} A^{-j} dT A^{*-k}.
\end{aligned}
$$

For $\ell \geq 1$, by (6.1) and Lemma 6.1,

$$
\begin{aligned}
Q_\ell &= \sum_{p+q=\ell-1} A^{-2} B^p dT B^{*q} A^{*-2} + \sum_{p+q=\ell} A^{-1} B^p dT B^{*q} A^{*-1} \\
&\quad - \sum_{p+q=\ell-1} A^{-1} B^p dT B^{*q} A^{*-1} \\
&= \sum_{p+q=\ell-1} \sum_{\mu,\nu=0}^{\ell} \binom{p}{\mu}\binom{q}{\nu} A^{-2\mu-2} dT A^{*-2\nu-2} \\
&\quad + \sum_{p+q=\ell} \sum_{\mu,\nu=0}^{\ell} \binom{p}{\mu}\binom{q}{\nu} A^{-2\mu-1} dT A^{*-2\nu-1} \\
&\quad - \sum_{p+q=\ell-1} \sum_{\mu,\nu=0}^{\ell} \binom{p}{\mu}\binom{q}{\nu} A^{-2\mu-1} dT A^{*-2\nu-1} \\
&= \sum_{\mu,\nu=0}^{\ell} \binom{\ell}{\mu+\nu+1} A^{-2\mu-2} dT A^{*-2\nu-2} \\
&\quad + \sum_{\mu,\nu=0}^{\ell} \binom{\ell+1}{\mu+\nu+1} A^{-2\mu-1} dT A^{*-2\nu-1} \\
&\quad - \sum_{\mu,\nu=0}^{\ell} \binom{\ell}{\mu+\nu+1} A^{-2\mu-1} dT A^{*-2\nu-1} \\
&= \sum_{\mu,\nu=0}^{\ell} \binom{\ell}{\mu+\nu+1} A^{-2\mu-2} dT A^{*-2\nu-2} \\
&\quad + \sum_{\mu,\nu=0}^{\ell} \binom{\ell}{\mu+\nu} A^{-2\mu-1} dT A^{*-2\nu-1}
\end{aligned}
$$

$$= \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2 \\ j,k \geq 1}} A^{-j} dT A^{*-k} .$$

The last expression agrees with $Q_0 = A^{-1} dT A^{*-1}$ when $\ell = 0$, and so we obtain the formula for $d\tau_0(t; A, \Phi_2)$. Finally, by (3.6),

$$\begin{aligned} A(I - At)^{-1} &= -t \sum_{p=0}^{\infty} \frac{(I + A^{-2})^p}{(1+t^2)^{p+1}} - A^{-1} \sum_{p=0}^{\infty} \frac{(I + A^{-2})^p}{(1+t^2)^{p+1}} \\ &\sim -t \sum_{p=0}^{\rho_0} \frac{(I + A^{-2})^p}{(1+t^2)^{p+1}} - A^{-1} \sum_{p=0}^{\rho_0 - 1} \frac{(I + A^{-2})^p}{(1+t^2)^{p+1}} \\ &= -S_0(t, A^{-1}), \end{aligned}$$

on Δ_0, which gives the formula for $\mathfrak{S}_0(t; A)$. □

Proof of Theorem 3.3. Calculate the residues using Lemmas 6.2 and 6.3. □

Proof of Theorem 3.4. Suppose that we have two representations $v(z) = v_{\boldsymbol{\tau}}(z) = v_{\widetilde{\boldsymbol{\tau}}}(z)$. Write $S_{\boldsymbol{\tau}}, S_{\widetilde{\boldsymbol{\tau}}}, \Phi_{1,\boldsymbol{\tau}}, \Phi_{1,\widetilde{\boldsymbol{\tau}}}$ for the operators (3.2) and (3.3) in the two representations. We show that

$$S_{\boldsymbol{\tau}} = S_{\widetilde{\boldsymbol{\tau}}} \qquad \text{and} \qquad \Phi_{1,\boldsymbol{\tau}} = \Phi_{1,\widetilde{\boldsymbol{\tau}}}. \tag{6.3}$$

The parts of $\boldsymbol{\tau}$ and $\widetilde{\boldsymbol{\tau}}$ coming from the nonreal poles of $v(z)$ are the same, and so we can assume that there are no nonreal poles. Then by (2.2),

$$\begin{aligned} \boldsymbol{\tau} &= \{\tau(t);\, \alpha_1, \dots, \alpha_r; -;\, \rho_0, \dots, \rho_r;\, \Delta_0, \dots, \Delta_r;\, R_0(z), \dots, R_r(z); -\}, \\ \widetilde{\boldsymbol{\tau}} &= \{\widetilde{\tau}(t);\, \widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{\widetilde{r}}; -;\, \widetilde{\rho}_0, \dots, \widetilde{\rho}_{\widetilde{r}};\, \widetilde{\Delta}_0, \dots, \widetilde{\Delta}_{\widetilde{r}};\, \widetilde{R}_0(z), \dots, \widetilde{R}_{\widetilde{r}}(z); -\}. \end{aligned}$$

By [8, Corollary 3.3], we can assume that $\widetilde{\tau}(t) = \tau(t)$ in the open intervals determined by the union of the points $\alpha_1, \dots, \alpha_r$ and $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{\widetilde{r}}$. We check (6.3) in three special cases.

Special Case A: $\widetilde{\boldsymbol{\tau}}$ is obtained from $\boldsymbol{\tau}$ by replacing one of the intervals $\Delta_1, \dots, \Delta_r$ by a smaller interval.

For example, suppose $\alpha_1 \in \widetilde{\Delta}_1 \subseteq \Delta_1$. Write $\widetilde{\Delta}_0 = \Delta_0 \cup E$, where $\widetilde{\Delta}_1 = \Delta_1 \setminus E$. Ignoring terms in (2.1) that do not change, we may take

$$\begin{aligned} v_{\boldsymbol{\tau}}(z) &= \int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z) \right] d\tau(t) + \int_{\Delta_1} \left[\frac{1}{t-z} - S_1(t,z) \right] d\tau(t) \\ &\quad + R_0(z) - R_1\left(\frac{1}{z - \alpha_1} \right), \\ v_{\widetilde{\boldsymbol{\tau}}}(z) &= \int_{\widetilde{\Delta}_0} \left[\frac{1}{t-z} - S_0(t,z) \right] d\tau(t) + \int_{\widetilde{\Delta}_1} \left[\frac{1}{t-z} - S_1(t,z) \right] d\tau(t) \\ &\quad + \widetilde{R}_0(z) - \widetilde{R}_1\left(\frac{1}{z - \alpha_1} \right), \end{aligned}$$

where because $v_{\boldsymbol{\tau}}(z) = v_{\widetilde{\boldsymbol{\tau}}}(z)$,

$$\widetilde{R}_0(z) = R_0(z) + \int_E S_0(t,z)\, d\tau(t),$$
$$\widetilde{R}_1\left(\frac{1}{z-\alpha_1}\right) = R_1\left(\frac{1}{z-\alpha_1}\right) + \int_E S_1(t,z)\, d\tau(t).$$

Theorems 3.2 and 3.3 allow us to explicitly calculate the operators in (6.3) and verify the equalities. We omit the routine calculations.

Special Case B: $\widetilde{\boldsymbol{\tau}}$ is obtained from $\boldsymbol{\tau}$ by adding a new point α_{r+1}.

By Special Case A, it can be presumed that the new point α_{r+1} lies in Δ_0. Choose any order ρ_{r+1} and any open interval $\widetilde{\Delta}_1$ which contains α_{r+1} and is contained in the interior of Δ_0. Take

$$v_{\boldsymbol{\tau}}(z) = \int_{\Delta_0}\left[\frac{1}{t-z} - S_0(t,z)\right] d\tau(t) + R_0(z),$$
$$v_{\widetilde{\boldsymbol{\tau}}}(z) = \int_{\widetilde{\Delta}_0}\left[\frac{1}{t-z} - S_0(t,z)\right] d\tau(t) + \int_{\Delta_{r+1}}\left[\frac{1}{t-z} - S_{r+1}(t,z)\right] d\tau(t)$$
$$+ \widetilde{R}_0(z) + \widetilde{R}_{r+1}\left(\frac{1}{z-\alpha_{r+1}}\right),$$

where $\widetilde{\Delta}_0 = \Delta_0 \setminus \widetilde{\Delta}_1$ and

$$\widetilde{R}_0(z) = R_0(z) - \int_{\Delta_{r+1}} S_0(t,z)\, d\tau(t),$$
$$\widetilde{R}_{r+1}\left(\frac{1}{z-\alpha_{r+1}}\right) = -\int_{\Delta_{r+1}} S_{r+1}(t,z)\, d\tau(t).$$

The identities (6.3) are again verified using Theorems 3.2 and 3.3.

From the first two special cases, it may be presumed that

$$\widetilde{\boldsymbol{\tau}} = \{\tau(t);\, \alpha_1,\dots,\alpha_r; -;\, \widetilde{\rho}_0,\dots,\widetilde{\rho}_r;\, \Delta_0,\dots,\Delta_r;\, \widetilde{R}_0(z),\dots,\widetilde{R}_{\widetilde{r}}(z); -\}.$$

To complete the proof, it remains to bring the orders ρ_j and $\widetilde{\rho}_j$, $j = 0,\dots,r$, to the same values; it then follows that $R_0(z) = \widetilde{R}_0(z),\dots,R_r(z) = \widetilde{R}_r(z)$ and $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}}$. Thus the proof is completed with one more special case.

Special Case C: $\widetilde{\boldsymbol{\tau}}$ is obtained from $\boldsymbol{\tau}$ by replacing one of the integers $\rho_0,\dots,\rho_r$ by a larger value.

For example, suppose that $\widetilde{\rho}_1 > \rho_1$ and

$$v_{\boldsymbol{\tau}}(z) = \int_{\Delta_1}\left[\frac{1}{t-z} - S_1(t,z)\right] d\tau(t),$$
$$v_{\widetilde{\boldsymbol{\tau}}}(z) = \int_{\Delta_1}\left[\frac{1}{t-z} - \widetilde{S}_1(t,z)\right] d\tau(t) - \widetilde{R}_1\left(\frac{1}{z-\alpha_1}\right),$$

where $\widetilde{S}_1(t,z)$ is given by (2.5) with ρ_1 replaced by $\widetilde{\rho}_1$ and

$$\widetilde{R}_1\left(\frac{1}{z-\alpha_1}\right) = -\int_{\Delta_0}\left[\widetilde{S}_1(t,z) - S_1(t,z)\right] d\tau(t).$$

We verify (6.3) as before using Theorems 3.2 and 3.3. It remains to treat the possibility $\widetilde{\rho}_0 > \rho_0$ in Case 1 of Assumptions 3.1 (necessarily $\rho_0 = 0$ in Case 2). This is handled similarly. □

Proof of Theorem 3.5. We assume Case 1 in the proof. Case 2 is handled with minor modifications. The selfadjointness of $S = S_v$ follows from the explicit formulas in Theorems 3.2 and 3.3. The main problem is to verify (1.1) when S and Φ_1 are corresponding parts of (3.2) and (3.3). Suppose first that

$$S = \int_{\Delta_j}\left\{(I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} - d\tau_j(t;A,\Phi_2)\right\}, \tag{6.4}$$

$$\Phi_1 = -i\int_{\Delta_j}\left\{A(I-At)^{-1} - \mathfrak{S}_j(t;A)\right\}\Phi_2\,[d\tau(t)], \tag{6.5}$$

$j = 0,\dots,r$. Writing $dT = \Phi_2[d\tau(t)]\Phi_2^*$, we obtain

$$\begin{aligned} AS - SA^* = \int_{\Delta_j}\Big\{&\Big[A(I-At)^{-1}dT - dT(I-A^*t)^{-1}A^*\Big] \\ &- \Big[A\,d\tau_j(t;A,\Phi_2) - d\tau_j(t;A,\Phi_2)\,A^*\Big]\Big\}, \\ i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\right] = \int_{\Delta_j}\Big\{&\Big[A(I-At)^{-1}dT - dT(I-A^*t)^{-1}A^*\Big] \\ &- \Big[\mathfrak{S}_j(t,A)dT - dT\mathfrak{S}_j^*(t,A)\Big]\Big\}. \end{aligned}$$

We show that for all $j = 0,1,\dots,r$,

$$A\,d\tau_j(t;A,\Phi_2) - d\tau_j(t;A,\Phi_2)\,A^* = \mathfrak{S}_j(t,A)dT - dT\mathfrak{S}_j^*(t,A). \tag{6.6}$$

First assume $j = 1,\dots,r$ and $\alpha_j \neq 0$. Set $B = I + \alpha_j A(I-\alpha_j A)^{-1}$. By Theorem 3.2 and the operator identity $\sum_{j+k=n}\left(L^{j+1}XR^k - L^jXR^{k+1}\right) = L^{n+1}X - XR^{n+1}$, which holds for all $n \geq 0$,

$$\begin{aligned} &A\,d\tau_j(t;A,\Phi_2) - d\tau_j(t;A,\Phi_2)\,A^* \\ &\quad = \sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q\geq 0}} \frac{(B-I)^{p+1}}{\alpha_j^{p+1}}\,dT\,\frac{(B^*-I)^q}{\alpha_j^q}\,[(B^*-I)+I] \\ &\qquad - \sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^\ell \sum_{\substack{p+q=\ell \\ p,q\geq 0}} [(B-I)+I]\,\frac{(B-I)^p}{\alpha_j^p}\,dT\,\frac{(B^*-I)^{q+1}}{\alpha_j^{q+1}} \end{aligned}$$

$$= \sum_{\ell=0}^{2\rho_j-1} \frac{(t-\alpha_j)^\ell}{\alpha^{\ell+1}} \sum_{\substack{p+q=\ell \\ p,q\geq 0}} \Big[(B-I)^{p+1}\, dT\, (B^*-I)^q - (B-I)^p\, dT\, (B^*-I)^{q+1} \Big]$$

$$= \sum_{\ell=0}^{2\rho_j-1} \frac{(t-\alpha_j)^\ell}{\alpha^{\ell+1}} \Big[(B-I)^{\ell+1} dT - dT(B^*-I)^{\ell+1} \Big]$$

$$= \mathfrak{S}_j(t,A) dT - dT \mathfrak{S}_j^*(t,A).$$

The case $\alpha_j = 0$ follows by continuity. Thus (6.6) holds for $j = 1, \dots, r$. A similar argument verifies (6.6) for $j = 0$. Hence (1.1) holds when S and Φ_1 are defined by (6.4) and (6.5).

The discrete parts in (3.2) and (3.3) come in two types by Theorem 3.3. One type is

$$S = \sum_{\substack{\mu+\nu=p+1 \\ \mu,\nu\geq 1}} A^{-\mu}\Phi_2 X \Phi_2^* A^{*-\nu}\,, \qquad \Phi_1 = i\, A^{-p}\Phi_2 X\,,$$

where $p \geq 0$ and $X = X^*$. If $p = 0$, then $S = 0$ and $\Phi_1 = i\Phi_2 X$ and both sides of (1.1) reduce to zero. For $p \geq 1$,

$$AS - SA^* = \sum_{\substack{\mu+\nu=p+1 \\ \mu,\nu\geq 1}} \Big(A^{-\mu+1}\Phi_2 X \Phi_2^* A^{*-\nu} - A^{-\mu}\Phi_2 X \Phi_2^* A^{*-\nu+1} \Big)$$
$$= \Phi_2 X \Phi_2^* A^{*-p} - A^{-p}\Phi_2 X \Phi_2^*. = i\left[\Phi_1\Phi_2^* + \Phi_2\Phi_1^* \right],$$

which verifies (1.1). The other type has the form

$$S = \sum_{\substack{\mu+\nu=p+1 \\ \mu,\nu\geq 1}} \Big[A^{\mu-1}(I-\lambda A)^{-\mu}\Phi_2 X \Phi_2^* (I-\lambda A^*)^{-\nu} A^{*\nu-1} + A^{\mu-1}(I-\bar\lambda A)^{-\mu}\Phi_2 X^* \Phi_2^* (I-\bar\lambda A^*)^{-\nu} A^{*\nu-1} \Big],$$

$$\Phi_1 = -i\Big[A^p (I-\lambda A)^{-p}\Phi_2 X + A^p (I-\bar\lambda A)^{-p}\Phi_2 X^* \Big],$$

where $p \geq 1$ and X is not necessarily selfadjoint. We verify (1.1) in this case in a similar way.

We omit a proof that $\varkappa_{S_v} < \infty$ here because a more general result will be proved later (independently) in Theorem 4.5. □

Proof of Theorem 3.7. The Assumptions 3.1, Case 1, are met. It is sufficient to prove the formula for $\Phi_{1,v}$. For if this is known and

$$\widetilde{S} = \left[\frac{w(z_\mu) - w(z_\nu)^*}{z_\mu - \bar z_\nu} \right]_{\mu,\nu=1}^n,$$

then $A\widetilde{S} - \widetilde{S}A^* = i\left[\Phi_{1,v}\Phi_2^* + \Phi_2\Phi_{1,v}^*\right] = AS_v - S_vA^*$ by Theorem 3.5. Hence $A(\widetilde{S} - S_v) - (\widetilde{S} - S_v)A^* = 0$. Since A and A^* have disjoint spectra, $\widetilde{S} - S_v = 0$, and the formula for S_v follows. We prove the formula for $\Phi_{1,v}$ for corresponding parts of (3.3) and

$$w(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{z}{1-tz} + S_j(t, z^{-1}) \right] d\tau(t) - R_0(z^{-1})$$
$$+ \sum_{j=1}^{r} R_j \left(\frac{z}{1-\alpha_j z} \right) + \sum_{k=1}^{s} \left[M_k \left(\frac{\bar{z}}{1-\beta_k \bar{z}} \right)^* + M_k \left(\frac{z}{1-\beta_k z} \right) \right].$$

Suppose first that

$$i\Phi_{1,v} = \int_{-\infty}^{\infty} \left\{ A(I-At)^{-1} - \sum_{j=0}^{r} \mathfrak{S}_j(t; A) \right\} \Phi_2 \, [d\tau(t)],$$
$$w(z) = \int_{-\infty}^{\infty} \left\{ \frac{z}{1-zt} + \sum_{j=0}^{r} S_j(t, z^{-1}) \right\} d\tau(t).$$

For each $\mu = 1, \dots, n$ let P_μ be the projection of $\mathfrak{H} = \mathbf{C}^m \oplus \cdots \oplus \mathbf{C}^m$ onto the μ-th component. Then by (3.12),

$$P_\mu(i\Phi_{1,v}) = \int_{-\infty}^{\infty} \left\{ \frac{z_\mu}{1-z_\mu t} + \sum_{j=0}^{r} S_j(t, z_\mu^{-1}) \right\} d\tau(t) = w(z_\mu),$$

yielding the formula for $\Phi_{1,v}$. Next let

$$i\Phi_{1,v} = \widehat{\mathfrak{R}}_0 \,, \qquad w(z) = -R_0(z^{-1}) \,.$$

If $R_0(z) = \sum_{p=0}^{2\rho_0+1} C_p z^p$, then by (3.7),

$$P_\mu\big(i\Phi_{1,v}\big) = -\operatorname*{Res}_{\lambda=0} \; P_\mu A(A-\lambda I)^{-1}\Phi_2 R_0(\lambda^{-1})\lambda^{-1}$$
$$= -\operatorname*{Res}_{\lambda=0} \sum_{n=0}^{\infty} z_\mu^{-n}\lambda^n \sum_{p=0}^{2\rho_0+1} C_p \lambda^{-p-1} = -\sum_{p=0}^{2\rho_0+1} C_p z_\mu^{-p} = w(z_\mu) \,,$$

as required. The remaining cases

$$i\Phi_{1,v} = \widehat{\mathfrak{R}}_j \,, \qquad w(z) = R_j \left(\frac{z}{1-\alpha_j z} \right) \,,$$

and

$$i\Phi_{1,v} = \widehat{\mathfrak{M}}_{1k} + \widehat{\mathfrak{M}}_{2k} \,, \qquad w(z) = M_k \left(\frac{\bar{z}}{1-\beta_k \bar{z}} \right)^* + M_k \left(\frac{z}{1-\beta_k z} \right) \,,$$

are handled similarly. □

Proof of Theorem 4.1. We prove (3.11) as in the definite case [12]:

$$\mathfrak{A}(\bar{\zeta}) J \mathfrak{A}(\bar{z})^* - J$$
$$= \big[I - i\bar{\zeta}\Pi^*(I-\bar{\zeta}A^*)^{-1}S^{-1}\Pi J \big] J \big[I + izJ\Pi^* S^{-1}(I-zA)^{-1}\Pi \big] - J$$

$$
\begin{aligned}
&= izJ\Pi^* S^{-1}(I - zA)^{-1}\Pi - i\bar\zeta\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}\Pi \\
&\qquad + \bar\zeta z\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}\frac{AS - SA^*}{i}S^{-1}(I - zA)^{-1}\Pi \\
&= izJ\Pi^* S^{-1}(I - zA)^{-1}\Pi - i\bar\zeta\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}\Pi \\
&\qquad - i\bar\zeta\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}(zA - I + I)(I - zA)^{-1}\Pi \\
&\qquad + iz\Pi^*(I - \bar\zeta A^*)^{-1}(\bar\zeta A^* - I + I)S^{-1}(I - zA)^{-1}\Pi \\
&= -i(\bar\zeta - z)\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}(I - zA)^{-1}\Pi.
\end{aligned}
$$

To obtain the last statement, apply (3.11) with $\zeta = \bar z$. □

Proof of Theorem 4.2. Setting $\Phi_v(z) = \begin{bmatrix} -iv(z) \\ I \end{bmatrix}$, we obtain

$$
\frac{v(z) - v(\zeta)^*}{z - \bar\zeta} = i\,\frac{\Phi_v(\zeta)^* J\Phi_v(z)}{z - \bar\zeta}, \qquad B_v(z) = (I - zA)^{-1}\Pi J\Phi_v(z),
$$

by (4.3). Hence by Theorem 4.1,

$$
\begin{aligned}
D_v(z,\zeta) &= i\,\frac{\Phi_v(\zeta)^* J\Phi_v(z)}{z - \bar\zeta} - \Phi_v(\zeta)^* J\Pi^*(I - \bar\zeta A^*)^{-1}S^{-1}(I - zA)^{-1}\Pi J\Phi_v(z) \\
&= \Phi_v(\zeta)^*\,\frac{J\mathfrak{A}(\bar\zeta)J\mathfrak{A}(\bar z)^*J}{i(\bar\zeta - z)}\,\Phi_v(z) \\
&= \Phi_v(\zeta)^*\,\frac{\mathfrak{A}(\zeta)^{*-1}J\mathfrak{A}(z)^{-1}}{i(\bar\zeta - z)}\,\Phi_v(z) \\
&= i\,\frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{z - \bar\zeta}\,,
\end{aligned}
$$

which is (4.12). Again by Theorem 4.1,

$$
\begin{aligned}
P(\bar z)^*Q(z) + Q(\bar z)^*P(z) &= \begin{bmatrix} P(\bar z)^* & Q(\bar z)^* \end{bmatrix} J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \\
= \begin{bmatrix} iv(\bar z)^* & I \end{bmatrix} \mathfrak{A}(\bar z)^{*-1}J\mathfrak{A}(z)^{-1} \begin{bmatrix} -iv(z) \\ I \end{bmatrix} &= \begin{bmatrix} iv(\bar z)^* & I \end{bmatrix} J \begin{bmatrix} -iv(z) \\ I \end{bmatrix} = 0,
\end{aligned}
$$

and the result follows. □

Proof of Theorem 4.4. The function $v(z)$ is defined and analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$ except at isolated points. Set

$$
\begin{bmatrix} H(z) \\ K(z) \end{bmatrix} = \mathfrak{A}(z)\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} a(z)P(z) + b(z)Q(z) \\ c(z)P(z) + d(z)Q(z) \end{bmatrix}.
$$

Then $v(z) = iH(z)K(z)^{-1}$,

$$
\mathfrak{A}(z)\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \begin{bmatrix} -iv(z) \\ I \end{bmatrix} K(z), \tag{6.7}
$$

and so

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} = \mathfrak{A}(z)^{-1} \begin{bmatrix} -iv(z) \\ I \end{bmatrix} = J\mathfrak{A}(\bar{z})^* \begin{bmatrix} I \\ -iv(z) \end{bmatrix}$$

on $\mathbf{C}_+ \cup \mathbf{C}_-$ except at isolated points. We obtain

$$\begin{aligned} &\begin{bmatrix} I & iv(\zeta)^* \end{bmatrix} \mathfrak{A}(\bar{\zeta}) JJJ\mathfrak{A}(\bar{z})^* \begin{bmatrix} I \\ -iv(z) \end{bmatrix} \\ &\qquad = K(\zeta)^{*-1} \begin{bmatrix} P(\zeta)^* & Q(\zeta)^* \end{bmatrix} J \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} \\ &\qquad = K(\zeta)^{*-1} \left[P(\zeta)^* Q(z) + Q(\zeta)^* P(z) \right] K(z)^{-1}. \end{aligned}$$

On the other hand, by (4.9),

$$\begin{aligned} &\begin{bmatrix} I & iv(\zeta)^* \end{bmatrix} J \begin{bmatrix} I \\ -iv(z) \end{bmatrix} \\ &\quad = \begin{bmatrix} I & iv(\zeta)^* \end{bmatrix} \mathfrak{A}(\bar{\zeta}) J \mathfrak{A}(\bar{z})^* \begin{bmatrix} I \\ -iv(z) \end{bmatrix} \\ &\qquad + i(\bar{\zeta} - z) \begin{bmatrix} I & iv(\zeta)^* \end{bmatrix} \Pi^* (I - \bar{\zeta} A^*)^{-1} S^{-1} (I - zA)^{-1} \Pi \begin{bmatrix} I \\ -iv(z) \end{bmatrix}. \end{aligned}$$

It follows that

$$\frac{v(z) - v(\zeta)^*}{z - \bar{\zeta}} = K(\zeta)^{*-1}\, i\, \frac{P(\zeta)^* Q(z) + Q(\zeta)^* P(z)}{z - \bar{\zeta}} K(z)^{-1} + \Lambda(\zeta)^* S^{-1} \Lambda(z),$$

where

$$\Lambda(z) = (I - zA)^{-1} \Pi \begin{bmatrix} I \\ -iv(z) \end{bmatrix}.$$

To see that $v(z) = v(\bar{z})^*$, multiply the last identity by $z - \bar{\zeta}$, then take $\zeta = \bar{z}$ and use the condition (i) in Definition 4.3. By our assumption that $\varkappa_S < \infty$ and condition (iii) in Definition 4.3, we deduce that $v(z) \in \mathbf{N}_{\varkappa_v}$, where $\varkappa_v \le \varkappa_{P,Q} + \varkappa_S$. The equality $\varkappa_{P,Q} + \varkappa_S = \varkappa_{L_v}$ follows from (4.11).

By (4.3), $\Lambda(z) = B_v(z)$ and (4.16) follows. The identity (4.17) is proved by a straightforward algebraic calculation, which we omit. □

Proof of Theorem 4.5. First assume Case 1: $0 \notin \sigma(A)$. In the proof, for brevity we drop the subscript v and write (4.2) more simply as

$$L(z,\zeta) = \begin{bmatrix} S & B(z) \\ B(\zeta)^* & C(z,\zeta) \end{bmatrix}.$$

It is sufficient to show that $\varkappa_L < \infty$ when $L(z,\zeta)$ is calculated from corresponding parts of (2.1). We distinguish five subcases (a)–(e).

Case 1: (a) Fix $j = 1, \dots, r$, and let

$$v(z) = \int_{\Delta_j} \left\{ \frac{1}{t - z} - S_j(t, z) \right\} d\tau(t),$$

$$S=\int_{\Delta_j}\Big\{(I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1}-d\tau_j(t;A,\Phi_2)\Big\},$$
$$\Phi_1=-i\int_{\Delta_j}\Big\{A(I-At)^{-1}-\mathfrak{S}_j(t;A)\Big\}\,\Phi_2\,[d\tau(t)]\,.$$

To calculate $L(z,\zeta)$ in this subcase, first use Theorem 3.2 to obtain

$$S=\int_{\Delta_j}\Big\{(I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} \\ -\sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^\ell\sum_{p+q=\ell}A_p(\alpha_j)\Phi_2[d\tau(t)]\Phi_2^*A_q(\alpha_j)^*\Big\}.$$

Using Lemmas 6.4 and 6.3, we get

$$\begin{aligned}B(z)&=(I-zA)^{-1}\big[\Phi_{1,v}-i\Phi_2v(z)\big]\\
&=-i\int_{\Delta_j}(I-zA)^{-1}\Big\{A(I-At)^{-1}+\frac{I}{t-z}\\
&\qquad\qquad-\mathfrak{S}_j(t;A)-S_j(t,z)I\Big\}\,\Phi_2\,[d\tau(t)]\\
&=-i\int_{\Delta_j}\Big\{\frac{(I-At)^{-1}-(I-Az)^{-1}}{t-z}+\frac{(I-Az)^{-1}}{t-z}\\
&\qquad+\operatorname*{Res}_{\lambda=\alpha_j}\Big[\frac{(I-A\lambda)^{-1}-(I-Az)^{-1}}{\lambda-z}\,S(t,\lambda)\Big]\\
&\qquad-(I-Az)^{-1}S_j(t,z)\Big\}\,\Phi_2\,[d\tau(t)]\\
&=-i\int_{\Delta_j}\Big\{\frac{(I-tA)^{-1}}{t-z}+\operatorname*{Res}_{\lambda=\alpha_j}\Big[\frac{(I-\lambda A)^{-1}}{\lambda-z}S_j(t,\lambda)\Big]\Big\}\,\Phi_2\,[d\tau(t)]\\
&=-i\int_{\Delta_j}\Big\{\frac{(I-tA)^{-1}}{t-z}+\sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^\ell\sum_{p+q=\ell}\frac{A_q(\alpha_j)}{(z-\alpha_j)^{p+1}}\Big\}\,\Phi_2\,[d\tau(t)]\,.\end{aligned}$$

We obtain

$$C(z,\zeta)=\int_{\Delta_j}\Big\{\frac{1}{(t-z)(t-\bar\zeta)} \\ -\sum_{\ell=0}^{2\rho_j-1}(t-\alpha_j)^\ell\sum_{p+q=\ell}\frac{1}{(z-\alpha_j)^{q+1}(\bar\zeta-\alpha_j)^{p+1}}\Big\}\,d\tau(t)$$

by means of the identity

$$\frac{S_j(t,z)-\overline{S_j(t,\zeta)}}{z-\bar\zeta}=\sum_{\ell=0}^{2\rho_j-1}\frac{(t-\alpha_j)^\ell}{(z-\alpha_j)^{\ell+1}(\bar\zeta-\alpha_j)^{\ell+1}}\,\frac{(z-\alpha_j)^{\ell+1}-(\bar\zeta-\alpha_j)^{\ell+1}}{(z-\alpha_j)-(\bar\zeta-\alpha_j)}$$

$$= \sum_{\ell=0}^{2\rho_j-1} (t-\alpha_j)^\ell \sum_{p+q=\ell} \frac{1}{(z-\alpha_j)^{q+1}(\bar\zeta-\alpha_j)^{p+1}}\,.$$

Thus

$$L(z,\zeta) = \int_{\Delta_j} \left\{ \begin{bmatrix} (I-At)^{-1}\Phi_2 \\ \dfrac{-iI}{\bar\zeta - t} \end{bmatrix} d\tau(t) \begin{bmatrix} \Phi_2^*(I-A^*t)^{-1} & \dfrac{iI}{z-t} \end{bmatrix} \right.$$

$$\left. - \sum_{\ell=0}^{2\rho_j-1} (t-\alpha_j)^\ell \sum_{p+q=\ell} \begin{bmatrix} A_p(\alpha_j)\Phi_2 \\ \dfrac{-iI}{(\bar\zeta-\alpha_j)^{p+1}} \end{bmatrix} d\tau(t) \begin{bmatrix} \Phi_2 A_q(\alpha_j)^* & \dfrac{iI}{(z-\alpha_j)^{q+1}} \end{bmatrix} \right\}.$$

To see that $\varkappa_L < \infty$, approximate $\tau(t)$ by functions $\tau_\varepsilon(t)$ that are constant in intervals $(\alpha_j-\varepsilon,\alpha_j)$ and $(\alpha_j,\alpha_j+\varepsilon)$ and define $L_\varepsilon(z,\zeta)$ by the same expression with $\tau(t)$ replaced by $\tau_\varepsilon(t)$. Then

$$\lim_{\varepsilon\downarrow 0} L_\varepsilon(z,\zeta) = L(z,\zeta)$$

pointwise. Since $\tau_\varepsilon(t)$ is constant to the left and right of α_j, the integrations in $L_\varepsilon(z,\zeta)$ can be carried out term by term. Then the first summand in $L_\varepsilon(z,\zeta)$ is nonnegative, and the number of negative squares in what remains has a finite bound independent of ε by the matrix identity

$$\sum_{\ell=0}^{\nu} \sum_{\substack{p+q=\ell+1\\ p,q\ge 1}} X_p^* H_\ell Y_p = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_\nu \end{bmatrix}^* \begin{bmatrix} H_1 & H_2 & \cdots & H_{\nu-1} & H_\nu \\ H_2 & H_3 & \cdots & H_\nu & 0 \\ & & \cdots & & \\ H_\nu & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_\nu \end{bmatrix}. \tag{6.8}$$

Therefore $\varkappa_L < \infty$.

Case 1: (b) Next assume that

$$\begin{aligned} v(z) &= \int_{\Delta_0} \left\{ \frac{1}{t-z} - S_0(t,z) \right\} d\tau(t)\,, \\ S &= \int_{\Delta_0} \left\{ (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} - d\tau_0(t;A,\Phi_2) \right\}, \\ \Phi_1 &= -i \int_{\Delta_0} \left\{ A(I-At)^{-1} - \mathfrak{S}_0(t;A) \right\} \Phi_2\,[d\tau(t)]\,. \end{aligned}$$

By Theorem 3.2,

$$S = \int_{\Delta_0} \left\{ (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*\,(I-A^*t)^{-1} \right.$$

$$- \sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\ge 1}} A^{-j}\Phi_2[d\tau(t)]\Phi_2^* A^{*-k}$$

$$- \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} A^{-j}\Phi_2[d\tau(t)]\Phi_2^* A^{*-k} \Bigg\}.$$

Calculating as above, we get by (2.6) and Lemma 6.4,

$$\begin{aligned}
B(z) &= (I-zA)^{-1}\big[\Phi_{1,v} - i\Phi_2 v(z)\big] \\
&= -i\int_{\Delta_0} \left\{ \frac{(I-tA)^{-1}}{t-z} + \operatorname*{Res}_{\lambda=0}\left[\frac{(A-\lambda I)^{-1}}{1-\lambda z}\, S_0(t,\lambda^{-1}) \right] \right\} \Phi_2\,[d\tau(t)] \\
&= -i\int_{\Delta_0} \Bigg\{ \frac{(I-tA)^{-1}}{t-z} + \sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\geq 1}} A^{-j} z^{k-1} \\
&\qquad + \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} A^{-j} z^{k-1} \Bigg\} \Phi_2\,[d\tau(t)]\,.
\end{aligned}$$

For $C(z,\zeta) = [v(z) - v(\zeta)^*]/(z-\bar{\zeta})$, we get

$$\begin{aligned}
C(z,\zeta) = \int_{\Delta_0} \Bigg\{ &\frac{1}{(t-z)(t-\bar{\zeta})} - \sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\geq 1}} z^{k-1}\bar{\zeta}^{j-1} \\
&- \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} z^{k-1}\bar{\zeta}^{j-1} \Bigg] \Bigg\} d\tau(t)
\end{aligned}$$

from the identity

$$\begin{aligned}
\frac{S_0(t,z) - \overline{S_0(t,\zeta)}}{z-\bar{\zeta}} &= \sum_{\ell=0}^{\rho_0} \frac{t}{(1+t^2)^{\ell+1}}\, \frac{(1+z^2)^\ell - (1+\bar{\zeta}^2)^\ell}{z-\bar{\zeta}} \\
&\qquad + \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}}\, \frac{z\,(1+z^2)^\ell - \bar{\zeta}\,(1+\bar{\zeta}^2)^\ell}{z-\bar{\zeta}} \\
&= \sum_{\ell=0}^{\rho_0} \frac{t}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \frac{z^{2p} - \bar{\zeta}^{2p}}{z-\bar{\zeta}} \\
&\qquad + \sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \frac{z^{2p+1} - \bar{\zeta}^{2p+1}}{z-\bar{\zeta}} \\
&= \sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\geq 1}} z^{k-1}\bar{\zeta}^{j-1}
\end{aligned}$$

$$+\sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} z^{k-1}\bar{\zeta}^{j-1}\,.$$

Thus

$$L(z,\zeta)=\int_{\Delta_0}\Bigg\{\begin{bmatrix}(I-At)^{-1}\Phi_2\\ \dfrac{-iI}{\bar{\zeta}-t}\end{bmatrix} d\tau(t)\begin{bmatrix}\Phi_2^*(I-A^*t)^{-1} & \dfrac{iI}{z-t}\end{bmatrix}$$

$$-\sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\geq 1}} \begin{bmatrix}A^{-j}\Phi_2\\ -i\,\bar{\zeta}^{j-1}\end{bmatrix} d\tau(t)\begin{bmatrix}\Phi_2^*A^{*-k} & i\,z^{k-1}\end{bmatrix}$$

$$-\sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\geq 1}} \begin{bmatrix}A^{-j}\Phi_2\\ -i\,\bar{\zeta}^{j-1}\end{bmatrix} d\tau(t)\begin{bmatrix}\Phi_2^*A^{*-k} & i\,z^{k-1}\end{bmatrix}\Bigg\}.$$

As above, we deduce that $\varkappa_L<\infty$.

Case 1: (c) In the case $v(z)=R_0(z)$, $S=\mathfrak{R}_0$, $\Phi_1=-i\widehat{\mathfrak{R}}_0$, the polynomial $R_0(z)=\sum_{\ell=0}^{2\rho_0+1}R_{0\ell}z^\ell$ has selfadjoint matrix coefficients, and

$$S=\sum_{\ell=1}^{2\rho_0+1}\sum_{\substack{j+k=\ell+1\\ j,k\geq 1}} A^{-j}\Phi_2R_{0\ell}\Phi_2^*A^{*-k}$$

by Theorem 3.3. By (3.7) and Lemma 6.4,

$$\begin{aligned}
B(z)&=(I-zA)^{-1}\big[\Phi_1-i\Phi_2v(z)\big]\\
&=i\,\operatorname*{Res}_{\lambda=0}\Big[(I-zA)^{-1}A(A-\lambda I)^{-1}\Phi_2R_0(\lambda^{-1})\lambda^{-1}\Big]\\
&\qquad-i\,(I-zA)^{-1}\Phi_2R_0(z)\\
&=i\,\operatorname*{Res}_{\lambda=0}\left[\frac{(I-zA)^{-1}-(I-\lambda^{-1}A)^{-1}}{1-\lambda z}\,\Phi_2R_0(\lambda^{-1})\lambda^{-1}\right]\\
&\qquad-i\,(I-zA)^{-1}\Phi_2R_0(z)\\
&=-i\,\operatorname*{Res}_{\lambda=0}\left[\frac{(I-\lambda^{-1}A)^{-1}}{1-\lambda z}\,\Phi_2R_0(\lambda^{-1})\lambda^{-1}\right]\\
&=i\sum_{\ell=1}^{2\rho_0+1}\sum_{\substack{j+k=\ell+1\\ j,k\geq 1}} A^{-j}\Phi_2R_{0\ell}z^{k-1}\,.
\end{aligned}$$

A straightforward calculation of $C(z,\zeta)$ yields

$$L(z,\zeta)=\sum_{\ell=1}^{2\rho_0+1}\sum_{\substack{j+k=\ell+1\\ j,k\geq 1}} \begin{bmatrix}A^{-j}\Phi_2\\ -i\,\bar{\zeta}^{j-1}\end{bmatrix} R_{0\ell}\begin{bmatrix}\Phi_2^*A^{*-k} & iz^{k-1}\end{bmatrix}\,,$$

and $\varkappa_L < \infty$ by (6.8).

Case 1: (d) Let $j = 1, \dots, r$, and suppose that

$$v(z) = -R_j\Big(\frac{1}{z-\alpha_j}\Big), \qquad S = \mathfrak{R}_j, \qquad \Phi_1 = -i\,\widehat{\mathfrak{R}}_j\,,$$

where $R_j(z) = \sum_{\ell=1}^{2\rho_j+1} R_{j\ell} z^\ell$ has selfadjoint matrix coefficients and constant term zero. By Theorem 3.3,

$$S = \sum_{\ell=1}^{2\rho_j+1} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} A_{p-1}(\alpha_j)\Phi_2 R_{j\ell}\Phi_2^* A_{q-1}(\alpha_j)^*\,.$$

By (3.8) and Lemmas 6.4 and 6.3,

$$\begin{aligned}
B(z) &= (I-zA)^{-1}\big[\Phi_1 - i\Phi_2 v(z)\big] \\
&= -i \operatorname*{Res}_{\lambda=\alpha_j} \Big[(I-zA)^{-1}A(I-\lambda A)^{-1}\Phi_2 R_j\Big(\frac{1}{\lambda-\alpha_j}\Big)\Big] \\
&\qquad + i(I-zA)^{-1}\Phi_2 R_j\left(\frac{1}{z-\alpha_j}\right) \\
&= -i \operatorname*{Res}_{\lambda=\alpha_j} \left[\frac{(I-\lambda A)^{-1}-(I-zA)^{-1}}{\lambda - z}\,\Phi_2 R_j\Big(\frac{1}{\lambda-\alpha_j}\Big)\right] \\
&\qquad + i(I-zA)^{-1}\Phi_2 R_j\left(\frac{1}{z-\alpha_j}\right) \\
&= i \operatorname*{Res}_{\lambda=\alpha_j} \left[\frac{(I-\lambda A)^{-1}}{z-\lambda}\,\Phi_2 R_j\Big(\frac{1}{\lambda-\alpha_j}\Big)\right] \\
&= i \sum_{\ell=1}^{2\rho_j+1} \sum_{\substack{p+q=\ell-1\\ p,q\geq 0}} \frac{A_q(\alpha_j)\Phi_2 R_{j\ell}}{(z-\alpha_j)^{p+1}} \\
&= i \sum_{\ell=1}^{2\rho_j+1} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \frac{A_{q-1}(\alpha_j)\Phi_2 R_{j\ell}}{(z-\alpha_j)^{p}}\,.
\end{aligned}$$

A short calculation of $C(z,\zeta)$ yields

$$L(z,\zeta) = \sum_{\ell=1}^{2\rho_j+1} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \begin{bmatrix} A_{p-1}(\alpha_j)\Phi_2 \\ \dfrac{-iI}{(\bar\zeta-\alpha_j)^p} \end{bmatrix} R_{j\ell} \begin{bmatrix} \Phi_2^* A_{q-1}(\alpha_j)^* & \dfrac{iI}{(z-\alpha_j)^q} \end{bmatrix},$$

and we again obtain $\varkappa_L < \infty$ by (6.8).

Case 1: (e) Let $k = 1, \dots, s$, and assume that

$$v(z) = -M_k\Big(\frac{1}{z-\beta_k}\Big) - M_k\Big(\frac{1}{\bar z-\beta_k}\Big)^*,$$

$$S = \mathfrak{M}_{1k} + \mathfrak{M}_{2k},$$
$$\Phi_1 = -i\left[\widehat{\mathfrak{M}}_{1k} + \widehat{\mathfrak{M}}_{2k}\right],$$

where $M_k(z) = \sum_{\ell=1}^{\sigma_k} M_{k\ell} z^\ell$ is a polynomial with matrix coefficients and constant term zero. Calculations similar to those above yield

$$S = \sum_{\ell=1}^{\sigma_k} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \Big[A_{p-1}(\beta_k)\Phi_2 M_{kl}\Phi_2^* A_{q-1}(\bar\beta_k)^* + A_{p-1}(\bar\beta_k)\Phi_2 M_{kl}^*\Phi_2^* A_{q-1}(\beta_k)^*\Big],$$

$$B(z) = i\sum_{\ell=1}^{\sigma_k} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \left[\frac{A_{p-1}(\beta_k)\Phi_2 M_{kl}}{(z-\beta_k)^q} + \frac{A_{p-1}(\bar\beta_k)\Phi_2 M_{kl}^*}{(z-\bar\beta_k)^q}\right],$$

$$C(z,\zeta) = \sum_{\ell=1}^{\sigma_k} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \left[\frac{M_{k\ell}}{(z-\beta_k)^q(\bar\zeta-\beta_k)^p} + \frac{M_{k\ell}^*}{(z-\bar\beta_k)^q(\bar\zeta-\bar\beta_k)^p}\right].$$

We again obtain a kernel,

$$L(z,\zeta) = \sum_{\ell=1}^{\sigma_k} \sum_{\substack{p+q=\ell+1\\ p,q\geq 1}} \begin{bmatrix} A_{p-1}(\beta_k)\Phi_2 & A_{p-1}(\bar\beta_k)\Phi_2 \\ \dfrac{-iI}{(\bar\zeta-\beta_k)^p} & \dfrac{-iI}{(\bar\zeta-\bar\beta_k)^p}\end{bmatrix} \begin{bmatrix} 0 & M_{k\ell} \\ M_{k\ell}^* & 0\end{bmatrix} \cdot \begin{bmatrix} \Phi_2^* A_{q-1}(\beta_k)^* & \dfrac{iI}{(z-\bar\beta_k)^q} \\ \Phi_2^* A_{q-1}(\bar\beta_k)^* & \dfrac{iI}{(z-\beta_k)^q}\end{bmatrix},$$

which has a finite number of negative squares. This verifies the conclusion in each of the subcases (a)–(e), and so Theorem 4.5 follows in Case 1.

Assume Case 2: $0 \notin \sigma(A)$. We show that $\varkappa_L < \infty$ in the same subcases (a)–(e). Recall that in Case 2, $\rho_0 = 0$ and $R_0(z) = C_0$ is constant.

Case 2: (a),(d),(e) There is no change here from Case 1.

Case 2: (b) By (3.11), we now have

$$v(z) = \int_{\Delta_0}\left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\tau(t),$$
$$S = \int_{\Delta_0} (I-At)^{-1}\Phi_2[d\tau(t)]\Phi_2^*(I-A^*t)^{-1},$$
$$\Phi_1 = -i\int_{\Delta_0}\left[A(I-At)^{-1} + \frac{tI}{1+t^2}\right]\Phi_2\,[d\tau(t)].$$

A short calculation gives

$$B(z) = -i \int_{\Delta_0} \frac{(I-tA)^{-1}}{t-z}\, \Phi_2\,[d\tau(t)],$$
$$C(z,\zeta) = \int_{\Delta_0} \frac{d\tau(t)}{(t-z)(t-\bar\zeta)}.$$

The kernel

$$L(z,\zeta) = \int_{\Delta_0} \begin{bmatrix} (I-zA)^{-1}\Phi_2 \\ \dfrac{iI}{t-\bar\zeta} \end{bmatrix} d\tau(t) \begin{bmatrix} \Phi_2^*(I-zA^*)^{-1} & \dfrac{-iI}{t-z} \end{bmatrix}$$

is nonnegative in this subcase.

Case 2: (c) Here $v(z) = R_0(z) = C_0$ is constant, $S = \mathfrak{R}_0 = 0$, and $\Phi_1 = -i\widehat{\mathfrak{R}}_0 = i\Phi_2 C_0$ by (3.11). Thus $L(z,\zeta) = 0$ is a nonegative kernel.

So $\varkappa_L < \infty$ in all subcases (a)–(e) in Case 2, and the result follows. □

Proof of Theorem 4.6. We use the notation in the proof of Theorem 4.5 and verify the conclusion in the same subcases (a)–(e).

Case 1: (a) In this subcase, our previous formula for $B(z)$ can be written

$$B(z) = \int_{\Delta_j} F(t,z)\, d\tau(t), \tag{6.9}$$

where

$$F(t,z) = -i\left[\frac{(I-tA)^{-1}}{t-z} + \sum_{\ell=0}^{2\rho_j-1} (t-\alpha_j)^\ell \sum_{p+q=\ell} \frac{A_q(\alpha_j)}{(z-\alpha_j)^{p+1}}\right]\Phi_2 \tag{6.10}$$
$$= i\sum_{\ell=2\rho_j}^{\infty} (t-\alpha_j)^\ell \sum_{p+q=\ell} \frac{A_q(\alpha_j)}{(z-\alpha_j)^{p+1}}\, \Phi_2\,.$$

The last series converges uniformly for t in a neighborhood of α_j for z in any compact subset of $\mathbf{C}_+ \cup \mathbf{C}_-$. It follows that $B(z)$ is analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$.

Case 1: (b) We now obtain a representation (6.9) with $j = 0$ and

$$\begin{aligned} F(t,z) = -i\Bigg[&\frac{(I-tA)^{-1}}{t-z} \\ &+\sum_{\ell=0}^{\rho_0-1} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1\\ j,k\ge 1}} A^{-j} z^{k-1} \\ &+\sum_{\ell=0}^{\rho_0-1} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2\\ j,k\ge 1}} A^{-j} z^{k-1}\Bigg]\Phi_2 \end{aligned} \tag{6.11}$$

$$= i \left[\sum_{\ell=\rho_0}^{\infty} \frac{t}{(1+t^2)^{\ell+2}} \sum_{p=1}^{\ell+1} \binom{\ell+1}{p} \sum_{\substack{j+k=2p+1 \\ j,k\geq 1}} A^{-j} z^{k-1} \right.$$

$$\left. + \sum_{\ell=\rho_0}^{\infty} \frac{1}{(1+t^2)^{\ell+1}} \sum_{p=0}^{\ell} \binom{\ell}{p} \sum_{\substack{j+k=2p+2 \\ j,k\geq 1}} A^{-j} z^{k-1} \right] \Phi_2$$

The two series in the last expression converge in a neighborhood of infinity for z in any compact subset of $\mathbf{C}_+ \cup \mathbf{C}_-$. Again $B(z)$ is analytic on $\mathbf{C}_+ \cup \mathbf{C}_-$.

Case 1: (c),(d),(e) Our previous expressions for $B(z)$ here are rational functions whose only nonreal poles are at the points $\beta_k, \bar{\beta}_k$, $k = 1, \dots, s$.

The conclusion holds in all subcases, and the result follows. □

Proof of Theorem 4.7. Notation is as in the proof of Theorem 4.5. We check (4.18) and (4.19) in each of the subcases (a)–(e).

We first prove (4.18).

Case 2: (a),(c),(d),(e) Here, in fact, $\|B(z)\| = \mathcal{O}(1/|z|)$ as $|z| \to \infty$. For (a) this follows from (6.9) and (6.10). In (c), $B(z) \equiv 0$. The assertion is clear for (d) and (e).

Case 2: (b) By the proof of Theorem 4.5, Case 2,

$$B(z) = -i \int_{\Delta_0} \frac{(I - tA)^{-1}}{t - z} \, \Phi_2 \, [d\tau(t)] \, . \tag{6.12}$$

Define $L^2(d\tau) = L^2(\Delta_0, d\tau)$ as in Appendix 1. If $g \in \mathbf{C}^m$ and $h \in \mathfrak{H}$, then

$$\langle B(z)g, h \rangle = -i \left\langle \frac{\sqrt{t^2+1}}{t-z} \frac{g}{\sqrt{t^2+1}} \, , \Phi_2^*(I - tA^*)^{-1} h \right\rangle_{L^2(d\tau)} . \tag{6.13}$$

Here $g/\sqrt{t^2+1} \in L^2(d\tau)$ by Theorem 2.1(2°), and $\Phi_2^*(I - tA^*)^{-1}h \in L^2(d\tau)$ by (3.1). For $z = x + iy \in D_\delta$, $x^2 \leq cy^2$ for some $c > 0$ and

$$\left| \frac{z}{t-z} \right|^2 = \frac{x^2+y^2}{(t-x)^2+y^2} \leq \frac{(c+1)y^2}{y^2} \leq c+1. \tag{6.14}$$

If also $|y| \geq 1$, then

$$\left| \frac{\sqrt{t^2+1}}{t-z} \right| \leq \frac{|t|+1}{|t-z|} = \frac{|t-z+z|+1}{|t-z|}$$

$$\leq 1 + \left| \frac{z}{t-z} \right| + \frac{1}{|t-z|} \leq 1 + \sqrt{c+1} + 1 \, .$$

To deduce that $\|B(z)\| = \mathcal{O}(1)$ as $|z| \to \infty$ in D_δ, we apply the Schwarz inequality in (6.13) for fixed g and h and then appeal to the principle of uniform boundedness. This completes the proof of (4.18).

We next prove (4.19). Set $B_T(z) = B_{v,T}(z)$.

Case 2: (a) By the formulas for S and $B(z)$ in the proof of Theorem 4.5,

$$\begin{aligned}
B_T(z) &= SA^*(I - zA^*)^{-1} + iB(z)\Phi_2^*(I - zA^*)^{-1} \\
&= \int_{\Delta_j} \Bigg\{ (I - At)^{-1}\, dT\, (I - A^*t)^{-1} A^*(I - zA^*)^{-1} \\
&\quad - \sum_{\ell=0}^{2\rho_j - 1} (t - \alpha_j)^\ell \sum_{p+q=\ell} A_p(\alpha_j)\, dT\, A_q(\alpha_j)^*\, A^*(I - zA^*)^{-1} \Bigg\} \\
&\quad + \int_{\Delta_j} \Bigg\{ \frac{(I - tA)^{-1}}{t - z}\, dT\, (I - zA^*)^{-1} \\
&\quad + \sum_{\ell=0}^{2\rho_j - 1} (t - \alpha_j)^\ell \sum_{p+q=\ell} \frac{A_q(\alpha_j)}{(z - \alpha_j)^{p+1}}\, dT\, (I - zA^*)^{-1} \Bigg\}.
\end{aligned}$$

where $dT = \Phi_2\, [d\tau(t)]\, \Phi_2^*$. Using the identities

$$(I - A^*t)^{-1} A^*(I - zA^*)^{-1} = \frac{(I - A^*t)^{-1} - (I - zA^*)^{-1}}{t - z}, \tag{6.15}$$

$$\begin{aligned}
A_q(\lambda)^*\, A^*(I - zA^*)^{-1} &= \frac{(I - zA^*)^{-1}}{(z - \bar\lambda)^{q+1}} - \frac{A_q(\lambda)^*}{z - \bar\lambda} \\
&\quad - \frac{A_{q-1}(\lambda)^*}{(z - \bar\lambda)^2} - \cdots - \frac{A_0(\lambda)^*}{(z - \bar\lambda)^{q+1}},
\end{aligned} \tag{6.16}$$

and

$$\frac{(I - At)^{-1}\, dT\, (I - A^*t)^{-1}}{t - z} = -\sum_{\ell=0}^{\infty} (t - \alpha_j)^\ell \sum_{p+q+\mu=\ell} \frac{A_p(\alpha_j)\, dT\, A_q(\alpha_j)^*}{(z - \alpha_j)^{\mu+1}},$$

we obtain

$$B_T(z) = \int_{\Delta_j} dG(t, z),$$

where

$$dG(t, z) = -\sum_{\ell=2\rho_j}^{\infty} (t - \alpha_j)^\ell \sum_{p+q+\mu=\ell} \frac{A_p(\alpha_j)\, dT\, A_q(\alpha_j)^*}{(z - \alpha_j)^{\mu+1}}$$

in a neighborhood of α_j. Straightforward estimates show that $\|B_T(z)\| = \mathcal{O}(1/|z|)$ as $|z| \to \infty$.

Case 2: (b) By (6.12),

$$\begin{aligned}
B_T(z) &= SA^*(I - zA^*)^{-1} + iB(z)\Phi_2^*(I - zA^*)^{-1} \\
&= \int_{\Delta_0} (I - tA)^{-1} \Phi_2\, [d\tau(t)]\, \Phi_2^*(I - tA^*)^{-1}.
\end{aligned}$$

Define $L^2(d\tau) = L^2(\Delta_0, d\tau)$ as in Appendix 1. By (3.1) and the closed graph theorem, the mapping $h \to \Phi_2^*(I - tA^*)^{-1}h$ is a bounded operator from $\mathfrak{H}$ into $L^2(d\tau)$, and hence

$$\left\|\Phi_2^*(I - tA^*)^{-1}h\right\|_{L^2(d\tau)} \le K\|h\|_{\mathfrak{H}}, \qquad h \in \mathfrak{H}, \tag{6.17}$$

for some positive constant K. Thus for any $h_1, h_2 \in \mathfrak{H}$ and $z \in D_\delta$,

$$\langle B_T(z)h_1, h_2\rangle_{\mathfrak{H}} = \left\langle \frac{(I - tA^*)^{-1}h_1}{t - z}, (I - tA^*)^{-1}h_2 \right\rangle_{L^2(d\tau)}.$$

By (6.14), $1/|t-z| \le \eta/|z|$, $z \in D_\delta$, for some positive constant η. Hence for $z \in D_\delta$, by (6.17) and the Schwarz inequality,

$$\begin{aligned}\left|\langle B_T(z)h_1, h_2\rangle_{\mathfrak{H}}\right| &\le \left\|\frac{\Phi_2(I - tA^*)^{-1}h_1}{t - z}\right\|_{L^2(d\tau)} \left\|\Phi_2(I - tA^*)^{-1}h_2\right\|_{L^2(d\tau)} \\ &\le \frac{\eta K^2}{|z|}\, \|h_1\|_{\mathfrak{H}}\, \|h_2\|_{\mathfrak{H}}.\end{aligned}$$

By the arbitrariness of h_1 and h_2, $\|B_T(z)\| = \mathcal{O}(1/|z|)$ as $z \to \infty$ inside D_δ.

Case 2: (c) Here by (3.11), $S = 0$ and $B(z) \equiv 0$. Hence $B_T(z) \equiv 0$.

Case 2: (d) Let $R_j(z) = \sum_{p=1}^{2\rho_j+1} R_{jp}z^p$. We use the formula for $S = \mathfrak{R}_j$ in Theorem 3.3 and the formula for $B(z)$ in the proof of Theorem 4.5 to obtain

$$\begin{aligned}B_T(z) &= SA^*(I - zA^*)^{-1} + iB(z)\Phi_2^*(I - zA^*)^{-1} \\ &= \sum_{p=1}^{2\rho_j+1} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\ge 1}} A_{\mu-1}(\alpha_j)\Phi_2 R_{jp}\Phi_2^* A_{\nu-1}(\alpha_j)^* A^*(I - zA^*)^{-1} \\ &\quad - \sum_{p=1}^{2\rho_j+1} \sum_{\substack{\mu+\nu=p+1\\ \mu,\nu\ge 1}} \frac{A_{\mu-1}(\alpha_j)\Phi_2 R_{jp}}{(z - \alpha_j)^\nu}\, \Phi_2^*(I - zA^*)^{-1}\end{aligned}$$

With the aid of (6.16) we bring this to the form

$$B_T(z) = -\sum_{p=1}^{2\rho_j+1} \sum_{\substack{\mu+m+n=p-1\\ \mu,m,n\ge 0}} \frac{A_\mu(\alpha_j)\Phi_2 R_{jp}\Phi_2^* A_m(\alpha_j)^*}{(z - \alpha_j)^{n+1}},$$

which obviously is $\mathcal{O}(1/|z|)$ as $|z| \to \infty$.

Case 2: (e) This is similar to (d). Let $M_k(z) = \sum_{p=1}^{\sigma_k} M_{kp}z^p$. The formula for $S = \mathfrak{M}_{1k} + \mathfrak{M}_{2k}$ obtained from Theorem 3.3, together with the formula for $B(z)$ in the proof of Theorem 4.5, now yield

$$B_T(z) = -\sum_{p=1}^{\sigma_k} \Bigg[\sum_{\substack{\mu+m+n=p-1\\ \mu,m,n\ge 0}} \frac{A_\mu(\beta_k)\Phi_2 M_{kp}\Phi_2^* A_m(\bar\beta_k)^*}{(z - \beta_k)^{n+1}}$$

$$+ \sum_{\substack{\mu+m+n=p-1 \\ \mu,m,n\geq 0}} \frac{A_\mu(\bar{\beta}_k)\Phi_2 M_{kp}^* \Phi_2^* A_m(\beta_k)^*}{(z-\bar{\beta}_k)^{n+1}} \Bigg].$$

Again clearly this is $\mathcal{O}(1/|z|)$ as $|z| \to \infty$. □

Proof of Theorem 5.1. (1) Suppose $v(z) \in \mathbf{N}(\mathfrak{A})$ and satisfies conditions (i) and (ii). By Theorem 4.4, $v(z)$ is a generalized Nevanlinna function. We verify Assumptions 3.1, Case 1, for any representation (2.1). By assumption, $\sigma(A)$ is a finite set, and $\sigma(A)$ contains no real point because $\sigma(A) \cap \sigma(A^*) = \emptyset$. We claim that $\sigma(A)$ contains no point $1/\beta_k$, $1/\bar{\beta}_k$, $k = 1, \dots, s$. In fact, if $1/\beta_k \in \sigma(A)$ (resp. $1/\bar{\beta}_k \in \sigma(A)$), then by (i), $v(z)$ has at most a removable singularity at β_k (resp. $\bar{\beta}_k$). But β_k and $\bar{\beta}_k$ are poles of $v(z)$ by condition (4°) in Theorem 2.1, which is impossible. The claim follows, and therefore operators S_v and $\Phi_{1,v}$ are defined under Case 1 of Assumptions 3.1.

We show that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$. By Theorem 3.5, in addition to the given identity (1.1), we also have

$$AS_v - S_v A^* = i\left[\Phi_{1,v}\Phi_2^* + \Phi_2\Phi_{1,v}^*\right]. \tag{6.18}$$

Define $\widetilde{L}_v(z,\zeta)$ and $\widetilde{B}_v(z)$ by (4.2) and (4.3) but with S and Φ_1 replaced by S_v and $\Phi_{1.v}$. Since

$$B_v(z) = (I - zA)^{-1}\big[\Phi_1 - i\Phi_2 v(z)\big],$$
$$\widetilde{B}_v(z) = (I - zA)^{-1}\big[\Phi_{1,v} - i\Phi_2 v(z)\big],$$

the function

$$F(z) \stackrel{\text{def}}{=} B_v(z) - \widetilde{B}_v(z) = (I - zA)^{-1}\big[\Phi_1 - \Phi_{1,v}\big] \tag{6.19}$$

is analytic in the complex plane except perhaps at a finite number of nonreal points λ such that $1/\lambda \in \sigma(A)$, and it vanishes at infinity because A is invertible. By (ii), $B_v(z)$ has at most a removable singularity at any point λ such that $1/\lambda \in \sigma(A)$, and by Theorem 4.6, $\widetilde{B}_v(z)$ is analytic at any point λ such that $1/\lambda \in \sigma(A)$. It follows that $F(z)$ is entire and therefore $F(z) \equiv 0$. This is only possible if $\Phi_1 = \Phi_{1,v}$. Finally, by (1.1) and (6.18),

$$A(S - S_v) - (S - S_v)A^* = i\big(\Phi_1\Phi_2^* + \Phi_2\Phi_1^*\big) - i\big(\Phi_{1,v}\Phi_2^* + \Phi_2\Phi_{1,v}^*\big) = 0.$$

Since $\sigma(A) \cap \sigma(A^*) = \emptyset$, by [6] the operator equation $AX - XA^* = 0$ has only the trivial solution, and hence $S - S_v = 0$.

(2) Conversely, let $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ for some generalized Nevanlinna function $v(z)$ having a representation (2.1) which satisfies Assumptions 3.1, Case 1. We show that $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$ and satisfies conditions (i) and (ii) in (1). Define $P(z)$ and $Q(z)$ for $v(z)$ as in Theorem 4.2. Then $P(\bar{z})^*Q(z) + Q(\bar{z})^*P(z) \equiv 0$. By (4.10), $v(z)$ has the representation (4.14) with $c(z)P(z) + d(z)Q(z) \equiv I$.

Conditions (i) and (ii) in Definition 4.3 thus hold. To see that condition (iii) in Definition 4.3 is satisfied, we use (4.11) and Theorem 4.5 to obtain

$$\operatorname{sq}_-\, i\,\frac{P(\zeta)^*Q(z)+Q(\zeta)^*P(z)}{z-\bar\zeta}\le \operatorname{sq}_-\,\widetilde{L}_v(z,\zeta)<\infty,$$

where sq_- denotes the number of negative squares of the kernel. Therefore $v(z)\in\mathbf{N}(\mathfrak{A})$. Since the only poles of $v(z)$ in $\mathbf{C}_+\cup\mathbf{C}_-$ are at the points $\beta_k,\bar\beta_k$, $k=1,\dots,s$, and according to Assumptions 3.1, $\sigma(A)$ contains no point $1/\beta_k, 1/\bar\beta_k$, $k=1,\dots,s$, we conclude that $v(z)$ satisfies condition (i) in (1). By Theorem 4.6, $v(z)$ satisfies condition (ii) in (1) as well. □

Proof of Theorem 5.2. Suppose that $v(z)$ has the form (4.14), and that every point λ satisfying $1/\lambda\in\sigma(A)$ belongs to the domain of holomorphy of $P(z)$ and $Q(z)$ and $c(\lambda)P(\lambda)+d(\lambda)Q(\lambda)$ is invertible. Then $v(z)$ is defined and analytic at every point λ such that $1/\lambda\in\sigma(A)$. Thus (i) holds.

To verify (ii), write $B_v(z)$ in the form

$$B_v(z)=(I-zA)^{-1}\Pi J\begin{bmatrix}-iv(z)\\ I\end{bmatrix},$$

where Π and J are defined by (4.1). By (6.7),

$$B_v(z)=(I-zA)^{-1}\Pi J\,\mathfrak{A}(z)\begin{bmatrix}P(z)\\ Q(z)\end{bmatrix}\big[c(z)P(z)+d(z)Q(z)\big]^{-1}.$$

By (4.8) and (1.1),

$$\begin{aligned}(I-zA)^{-1}\Pi J\mathfrak{A}(z)&=(I-zA)^{-1}\Pi J\big[I-iz\Pi^*(I-zA^*)^{-1}S^{-1}\Pi J\big]\\ &=(I-zA)^{-1}\Pi J\\ &\qquad -iz(I-zA)^{-1}\frac{AS-SA^*}{i}(I-zA^*)^{-1}S^{-1}\Pi J\\ &=(I-zA)^{-1}\Pi J\\ &\qquad-(I-zA)^{-1}\big[(zA-I+I)S\\ &\qquad\qquad-S(zA^*-I+I)\big](I-zA^*)^{-1}S^{-1}\Pi J\\ &=(I-zA)^{-1}\Pi J+S(I-zA^*)^{-1}S^{-1}\Pi J\\ &\qquad-(I-zA)^{-1}SS^{-1}\Pi J\\ &=S(I-zA^*)^{-1}S^{-1}\Pi J.\end{aligned}$$

Therefore

$$B_v(z)=S(I-zA^*)^{-1}S^{-1}\Pi J\begin{bmatrix}P(z)\\ Q(z)\end{bmatrix}\big[c(z)P(z)+d(z)Q(z)\big]^{-1}.$$

Since $\sigma(A)\cap\sigma(A^*)=\emptyset$, the last formula and our assumptions on $P(z)$ and $Q(z)$ show that $B_v(z)$ is analytic at every point λ such that $1/\lambda\in\sigma(A)$. This verifies (ii). □

Lemma 6.5. *Let* (1.1) *be an operator identity such that* $\sigma(A) = \{0\}$*, and let* $v(z)$ *be an* $m \times m$ *matrix-valued function in* $\mathbf{N}_{\varkappa}$ *which has the Kreĭn-Langer representation* (2.1)*. Define* $B_{v,T}(z)$ *using the operators in* (1.1) *as in* (4.4)*–*(4.5)*. If* $z = x + iy$*, then*

$$\frac{B_{v,T}(z) - B_{v,T}(z)^*}{2i} = (I - xA)^{-1}\Phi_2 \frac{v(z) - v(z)^*}{2i} \Phi_2^*(I - xA^*)^{-1} + G(z),$$

where $G(z)$ *is continuous in* $\mathbf{C}_+$ *and for any interval* $[a, b]$ *which contains no point* $\alpha_1, \dots, \alpha_r$*,* $G(x + iy)$ *is bounded for* $a \le x \le b$ *and* $0 < y \le 1$*, and* $G(x + i0) = 0$ *strongly a.e. with respect to Lebesgue measure on* $(-\infty, \infty)$*.*

Proof of Lemma 6.5. Since $B_{v,T}(z) = i\big[SL_0(\bar z)^* + B_v(z)L_2^*(\bar z)\big]$,

$$\begin{aligned}\frac{B_{v,T}(z) - B_{v,T}(z)^*}{2i} &= \frac{1}{2}\Big[SL_0(\bar z)^* + B_v(z)L_2(\bar z)^* + L_0(\bar z)S + L_2(\bar z)B_v(z)^*\Big] \\ &= \frac{1}{2}\Big[S(-i)(I - zA^*)^{-1}A^* \\ &\qquad + (I - zA)^{-1}\big[\Phi_1 - i\Phi_2 v(z)\big]\Phi_2^*(I - zA^*)^{-1} \\ &\qquad + iA(I - \bar z A)^{-1}S \\ &\qquad + (I - \bar z A)^{-1}\Phi_2\big[\Phi_1^* + iv(z)^*\Phi_2^*\big](I - \bar z A^*)^{-1}\Big] \\ &= G_1(z) + G_2(z),\end{aligned}$$

where

$$\begin{aligned}G_1(z) &= \frac{1}{2}\Big\{ -iSA^*(I - zA^*)^{-1} + i(I - \bar z A)^{-1}AS \\ &\qquad + (I - zA)^{-1}\Phi_1\Phi_2^*(I - zA^*)^{-1} \\ &\qquad + (I - \bar z A)^{-1}\Phi_2\Phi_1^*(I - \bar z A^*)^{-1}\Big\}, \\ G_2(z) &= \frac{1}{2i}(I - zA)^{-1}\Phi_2 v(z)\Phi_2^*(I - zA^*)^{-1} \\ &\qquad - \frac{1}{2i}(I - \bar z A)^{-1}\Phi_2 v(z)^*\Phi_2^*(I - \bar z A^*)^{-1}.\end{aligned}$$

Since $\sigma(A) = \{0\}$, $G_1(z)$ is continuous in the complex plane, and $G_1(x) = 0$ for all real x by (1.1). For the other part, we have

$$\begin{aligned}G_2(z) = &\frac{1}{2i}\Big[(I - zA)^{-1} - (I - xA)^{-1}\Big]\Phi_2 v(z)\Phi_2^*(I - zA^*)^{-1} \\ &+ \frac{1}{2i}(I - xA)^{-1}\Phi_2 v(z)\Phi_2^*\Big[(I - zA^*)^{-1} - (I - xA^*)^{-1}\Big] \\ &+ \frac{1}{2i}(I - xA)^{-1}\Phi_2\Big[v(z) - v(z)^*\Big]\Phi_2^*(I - xA^*)^{-1} \\ &+ \frac{1}{2i}(I - xA)^{-1}\Phi_2 v(z)^*\Phi_2^*\Big[(I - xA^*)^{-1} - (I - \bar z A^*)^{-1}\Big] \\ &+ \frac{1}{2i}\Big[(I - xA)^{-1} - (I - \bar z A)^{-1}\Big]\Phi_2 v(z)^*\Phi_2^*(I - \bar z A^*)^{-1}\end{aligned}$$

$$= G_3(z) + \frac{1}{2i}\,(I - xA)^{-1}\Phi_2\Big[v(z) - v(z)^*\Big]\Phi_2^*(I - xA^*)^{-1}.$$

Using [8, Proposition 3.4], we see that $G_3(z)$ is bounded for $a \le x \le b$ and $0 < y \le 1$. Since also $G_3(x + i0) = 0$ strongly a.e. on $(-\infty, \infty)$, the result follows with $G(z) = G_1(z) + G_3(z)$. □

Proof of Theorem 5.3. (1) Assume that $v(z) \in \mathbf{N}(\mathfrak{A})$ and conditions (i)–(iii) hold. By Theorem 4.4, $v(z)$ is a generalized Nevanlinna function. We show that a representation (2.1) can be chosen such that the conditions of Assumptions 3.1, Case 2, are satisfied.

By [8, Theorem 4.1], (i) allows us to choose a representation (2.1) such that $\rho_0 = 0$ and $R_0(z)$ is constant. We verify (3.1). Define $L_v(z,\zeta)$ and $L_{v,T}(z,\zeta)$ by (4.2) and (4.4). Fix h in $\mathfrak{H}$, and set

$$v_h(z) = \langle B_{v,T}(z)h, h\rangle\ .$$

We show that $v_h(z)$ belongs to $\mathbf{N}_\varkappa$ for some $\varkappa \ge 0$. Since $v(z) \in \mathbf{N}(\mathfrak{A})$, it has a representation (4.14), and hence

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} K(z)^{-1} = \mathfrak{A}(z)^{-1} \begin{bmatrix} -iv(z) \\ I \end{bmatrix}\ ,$$

where $K(z) = c(z)P(z) + d(z)Q(z)$. Therefore by Theorem 4.2,

$$L_v(z,\zeta) = \begin{bmatrix} I & 0 \\ B_v(\zeta)^* S^{-1} & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D_v(z,\zeta) \end{bmatrix} \begin{bmatrix} I & S^{-1}B_v(z) \\ 0 & I \end{bmatrix}\ ,$$

where

$$D_v(z,\zeta) = K(\zeta)^*\ i\ \frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{z - \bar\zeta}\ K(z)\,.$$

Hence $\varkappa_{L_v} = \varkappa_S + \varkappa_{P,Q}$. By (4.7), if

$$C_{v,T}(z,\zeta) = \frac{B_{v,T}(z) - B_{v,T}(\zeta)^*}{z - \bar\zeta}\ ,$$

then

$$\begin{bmatrix} S & -iB_{v,T}(z) \\ iB_{v,T}(\zeta)^* & C_{v,T}(z,\zeta) \end{bmatrix} = L_{v,T}(z,\zeta)$$

$$= \begin{bmatrix} I & 0 \\ L_0(\bar\zeta) & L_2(\bar\zeta) \end{bmatrix} L_v(z,\zeta) \begin{bmatrix} I & L_0(\bar z)^* \\ 0 & L_2(\bar z)^* \end{bmatrix}\ ,$$

where $L_0(z)$ and $L_2(z)$ are defined by (4.6). It then follows from (4.15) that

$$\varkappa_{C_{v,T}} \le \varkappa_{L_{v,T}} \le \varkappa_{L_v} \le \varkappa_S + \varkappa_{P,Q} < \infty\,.$$

In particular, $v_h(z)$ belongs to $\mathbf{N}_\varkappa$ for some $\varkappa \ge 0$.

By condition (iii) and [8, Theorem 4.2], the Kreĭn-Langer representation of $v_h(z)$ can be reduced to the form

$$\begin{aligned} v_h(z) &= \Bigg\{ \sum_{j=1}^{r_h} \int_{\Delta_{j,h}} \left[\frac{1}{t-z} - S_j(t,z) \right] d\tau_h(t) - \sum_{j=1}^{r_h} R_{j,h} \left(\frac{1}{z-\alpha_j} \right) \\ &\qquad - \sum_{k=1}^{s} \left[M_{k,h} \left(\frac{1}{z-\beta_k} \right) + M_{k,h} \left(\frac{1}{\bar z - \beta_k} \right)^* \right] \Bigg\} + \int_{\Delta_{0,h}} \frac{d\sigma_h(t)}{t-z} \\ &= \tilde v_h(z) + \int_{\Delta_{0,h}} \frac{d\sigma_h(t)}{t-z}, \end{aligned}$$

where $\tilde v_h(z)$ is analytic across the interior of $\Delta_{0,h}$ and real on this set, and $\sigma_h(t)$ is a nondecreasing function satisfying

$$\int_{\Delta_{0,h}} d\sigma_h(t) < \infty. \tag{6.20}$$

Suppose that $[a,b] \subseteq \Delta_{0,h} \cap \Delta_0$ and a and b are points of continuity of $\sigma_h(t)$ and $\tau(t)$. By the Stieltjes inversion formula and Lemma 6.5,

$$\begin{aligned} \int_a^b d\sigma_h(t) &= \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^b \operatorname{Im} v_h(t+iy)\, dt \\ &= \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^b \left\langle \frac{B_{v,T}(t+iy) - B_{v,T}(t+iy)^*}{2i}\, h, h \right\rangle dt \\ &= \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^b \left\langle (I-tA)^{-1} \Phi_2 \frac{v(t+iy) - v(t+iy)^*}{2i}\, \Phi_2^* (I-tA^*)^{-1} h, h \right\rangle dt. \end{aligned}$$

Hence by [8, Theorem 3.1],

$$\int_{\Delta_{0,h}\cap\Delta_0} \langle d\tau(t)\, \Phi_2^*(I-A^*t)^{-1}h, \Phi_2^*(I-A^*t)^{-1}h \rangle = \int_{\Delta_{0,h}\cap\Delta_0} d\sigma_h(t) < \infty,$$

and therefore

$$\int_{\Delta_0} \langle d\tau(t)\, \Phi_2^*(I-A^*t)^{-1}h, \Phi_2^*(I-A^*t)^{-1}h \rangle < \infty.$$

This verifies (3.1), and therefore operators S_v and $\Phi_{1,v}$ are defined.

It remains to show that $\Phi_1 = \Phi_{1,v}$ and $S = S_v$. Recall that $B_v(z)$ and $B_{v,T}(z)$ are defined using the operators A, S, Φ_1, Φ_2 from the given operator identity (1.1). By Theorem 3.5, we have a second operator identity,

$$AS_v - S_v A^* = i\left[\Phi_{1,v}\Phi_2^* + \Phi_2 \Phi_{1,v}^* \right]. \tag{6.21}$$

Define $\widetilde L_v(z,\zeta)$ and $\widetilde B_v(z)$ by (4.2) and (4.3) but with S and Φ_1 replaced by S_v and $\Phi_{1.v}$. Analogously, define $\widetilde L_{v,T}(z,\zeta)$ and $\widetilde B_{v,T}(z)$ using the transformed kernel (4.4) with S and Φ_1 replaced by S_v and $\Phi_{1.v}$. Thus

$$\begin{aligned} \widetilde B_v(z) &= (I-zA)^{-1}\left[\Phi_{1,v} - i\Phi_2 v(z) \right], \\ \widetilde B_{v,T}(z) &= i\left[S_v L_0(\bar z)^* + \widetilde B_v(z) L_2(\bar z)^* \right], \end{aligned}$$

where $L_0(z)$ and $L_2(z)$ are given by (4.6). In particular,

$$B_v(z) - \widetilde{B}_v(z) = (I - zA)^{-1}\big[\Phi_1 - \Phi_{1,v}\big] .$$

Hence for any $g \in \mathbf{C}^m$, by (ii) and Theorem 4.7,

$$(I - iyA)^{-1}\big[\Phi_1 - \Phi_{1,v}\big]g = \mathcal{O}(1), \qquad |y| \to \infty.$$

Since we assume that the only f in $\mathfrak{H}$ such that $\|(I-iyA)^{-1}f\| = \mathcal{O}(1)$ as $|y| \to \infty$ is $f = 0$, we deduce that $[\Phi_1 - \Phi_{1,v}]g = 0$, and so $\Phi_1 = \Phi_{1,v}$.

By what we have shown so far, $B_v(z) = \widetilde{B}_v(z)$. Therefore

$$B_{v,T}(z) - \widetilde{B}_{v,T}(z) = i(S - S_v)L_0(\bar{z})^* = (S - S_v)A^*(I - zA^*)^{-1}.$$

Hence for any $h \in \mathfrak{H}$, by (iii) and Theorem 4.7,

$$A(I - iyA)^{-1}(S - S_v)h = \mathcal{O}(1/|y|), \qquad |y| \to \infty.$$

By the identity $(I - iyA)^{-1} = iyA(I - iyA)^{-1} + I$, we get

$$(I - iyA)^{-1}(S - S_v)h = \mathcal{O}(1), \qquad |y| \to \infty.$$

As above, our assumptions on A imply that $(S - S_v)h = 0$, and therefore $S = S_v$.

(2) Conversely, assume that $S = S_v$ and $\Phi_1 = \Phi_{1,v}$, where $v(z)$ is a generalized Nevanlinna function having a representation (2.1) satisfying Assumptions 3.1, Case 2. To see that $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$, set

$$\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = \mathfrak{A}(z)^{-1}\begin{bmatrix} -iv(z) \\ I \end{bmatrix} .$$

Then $P(z)$ and $Q(z)$ are meromorphic on $\mathbf{C}_+ \cup \mathbf{C}_-$ and by (4.13),

$$\begin{aligned} v(z) &= i\big[a(z)P(z) + b(z)Q(z)\big], \\ I &= c(z)P(z) + d(z)Q(z) , \end{aligned}$$

and hence (4.14) holds. By Theorem 4.2, $P(\bar{z})^*Q(z) + Q(\bar{z})^*P(z) = 0$ on $\mathbf{C}_+ \cup \mathbf{C}_-$. Since $S = S_v$ and $\Phi_1 = \Phi_{1,v}$ by assumption, the kernel $L_v(z, \zeta)$ has a finite number of negative squares by Theorem 4.5. Hence by (4.11) and (4.12), the kernel

$$i\,\frac{P(\zeta)^*Q(z) + Q(\zeta)^*P(z)}{z - \bar{\zeta}}$$

has a finite number of negative squares. It follows that $v(z)$ belongs to $\mathbf{N}(\mathfrak{A})$.

We obtain (i) by the converse part of [8, Theorem 4.1]. Conditions (ii) and (iii) follow from Theorem 4.7. □

Proof of Theorem 5.4. The only change in the proof of Theorem 5.3 is in the proofs that $\Phi_1 = \Phi_{1,v}$ and $S = S_v$ in part (1) of the theorem. As before, for any $g \in \mathbf{C}^m$, $(I - iyA)^{-1}\big[\Phi_1 - \Phi_{1,v}\big]g = \mathcal{O}(1)$ as $|y| \to \infty$. By the identity $(I - iyA)^{-1} = iyA(I - iyA)^{-1} + I$,

$$y\,(I - iyA)^{-1}A\big[\Phi_1 - \Phi_{1,v}\big]g = \mathcal{O}(1), \qquad |y| \to \infty.$$

Our assumptions on A imply that $A\big[\Phi_1-\Phi_{1,v}\big]g=0$. Since $\ker A=\{0\}$, we deduce that $\Phi_1=\Phi_{1,v}$. Again as before, for any $h\in\mathfrak{H}$, $A(I-iyA)^{-1}(S-S_v)h=\mathcal{O}(1/|y|)$ as $|y|\to\infty$. By our assumptions on A, $A(S-S_v)h=0$. Since $\ker A=\{0\}$, $S=S_v$. □

Acknowledgement. The authors thank A. L. Sakhnovich for useful discussions of this paper.

References

[1] D. Alpay, A. Dijksma, J. Rovnyak, and H. S. V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory Adv. Appl., vol. 96, Birkhäuser Verlag, Basel, 1997.

[2] K. Daho and H. Langer, *Matrix functions of the class N_κ*, Math. Nachr. **120** (1985), 275–294.

[3] V. E. Katsnelson, *Methods of J-theory in continuous interpolation problems of analysis. Part I*, T. Ando Hokkaido University, Sapporo, 1985, Translated from the Russian and with a foreword by T. Ando.

[4] I. V. Kovalishina and V. P. Potapov, *Integral representation of Hermitian positive functions*, T. Ando Hokkaido University, Sapporo, 1982, Translated from the Russian by T. Ando.

[5] M. G. Kreĭn and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. **77** (1977), 187–236.

[6] M. Rosenblum, *On the operator equation $BX-XA=Q$*, Duke Math. J. **23** (1956), 263–269.

[7] J. Rovnyak and L. A. Sakhnovich, *Some indefinite cases of spectral problems for canonical systems of difference equations*, Linear Algebra Appl. **343/344** (2002), 267–289.

[8] ———, *On the Kreĭn-Langer integral representation of generalized Nevanlinna functions*, Electron. J. Linear Algebra **11** (2004), 1–15 (electronic).

[9] ———, *Spectral problems for some indefinite cases of canonical differential equations*, J. Operator Theory **51** (2004), 115–139.

[10] A. L. Sakhnovich, *Modification of V. P. Potapov's scheme in the indefinite case*, Matrix and operator valued functions, Oper. Theory Adv. Appl., vol. 72, Birkhäuser, Basel, 1994, pp. 185–201.

[11] L. A. Sakhnovich, *Integral equations with difference kernels on finite intervals*, Oper. Theory Adv. Appl., vol. 84, Birkhäuser Verlag, Basel, 1996.

[12] ———, *Interpolation theory and its applications*, Kluwer, Dordrecht, 1997.

[13] ———, *Spectral theory of canonical differential systems. Method of operator identities*, Oper. Theory Adv. Appl., vol. 107, Birkhäuser Verlag, Basel, 1999.

J. Rovnyak
University of Virginia, Department of Mathematics, P. O. Box 400137, Charlottesville, VA 22904
e-mail: rovnyak@Virginia.EDU

L. A. Sakhnovich
University of Connecticut at Storrs. Address: 735 Crawford Avenue, Brooklyn, NY 11223
e-mail: Lev.Sakhnovich@verizon.net