Linear Passive Stationary Scattering Systems with Pontryagin State Spaces

D. Z. Arov[*] [1], **J. Rovnyak**[**2], and **S. M. Saprikin**[***1]

[1] Department of Physics and Mathematics, Division of Mathematical Analysis, South-Ukrainian Pedagogical University, Staroportofrankovskaya 26, Odessa 65020, Ukraine
[2] Department of Mathematics, University of Virginia, P. O. Box 400137, Charlottesville, VA 22904–4137, U.S.A.

Received xxx, revised xxx, accepted xxx
Published online xxx

Key words Scattering system, passive, conservative, transfer function, Krein-Langer factorization, Pontryagin space, minimal system, dilation, Julia operator.
MSC (2000) Primary: 47A48; Secondary 47A45, 47A20, 46C20, 47B50, 47N70, 93B28

Passive scattering systems having Pontryagin state spaces and their minimal conservative dilations are investigated. The transfer functions of passive scattering systems are generalized Schur functions. In the case of a simple conservative system, the right and left Kreĭn-Langer factorizations of the transfer function correspond to natural cascade syntheses of systems. A generalization of Sz.-Nagy and Foias criteria for a cascade synthesis of two simple conservative systems to be simple is obtained for systems with Pontryagin state spaces. It is shown that the state space of a simple passive system admits certain unique fundamental decompositions, which give rise to a notion of stability and a characterization of simple conservative systems whose transfer functions have unitary boundary values a.e. on the unit circle.

Contents

1 Introduction

In this paper we study linear stationary scattering systems

$$
\begin{aligned}
x(n+1) &= Ax(n) + Bu(n),\\
y(n) &= Cx(n) + Du(n),
\end{aligned}
$$

* e-mail: aspect@farlep.net
** Corresponding author: e-mail: rovnyak@Virginia.edu
*** e-mail: sergey_saprikin@ukr.net

$n = 0, 1, 2, \ldots$, whose states $x(n)$ belong to a Pontryagin space $\mathcal{X}$ and whose inputs $u(n)$ and outputs $y(n)$ belong to Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. The transfer function of such a system is defined by

$$\Theta_\Sigma(z) = D + zC(I - zA)^{-1}B$$

whenever the inverse exists. The case in which the state space $\mathcal{X}$ is a Hilbert space is classical. In several recent papers, the classical theory, as it appears, for example, in [3], has been extended to Pontryagin state spaces. Basic properties of passive scattering systems with Pontryagin state space, including the notions of dilation and embedding of systems, are discussed in [4] and [23]. Results on Darlington representations of systems having Pontryagin state space are derived in the appendix of the English translation in [3]. In a related work, S. A. Kuzhel′ [21] generalizes the abstract Lax-Phillips conservative scattering scheme to Pontryagin spaces. The purpose of this paper is to continue the study of passive scattering systems having Pontryagin state spaces by investigating further properties of dilations and embeddings, minimal systems, cascade representations and invariant subspaces, and a generalization of the notion of stability.

We summarize background in Sections 2 and 3. This includes properties of Pontryagin spaces, contraction operators, and Julia operators that are used in the paper. Systems (3.1)–(3.3) are viewed as colligations $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$, and they are classified as passive scattering systems or conservative scattering systems according as the system operator

$$V_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is contractive or unitary. The definitions of controllable, observable, and simple systems are parallel to the Hilbert space case. Among several equivalent definitions of a minimal system for Hilbert state spaces, we choose one that is appropriate for Pontryagin state spaces. In Section 4 we summarize known results, mainly from [23], on passive and conservative systems and their dilations and embeddings. An important condition on indices is introduced here. The transfer function $\Theta_\Sigma(z)$ of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ belongs to some generalized Schur class $\mathbf{S}_{\varkappa'}(\mathcal{U}, \mathcal{Y})$ with $\varkappa' \leq \varkappa$. Many theorems use the hypothesis that $\Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$, that is, $\varkappa' = \varkappa$. Sufficient conditions for the index condition to hold are identified.

Section 5 constructs systems associated with the Schur complements $D - CA^{-1}B$ and $A - BD^{-1}C$ in the system operator. Such constructions, of course, require that the operator inverses exist.

The form of a minimal conservative dilation $\widetilde{\Sigma}$ of a passive system Σ is determined in Section 6. If $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$, there are many ways in which one can form a unitary operator

$$U_\Sigma = \begin{bmatrix} V_\Sigma & E \\ F & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \oplus \mathcal{U} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \oplus \mathcal{Y} \\ \mathcal{F} \end{bmatrix},$$

and each such unitary operator induces a conservative dilation $\widetilde{\Sigma}$. It is shown that $\widetilde{\Sigma}$ is a minimal conservative dilation if and only if U_Σ is a Julia operator. Passive systems which admit *simple* conservative dilations are said to have minimal losses. The results of Section 6 determine the form of a minimal conservative dilation of a passive system, namely, they are induced by Julia operators for the system operator.

In Section 7 we show that the right and left Kreĭn-Langer factorizations of the transfer function of a simple conservative system Σ correspond to natural cascade syntheses of the system. These results are applied to simple passive systems Σ. In this case, the semi-definite invariant subspaces of the main operator A, whose existence is assured by the classical theory of contraction operators on a Pontryagin space, are regular; in fact, they are unique and determine natural fundamental decompositions $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ and $\mathcal{X} = \mathcal{X}'_+ \oplus \mathcal{X}'_-$ of the state space with $A\mathcal{X}_+ \subseteq \mathcal{X}_+$ and $A\mathcal{X}'_- \subseteq \mathcal{X}'_-$.

A key step in Section 7 is to show that the cascade synthesis of two particular simple conservative systems is simple. In general, such a cascade synthesis is not always simple, even with Hilbert state spaces. Section 8 is concerned with the problem to determine when the cascade synthesis of two simple conservative systems having Pontryagin state spaces is simple. We generalize a well-known analytical condition, which is related to the notion of a regular factorization of an operator-valued function, from the case of Hilbert state spaces to Pontryagin state spaces.

In Section 9 we examine the notion of stability. In the case of Hilbert state spaces, we call a system stable if it is bi-stable in the sense that both of the semigroups A^n and A^{*n} are stable, that is, they tend to zero strongly as $n \to \infty$. This notion does not extend directly to Pontryagin state spaces due to the existence of eigenvalues of the main operator of modulus greater than one. In place of stable systems, we introduce classes $\mathbf{P}_{00}^{\varkappa}$ and $\mathbf{C}_{00}^{\varkappa}$ of passive and conservative systems that are "partially stable" in a sense that depends on the fundamental decompositions $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ and $\mathcal{X} = \mathcal{X}'_+ \oplus \mathcal{X}'_-$ of the state space introduced in Section 7; in the definitions of these classes we require, roughly, that the parts of the system corresponding to the positive subspaces are stable in the usual Hilbert space sense. It is shown, for example, that a simple conservative system belongs to $\mathbf{C}_{00}^{\varkappa}$ if and only if its transfer function has unitary boundary values a.e. on the unit circle.

The concluding Section 10 shows how the class $\mathbf{C}_{00}^{\varkappa}$ can be described in terms of canonical models.

2 Pontryagin spaces and linear operators

Mainly only elementary notions concerning Pontryagin spaces and operators which act on them are used here. For example, see [5, 8, 16, 18]. We recall a few basic ideas in order to fix terminology and notation.

All Hilbert spaces are assumed to be separable. A **Pontryagin space** is a complex vector space $\mathcal{X}$ together with a linear and symmetric inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{X}}$ which admits a representation $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$, where $(\mathcal{X}_\pm, \pm\langle\cdot,\cdot\rangle_{\mathcal{X}})$ are Hilbert spaces and $\dim \mathcal{X}_- < \infty$. We call such a representation a **fundamental decomposition**. A fundamental decomposition is in general not unique, but it determines a unique strong topology. The dimensions $\operatorname{ind}_\pm \mathcal{X} = \dim \mathcal{X}_\pm$ are independent of the choice of fundamental decomposition and called the **positive** and **negative indices** of $\mathcal{X}$. The terms regular subspace and Hilbert subspace are used as in standard sources; the orthogonal projection operator whose range is a regular subspace $\mathcal{M}$ is written $P_{\mathcal{M}}$. We shall use the following result from [23, Lemma 3.1].

Lemma 2.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be regular subspaces of a Pontryagin space $\mathcal{X}$ which has negative index $\varkappa$. If $\mathcal{M}$ and $\mathcal{N}$ have negative index $\varkappa$, then $\overline{P_{\mathcal{M}}\mathcal{N}}$ and $\overline{P_{\mathcal{N}}\mathcal{M}}$ are regular subspaces of $\mathcal{X}$ having negative index $\varkappa$.*

We write $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X},\mathcal{Y})$ for the spaces of continuous linear operators on a Pontryagin space $\mathcal{X}$ into itself and into a Pontryagin space $\mathcal{Y}$, respectively. An operator is **invertible** if it has an everywhere defined and continuous inverse. The **adjoint** of an operator $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is the operator $A^* \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ such that $\langle Ax, y\rangle_{\mathcal{Y}} = \langle x, A^*y\rangle_{\mathcal{X}}$ for all x in $\mathcal{X}$ and y in $\mathcal{Y}$. Classes of selfadjoint, isometric, and unitary operators are defined as for Hilbert spaces. We call $A \in \mathcal{L}(\mathcal{X})$ **nonnegative** and write $A \geq 0$ if $\langle Ax, x\rangle_{\mathcal{X}} \geq 0$ for every $x \in \mathcal{X}$. An operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is a **contraction** if $I - T^*T \geq 0$. If $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is a contraction and $\operatorname{ind}_- \mathcal{X} = \operatorname{ind}_- \mathcal{Y}$, then $T^* \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ is also a contraction [16, Corollary 2.5].

Let $\mathcal{X}$ and $\mathcal{Y}$ be Pontryagin spaces such that $\operatorname{ind}_- \mathcal{X} = \operatorname{ind}_- \mathcal{Y}$. If $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is a contraction, there exist Hilbert spaces $\mathcal{E}, \mathcal{F}$ and operators E, F, G acting on appropriate spaces such that the operator

$$U = \begin{bmatrix} T & E \\ F & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{F} \end{bmatrix}$$

is unitary. In this case, the conditions $\ker E = \{0\}$ and $\ker F^* = \{0\}$ are equivalent, and when these conditions are satisfied we call U a **Julia operator** for T. A Julia operator always exists and is essentially unique [16, Theorems 2.3 and 2.6]. Here and below, various notions of essential uniqueness appear, and we leave it to the reader to construct definitions analogous to the Hilbert space case.

A subspace $\mathcal{N}$ of a Pontryagin space $\mathcal{X}$ is called **nonnegative** (**nonpositive**) if $\langle x, x\rangle_{\mathcal{X}}$ is nonnegative (nonpositive) for all x in $\mathcal{N}$. We say that $\mathcal{N}$ is semi-definite if it is either nonnegative or nonpositive. A semi-definite subspace may contain nonzero vectors x such that $\langle x, x\rangle_{\mathcal{X}} = 0$. Notions of **maximal nonnegative** and **maximal nonpositive** subspaces are defined in the usual way relative to inclusion of subspaces. If $\mathcal{X}$ has negative index $\varkappa$, a nonpositive subspace $\mathcal{N}$ is maximal nonpositive if and only if $\dim \mathcal{N} = \varkappa$.

Theorem 2.2 *Let $\mathcal{X}$ be a Pontryagin space of negative index $\varkappa$, and let $T \in \mathcal{L}(\mathcal{X})$ be a contraction operator.*

(i) *The part of the spectrum of T in $\{z\colon |z| > 1\}$ consists of isolated eigenvalues which have finite-dimensional root subspaces.*

(ii) *There exists a maximal nonpositive subspace $\mathcal{N}$ of $\mathcal{X}$ which is invariant under T and which contains all of the root subspaces for the eigenvalues of T in $\{z\colon |z| > 1\}$.*

(iii) *There exists a maximal nonnegative subspace $\mathcal{P}$ of $\mathcal{X}$ which is invariant under T and which is orthogonal to all of the root subspaces for the eigenvalues of T^* in $\{z\colon |z|>1\}$.*

In particular, the part of the spectrum of T in $\{z\colon |z|>1\}$ consists of at most $\varkappa$ eigenvalues.

See [5, Chapter 3], [15, Section 2], and [18, Section 11] for additional results and historical notes on theorems of this type. Concerning the statement (i) in Theorem 2.2, see also Kuzhel′ [21, Section 4, Assertion 6], where unitary dilations are used.

Sketch of the proof of Theorem 2.2. (i) We follow [18, Lemma 11.1]. Choose a fundamental decomposition $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ of $\mathcal{X}$. Set $\mathcal{H}_+ = \mathcal{X}_+$, let $\mathcal{H}_-$ be the antispace of $\mathcal{X}_-$, and write

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}.$$

Since $\langle Tx, Tx\rangle_{\mathcal{X}} \le \langle x, x\rangle_{\mathcal{X}}$ for all x in $\mathcal{X}$, for any $x_+ \in \mathcal{X}_+$,

$$\|T_{11}x_+\|^2 - \|T_{21}x_+\|^2 \le \|x_+\|^2.$$

Thus $T_{11}^*T_{11} \le I + T_{21}^*T_{21} = R^2$, $R = \left[I + T_{21}^*T_{21}\right]^{1/2}$, and so $T_{11} = SR$, where S is a contraction operator on $\mathcal{H}_+$. Since $T_{21}^*T_{21}$ has finite rank, by the spectral theorem $R = I + R_0$ where R_0 has finite rank. Hence T differs from the contraction operator $S \oplus 0$ on $\mathcal{H}_+ \oplus \mathcal{H}_-$ by an operator of finite rank. Therefore by [17, Lemma 5.2], the part of the spectrum of T outside the unit circle consists of isolated eigenvalues having finite-dimensional root subspaces.

(ii) See [18, Theorem 11.2].

(iii) The adjoint operator T^* is also a contraction, so by (ii) it has a maximal nonpositive invariant subspace $\mathcal{M}$. Then $\mathcal{P} = \mathcal{M}^\perp = \{x\colon \langle x, y\rangle_{\mathcal{X}} = 0, y \in \mathcal{M}\}$ is a maximal nonnegative invariant subspace for T having the required properties (see [16, Theorem 1.6]). □

3 Scattering systems with Pontryagin state spaces

In this paper we shall study conservative and passive scattering systems which have Pontryagin state spaces. The definitions of these notions are adapted from the Hilbert state space case.

Consider a **system**

$$\begin{cases} x(n+1) = Ax(n) + Bu(n), \\ \quad\; y(n) = Cx(n) + Du(n), \end{cases} \tag{3.1}$$

$n = 1, 2, \dots$, whose **states** $x(n)$ belong to a Pontryagin space $\mathcal{X}$ of negative index $\varkappa$ and whose **inputs** $u(n)$ and **outputs** $y(n)$ belong to Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. We call $\mathcal{X}$ the **state space** and $\mathcal{U}$ and $\mathcal{Y}$ the **input** and **output spaces**. The system (3.1) is equivalently viewed as an operator node

$$\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa). \tag{3.2}$$

The **system operator** $V_\Sigma \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$ is defined by

$$V_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}. \tag{3.3}$$

We call A, B, C the **main**, **input**, and **output operators** for the system. The **transfer function** of the system is defined by

$$\Theta_\Sigma(z) = D + zC(I - zA)^{-1}B$$

whenever the inverse exists. The **adjoint system** is $\Sigma^* = (A^*, C^*, B^*, D^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}; \varkappa)$. Thus $V_{\Sigma^*} = V_\Sigma^*$, and $\Theta_{\Sigma^*}(z) = \Theta_\Sigma(\bar z)^*$.

A system (3.1) is said to be a **passive scattering system** if

$$\langle x(n+1), x(n+1)\rangle_{\mathcal{X}} - \langle x(n), x(n)\rangle_{\mathcal{X}} \le \|u(n)\|^2_{\mathcal{U}} - \|y(n)\|^2_{\mathcal{Y}} \tag{3.4}$$

for all initial states $x(0)$ in $\mathcal{X}$ and all inputs $u(n)$, $n \ge 0$, in $\mathcal{U}$. We call (3.1) a **conservative scattering system** if equality always holds in (3.4) and if the adjoint system Σ^* has the same property. In what follows, the term "scattering" will usually be omitted, and we shall speak more simply of "passive systems" and "conservative systems" because other types of systems will not be considered. A system Σ is passive (conservative) if and only if the system operator V_Σ is a contraction (unitary) operator. If Σ is passive or conservative, the adjoint system Σ^* has the same property.

For any system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$, set $\mathcal{X}^c_\Sigma = \bigvee_0^\infty A^n B\mathcal{U}$ and $\mathcal{X}^o_\Sigma = \bigvee_0^\infty A^{*n} C^*\mathcal{Y}$. Then Σ is said to be (i) **controllable**, (ii) **observable**, (iii) **simple**, or (iv) **minimal** according as

(i) $\mathcal{X} = \mathcal{X}^c_\Sigma$,

(ii) $\mathcal{X} = \mathcal{X}^o_\Sigma$,

(iii) $\mathcal{X} = \mathcal{X}^c_\Sigma \vee \mathcal{X}^o_\Sigma$, or

(iv) $\mathcal{X} = \mathcal{X}^c_\Sigma$ and $\mathcal{X} = \mathcal{X}^o_\Sigma$.

It is easy to see that $\mathcal{X}^c_{\Sigma^*} = \mathcal{X}^o_\Sigma$ and $\mathcal{X}^o_{\Sigma^*} = \mathcal{X}^c_\Sigma$.

Remark. We have defined a minimal system as a system that is controllable and observable. This definition differs from the usual definition of a minimal system with a Hilbert state space. In the case of a Hilbert state space, a system is often called minimal if it is not a nontrivial dilation (as defined below) of another system. These two definitions are equivalent in the Hilbert state space case (see [2, Proposition 3]), but for systems with Pontryagin state spaces they are not the same. However, a controllable and observable system cannot be a nontrivial dilation of another system, and in the special case of passive systems, which is of our main interest, the converse statement is also true (see Corollary 4.8).

We call two systems $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}; \varkappa)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \varkappa)$ **equivalent** and write $\Sigma_1 \cong \Sigma_2$ if

$$A_2 = WA_1W^{-1}, \qquad B_2 = WB_1, \qquad C_2 = C_1W^{-1}, \qquad D_2 = D_1.$$

for some unitary operator $W \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$.

We use two types of extensions of a system Σ to a larger system $\widetilde{\Sigma}$. One does not change the state space and main operator and enlarges the input and output spaces. A system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is **embedded** in a system $\widetilde{\Sigma} = (A, \widetilde{B}, \widetilde{C}, \widetilde{D}; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}; \varkappa)$ if there exist Hilbert spaces $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{Y}}$ such that $\widetilde{\mathcal{U}} = \widehat{\mathcal{U}} \oplus \mathcal{U}$ and $\widetilde{\mathcal{Y}} = \mathcal{Y} \oplus \widehat{\mathcal{Y}}$, and

$$V_{\widetilde{\Sigma}} = \begin{bmatrix} V_\Sigma & E \\ F & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \oplus \mathcal{U} \\ \widehat{\mathcal{U}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \oplus \mathcal{Y} \\ \widehat{\mathcal{Y}} \end{bmatrix} \tag{3.5}$$

for some operators E, F, G. Equivalently,

$$V_{\widetilde{\Sigma}} = \begin{bmatrix} A & B & E_1 \\ C & D & E_2 \\ F_1 & F_2 & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \widehat{\mathcal{U}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \widehat{\mathcal{Y}} \end{bmatrix}$$

or

$$V_{\widetilde{\Sigma}} = \begin{bmatrix} A & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \begin{bmatrix} A & \begin{bmatrix} E_1 & B \end{bmatrix} \\ \begin{bmatrix} C \\ F_1 \end{bmatrix} & \begin{bmatrix} E_2 & D \\ G & F_2 \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \widehat{\mathcal{U}} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \widehat{\mathcal{Y}} \end{bmatrix}. \tag{3.6}$$

In this case,

$$\Theta_{\widetilde{\Sigma}}(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}, \qquad \Theta_\Sigma(z) = \Theta_{12}(z). \tag{3.7}$$

If V_Σ is a contraction operator (that is, Σ is a passive system) and (3.5) is a Julia operator for V_Σ (see Section 2), we say that the embedding is a **Julia embedding**.

A second type of extension of a system Σ to a larger system $\widetilde{\Sigma}$ uses the same input and output spaces and enlarges the state space, but without increasing its negative index. We call $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ a **dilation** of $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ if $\widetilde{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$, where $\mathcal{D}_+$ and $\mathcal{D}_-$ are Hilbert spaces and

$$V_{\widetilde{\Sigma}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} * & 0 & 0 \\ * & A & 0 \\ * & * & * \end{bmatrix} & \begin{bmatrix} 0 \\ B \\ * \end{bmatrix} \\ \begin{bmatrix} * & C & 0 \end{bmatrix} & D \end{bmatrix} : \begin{bmatrix} \mathcal{D}_- \\ \mathcal{X} \\ \mathcal{D}_+ \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_- \\ \mathcal{X} \\ \mathcal{D}_+ \\ \mathcal{Y} \end{bmatrix}. \tag{3.8}$$

The form of the system operator is equivalent to the relations

$$\widetilde{A}\mathcal{D}_+ \subseteq \mathcal{D}_+, \qquad \widetilde{A}^*\mathcal{D}_- \subseteq \mathcal{D}_-, \qquad \widetilde{C}\mathcal{D}_+ = \{0\}, \qquad \widetilde{B}^*\mathcal{D}_- = \{0\}.$$

Using this form, we easily obtain $\Theta_{\widetilde{\Sigma}}(z) = \Theta_\Sigma(z)$. A dilation $\widetilde{\Sigma}$ of Σ is called **conservative** or **simple** according as $\widetilde{\Sigma}$ is a conservative or simple system. If $\widetilde{\Sigma}$ is a dilation of Σ, we also call Σ a **restriction** of $\widetilde{\Sigma}$; we say that Σ a **proper restriction** of $\widetilde{\Sigma}$ if it is a restriction of $\widetilde{\Sigma}$ and $\Sigma \neq \widetilde{\Sigma}$.

It is not hard to see that the main operator $A \in \mathcal{L}(\mathcal{X})$ of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is a contraction. For since $\mathcal{Y}$ is a Hilbert space, from the inequality

$$\langle V_\Sigma(x \oplus 0), V_\Sigma(x \oplus 0)\rangle_{\mathcal{X}\oplus\mathcal{Y}} \leq \langle x \oplus 0, x \oplus 0\rangle_{\mathcal{X}\oplus\mathcal{U}}$$

for any x in $\mathcal{X}$, we obtain $\langle Ax, Ax\rangle_{\mathcal{X}} \leq \langle Ax, Ax\rangle_{\mathcal{X}} + \langle Cx, Cx\rangle_{\mathcal{Y}} \leq \langle x, x\rangle_{\mathcal{X}}$. Moreover, every contraction operator A on a Pontryagin space $\mathcal{X}$ may be viewed as the main operator of a conservative system by considering a Julia operator for A. We also remark that if a system Σ is embedded in any way in a system $\widetilde{\Sigma}$, and if $\widetilde{\Sigma}$ is passive, then Σ is passive too. Analogously, every restriction Σ of a passive system $\widetilde{\Sigma}$ is passive.

4 A brief survey of known results

The only systems that concern us are conservative or at least passive. As noted above, the main operator A of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is a contraction operator. In the classical case $\varkappa = 0$, $\mathcal{X}$ is a Hilbert space and the inverse $(I - zA)^{-1}$ exists at all points of the open unit disk $\mathbf{D}$. In this case, the transfer function $\Theta_\Sigma(z)$ is defined everywhere in $\mathbf{D}$ and, as is well known, it is an operator-valued Schur function, that is, it is holomorphic and bounded by one in $\mathbf{D}$. When the state space $\mathcal{X}$ is a Pontryagin space, then according to Theorem 2.2, the inverse $(I - zA)^{-1}$ exists for all but at most $\varkappa$ nonzero points in $\mathbf{D}$. We shall usually consider the domain of the transfer function of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ to be the open unit disk with at most $\varkappa$ nonzero points deleted. In some places (for example, see Section 5) we shall also consider systems Σ with invertible main operator A; in this case, the transfer function $\Theta_\Sigma(z)$ is holomorphic at ∞ as well, and its restriction to a neighborhood of ∞ plays a role.

Given Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, the **generalized Schur class** $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ is the set of functions $S(z)$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ which are holomorphic in a neighborhood of the origin and meromorphic in $\mathbf{D}$ such that the kernel $K(w, z) = [I - S(z)S(w)^*]/(1 - z\bar{w})$ has $\varkappa$ negative squares ($\varkappa = 0, 1, 2, \dots$). This means that for any finite set of points $w_1, \dots, w_n$ in the domain of holomorphy of $S(z)$ in $\mathbf{D}$, the selfadjoint operator on $\mathcal{Y} \oplus \cdots \oplus \mathcal{Y}$ defined by the block Hermitian matrix $\big[K(w_j, w_k)\big]_{j,k=1}^n$ has at most a $\varkappa$-dimensional invariant subspace in which the operator has spectrum contained in $(-\infty, 0)$, and there exists at least one choice of points $w_1, \dots, w_n$ such that the operator has such an invariant subspace of dimension equal to $\varkappa$. By the Kreĭn-Langer factorization (see Section 7), every $S(z)$ in $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ has strong nontangential boundary values $S(\zeta)$ a.e. on the unit circle $|\zeta| = 1$, and these boundary values are contractions a.e. Let $\mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$ be the subclass of functions in $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ whose boundary values are unitary a.e. For $\varkappa = 0$, we write more simply $\mathbf{S}(\mathcal{U}, \mathcal{Y})$ for $\mathbf{S}_0(\mathcal{U}, \mathcal{Y})$, and $\mathbf{U}(\mathcal{U}, \mathcal{Y})$ for $\mathbf{U}_0(\mathcal{U}, \mathcal{Y})$.

From [1, Theorems 2.1.2, 2.3.1, and 2.1.3], we obtain:

Theorem 4.1 (i) *The transfer function $\Theta_\Sigma(z)$ of a simple conservative system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ belongs to the class $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$. Conversely, every $\Theta(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ has the form $\Theta(z) = \Theta_\Sigma(z)$ for some simple conservative system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$.*

(ii) *Two simple conservative systems Σ_1 and Σ_2 are equivalent if and only if $\Theta_{\Sigma_1}(z) = \Theta_{\Sigma_2}(z)$ in a neighborhood of the origin.*

It may happen that the transfer function of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is a generalized Schur function, but for a possibly smaller index than $\varkappa$. The next result, which is taken from [23, Theorems 2.2 and 2.3], includes a sufficient condition for equality.

Theorem 4.2 *The transfer function $\Theta_\Sigma(z)$ of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ belongs to $\mathbf{S}_{\varkappa'}(\mathcal{U}, \mathcal{Y})$ for some $\varkappa' \le \varkappa$. If Σ is minimal, then $\varkappa' = \varkappa$.*

A passive system can be embedded in a conservative system in many ways. In particular, a Julia embedding always exists and is essentially unique.

Theorem 4.3 *Let Σ be a passive system which is embedded in a conservative system $\widetilde{\Sigma}$. If Σ is simple, then $\widetilde{\Sigma}$ is simple.*

Proof. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ and $\widetilde{\Sigma} = (A, \widetilde{B}, \widetilde{C}, \widetilde{D}; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}; \varkappa)$. Since $\widetilde{B}\widetilde{\mathcal{U}} \supseteq B\mathcal{U}$ and $\widetilde{C}^*\widetilde{\mathcal{Y}} \supseteq C^*\mathcal{Y}$,

$$\mathcal{X}^c_{\widetilde{\Sigma}} = \bigvee_0^\infty A^n \widetilde{B}\widetilde{\mathcal{U}} \supseteq \bigvee_0^\infty A^n B\mathcal{U} = \mathcal{X}^c_\Sigma,$$

$$\mathcal{X}^o_{\widetilde{\Sigma}} = \bigvee_0^\infty A^{*n} \widetilde{C}^*\widetilde{\mathcal{Y}} \supseteq \bigvee_0^\infty A^{*n} C^*\mathcal{Y} = \mathcal{X}^o_\Sigma.$$

Therefore $\mathcal{X}^c_{\widetilde{\Sigma}} \vee \mathcal{X}^o_{\widetilde{\Sigma}} \supseteq \mathcal{X}^c_\Sigma \vee \mathcal{X}^o_\Sigma$, and hence if Σ is simple, so is $\widetilde{\Sigma}$. □

It should be emphasized that if $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is a passive system, then $\Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ only under special conditions; two sufficient conditions for the inclusion $\Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ are identified in Theorems 4.1 and 4.2. The property that $\Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ is a hypothesis in the next two results. Theorem 4.4 is from [23, Lemma 2.5], and Theorem 4.5 is from [23, Proposition 2.6].

Theorem 4.4 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a passive system whose transfer function $\Theta_\Sigma(z)$ belongs to $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$. Then each of the spaces $\mathcal{X}^c_\Sigma$, $\mathcal{X}^o_\Sigma$, and $\mathcal{X}^c_\Sigma \vee \mathcal{X}^o_\Sigma$ is regular and has negative index $\varkappa$. Hence $(\mathcal{X}^c_\Sigma)^\perp$, $(\mathcal{X}^o_\Sigma)^\perp$, and $(\mathcal{X}^c_\Sigma \vee \mathcal{X}^o_\Sigma)^\perp$ are Hilbert subspaces of $\mathcal{X}$.*

Theorem 4.5 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a conservative system whose transfer function $\Theta_\Sigma(z)$ belongs to $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$. Then*

(i) *$(\mathcal{X}^o_\Sigma)^\perp$ is the largest Hilbert subspace $\mathcal{M}$ of $\mathcal{X}$ which is invariant under A and contained in $\ker C$, such that $A|\mathcal{M}$ is isometric;*

(ii) *$(\mathcal{X}^c_\Sigma)^\perp$ is the largest Hilbert subspace $\mathcal{K}$ of $\mathcal{X}$ which is invariant under A^* and contained in $\ker B^*$, such that $A^*|\mathcal{K}$ is isometric;*

(iii) *$\left(\mathcal{X}^c_\Sigma \vee \mathcal{X}^o_\Sigma\right)^\perp$ is the largest Hilbert subspace $\mathcal{L}$ of $\mathcal{X}$ which is invariant under A and contained in $\ker C \cap \ker B^*$, such that $A|\mathcal{L}$ is unitary.*

We include the construction of a conservative dilation for any passive system from [23, Theorem 2.1] since this construction will be needed later.

Theorem 4.6 *Every passive system Σ has a conservative dilation $\widetilde{\Sigma}$.*

Proof. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$. Since Σ is passive, we can choose Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ and a unitary operator of the form

$$\begin{bmatrix} V_\Sigma & E \\ F & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \oplus \mathcal{U} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \oplus \mathcal{Y} \\ \mathcal{F} \end{bmatrix}, \tag{4.1}$$

or, equivalently,

$$\begin{bmatrix} A & B & E_1 \\ C & D & E_2 \\ F_1 & F_2 & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{F} \end{bmatrix}. \tag{4.2}$$

This can be done in many ways; for example, we can choose (4.1) to be a Julia operator for V_Σ. Set $\widetilde{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$, where $\mathcal{D}_- = \ell^2_-(\mathcal{E})$ is the Hilbert space of square summable sequences $\{\dots, e_{-2}, e_{-1}\}$ with entries in $\mathcal{E}$, and $\mathcal{D}_+ = \ell^2_+(\mathcal{F})$ is the analogous space of sequences $\{f_0, f_1, \dots\}$ with entries in $\mathcal{F}$. Set

$$\begin{cases} \widetilde{A} = \begin{bmatrix} V_-^* & 0 & 0 \\ E_1 P_{-1} & A & 0 \\ Q_0 G P_{-1} & Q_0 F_1 & V_+ \end{bmatrix}, & \widetilde{B} = \begin{bmatrix} 0 \\ B \\ Q_0 F_2 \end{bmatrix}, \\ \widetilde{C} = \begin{bmatrix} E_2 P_{-1} & C & 0 \end{bmatrix}, & \widetilde{D} = D, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} P_{-1} &: \{\dots, e_{-3}, e_{-2}, e_{-1}\} \to e_{-1}, \\ Q_0 &: f_0 \to \{f_0, 0, 0, \dots\}, \\ V_- &: \{\dots, e_{-3}, e_{-2}, e_{-1}\} \to \{\dots, e_{-2}, e_{-1}, 0\}, \\ V_+ &: \{f_0, f_1, f_2, \dots\} \to \{0, f_0, f_1, \dots\}, \end{aligned}$$

on arbitrary elements of the appropriate spaces. The system $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ has the required properties. □

In the other direction, we can ask if a given system has restrictions with special properties. Theorem 4.7 and Corollary 4.8 also use the condition of equality $\varkappa' = \varkappa$ in Theorem 4.2.

Theorem 4.7 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a passive system whose transfer function $\Theta_\Sigma(z)$ belongs to $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$. Define subspaces*

$$\mathcal{X}_1 = \mathcal{X}^o_\Sigma \cap (\mathcal{X}^c_\Sigma)^\perp, \quad \mathcal{X}_2 = \overline{P_{\mathcal{X}^o_\Sigma} \mathcal{X}^c_\Sigma}, \quad \mathcal{X}_3 = (\mathcal{X}^o_\Sigma)^\perp.$$

Then $\mathcal{X}_1$ and $\mathcal{X}_3$ are Hilbert spaces, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$, and the system operator for Σ has the form

$$V_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} & \begin{bmatrix} 0 \\ B_2 \\ B_3 \end{bmatrix} \\ \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix} & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_3 \\ \mathcal{X}_2 \\ \mathcal{X}_1 \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_3 \\ \mathcal{X}_2 \\ \mathcal{X}_1 \\ \mathcal{Y} \end{bmatrix}.$$

Theorem 4.7 is given in [23, Theorem 3.2]. Notice that by Theorem 4.4 and Lemma 2.1, the subspace $\mathcal{X}_2$ in Theorem 4.7 is regular and has negative index $\varkappa$. The conclusion of Theorem 4.7 is that Σ is a dilation of the system

$$\Sigma_{res,1} = (P_{\mathcal{X}_2} A|\mathcal{X}_2, P_{\mathcal{X}_2} B, C|\mathcal{X}_2, D; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; \varkappa),$$

which is called the **first restriction** of Σ. It is shown in [23, p. 204] that $\Sigma_{res,1}$ is minimal, and thus we obtain:

Corollary 4.8 *Every passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ with transfer function $\Theta_\Sigma(z)$ in $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ is the dilation of a minimal passive system, namely, its first restriction.*

5 Schur complements and associated systems

Consider a system Σ with system operator (3.3). Following standard matrix terminology, we call $D - CA^{-1}B$ the Schur complement of A in V_Σ whenever A is invertible, and $A - BD^{-1}C$ the Schur complement of D in V_Σ whenever D is invertible. In this section, we construct systems Σ' and Σ'' associated with these operators.

Theorem 5.1 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a conservative system. Then A is invertible if and only if D is invertible.*

Proof. Since Σ is conservative,

$$A^*A = I - C^*C, \qquad AA^* = I - BB^*,$$
$$D^*D = I - B^*B, \qquad DD^* = I - CC^*.$$

The invertibility of A is thus equivalent to the invertibility of both $I - C^*C$ and $I - BB^*$, which by [1, (1.3.15)] is equivalent to the invertibility of both $I - CC^*$ and $I - B^*B$. Hence A is invertible if and only if D is invertible. □

Theorem 5.2 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a system. Assume that A is invertible and that $\dim \mathcal{X} = n < \infty$. Define a system $\Sigma' = (A', B', C', D'; \mathcal{X}', \mathcal{U}, \mathcal{Y}; \varkappa')$, $\varkappa' = n - \varkappa$, whose state space $\mathcal{X}'$ is the antispace of $\mathcal{X}$ by setting*

$$\begin{cases} A' = A^{-1}, & B' = -A^{-1}B, \\ C' = CA^{-1}, & D' = D - CA^{-1}B. \end{cases} \tag{5.1}$$

Then

(i) *Σ is conservative if and only if Σ' is conservative;*

(ii) *$\Theta_\Sigma(z)$ is analytic at infinity, $\Theta_\Sigma(\infty) = D'$, and the identity $\Theta_{\Sigma'}(z) = \Theta_\Sigma(1/z)$ holds wherever these functions are defined, and in particular it holds in a neighborhood of the origin and in a neighborhood of infinity.*

Proof. (i) Assume that Σ is conservative. For the purpose of computing adjoints, first write (5.1) more correctly as

$$\begin{cases} A' = \sigma A^{-1}\sigma^{-1}, & B' = -\sigma A^{-1}B, \\ C' = CA^{-1}\sigma^{-1}, & D' = D - CA^{-1}B, \end{cases} \tag{5.2}$$

where $\sigma \colon \mathcal{X} \to \mathcal{X}'$ is the identity mapping. Then $\sigma^* = -\sigma^{-1}$. The identities

$$A'^*A' + C'^*C' = I_{\mathcal{X}'}, \quad A'^*B' + C'^*D' = 0, \quad B'^*B' + D'^*D' = I_{\mathcal{U}},$$
$$A'A'^* + B'B'^* = I_{\mathcal{X}'}, \quad A'C'^* + B'D'^* = 0, \quad C'C'^* + D'D'^* = I_{\mathcal{Y}},$$

are checked by straightforward algebraic calculations. For example,

$$\begin{aligned} A'^*A' + C'^*C' &= \big[-\sigma A^{*-1}(-\sigma^{-1})\big]\big[\sigma A^{-1}\sigma^{-1}\big] + \big[-\sigma A^{*-1}C^*\big]\big[CA^{-1}\sigma^{-1}\big] \\ &= \sigma\big[A^{*-1}A^{-1} - A^{*-1}\big(I_{\mathcal{X}} - A^*A\big)A^{-1}\big]\sigma^{-1} = I_{\mathcal{X}'}, \end{aligned}$$

yielding the first identity. The other five identities follow similarly. Hence Σ' is conservative. These steps are reversible.

(ii) The invertibility of A implies that $\Theta_\Sigma(z)$ is analytic at infinity. For any nonzero z the invertibility of $I_{\mathcal{X}'} - zA'$ is equivalent to the invertibility of $I_{\mathcal{X}} - z^{-1}A$. For any such point,

$$\begin{aligned} \Theta_{\Sigma'}(z) &= D' + zC'(I_{\mathcal{X}'} - zA')^{-1}B' \\ &= (D - CA^{-1}B) + zCA^{-1}(I_{\mathcal{X}} - zA^{-1})^{-1}(-A^{-1}B) \\ &= D + C\big[-A^{-1} + zA^{-1}(I_{\mathcal{X}} - zA^{-1})^{-1}(-A^{-1})\big]B \\ &= D + z^{-1}C(I_{\mathcal{X}} - z^{-1}A)^{-1}B, \end{aligned}$$

and (ii) follows. □

Theorem 5.3 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a system. Assume that D is invertible and that $\dim \mathcal{X} = n < \infty$. Define a system $\Sigma'' = (A'', B'', C'', D''; \mathcal{X}'', \mathcal{Y}, \mathcal{U}; \varkappa'')$, $\varkappa'' = n - \varkappa$, whose state space $\mathcal{X}''$ is the antispace of $\mathcal{X}$ by setting*

$$\begin{cases} A'' = A - BD^{-1}C, & B'' = BD^{-1}, \\ C'' = -D^{-1}C, & D'' = D^{-1}. \end{cases} \tag{5.3}$$

Then

(i) *Σ is conservative if and only if Σ'' is conservative;*

(ii) *$\mathcal{X}^c_{\Sigma''} = \mathcal{X}^c_{\Sigma}$ and $\mathcal{X}^o_{\Sigma''} = \mathcal{X}^o_{\Sigma}$;*

(iii) *$\Theta_{\Sigma''}(z) = \Theta_{\Sigma}(z)^{-1}$ at all points where $\Theta_{\Sigma}(z)$ and $\Theta_{\Sigma''}(z)$ are defined, and in particular this identity holds in a neighborhood of the origin.*

Proof. (i) This is verified by calculations as in the proof of Theorem 5.2.

(ii) For any positive integer N,

$$\bigvee_{n=0}^{N} (A'')^n B'' \mathcal{Y} = \bigvee_{n=0}^{N} \left(A - BD^{-1}C\right)^n B\mathcal{U} \subseteq \bigvee_{n=0}^{N} A^n B\mathcal{U},$$

and similarly $\bigvee_0^N (A''^*)^n C''^* \mathcal{U} \subseteq \bigvee_0^N A^{*n} C^* \mathcal{Y}$. Therefore $\mathcal{X}^c_{\Sigma''} \subseteq \mathcal{X}^c_{\Sigma}$ and $\mathcal{X}^o_{\Sigma''} \subseteq \mathcal{X}^o_{\Sigma}$. The reverse inclusions follow since the same construction applied to Σ'' yields the system Σ. This yields (ii).

(iii) For all z for which both functions $\Theta_{\Sigma}(z)$ and $\Theta_{\Sigma''}(z)$ are defined,

$$\begin{aligned}
\Theta_{\Sigma''}(z)\Theta_{\Sigma}(z) &= \left[D'' + zC''(I - zA'')^{-1}B''\right]\left[D + zC(I - zA)^{-1}B\right] \\
&= \left[D^{-1} - zD^{-1}C\big(I - z(A - BD^{-1}C)\big)^{-1}BD^{-1}\right] \cdot \\
&\qquad\qquad \cdot \left[D + zC(I - zA)^{-1}B\right] \\
&= I + zD^{-1}C(I - zA)^{-1}B \\
&\quad - zD^{-1}C\big(I - z(A - BD^{-1}C)\big)^{-1}BD^{-1}D \\
&\quad - z^2D^{-1}C\big(I - z(A - BD^{-1}C)\big)^{-1}BD^{-1}C(I - zA)^{-1}B \\
&= I + zD^{-1}C\big(I - z(A - BD^{-1}C)\big)^{-1} \cdot \\
&\quad \cdot \left[(I - zA + zBD^{-1}C) - (I - zA) - zBD^{-1}C\right](I - zA)^{-1}B \\
&= I.
\end{aligned}$$

Similarly, $\Theta_{\Sigma}(z)\Theta_{\Sigma''}(z) = I$ at all points where the two functions are defined. □

It should be noted that the hypothesis $\dim \mathcal{X} < \infty$ in Theorems 5.2 and 5.3 is only used to assure that the antispace of $\mathcal{X}$ has finite negative index and hence is a Pontryagin space (all of our definitions presume that the state space is a Pontryagin space). The algebraic manipulations in no way make any use of this assumption, however.

6 Conservative dilations and systems with minimal losses

In this section we show that every passive system has a minimal conservative dilation in the sense of Definition 6.1 below, and we determine the form of such a dilation in terms of the construction in Theorem 4.6 (see Theorem 6.4).

Definition 6.1 A conservative dilation $\widetilde{\Sigma}$ of a passive system Σ is said to be a **minimal conservative dilation** if there is no conservative dilation of Σ which is a proper restriction of $\widetilde{\Sigma}$.

Theorem 6.2 *Let $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a minimal conservative dilation of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$. Then the operators $V_+ = \widetilde{A}|\mathcal{D}_+$ and $V_- = \widetilde{A}^*|\mathcal{D}_-$ in the representation $\widetilde{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$ are simple isometries.*

By a **simple isometry** on a Hilbert space $\mathcal{H}$ we mean an isometry $V \in \mathcal{L}(\mathcal{H})$ such that $\cap_{n=0}^{\infty} V^n \mathcal{H} = \{0\}$.

Proof. By the definition of a dilation, $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ have the form

$$\begin{cases} \widetilde{A} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, & \widetilde{B} = \begin{bmatrix} 0 \\ B \\ B_3 \end{bmatrix}, \\ \widetilde{C} = \begin{bmatrix} C_1 & C & 0 \end{bmatrix}, & \widetilde{D} = D. \end{cases} \tag{6.1}$$

Hence $\mathcal{D}_-$ is invariant under $\widetilde{A}^*$ and $\mathcal{D}_+$ is invariant under $\widetilde{A}$. Since $V_{\widetilde{\Sigma}}$ is unitary, the operators $V_+ = \widetilde{A}|\mathcal{D}_+$ and $V_- = \widetilde{A}^*|\mathcal{D}_-$ are isometric. We show that the subspace $\mathcal{L}_- = \bigcap_{n=0}^{\infty} \widetilde{A}^{*n}\mathcal{D}_-$ reduces to $\{0\}$. In fact, since $V_{\widetilde{\Sigma}}$ is a unitary operator from $\widetilde{\mathcal{X}} \oplus \mathcal{U}$ onto $\widetilde{\mathcal{X}} \oplus \mathcal{Y}$ and $\mathcal{D}_- \supseteq \widetilde{A}^*\mathcal{D}_- \supseteq \widetilde{A}^{*2}\mathcal{D}_- \supseteq \dots$, we have $V_{\widetilde{\Sigma}}^*\mathcal{L}_- = \mathcal{L}_-$ and hence $V_{\widetilde{\Sigma}}\mathcal{L}_- = \mathcal{L}_-$. Now we can rewrite (6.1) according to the decomposition $\widetilde{\mathcal{X}} = \mathcal{L}_- \oplus \mathcal{L}_-^{\perp} \oplus \mathcal{X} \oplus \mathcal{D}_+$:

$$\begin{cases} \widetilde{A} = \begin{bmatrix} A_{111} & 0 & 0 & 0 \\ 0 & A_{112} & 0 & 0 \\ 0 & A_{21} & A & 0 \\ 0 & A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ 0 \\ B \\ B_3 \end{bmatrix}, \\ \widetilde{C} = \begin{bmatrix} 0 & C_1|\mathcal{L}_-^{\perp} & C & 0 \end{bmatrix}, \quad \widetilde{D} = D. \end{cases} \tag{6.2}$$

Since $\widetilde{\Sigma}$ is a minimal conservative dilation of Σ, $\mathcal{L}_- = \{0\}$. In a similar way, we obtain $\bigcap_{n=0}^{\infty} \widetilde{A}^n\mathcal{D}_+ = \{0\}$. □

Theorem 6.3 *Any minimal conservative dilation $\widetilde{\Sigma}$ of a passive system Σ has the form constructed in Theorem* 4.6*.*

Proof. Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be any passive system, and let $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a minimal conservative dilation of Σ. Write $\widetilde{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$ for some Hilbert spaces $\mathcal{D}_\pm$, and relative to this decomposition let the operators $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ be given as in (6.1). We first show that

$$\widetilde{A}\widetilde{A}^*\mathcal{D}_- = \mathcal{D}_-, \qquad \widetilde{A}^*\widetilde{A}\mathcal{D}_+ = \mathcal{D}_+. \tag{6.3}$$

Since the system operator $V_{\widetilde{\Sigma}}$ is unitary,

$$\begin{aligned} A_{31}^*A_{33} &= 0, & A_{32}^*A_{33} &= 0, & A_{33}^*A_{33} &= I, \\ A_{11}A_{11}^* &= I, & A_{21}A_{11}^* &= 0, & A_{31}A_{11}^* &= 0. \end{aligned}$$

It follows that for all $d_- \in \mathcal{D}_-$ and $d_+ \in \mathcal{D}_+$,

$$\widetilde{A}\widetilde{A}^* \begin{bmatrix} d_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} A_{11}^*d_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_- \\ 0 \\ 0 \end{bmatrix},$$

$$\widetilde{A}^*\widetilde{A} \begin{bmatrix} 0 \\ 0 \\ d_+ \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{21}^* & A_{31}^* \\ 0 & A^* & A_{32}^* \\ 0 & 0 & A_{33}^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ A_{33}d_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_+ \end{bmatrix},$$

yielding (6.3). By Theorem 6.2 we can assume that $\mathcal{D}_- = \ell_-^2(\mathcal{E})$ and $\mathcal{D}_+ = \ell_+^2(\mathcal{F})$ for some Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ and that $V_+ = \widetilde{A}|\mathcal{D}_+$ and $V_- = \widetilde{A}^*|\mathcal{D}_-$ are the canonical shift operators on these spaces.

Claim 1: $A_{21}V_- = 0$, $A_{31}V_- = 0$, and $C_1V_- = 0$.

For since $\widetilde{A}\widetilde{A}^*\mathcal{D}_- = \mathcal{D}_-$,

$$\begin{bmatrix} V_-^* & 0 & 0 \\ A_{21} & A & 0 \\ A_{31} & A_{32} & V_+ \end{bmatrix} \begin{bmatrix} V_-d_- \\ 0 \\ 0 \end{bmatrix} \in \mathcal{D}_-, \qquad d_- \in \mathcal{D}_-,$$

and hence A_{21} and A_{31} annihilate $V_-\mathcal{D}_-$, that is, $A_{21}V_- = 0$ and $A_{31}V_- = 0$. But then for all $d_- \in \mathcal{D}_-$,

$$V_{\widetilde{\Sigma}} \begin{bmatrix} V_-d_- \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} V_-^* & * & * & * \\ A_{21} & * & * & * \\ A_{31} & * & * & * \\ C_1 & * & * & * \end{bmatrix} \begin{bmatrix} V_-d_- \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_- \\ 0 \\ 0 \\ C_1V_-d_- \end{bmatrix}.$$

Since V_- and $V_{\widetilde{\Sigma}}$ are isometric, $\|C_1V_-d_-\|_{\mathcal{Y}} = 0$. Thus $C_1V_- = 0$.

Claim 2: $A_{31}^*V_+ = 0$, $A_{32}^*V_+ = 0$, and $B_3^*V_+ = 0$.

Since $\widetilde{A}^*\widetilde{A}\mathcal{D}_+ = \mathcal{D}_+$,

$$\begin{bmatrix} V_- & A_{21}^* & A_{31}^* \\ 0 & A^* & A_{32}^* \\ 0 & 0 & V_+^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_+d_+ \end{bmatrix} \in \mathcal{D}_+, \qquad d_+ \in \mathcal{D}_+,$$

and therefore $A_{31}^*V_+ = 0$ and $A_{32}^*V_+ = 0$. Then for all $d_+ \in \mathcal{D}_+$,

$$V_{\widetilde{\Sigma}}^* \begin{bmatrix} 0 \\ 0 \\ V_+d_+ \\ 0 \end{bmatrix} = \begin{bmatrix} * & * & A_{31}^* & * \\ * & * & A_{32}^* & * \\ * & * & V_+^* & * \\ * & * & B_3^* & * \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ V_+d_+ \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_+ \\ B_3^*V_+d_+ \end{bmatrix}.$$

Since V_+ and $V_{\widetilde{\Sigma}}^*$ are isometric, $\|B_3^*V_+d_+\|_{\mathcal{U}} = 0$. Therefore $B_3^*V_+ = 0$.

By the two claims, $A_{21} = E_1P_{-1}$, $A_{31} = Q_0GP_{-1}$, $A_{32} = Q_0F_1$, $B_3 = Q_0F_2$, and $C_1 = E_2P_{-1}$ for some operators

$$E_1 \in \mathcal{L}(\mathcal{E},\mathcal{X}), \quad E_2 \in \mathcal{L}(\mathcal{E},\mathcal{Y}), \quad G \in \mathcal{L}(\mathcal{E},\mathcal{F}), \quad F_1 \in \mathcal{L}(\mathcal{X},\mathcal{F}), \quad F_2 \in \mathcal{L}(\mathcal{U},\mathcal{F}).$$

In other words, $\widetilde{\Sigma}$ has the form (4.3). Straightforward calculations show that the unitarity of $V_{\widetilde{\Sigma}}$ is equivalent to that of (4.2). Therefore the dilation $\widetilde{\Sigma}$ has the form constructed in Theorem 4.6. □

Theorem 6.4 *Let $\Sigma = (A,B,C,D;\mathcal{X},\mathcal{U},\mathcal{Y};\varkappa)$ be a passive system. Assume that a conservative dilation $\widetilde{\Sigma} = (\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D};\widetilde{\mathcal{X}},\mathcal{U},\mathcal{Y};\varkappa)$ for Σ is constructed from a unitary operator*

$$\begin{bmatrix} V_\Sigma & E \\ F & G \end{bmatrix} = \begin{bmatrix} A & B & E_1 \\ C & D & E_2 \\ F_1 & F_2 & G \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{F} \end{bmatrix} \tag{6.4}$$

as in Theorem 4.6. Then $\widetilde{\Sigma}$ is a minimal conservative dilation of Σ if and only if (6.4) is a Julia operator.

P r o o f. In both the necessity and sufficiency parts of the theorem, we use the same notation as in the proof of Theorem 4.6.

Assume first that $\widetilde{\Sigma}$ is a minimal conservative dilation of Σ. We prove that (6.4) is a Julia operator by proving the equivalent relations $\ker E = \{0\}$ and $\ker F^* = \{0\}$. Set $\ker E = \mathcal{E}_\circ$ and $\ker F^* = \mathcal{F}_\circ$, and write $\mathcal{E} = \mathcal{E}_\circ \oplus \mathcal{E}_\bullet$, $\mathcal{F} = \mathcal{F}_\circ \oplus \mathcal{F}_\bullet$. Then the restriction of G to $\mathcal{E}_\circ$ maps $\mathcal{E}_\circ$ isometrically onto $\mathcal{F}_\circ$, and $G\mathcal{E}_\bullet \subseteq \mathcal{F}_\bullet$. For as in [1, p. 20], (6.4) acts as a unitary operator from $\mathcal{E}_\circ$ onto $\mathcal{F}_\circ$, and its adjoint maps $\mathcal{F}_\circ$ onto $\mathcal{E}_\circ$; moreover (6.4) and its adjoint coincide with G and G^* on $\mathcal{E}_\circ$ and $\mathcal{F}_\circ$, respectively, by the definitions of these subspaces. The assertion follows.

Write $\mathcal{D}_- = \mathcal{D}_{-\circ} \oplus \mathcal{D}_{-\bullet}$ and $\mathcal{D}_+ = \mathcal{D}_{+\bullet} \oplus \mathcal{D}_{+\circ}$, where

$$\mathcal{D}_{-\circ} = \ell^2_-(\mathcal{E}_\circ), \qquad \mathcal{D}_{-\bullet} = \ell^2_-(\mathcal{E}_\bullet),$$
$$\mathcal{D}_{+\bullet} = \ell^2_+(\mathcal{E}_\bullet), \qquad \mathcal{D}_{+\circ} = \ell^2_+(\mathcal{E}_\circ).$$

Using the definitions of the operators V_+, V_-, P_{-1}, and Q_0 we easily obtain that $V_- = V_-|\mathcal{D}_{-\circ} \oplus V_-|\mathcal{D}_{-\bullet}$, $V_+ = V_+|\mathcal{D}_{-\bullet} \oplus V_+|\mathcal{D}_{-\circ}$, and

$$E_1P_{-1}\mathcal{D}_{-\circ} = \{0\}, \qquad F_1^*Q_0^*\mathcal{D}_{+\circ} = \{0\},$$
$$Q_0GP_{-1}\mathcal{D}_{-\circ} \subseteq \mathcal{D}_{+\circ}, \qquad Q_0GP_{-1}\mathcal{D}_{-\bullet} \subseteq \mathcal{D}_{+\bullet},$$
$$F_2^*Q_0^*\mathcal{D}_{+\circ} = \{0\}, \qquad E_2P_{-1}\mathcal{D}_{-\circ} = \{0\}.$$

Hence by (4.3), relative to the decomposition $\widetilde{\mathcal{X}} = \mathcal{D}_{-\circ} \oplus \mathcal{D}_{-\bullet} \oplus \mathcal{X} \oplus \mathcal{D}_{+\bullet} \oplus \mathcal{D}_{+\circ}$,

$$\widetilde{A} = \begin{bmatrix} V_-^*|\mathcal{D}_{-\circ} & 0 & 0 & 0 & 0 \\ 0 & V_-^*|\mathcal{D}_{-\bullet} & 0 & 0 & 0 \\ 0 & * & A & 0 & 0 \\ 0 & * & * & V_+|\mathcal{D}_{+\bullet} & 0 \\ * & 0 & 0 & 0 & V_+|\mathcal{D}_{+\circ} \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} 0 \\ 0 \\ B \\ * \\ 0 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} 0 & * & C & 0 & 0 \end{bmatrix}, \qquad \widetilde{D} = D.$$

It is clear now that $\widetilde{\Sigma}$ is a dilation of a system

$$\widehat{\Sigma} = \left(\begin{bmatrix} * & 0 & 0 \\ * & A & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 \\ B \\ * \end{bmatrix}, \begin{bmatrix} * & C & 0 \end{bmatrix}, D; \mathcal{D}_{-\bullet} \oplus \mathcal{X} \oplus \mathcal{D}_{+\bullet}, \mathcal{U}, \mathcal{Y}; \varkappa \right),$$

which is a conservative dilation of Σ. Since $\widetilde{\Sigma}$ is a minimal conservative dilation of Σ, it follows that $\mathcal{E}_\circ = \{0\}$ and $\mathcal{F}_\circ = \{0\}$, and thus (6.4) is a Julia operator.

Conversely, assume that (6.4) is a Julia operator. Consider any conservative dilation $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ of Σ which is a restriction of $\widetilde{\Sigma}$. We show that $\widehat{\Sigma} = \widetilde{\Sigma}$. By the definition of dilations and restrictions,

$$\widehat{\mathcal{X}} = \mathcal{D}'_- \oplus \mathcal{X} \oplus \mathcal{D}'_+, \qquad \widetilde{\mathcal{X}} = \mathcal{D}''_- \oplus \widehat{\mathcal{X}} \oplus \mathcal{D}''_+,$$

where $\mathcal{D}'_\pm$ and $\mathcal{D}''_\pm$ are Hilbert spaces. We show that the resulting decomposition

$$\widetilde{\mathcal{X}} = \left(\mathcal{D}''_- \oplus \mathcal{D}'_-\right) \oplus \mathcal{X} \oplus \left(\mathcal{D}'_+ \oplus \mathcal{D}''_+\right). \tag{6.5}$$

coincides with the decomposition $\widetilde{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$ in the proof of Theorem 4.6. Relative to (6.5), $V_{\widetilde{\Sigma}}$ has the form

$$V_{\widetilde{\Sigma}} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ A_{31} & A_{32} & A & 0 & 0 & B \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & B_4 \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & B_5 \\ C_1 & C_2 & C & 0 & 0 & D \end{bmatrix}. \tag{6.6}$$

Since the system operator

$$V_{\widehat{\Sigma}} = \begin{bmatrix} A_{22} & 0 & 0 & 0 \\ A_{32} & A & 0 & B \\ A_{42} & A_{43} & A_{44} & B_4 \\ C_2 & C & 0 & D \end{bmatrix}$$

is also unitary $A_{21}, A_{31}, A_{41}, A_{52}, A_{53}, A_{54}, C_1, B_5$ are zero operators. So $\widetilde{A}$ has the form

$$\widetilde{A} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 \\ 0 & A_{32} & A & 0 & 0 \\ 0 & A_{42} & A_{43} & A_{44} & 0 \\ A_{51} & 0 & 0 & 0 & A_{55} \end{bmatrix}$$

relative to the decomposition (6.5). The subspace $\mathcal{D}'_+ \oplus \mathcal{D}''_+$ is an $\widetilde{A}$-invariant subspace of $\widetilde{\mathcal{X}} \ominus \mathcal{X}$, and it is contained in the kernel of $\widetilde{C} = P_{\mathcal{Y}} V_{\widetilde{\Sigma}} | \widetilde{X}$. Every vector $\widetilde{x} \in \mathcal{D}'_+ \oplus \mathcal{D}''_+$ has the form

$$\{\ldots,\ e_{-2},\ e_{-1},\ 0,\ f_0, f_1,\ f_2,\ \ldots\} \in \ell^2_-(\mathcal{E}) \oplus \mathcal{X} \oplus \ell^2_+(\mathcal{F}).$$

Since $\widetilde{A}\widetilde{x} \in \mathcal{D}'_+ \oplus \mathcal{D}''_+$, $P_{\mathcal{X}} \widetilde{A}\widetilde{x} = 0$, and therefore by the form of $\widetilde{A}$ in (4.3), $E_1 e_{-1} = 0$. Since also $\widetilde{C}\widetilde{x} = 0$, by the form of $\widetilde{C}$ in (4.3), $E_2 e_{-1} = 0$. Therefore $e_{-1} \in \ker E$ and $e_{-1} = 0$ since (6.4) is a Julia operator. Thus $e_{-n} = 0$ for every $n \geq 1$, since $\mathcal{D}'_+ \oplus \mathcal{D}''_+$ is $\widetilde{A}$-invariant. It follows that $\mathcal{D}'_+ \oplus \mathcal{D}''_+ \subseteq \mathcal{D}_+$. Analogously $\mathcal{D}''_- \oplus \mathcal{D}'_- \subseteq \mathcal{D}_-$. Since $\mathcal{D}''_- \oplus \mathcal{D}'_- \oplus \mathcal{D}'_+ \oplus \mathcal{D}''_+ = \mathcal{D}_- \oplus \mathcal{D}_+$, we obtain $\mathcal{D}'_+ \oplus \mathcal{D}''_+ = \mathcal{D}_+$ and $\mathcal{D}''_- \oplus \mathcal{D}'_- = \mathcal{D}_-$.

By the construction of a unitary dilation in Theorem 4.6, the operator

$$V_- = \widetilde{A}^* | \mathcal{D}_- = \begin{bmatrix} A^*_{11} & 0 \\ 0 & A^*_{22} \end{bmatrix}$$

is a simple isometry on $\mathcal{D}_- = \ell^2_-(\mathcal{E})$ with the reducing subspaces $\mathcal{D}''_-$ and $\mathcal{D}'_-$. Hence there is a decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ such that $\mathcal{D}''_- = \ell^2_-(\mathcal{E}_1)$ and $\mathcal{D}'_- = \ell^2_-(\mathcal{E}_2)$. Recalling that the operators A_{31} and C_1 in (6.6) are zero, we see that $P_{\mathcal{X}} \widetilde{A} \mathcal{D}''_- = \{0\}$ and $\widetilde{C}\mathcal{D}''_- = \{0\}$; hence by (4.3), $E_1 \mathcal{E}_1 = \{0\}$ and $E_2 \mathcal{E}_1 = \{0\}$. Thus $E\mathcal{E}_1 = \{0\}$. Since (6.4) is a Julia operator, $\mathcal{E}_1 = \{0\}$ and hence $\mathcal{D}''_- = \{0\}$. In a similar way, $\mathcal{D}''_+ = \{0\}$. Therefore $\widehat{\Sigma} = \widetilde{\Sigma}$, and we have shown that $\widetilde{\Sigma}$ is a minimal conservative dilation of Σ. □

Every passive system has a conservative dilation, but not every passive system has a *simple* conservative dilation. Below we give an example with $\varkappa = 0$ of a minimal passive system which does not have a simple conservative dilation.

Definition 6.5 We say that a passive system Σ has **minimal losses** if it has a simple conservative dilation $\widetilde{\Sigma}$.

If a simple conservative dilation $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ of a passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ exists, then its transfer function $\Theta_{\widetilde{\Sigma}}(z)$ belongs to $\mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ by Theorem 4.1(i). By Theorem 4.1(ii), $\widetilde{\Sigma}$ is essentially unique because any two such dilations have the same transfer function $\Theta_\Sigma(z) = \Theta_{\widetilde{\Sigma}}(z)$.

Theorem 6.6 *A simple conservative dilation* $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$ *of a given passive system* $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ *is a minimal conservative dilation.*

Proof. Assume that $\widetilde{\Sigma}$ is a dilation of a conservative system $\widehat{\Sigma} = (\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}; \widehat{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; \varkappa)$, which is in turn a dilation of Σ. Then $\widehat{\mathcal{X}} = \mathcal{D}_- \oplus \mathcal{X} \oplus \mathcal{D}_+$ and $\widetilde{\mathcal{X}} = \widehat{\mathcal{D}}_- \oplus \widehat{\mathcal{X}} \oplus \widehat{\mathcal{D}}_+$, and relative to the latter decomposition the operators $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ have the form

$$\begin{cases} \widetilde{A} = \begin{bmatrix} \widehat{A}_{11} & 0 & 0 \\ \widehat{A}_{21} & \widehat{A} & 0 \\ \widehat{A}_{31} & \widehat{A}_{32} & \widehat{A}_{33} \end{bmatrix}, & \widetilde{B} = \begin{bmatrix} 0 \\ \widehat{B} \\ \widehat{B}_3 \end{bmatrix}, \\ \widetilde{C} = \begin{bmatrix} \widehat{C}_1 & \widehat{C} & 0 \end{bmatrix}, & \widetilde{D} = \widehat{D}. \end{cases} \tag{6.7}$$

Since both operators

$$M_{\widetilde{\Sigma}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \qquad \text{and} \qquad M_{\widehat{\Sigma}} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix}$$

are unitary, $\widehat{A}_{21} = 0$, $\widehat{A}_{31} = 0$, $\widehat{A}_{32} = 0$, $\widehat{C}_1 = 0$, $\widehat{B}_3 = 0$. Then $\bigvee_{n\geq 0}\widetilde{A}^n\widetilde{B}\mathcal{U} \subseteq \widehat{\mathcal{X}}$ and $\bigvee_{n\geq 0}\widetilde{A}^{*n}\widetilde{C}^*\mathcal{Y} \subseteq \widehat{\mathcal{X}}$, so $\widetilde{\Sigma}$ is simple only if $\widehat{D}_- = \{0\}$ and $\widehat{D}_+ = \{0\}$. □

Theorems 6.3, 6.4 and 6.6 yield the following corollary.

Corollary 6.7 *Let Σ be an arbitrary passive system with minimal losses. Then a simple conservative dilation $\widetilde{\Sigma}$ of Σ has the form of the dilation constructed in Theorem* 4.6 *with Julia operator* (4.1).

Example of a minimal passive system without minimal losses. Let $\mathcal{X} = \mathcal{U} = \mathcal{Y} = \mathbf{C}$ in the Euclidean metric. Choose a system $\sigma = (a, b, c, d; \mathcal{X}, \mathcal{U}, \mathcal{Y}; 0)$ whose system operator

$$V_\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a contraction such that $abc \neq 0$, $|a| < 1$, and $I - V_\sigma^* V_\sigma$ and $I - V_\sigma V_\sigma^*$ are invertible. Then σ is a minimal passive system. We show that σ does not have a simple conservative dilation. Argue by contradiction, assuming that σ has a simple conservative dilation $\widetilde{\Sigma} = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \widetilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y}; 0)$. By Corollary 6.7, $\widetilde{\Sigma}$ is constructed from a Julia operator (6.4) for V_σ. Since a Julia operator is essentially unique, we can choose $\mathcal{E} = \mathcal{F} = \mathbf{C}^2$, $E = (I - V_\sigma V_\sigma^*)^{1/2}$, $F = (I - V_\sigma^* V_\sigma)^{1/2}$, and $G = -V_\sigma^*$. We construct a nonzero vector

$$\widetilde{x} = \begin{bmatrix} \{\dots, e_{-2}, e_{-1}\} \\ x \\ \{f_0, f_1, \dots\} \end{bmatrix} \tag{6.8}$$

in $\widetilde{\mathcal{X}} = \ell^2_-(\mathcal{E}) \oplus \mathcal{X} \oplus \ell^2_+(\mathfrak{F})$ which is orthogonal to all vectors

$$\widetilde{A}^n\widetilde{B}u = \begin{bmatrix} 0 \\ a^n bu \\ \{F_1 a^{n-1}bu, \dots, F_1 bu, F_2 u, 0, 0, \dots\} \end{bmatrix} \tag{6.9}$$

and

$$\widetilde{A}^{*n}\widetilde{C}^* y = \begin{bmatrix} \{\dots, 0, 0, E_2^* y, E_1^*\bar{c}y, \dots, E_1^*\bar{a}^{n-1}\bar{c}y\} \\ \bar{a}^n\bar{c}y \\ 0 \end{bmatrix}, \tag{6.10}$$

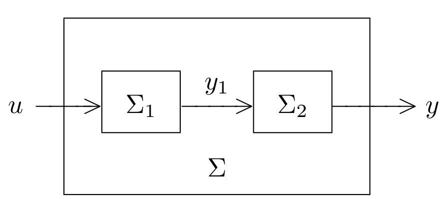
where $u \in \mathcal{U}$, $y \in \mathcal{Y}$, and $n \geq 0$. Since $\dim E_1^*\mathcal{X} = \dim F_1\mathcal{X} = 1$, there exist unit vectors $\varphi \in \mathcal{E} \ominus E_1^*\mathcal{X}$ and $\psi \in \mathcal{F} \ominus F_1\mathcal{X}$. To construct $\widetilde{x}$, we choose $x = 1$ in (6.8) and seek square summable scalars $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ such that the vectors $e_{-n-1} = \alpha_n\varphi$ and $f_n = \beta_n\psi$ meet the required orthogonality conditions. It is sufficient to take $u = 1$ and $y = 1$ in (6.9) and (6.10). Without difficulty, we find that such scalars are uniquely determined by the orthogonality conditions and given by

$$\alpha_n = -\frac{a^n c}{\langle\varphi, E_2^* 1\rangle}, \quad \text{and} \quad \beta_n = -\frac{\bar{a}^n\bar{b}}{\langle\psi, F_2 1\rangle}, \qquad n \geq 0.$$

The denominators here do not vanish under our assumptions, and the sequences are square summable because $|a| < 1$. Therefore $\widetilde{\Sigma}$ is not simple, and hence σ is without minimal losses.

7 Cascade synthesis and Kreĭn-Langer factorizations

The cascade synthesis of the two systems $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \varkappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \varkappa_2)$ is a system $\Sigma = \Sigma_2 \circ \Sigma_1$ which uses the output



from Σ_1 as input for Σ_2: for all $n \geq 0$,

$$\Sigma_1 : \quad \begin{cases} x_1(n+1) = A_1 x_1(n) + B_1 u(n), \\ \quad\; y_1(n) = C_1 x_1(n) + D_1 u(n), \end{cases}$$

$$\Sigma_2 : \quad \begin{cases} x_2(n+1) = A_2 x_2(n) + B_2 u(n), \\ \quad\;\; y(n) = C_2 x_2(n) + D_2 u(n), \end{cases}$$

and hence

$$\Sigma : \quad \begin{cases} \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(n), \\ \qquad\qquad y(n) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + D_2 D_1 u(n). \end{cases}$$

Therefore $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$, where $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\varkappa = \varkappa_1 + \varkappa_2$, and

$$\begin{cases} A = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}, & B = \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}, \\ C = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}, & D = D_2 D_1. \end{cases} \tag{7.1}$$

In particular, $A\mathcal{X}_2 \subseteq \mathcal{X}_2$. We note the following elementary properties:

(i) If $\Sigma = \Sigma_2 \circ \Sigma_1$, then $\Theta_\Sigma(z) = \Theta_{\Sigma_2}(z)\Theta_{\Sigma_1}(z)$.

(ii) If $\Sigma = \Sigma_2 \circ \Sigma_1$, then $\Sigma^* = \Sigma_1^* \circ \Sigma_2^*$.

(iii) Given a third system Σ_3, $\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1) = (\Sigma_3 \circ \Sigma_2) \circ \Sigma_1$ provided that the operations are meaningful.

(iv) If $\Sigma = \Sigma_2 \circ \Sigma_1$ and Σ'', Σ_1'', and Σ_2'' are defined as in Theorem 5.3, then $\Sigma'' = \Sigma_1'' \circ \Sigma_2''$.

Theorem 7.1 *If $\Sigma = \Sigma_2 \circ \Sigma_1$ where Σ_1 and Σ_2 are passive (conservative) systems, then Σ is a passive (conservative) system.*

P r o o f. Use the identity

$$\begin{bmatrix} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ D_2 C_1 & C_2 & D_2 D_1 \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}_1} & 0 & 0 \\ 0 & A_2 & B_2 \\ 0 & C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I_{\mathcal{X}_2} & 0 \\ C_1 & 0 & D_1 \end{bmatrix}$$

and the fact that the product of two contraction (unitary) operators is a contraction (unitary) operator. □

We shall prove:

Theorem 7.2 *If $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ is a simple conservative system, then*

(i) $\Sigma = \Sigma_2 \circ \Sigma_1$, *where $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}; 0)$ is a simple conservative system with a Hilbert state space, and $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{U}; \varkappa)$ is a simple conservative system whose state space is a $\varkappa$-dimensional antispace of a Hilbert space;*

(ii) $\Sigma = \Sigma_2' \circ \Sigma_1'$, *where $\Sigma_2' = (A_2', B_2', C_2', D_2'; \mathcal{X}_2', \mathcal{Y}, \mathcal{Y}; \varkappa)$ is a simple conservative system whose state space is a $\varkappa$-dimensional antispace of a Hilbert space, and $\Sigma_1' = (A_1', B_1', C_1', D_1'; \mathcal{X}_1', \mathcal{U}, \mathcal{Y}; 0)$ is a simple conservative system with a Hilbert state space.*

This result will be established in more precise form in Theorems 7.3 and 7.6.

Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces. A **Blaschke-Potapov product** of degree $\varkappa$ with values in $\mathcal{L}(\mathcal{U})$ is a product of $\varkappa$ factors of the form

$$I - P + \rho\,\frac{z-w}{1-\bar{w}z}\,P,$$

where $P \in \mathcal{L}(\mathcal{U})$ is a rank-one projection operator, $w \in \mathbf{D}$, and $|\rho| = 1$. A **right Kreĭn-Langer factorization** of a function $S(z)$ in $\mathbf{S}_\varkappa(\mathcal{U},\mathcal{Y})$ is a representation

$$S(z) = S_r(z)B_r(z)^{-1}, \tag{7.2}$$

where $S_r(z)$ belongs to $\mathbf{S}(\mathcal{U},\mathcal{Y})$ and $B_r(z)$ is a Blaschke-Potapov product of degree $\varkappa$ with values in $\mathcal{L}(\mathcal{U})$ which is invertible at the origin; the factorization is **coprime** in the sense that if $S_r(w)u = 0$ and $B_r(w)u = 0$ for some w in $\mathbf{D}$ and $u \in \mathcal{U}$, then $u = 0$. A **left Kreĭn-Langer factorization** of $S(z)$ is a representation

$$S(z) = B_\ell(z)^{-1}S_\ell(z), \tag{7.3}$$

where $S_\ell(z)$ belongs to $\mathbf{S}(\mathcal{U},\mathcal{Y})$ and $B_\ell(z)$ is a Blaschke-Potapov product of degree $\varkappa$ with values in $\mathcal{L}(\mathcal{Y})$ which is invertible at the origin; the factorization is **coprime** in the sense that if $S_\ell(w)^*y = 0$ and $B_\ell(w)^*y = 0$ for some w in $\mathbf{D}$ and $y \in \mathcal{Y}$, then $y = 0$. Left and right Kreĭn-Langer factorizations are essentially unique. Conversely, an arbitrary product (7.2) in which $S_r(z)$ belongs to $\mathbf{S}(\mathcal{U},\mathcal{Y})$ and $B_r(z)$ is a Blaschke-Potapov product of degree $\varkappa$ with values in $\mathcal{L}(\mathcal{U})$ which is invertible at the origin represents a function in $\mathbf{S}_{\varkappa'}(\mathcal{U},\mathcal{Y})$ for some $\varkappa' \le \varkappa$, and $\varkappa' = \varkappa$ if the factorization is coprime. A parallel assertion holds for products (7.3). These results are due to Kreĭn and Langer [19]; an account is given in [1, Section 4.2].

Theorem 7.3 *Suppose that the function $\Theta(z) \in \mathbf{S}_\varkappa(\mathcal{U},\mathcal{Y})$ has the right Kreĭn-Langer factorization $\Theta(z) = \Theta_r(z)b_r(z)^{-1}$. Let*

$$\begin{aligned}\Sigma_-^r &= (A_-^r, B_-^r, C_-^r, D_-^r; \mathcal{X}_-^r, \mathcal{U}, \mathcal{U}; \varkappa),\\ \Sigma_+^r &= (A_+^r, B_+^r, C_+^r, D_+^r; \mathcal{X}_+^r, \mathcal{U}, \mathcal{Y}; 0),\end{aligned}$$

be simple conservative systems such that $\Theta_{\Sigma_+^r}(z) = \Theta_r(z)$ and $\Theta_{\Sigma_-^r}(z) = b_r(z)^{-1}$. Then $\Sigma^r = \Sigma_+^r \circ \Sigma_-^r$ is a simple conservative system with transfer function $\Theta(z)$.

Lemma 7.4 *Let $\Sigma = \Sigma_2 \circ \Sigma_1$, where $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \varkappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \varkappa_2)$. Then $(\mathcal{X}_\Sigma^c \vee \mathcal{X}_\Sigma^o)^\perp$ consists of all $x = x_1 \oplus x_2$ in $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ such that*

$$\begin{cases} \Theta_{\Sigma_2}(z)C_1(I - zA_1)^{-1}x_1 = -C_2(I - zA_2)^{-1}x_2,\\ \Theta_{\Sigma_1}(\bar{z})^*B_2^*(I - zA_2^*)^{-1}x_2 = -B_1^*(I - zA_1^*)^{-1}x_1,\end{cases} \tag{7.4}$$

in a neighborhood of the origin. Hence if the only vectors $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ which satisfy (7.4) *are $x_1 = 0$ and $x_2 = 0$, then $\Sigma = \Sigma_2 \circ \Sigma_1$ is simple.*

Lemma 7.5 *Let $\Theta_1(z) = b(z)^{-1}$, where $b(z)$ is a Blaschke product of degree $\varkappa$ which has values in $\mathcal{L}(\mathcal{U})$ for some Hilbert space $\mathcal{U}$ and which is invertible at the origin. Then $\Theta_1(z)$ is the transfer function of a simple conservative system $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{U}; \varkappa)$ for which the state space $\mathcal{X}_1$ is the $\varkappa$-dimensional Pontryagin space $\mathcal{H}(\Theta_1)$ having reproducing kernel $[I - \Theta_1(z)\Theta_1(w)^*]/(1 - z\bar{w})$, and*

$$\begin{cases} A_1\colon h(z) \to \dfrac{h(z) - h(0)}{z}, & B_1\colon u \to \dfrac{\Theta_1(z) - \Theta_1(0)}{z}\,u,\\ C_1\colon h(z) \to h(0), & D_1\colon u \to \Theta_1(0)u,\end{cases} \tag{7.5}$$

for all $h(z)$ in $\mathcal{H}(\Theta_1)$ and all u in $\mathcal{U}$. The space $\mathcal{X}_1$ is the antispace of a Hilbert space, and the identity

$$\left\langle \frac{h_1(z) - h_1(0)}{z}, \frac{h_1(z) - h_1(0)}{z} \right\rangle_{\mathcal{H}(\Theta_1)} = \langle h_1(z), h_1(z)\rangle_{\mathcal{H}(\Theta_1)} - \langle h_1(0), h_1(0)\rangle_{\mathcal{U}} \tag{7.6}$$

holds for all elements $h_1(z)$ of the space.

Proof of Lemma 7.4. Write $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$, where $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\varkappa = \varkappa_1 + \varkappa_2$, and A, B, C, D are given by (7.1). It is easy to see that the definitions of the subspaces $\mathcal{X}_\Sigma^c$ and $\mathcal{X}_\Sigma^o$ in Section 3 can be rewritten in the form

$$\mathcal{X}_\Sigma^c = \bigvee\nolimits_{z \in \Omega} (I - zA)^{-1} B \mathcal{U} \quad \text{and} \quad \mathcal{X}_\Sigma^o = \bigvee\nolimits_{z \in \Omega} (I - zA^*)^{-1} C^* \mathcal{Y},$$

where Ω is any small neighborhood of the origin. Therefore $x = x_1 \oplus x_2$ is orthogonal to $\mathcal{X}_\Sigma^c \vee \mathcal{X}_\Sigma^o$ if and only if for all $z \in \Omega$, $B^*(I - zA^*)^{-1} x = 0$ and $C(I - zA)^{-1} x = 0$, or equivalently (by (7.1))

$$\begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} (I - zA_1)^{-1} & 0 \\ z(I - zA_2)^{-1} B_2 C_1 (I - zA_1)^{-1} & (I - zA_2)^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

$$\begin{bmatrix} B_1^* & D_1^* B_2^* \end{bmatrix} \begin{bmatrix} (I - zA_1^*)^{-1} & z(I - zA_1^*)^{-1} C_1^* B_2^* (I - zA_2^*)^{-1} \\ 0 & (I - zA_2^*)^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

Expanding these identities and simplifying by means of the relations

$$\Theta_{\Sigma_2}(z) = D_2 + zC_2(I - zA_2)^{-1} B_2,$$
$$\Theta_{\Sigma_1}(\bar{z})^* = D_1^* + zB_1^*(I - zA_1^*)^{-1} C_1^*,$$

we obtain (7.4). □

Proof of Lemma 7.5. This follows from [1, Theorem A3] and the last statement in [1, Theorem 3.2.5]. □

Proof of Theorem 7.3. Set $\mathcal{Y}_1 = \mathcal{U}$, and write

$$\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \varkappa_1) = \Sigma_-^r, \qquad \varkappa_1 = \varkappa,$$
$$\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \varkappa_2) = \Sigma_+^r, \qquad \varkappa_2 = 0.$$

By Theorem 7.1, $\Sigma^r = \Sigma_+^r \circ \Sigma_-^r = \Sigma_2 \circ \Sigma_1$ is conservative. It has transfer function $\Theta_{\Sigma^r}(z) = \Theta_{\Sigma_2}(z)\Theta_{\Sigma_1}(z) = \Theta_r(z) b_r(z)^{-1} = \Theta(z)$ at all points in $\mathbf{D}$ where the functions are defined.

The main problem is to show that Σ^r is simple, and for this we use Lemma 7.4. Let $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ satisfy (7.4). Since all simple conservative realizations are equivalent by Theorem 4.1, without loss of generality we can assume that Σ_1 is given as in Lemma 7.5 with $\Theta_1(z) = b_r(z)^{-1}$. Thus $\mathcal{X}_1 = \mathcal{H}(b_r^{-1})$ is the antispace of a $\varkappa$-dimensional Hilbert space, and for every $h(z)$ in the space,

$$C_1(I - wA_1)^{-1} \colon h(z) \to h(w) \tag{7.7}$$

for all w in a neighborhood of the origin [1, p. 89]. We remark also that since $\varkappa_2 = 0$, A_2 is a contraction operator on the Hilbert space $\mathcal{X}_2$ and therefore $(I - zA_2)^{-1}$ exists for z in the unit disk $\mathbf{D}$.

We first show that $x_1 = 0$. Argue by contradiction, assuming $x_1 \neq 0$. Let $\mathcal{M}$ be the subspace of elements $h(z)$ in $\mathcal{H}(b_r^{-1})$ such that $\Theta_r(z)h(z) \in \mathrm{Hol}(\mathbf{D})$, that is, apart from removable singularities $\Theta_r(z)h(z)$ is holomorphic on $\mathbf{D}$. Then $x_1(z)$ belongs to $\mathcal{M}$ by (7.7) and the first relation in (7.4), and so $\mathcal{M} \neq \{0\}$. The subspace $\mathcal{M}$ is invariant under A_1, since if $\Theta_r(z)h(z) \in \mathrm{Hol}(\mathbf{D})$, then

$$\Theta_r(z)\, \frac{h(z) - h(0)}{z} = \frac{\Theta_r(z)h(z) - \Theta_r(z)h(0)}{z} \in \mathrm{Hol}(\mathbf{D}).$$

Since $\mathcal{M}$ is finite dimensional, $A_1|\mathcal{M}$ has an eigenvalue β_1. Thus $\mathcal{M}$ contains an element of the form

$$h_1(z) = \frac{u}{1 - \beta_1 z}, \qquad \|u\|_{\mathcal{U}} = 1.$$

The identity (7.6), together with the fact that $\mathcal{X}_1$ is the antispace of a Hilbert space, implies that $|\beta_1| > 1$. Set $\alpha_1 = 1/\beta_1$. By the definition of $\mathcal{M}$, $\Theta_r(z)h_1(z) \in \mathrm{Hol}(\mathbf{D})$, and this is only possible if $\Theta_r(\alpha_1)u = 0$. Writing

$$b_1(z) = I - P_1 + \frac{z - \alpha_1}{1 - \bar{\alpha}_1 z}\, P_1, \qquad P_1 = \langle \cdot, u \rangle_{\mathcal{U}} u,$$

we therefore have $\Theta_r(z)b_1(z)^{-1} \in \mathbf{S}(\mathcal{U},\mathcal{U})$. An argument in [1, p. 142] shows that

$$b_r(z)^{-1} = b_1(z)^{-1}S_1(z),$$

where $S_1(z)$ belongs to $\mathbf{S}_{\varkappa-1}(\mathcal{U},\mathcal{U})$. But then [1, Theorem 4.1.1]

$$\Theta(z) = \big[\Theta_r(z)b_1(z)^{-1}\big]S_1(z) \in \mathbf{S}_{\varkappa'}(\mathcal{U},\mathcal{Y}), \qquad \varkappa' \le \varkappa - 1,$$

which contradicts our assumption that $\Theta(z) \in \mathbf{S}_{\varkappa}(\mathcal{U},\mathcal{Y})$. Therefore $\mathcal{M} = \{0\}$ and hence $x_1 = 0$.

Returning to (7.4), we see that $C_2(I - zA_2)^{-1}x_2 = 0$ and $B_2^*(I - zA_2^*)^{-1}x_2 = 0$ in a neighborhood of the origin. Since Σ_2 is simple, it follows that $x_2 = 0$. Hence by Lemma 7.4, Σ is simple, as was to be shown. □

A parallel result holds for left Kreĭn-Langer factorizations.

Theorem 7.6 *Suppose that the function $\Theta(z) \in \mathbf{S}_{\varkappa}(\mathcal{U},\mathcal{Y})$ has the left Kreĭn-Langer factorization $\Theta(z) = b_\ell(z)^{-1}\Theta_\ell(z)$. Let*

$$\begin{aligned}\Sigma_+^\ell &= (A_+^\ell, B_+^\ell, C_+^\ell, D_+^\ell; \mathcal{X}_+^\ell, \mathcal{U}, \mathcal{Y}; 0),\\ \Sigma_-^\ell &= (A_-^\ell, B_-^\ell, C_-^\ell, D_-^\ell; \mathcal{X}_-^\ell, \mathcal{Y}, \mathcal{Y}; \varkappa),\end{aligned}$$

be simple conservative systems such that $\Theta_{\Sigma_+^\ell}(z) = \Theta_\ell(z)$ and $\Theta_{\Sigma_-^\ell}(z) = b_\ell(z)^{-1}$. Then $\Sigma^\ell = \Sigma_-^\ell \circ \Sigma_+^\ell$ is a simple conservative system with transfer function $\Theta(z)$.

Proof. Write $\widetilde{F}(z) = F(\bar{z})^*$ for any operator-valued function $F(z)$. Then $\widetilde{\Theta}(z) \in \mathbf{S}_{\varkappa}(\mathcal{Y},\mathcal{U})$ (for example, see [1, p. 68]), and we can define a right Kreĭn-Langer factorization of $\widetilde{\Theta}(z)$ by setting

$$\widetilde{\Theta}(z) = \Theta_r(z)b_r(z)^{-1}, \qquad \Theta_r(z) = \widetilde{\Theta}_\ell(z), \qquad b_r(z) = \widetilde{b}_\ell(z).$$

Then $\Sigma_+^{\ell\,*}$ and $\Sigma_-^{\ell\,*}$ are simple conservative systems with transfer functions

$$\begin{aligned}\Theta_{\Sigma_+^{\ell\,*}}(z) &= \widetilde{\Theta}_{\Sigma_+^\ell}(z) = \widetilde{\Theta}_\ell(z) = \Theta_r(z),\\ \Theta_{\Sigma_-^{\ell\,*}}(z) &= \widetilde{\Theta}_{\Sigma_-^\ell}(z) = \widetilde{b}_\ell(z)^{-1} = b_r(z)^{-1}.\end{aligned}$$

By Theorem 7.3, $\Sigma_+^{\ell*} \circ \Sigma_-^{\ell*}$ is a simple conservative system such that $\Theta_{\Sigma_+^{\ell\,*}\circ\Sigma_-^{\ell\,*}}(z) = \Theta_r(z)b_r(z)^{-1} = \widetilde{\Theta}(z)$. It follows that $\Sigma^\ell = \Sigma_-^\ell \circ \Sigma_+^\ell$ is a simple conservative system whose transfer function is given by $\Theta_\Sigma(z) = \Theta(z)$. □

Every contraction operator on a Pontryagin space admits semi-definite invariant subspaces (see Section 2). We obtain a stronger conclusion when the contraction operator is the main operator of a simple passive system.

Theorem 7.7 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be any simple passive system.*

(i) *The state space $\mathcal{X}$ has a unique fundamental decomposition $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ such that $A\mathcal{X}_+ \subseteq \mathcal{X}_+$.*

(ii) *The state space $\mathcal{X}$ has a unique fundamental decomposition $\mathcal{X} = \mathcal{X}_+' \oplus \mathcal{X}_-'$ such that $A\mathcal{X}_-' \subseteq \mathcal{X}_-'$.*

Remark. The statement of Theorem 7.7 for a simple conservative system is a consequence of [18, Lemma 11.5, p. 82], as described in Corollary 2.3 and the discussion in Section 4 of the paper [15]. For a simple passive system the statement of Theorem 7.7 then follows from this fact and an embedding into a simple conservative system as in our proof below. The authors learned of this connection from the referee's report, and they thank the referee for his remark.

Lemma 7.8 *The main operator of a simple passive system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ has no eigenvalue of modulus one.*

Proof of Lemma 7.8. By embedding Σ in a simple conservative system which has the same state space and main operator (see Theorem 4.3), without loss of generality we can assume that Σ is conservative, that is, the system operator V_Σ is unitary.

Suppose $x \in \mathcal{X}$, $|\lambda| = 1$, and $Ax = \lambda x$. Then

$$V_\Sigma \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda x \\ Cx \end{bmatrix}.$$

Since V_Σ is unitary and $|\lambda| = 1$,

$$\langle x, x \rangle_{\mathcal{X}} = |\lambda|^2 \langle x, x \rangle_{\mathcal{X}} + \|Cx\|^2_{\mathcal{Y}} = \langle x, x \rangle_{\mathcal{X}} + \|Cx\|^2_{\mathcal{Y}}, \tag{7.8}$$

and hence $Cx = 0$. Therefore $CA^n x = \lambda^n Cx = 0$, $n \geq 0$. Since V_Σ is unitary,

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = V_\Sigma^* \begin{bmatrix} \lambda x \\ 0 \end{bmatrix}.$$

This yields $A^* x = \bar{\lambda} x$ and $B^* x = 0$, and therefore $B^* A^{*n} x = \bar{\lambda}^n B^* x = 0$, $n \geq 0$. Hence since Σ is simple, $x = 0$. □

Proof of Theorem 7.7. As in the proof of Lemma 7.8, we can assume that Σ is a simple conservative system.

For the existence part of (i), let $\Theta(z) = \Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ have right Kreĭn-Langer factorization $\Theta(z) = \Theta_r(z) b_r(z)^{-1}$. By Theorem 7.3, we can assume that $\Sigma = \Sigma_+^r \circ \Sigma_-^r$ in the notation of that result. In particular, $\mathcal{X} = \mathcal{X}_-^r \oplus \mathcal{X}_+^r$, and we can choose $\mathcal{X}_+ = \mathcal{X}_+^r$ and $\mathcal{X}_- = \mathcal{X}_-^r$. The existence part of (ii) is handled in the same way, but using Theorem 7.6 in place of Theorem 7.3.

We prove uniqueness in (ii). By Theorem 7.6 we can assume that $\Sigma = \Sigma_-^\ell \circ \Sigma_+^\ell$, where Σ_+^ℓ and Σ_-^ℓ are simple conservative systems such that

$$\Theta_{\Sigma_+^\ell}(z) = \Theta_\ell(z), \qquad \Theta_{\Sigma_-^\ell}(z) = b_\ell(z)^{-1},$$

for some left Kreĭn-Langer factorization $\Theta(z) = b_\ell(z)^{-1} \Theta_\ell(z)$ of the function $\Theta(z) = \Theta_\Sigma(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$. Then in the notation of Theorem 7.6,

$$A = \begin{bmatrix} A_+^\ell & 0 \\ * & A_-^\ell \end{bmatrix}$$

relative to the decomposition $\mathcal{X} = \mathcal{X}_+^\ell \oplus \mathcal{X}_-^\ell$.

Consider any fundamental decomposition $\mathcal{X} = \mathcal{X}_+' \oplus \mathcal{X}_-'$ as in (ii). We show that $\mathcal{X}_-' \subseteq \mathcal{X}_-^\ell$, and hence $\mathcal{X}_-' = \mathcal{X}_-^\ell$ because the two spaces have the same finite dimension. It is sufficient to show that any nonzero root vector of $A|\mathcal{X}_-'$ belongs to $\mathcal{X}_-^\ell$. Let $x \in \mathcal{X}_-'$, $x \neq 0$, and assume that $(A - \lambda I)^n x = 0$ for some complex number λ and positive integer n. By Lemma 7.8, $|\lambda| \neq 1$; since $A|\mathcal{X}_-'$ is a contraction operator on the antispace of a Hilbert space, $|\lambda| > 1$. Write write $x = x_+ \oplus x_-$, where $x_\pm \in \mathcal{X}_\pm^\ell$. Then

$$\begin{bmatrix} (A_+^\ell - \lambda I)^n & 0 \\ * & (A_-^\ell - \lambda I)^n \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $(A_+ - \lambda I)^n x_+ = 0$. Since A_+^ℓ is a contraction operator on a Hilbert space and $|\lambda| > 1$, $x_+ = 0$. Hence $x = x_- \in \mathcal{X}_-^\ell$, as was to be shown.

Uniqueness in (i) can be proved similarly or by applying what has been shown to the adjoint system Σ^*. □

8 Application to simple conservative systems

If a simple system Σ is represented as the cascade synthesis $\Sigma = \Sigma_2 \circ \Sigma_1$ of two systems Σ_1 and Σ_2, it is well known that Σ_1 and Σ_2 are simple (for example, see [1, Theorem 1.2.1]). The converse is false in general, even for conservative systems and Hilbert state spaces. In the case of Hilbert state spaces, the condition for the cascade synthesis $\Sigma = \Sigma_2 \circ \Sigma_1$ of two simple conservative systems Σ_1 and Σ_2 to be simple is usually expressed in terms of the factorization $\Theta_\Sigma(z) = \Theta_{\Sigma_2}(z)\Theta_{\Sigma_1}(z)$ of the corresponding transfer functions: in standard terminology, the factorization is said to be regular if Σ is simple. This notion originates in the study of invariant subspaces. Invariant subspaces and regular factorizations are studied from the point of view of a functional model in Sz.-Nagy and Foias [24, Chapter VII]. In the latter work, necessary and sufficient conditions for a factorization to be regular are derived from an analysis of the unitary dilation of a contraction operator; another account is given in the survey of Ball and Cohen [7]. The theory of regular factorizations was put into the framework of operator nodes, again in the case of Hilbert state spaces, by M. S. Brodskiĭ [10], based in part on previous works by M. S. Brodskiĭ and V. M. Brodskiĭ [11] and V. M. Brodskiĭ and P. A. Švarcman [12]. Theorem 8.1 below generalizes [10, Theorem 8.2] to Pontryagin state spaces. We do not, however, go so far as to allow the input space $\mathcal{U}$ and output space $\mathcal{Y}$ to be Pontryagin spaces, even with $\varkappa = 0$. The latter situation occurs in the model theory of noncontractions (for example, see Kuzhel [20], Ball [6], and McEnnis [22]); Clark [13] defines a notion of regular factorization in such a situation, but this notion does not overlap with our account except in the classical case of Hilbert state, input, and output spaces.

Recall that every function $S(z) \in \mathbf{S}_\varkappa(\mathcal{U},\mathcal{Y})$ has strong boundary values $S(\zeta)$ a.e. on the circle $|\zeta| = 1$, and these boundary values are contraction operators. Set

$$\Delta_S(\zeta) = [I - S(\zeta)^* S(\zeta)]^{1/2}, \qquad \Delta_{*S}(\zeta) = [I - S(\zeta)S(\zeta)^*]^{1/2},$$

a.e. for $|\zeta| = 1$. Consider a cascade synthesis $\Sigma = \Sigma_2 \circ \Sigma_1$ of two simple conservative systems having transfer functions $\Theta_1(z) = \Theta_{\Sigma_1}(z)$ and $\Theta_2(z) = \Theta_{\Sigma_2}(z)$. Define an operator

$$V\colon \overline{\Delta_{\Theta_2\Theta_1} L^2(\mathcal{U})} \to \overline{\Delta_{\Theta_2} L^2(\mathcal{Y}_1)} \oplus \overline{\Delta_{\Theta_1} L^2(\mathcal{U})}$$

by setting

$$V\colon \Delta_{\Theta_2\Theta_1}(\zeta)u(\zeta) \to \Delta_{\Theta_2}(\zeta)\Theta_1(\zeta)u(\zeta) \oplus \Delta_{\Theta_1}(\zeta)u(\zeta)$$

for every $u(\zeta)$ in $L^2(\mathcal{U})$. We easily check that V is isometric. For fixed ζ on the circle $|\zeta| = 1$ with the exception of a set of measure zero, define

$$V_\zeta\colon \overline{\Delta_{\Theta_2\Theta_1}(\zeta)\mathcal{U}} \to \overline{\Delta_{\Theta_2}(\zeta)\mathcal{Y}_1} \oplus \overline{\Delta_{\Theta_1}(\zeta)\mathcal{U}}$$

by setting

$$V_\zeta\colon \Delta_{\Theta_2\Theta_1}(\zeta)u \to \Delta_{\Theta_2}(\zeta)\Theta_1(\zeta)u \oplus \Delta_{\Theta_1}(\zeta)u$$

for every $u \in \mathcal{U}$. Then V_ζ is isometric a.e. for $|\zeta| = 1$.

Theorem 8.1 *Let $\Sigma = \Sigma_2 \circ \Sigma_1$, where $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \varkappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \varkappa_2)$ are simple conservative systems with transfer functions $\Theta_1(z) = \Theta_{\Sigma_1}(z)$ and $\Theta_2(z) = \Theta_{\Sigma_2}(z)$. Then Σ is simple if and only if*

(i) $\Theta_2(z)\Theta_1(z) \in \mathbf{S}_{\varkappa_1+\varkappa_2}(\mathcal{U},\mathcal{Y})$, *and*

(ii) *one of the following conditions is satisfied*

  (a) $\Delta_{\Theta_2}(\zeta)\mathcal{Y}_1 \cap \Delta_{*\Theta_1}(\zeta)\mathcal{Y}_1 = \{0\}$ *a.e. for* $|\zeta| = 1$*;*

  (b) *the equality*

  $$\Theta_1(\zeta)^*\Delta_{\Theta_2}(\zeta)y_1(\zeta) + \Delta_{\Theta_1}(\zeta)u(\zeta) = 0$$

  with $y_1(\zeta) \in \overline{\Delta_{\Theta_2} L^2(\mathcal{Y}_1)}$ *and* $u(\zeta) \in \overline{\Delta_{\Theta_1} L^2(\mathcal{U})}$ *holds a.e. for* $|\zeta| = 1$ *only for the zero elements of the spaces;*

  (c) V *is unitary;*

  (d) V_ζ *is unitary a.e. for* $|\zeta| = 1$.

In this case, all of the conditions (a)–(d) *in* (ii) *hold.*

In the case of Hilbert state spaces, that is, when $\varkappa_1 = \varkappa_2 = 0$, condition (i) in Theorem 8.1 is automatically satisfied. This case of Theorem 8.1 is known from [10, Theorem 8.2]. For reference purposes, we state the result in the following form.

Lemma 8.2 *Let $\Sigma_+ = \Sigma_{2+} \circ \Sigma_{1+}$, where Σ_{1+} and Σ_{2+} are simple conservative systems with transfer functions $\Theta_{1r}(z)$ and $\Theta_{2\ell}(z)$ and input and output spaces $\mathcal{U}, \mathcal{Y}_1$ and $\mathcal{Y}_1, \mathcal{Y}$, respectively. Define operators V_+ and $V_{\zeta+}$ for $\Sigma_+ = \Sigma_{2+} \circ \Sigma_{1+}$ in the same way as V and V_ζ are defined above for $\Sigma = \Sigma_2 \circ \Sigma_1$. Then the following statements are equivalent:*

(a) $\Delta_{\Theta_{2\ell}}(\zeta)\mathcal{Y}_1 \cap \Delta_{*\Theta_{1r}}(\zeta)\mathcal{Y}_1 = \{0\}$ *a.e. for* $|\zeta| = 1$*;*

(b) *the equality*

$$\Theta_{1r}(\zeta)^* \Delta_{\Theta_{2\ell}}(\zeta) y_1(\zeta) + \Delta_{\Theta_{1r}}(\zeta) u(\zeta) = 0$$

with $y_1(\zeta) \in \overline{\Delta_{\Theta_{2\ell}} L^2(\mathcal{Y}_1)}$ and $u(\zeta) \in \overline{\Delta_{\Theta_{1r}} L^2(\mathcal{U})}$ holds a.e. for $|\zeta| = 1$ only for the zero elements of the spaces;

(c) V_+ *is unitary;*

(d) $V_{\zeta+}$ *is unitary a.e. for* $|\zeta| = 1$*;*

(e) Σ_+ *is simple.*

Lemma 8.3 *Let $\Sigma = \Sigma_2 \circ \Sigma_1$, where Σ_1 and Σ_2 are simple conservative systems whose input and output spaces are the same Hilbert space $\mathcal{U}$. If the transfer functions of Σ_1 and Σ_2 have the form $b_1(z)^{-1}$ and $b_2(z)^{-1}$, respectively, where $b_1(z)$ and $b_2(z)$ are finite Blaschke products which are invertible at the origin, then Σ is a simple conservative system whose transfer function is $b_2(z)^{-1} b_1(z)^{-1}$.*

Proof of Lemma 8.3. Everything is clear from Section 7 except that Σ is simple. To see this, construct the complementary systems Σ'', Σ_1'', and Σ_2'' as in Theorem 5.3. The state spaces of Σ_1 and Σ_2 are antispaces of Hilbert spaces by, for example, [1, Theorem A3, p. 197], and therefore the state spaces of Σ_1'', Σ_2'', and $\Sigma'' = \Sigma_1'' \circ \Sigma_2''$ are Hilbert spaces. By Theorem 5.3(iii), the transfer functions of Σ_1'' and Σ_2'' are $b_1(z)$ and $b_2(z)$, and these have unitary boundary values on the unit circle. Condition (a) in Lemma 8.2 is trivially satisfied for the cascade synthesis $\Sigma'' = \Sigma_1'' \circ \Sigma_2''$, and therefore Σ'' is simple. Hence by Theorem 5.3(ii), Σ is simple. □

Lemma 8.4 *Let $\Sigma = \Sigma_2 \circ \Sigma_1$, where $\Sigma_1 = (A_1, B_1, C_1, D_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}_1; \varkappa_1)$ and $\Sigma_2 = (A_2, B_2, C_2, D_2; \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}; \varkappa_2)$ are simple conservative systems with transfer functions $\Theta_1(z) = \Theta_{\Sigma_1}(z)$ and $\Theta_2(z) = \Theta_{\Sigma_2}(z)$. Let*

$$\Theta_1(z) = \Theta_{1r}(z) b_{1r}(z)^{-1} \qquad \text{and} \qquad \Theta_2(z) = b_{2\ell}(z)^{-1} \Theta_{2\ell}(z) \tag{8.1}$$

be right and left Kreĭn-Langer factorizations, respectively. Choose choose simple conservative systems Σ_{1+} and Σ_{2+} such that $\Theta_{1r}(z) = \Theta_{\Sigma_{1+}}(z)$ and $\Theta_{2\ell}(z) = \Theta_{\Sigma_{2+}}(z)$. Then Σ is simple if and only if

(i) $\Theta_2(z)\Theta_1(z) \in \mathbf{S}_{\varkappa_1+\varkappa_2}(\mathcal{U}, \mathcal{Y})$*, and*

(ii) $\Sigma_{2+} \circ \Sigma_{1+}$ *is simple.*

Proof of Lemma 8.4. Choose simple conservative systems Σ_{2-} and Σ_{1-} such that $b_{1r}(z)^{-1} = \Theta_{\Sigma_{1-}}(z)$ and $b_{2\ell}(z)^{-1} = \Theta_{\Sigma_{2-}}(z)$. Since a simple conservative realization is essentially unique by Theorem 4.1, by Theorems 7.3 and 7.6 without loss of generality we can assume that $\Sigma_1 = \Sigma_{1+} \circ \Sigma_{1-}$ and $\Sigma_2 = \Sigma_{2-} \circ \Sigma_{2+}$.

Assume that Σ is simple. Since $\Sigma = \Sigma_{2-} \circ (\Sigma_{2+} \circ \Sigma_{1+}) \circ \Sigma_{1-}$, it follows that $\Sigma_{2+} \circ \Sigma_{1+}$ is simple, that is, (ii) holds. Since $\Theta_\Sigma(z) = \Theta_2(z)\Theta_1(z)$ where Σ is a simple conservative system with state space $\mathcal{X}_1 \oplus \mathcal{X}_2$, $\Theta_2(z)\Theta_1(z) = \Theta_\Sigma(z) \in \mathbf{S}_{\varkappa_1+\varkappa_2}(\mathcal{U}, \mathcal{Y})$, and thus (i) holds.

Conversely, assume that (i) and (ii) hold. To see that $\Sigma = \Sigma_{2-} \circ \Sigma_{2+} \circ \Sigma_{1+} \circ \Sigma_{1-}$ is simple, we first argue that $\widehat{\Sigma} = \Sigma_{2-} \circ (\Sigma_{2+} \circ \Sigma_{1+})$ is simple. By (ii) and Theorem 7.6 it is sufficient to show that $\widehat{\Theta}(z) =$

$b_{2\ell}(z)^{-1}[\Theta_{2\ell}(z)\Theta_{1r}(z)]$ is a left Kreĭn-Langer factorization of $\widehat{\Theta}(z) = \Theta_{\widehat{\Sigma}}(z)$. In fact, since $b_{2\ell}(z)$ has degree $\varkappa_2$, $\widehat{\Theta}(z) \in \mathbf{S}_{\varkappa'}(\mathcal{U},\mathcal{Y})$ where $\varkappa' \le \varkappa_2$. Since $b_{1r}(z)$ has degree $\varkappa_1$, it follows that

$$\Theta_2(z)\Theta_1(z) = \widehat{\Theta}(z)b_{1r}(z)^{-1} \in \mathbf{S}_{\varkappa''}(\mathcal{U},\mathcal{Y}),$$

where $\varkappa'' \le \varkappa' + \varkappa_1 \le \varkappa_2 + \varkappa_1$. By (i), $\varkappa'' = \varkappa_1 + \varkappa_2$, and so $\varkappa' = \varkappa_2$. Thus $\widehat{\Theta}(z) = b_{2\ell}(z)^{-1}[\Theta_{2\ell}(z)\Theta_{1r}(z)]$ is a left Kreĭn-Langer factorization, and hence $\widehat{\Sigma} = \Sigma_{2-} \circ (\Sigma_{2+} \circ \Sigma_{1+})$ is simple by Theorem 7.6.

To see that $\Sigma = \widehat{\Sigma} \circ \Sigma_{1-}$ is simple, first choose a right Kreĭn-Langer factorization $\widehat{\Theta}(z) = \widehat{\Theta}_r(z)\widehat{b}_r(z)^{-1}$. By Theorem 7.3, we may assume that $\widehat{\Sigma} = \widehat{\Sigma}_+ \circ \widehat{\Sigma}_-$, where $\widehat{\Sigma}_+$ and $\widehat{\Sigma}_-$ are simple conservative systems having transfer functions $\widehat{\Theta}_r(z)$ and $\widehat{b}_r(z)^{-1}$. Thus $\Sigma = \widehat{\Sigma}_+ \circ (\widehat{\Sigma}_- \circ \Sigma_{1-})$. In view of Lemma 8.3, to prove that Σ is simple, it is sufficient to show that

$$\Theta_2(z)\Theta_1(z) = \widehat{\Theta}_r(z)[b_{1r}(z)\widehat{b}_r(z)]^{-1}$$

is a right Kreĭn-Langer factorization. This is clear because by (i), $\Theta_2(z)\Theta_1(z) \in \mathbf{S}_{\varkappa_1+\varkappa_2}(\mathcal{U},\mathcal{Y})$ and $b_{1r}(z)\widehat{b}_r(z)$ is a Blaschke product of degree $\varkappa_1 + \varkappa_2$. Hence Σ is simple by Theorem 7.3. □

Proof of Theorem 8.1. We use the Kreĭn-Langer factorizations (8.1) and notation of Lemma 8.4. By that lemma, Σ is simple if and only if $\Theta_2(z)\Theta_1(z) \in \mathbf{S}_{\varkappa_1+\varkappa_2}(\mathcal{U},\mathcal{Y})$ and $\Sigma_{2+} \circ \Sigma_{1+}$ is simple. Thus to complete the proof, it is sufficient to show that the four conditions (a)–(d) in Theorem 8.1(ii) are equivalent to their counterparts (a)–(d) in Lemma 8.2. For this we use the relations

$$\Delta_{\Theta_2}(z) = \Delta_{\Theta_{2\ell}}(\zeta), \qquad \Delta_{*\Theta_1}(z) = \Delta_{*\Theta_{1r}}(\zeta), \tag{8.2}$$

$$\Delta_{\Theta_1}(\zeta) = b_{1r}(\zeta)\Delta_{\Theta_{1r}}(\zeta)b_{1r}(\zeta)^{-1}, \tag{8.3}$$

which hold a.e. for $|\zeta| = 1$.

(a) The equivalence of the conditions (a) is immediate from (8.2).

(b) Assume that condition (b) in Lemma 8.2 holds. Suppose $y_1(\zeta) \in \overline{\Delta_{\Theta_2}L^2(\mathcal{Y}_1)}$, $u(\zeta) \in \overline{\Delta_{\Theta_1}L^2(\mathcal{U})}$, and $\Theta_1(\zeta)^*\Delta_{\Theta_2}(\zeta)y_1(\zeta) + \Delta_{\Theta_1}(\zeta)u(\zeta) = 0$ a.e. Then

$$b_{1r}(\zeta)^{-1}\big[\Theta_1(\zeta)^*\Delta_{\Theta_2}(\zeta)y_1(\zeta) + \Delta_{\Theta_1}(\zeta)u(\zeta)\big] = 0 \qquad a.e.,$$

and with the aid of (8.2) and (8.3) we obtain

$$\Theta_{1r}(\zeta)^*\Delta_{\Theta_{2\ell}}(\zeta)y_1(\zeta) + \Delta_{\Theta_{1r}}(\zeta)\widetilde{u}(\zeta) = 0 \qquad a.e.$$

where $y_1(\zeta) \in \overline{\Delta_{\Theta_2}L^2(\mathcal{Y}_1)} = \overline{\Delta_{\Theta_{2\ell}}L^2(\mathcal{Y}_1)}$ and

$$\widetilde{u}(\zeta) = b_{1r}(\zeta)^{-1}u(\zeta) \in b_{1r}(\zeta)^{-1}\Big(\overline{\Delta_{\Theta_1}L^2(\mathcal{U})}\Big) = \overline{\Delta_{\Theta_{1r}}L^2(\mathcal{U})}.$$

Thus $y_1(\zeta) = 0$ a.e. and $u(\zeta) = 0$ a.e., and hence condition (b) in Theorem 8.1 is satisfied. These steps are reversible, and so the conditions (b) are equivalent.

(c) In a routine way, we verify that an element $y_1(\zeta) \oplus u(\zeta)$ of $\overline{\Delta_{\Theta_2}L^2(\mathcal{Y}_1)} \oplus \overline{\Delta_{\Theta_1}L^2(\mathcal{U})}$ is orthogonal to the range of V if and only if the element $y_1(\zeta) \oplus b_{1r}(\zeta)^{-1}u(\zeta)$ of $\overline{\Delta_{\Theta_{2\ell}}L^2(\mathcal{Y}_1)} \oplus \overline{\Delta_{\Theta_{1r}}L^2(\mathcal{U})}$ is orthogonal to the range of V_+, which yields the equivalence of the conditions (c).

(d) The equivalence of the conditions (d) is a pointwise version of the preceding argument.

Since we assume the Hilbert space form of the theorem as stated in Lemma 8.2, the result follows. □

9 The classes $\mathbf{P}_{00}^{\varkappa}$ and $\mathbf{C}_{00}^{\varkappa}$

A system Σ is said to be bi-stable if the main operator A satisfies $A^n \xrightarrow{s} 0$ and $A^{*n} \xrightarrow{s} 0$. The class of contractive bi-stable operators in the Hilbert space case is denoted C_{00}. When state spaces are Pontryagin spaces, it is not possible to define stability in exactly the same way because in this case there are eigenvalues λ such that $|\lambda| > 1$. We introduce a corresponding notion which says, roughly, that states that can be stable are stable.

Definition 9.1 Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a simple passive system, and let $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ be the unique fundamental decomposition in Theorem 7.7(i) such that $A\mathcal{X}_+ \subseteq \mathcal{X}_+$.

(i) We say that Σ belongs to the class $\mathbf{P}_{00}^{\varkappa}$ if $A|\mathcal{X}_+ \in C_{00}$.

(ii) We say that Σ belongs to the class $\mathbf{C}_{00}^{\varkappa}$ if Σ is conservative $\Sigma \in \mathbf{P}_{00}^{\varkappa}$.

Theorem 9.2 *A simple conservative system $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ belongs to $\mathbf{C}_{00}^{\varkappa}$ if and only if $\Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$.*

Proof. The case $\varkappa = 0$ follows from [24, Proposition 3.5, p. 257]. In the general case, by Theorem 7.3 we can assume that $\Sigma = \Sigma_2 \circ \Sigma_1$, where Σ_1 and Σ_2 are simple conservative systems such that $\Theta_{\Sigma_2}(z) = \Theta_r(z)$ and $\Theta_{\Sigma_1}(z) = b_r(z)^{-1}$ for a Kreĭn-Langer factorization $\Theta_\Sigma(z) = \Theta_r(z)b_r(z)^{-1}$. Then $A|\mathcal{X}_+$ is the main operator of Σ_2; by the case $\varkappa = 0$, $A|\mathcal{X}_+ \in C_{00}$ if and only if $\Theta_r(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$. Since $b_r(z)^{-1}$ has unitary boundary values, $\Theta_r(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$ if and only if $\Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$, and the result follows. □

Theorem 9.3 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a simple passive system, and let $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ and $\mathcal{X} = \mathcal{X}'_+ \oplus \mathcal{X}'_-$ be the unique fundamental decompositions in Theorem 7.7 such that $A\mathcal{X}_+ \subseteq \mathcal{X}_+$ and $A\mathcal{X}'_- \subseteq \mathcal{X}'_-$. Then the conditions*

$$A|\mathcal{X}_+ \in C_{00} \quad \textit{and} \quad A^*|\mathcal{X}'_+ \in C_{00}$$

are equivalent. Hence if Σ belongs to $\mathbf{P}_{00}^{\varkappa}$ or $\mathbf{C}_{00}^{\varkappa}$, so does Σ^*.

Proof. Since we can embed any simple passive system into a simple conservative system having the same state space and main operator by Theorem 4.3, we can assume that Σ is conservative. By Theorem 9.2, $A|\mathcal{X}_+ \in C_{00}$ if and only if $\Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$, or equivalently $\Theta_{\Sigma^*}(z) = \Theta_\Sigma(\bar{z})^* \in \mathbf{U}_\varkappa(\mathcal{Y}, \mathcal{U})$. By Theorem 9.2 applied to Σ^*, the last condition is equivalent to $A^*|\mathcal{X}'_+ \in C_{00}$. □

Theorem 9.4 *Let $\Sigma = (A, B, C, D; \mathcal{X}, \mathcal{U}, \mathcal{Y}; \varkappa)$ be a simple passive system with transfer function $\Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U}, \mathcal{Y})$. Then Σ is conservative.*

In the case $\varkappa = 0$, this result is Theorem 1 in [3]. It follows from Theorem 10.2 that the system Σ in Theorem 9.4 is minimal.

Proof. Consider a Julia embedding of Σ into $\widetilde{\Sigma} = (A, \widetilde{B}, \widetilde{C}, \widetilde{D}; \mathcal{X}, \widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}; \varkappa)$. Write the system operator and transfer function for $\widetilde{\Sigma}$ in the forms (3.5), (3.6), and (3.7). By (3.6),

$$\begin{aligned}\Theta_{\widetilde{\Sigma}}(z) &= \begin{bmatrix} E_2 & D \\ G & F_2 \end{bmatrix} + z \begin{bmatrix} C \\ F_1 \end{bmatrix} (I - zA)^{-1} \begin{bmatrix} E_1 & B \end{bmatrix} \\ &= \begin{bmatrix} E_2 + zC(I - zA)^{-1}E_1 & D + zC(I - zA)^{-1}B \\ G + zF_1(I - zA)^{-1}E_1 & F_2 + z + F_1(I - zA)^{-1}B \end{bmatrix},\end{aligned}$$

and so

$$\begin{aligned} \Theta_{11}(z) &= E_2 + zC(I - zA)^{-1}E_1, & \Theta_{12}(z) &= D + zC(I - zA)^{-1}B, \\ \Theta_{21}(z) &= G + zF_1(I - zA)^{-1}E_1, & \Theta_{22}(z) &= F_2 + zF_1(I - zA)^{-1}B. \end{aligned}$$

Since the values of $\widetilde{\Theta}(\zeta)$ are contractive a.e. on the unit circle,

$$\begin{bmatrix} I_{\widehat{\mathcal{U}}} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} - \begin{bmatrix} \Theta_{11}(\zeta)^* & \Theta_{21}(\zeta)^* \\ \Theta_{12}(\zeta)^* & \Theta_{22}(\zeta)^* \end{bmatrix} \begin{bmatrix} \Theta_{11}(\zeta) & \Theta_{12}(\zeta) \\ \Theta_{21}(\zeta) & \Theta_{22}(\zeta) \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & I_{\widehat{Y}} \end{bmatrix} - \begin{bmatrix} \Theta_{11}(\zeta) & \Theta_{12}(\zeta) \\ \Theta_{21}(\zeta) & \Theta_{22}(\zeta) \end{bmatrix} \begin{bmatrix} \Theta_{11}(\zeta)^* & \Theta_{21}(\zeta)^* \\ \Theta_{12}(\zeta)^* & \Theta_{22}(\zeta)^* \end{bmatrix} \geq 0,$$

and hence

$$I_{\mathcal{U}} - \Theta_{12}(\zeta)^*\Theta_{12}(\zeta) - \Theta_{22}(\zeta)^*\Theta_{22}(\zeta) \geq 0,$$
$$I_{\mathcal{Y}} - \Theta_{11}(\zeta)\Theta_{11}(\zeta)^* - \Theta_{12}(\zeta)\Theta_{12}(\zeta)^* \geq 0,$$

a.e. Since $\Theta_{12}(z) = \Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U},\mathcal{Y})$ by assumption, it follows that $\Theta_{11}(\zeta) = 0$ and $\Theta_{22}(\zeta)^* = 0$ a.e. for $|\zeta| = 1$ and $\Theta_{11}(z) \equiv 0$ and $\Theta_{22}(z) \equiv 0$ on $\mathbf{D}$. Hence

$$E_2 = 0, \qquad CA^kE_1 = 0, \qquad k \geq 0, \tag{9.1}$$
$$F_2 = 0, \qquad F_1A^kB = 0, \qquad k \geq 0. \tag{9.2}$$

In particular, $E_1\widehat{\mathcal{U}} \perp \mathcal{X}^o_\Sigma$ and $F_1^*\widehat{\mathcal{Y}} \perp \mathcal{X}^c_\Sigma$. We shall show that $E_1\widehat{U} \perp \mathcal{X}^c_\Sigma$ and $F_1^*\widehat{\mathcal{Y}} \perp \mathcal{X}^o_\Sigma$ and hence, because Σ is simple, $E_1 = 0$ and $F_1^* = 0$. Granting this, we obtain $\widehat{\mathcal{U}} \subseteq \ker E_1 \cap \ker E_2 = \ker E = \{0\}$ and $\widehat{\mathcal{Y}} \subseteq \ker F_1^* \cap \ker F_2^* = \ker F^* = \{0\}$, and therefore $\Sigma = \widetilde{\Sigma}$ is conservative.

Thus to complete the proof, it remains to show that

$$B^*A^{*k}E_1 = 0 \quad \text{and} \quad CA^kF_1^* = 0, \quad k \geq 0. \tag{9.3}$$

From the relations

$$\begin{bmatrix} A^* & C^* & F_1^* \\ B^* & D^* & F_2^* \\ E_1^* & E_2^* & G^* \end{bmatrix}\begin{bmatrix} A & B & E_1 \\ C & D & E_2 \\ F_1 & F_2 & G \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 & 0 \\ 0 & I_{\mathcal{U}} & 0 \\ 0 & 0 & I_{\widehat{\mathcal{U}}} \end{bmatrix}, \tag{9.4}$$
$$\begin{bmatrix} A & B & E_1 \\ C & D & E_2 \\ F_1 & F_2 & G \end{bmatrix}\begin{bmatrix} A^* & C^* & F_1^* \\ B^* & D^* & F_2^* \\ E_1^* & E_2^* & G^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 & 0 \\ 0 & I_{\mathcal{Y}} & 0 \\ 0 & 0 & I_{\widehat{\mathcal{Y}}} \end{bmatrix}, \tag{9.5}$$

we obtain

$$B^*E_1 + D^*E_2 + F_2^*G = 0, \qquad CF_1^* + DF_2^* + E_2G^* = 0.$$

Since $E_2 = 0$ and $F_2 = 0$, (9.3) holds with $k = 0$. By (9.4) and (9.5) we also have

$$A^*E_1 + C^*E_2 + F_1^*G = 0, \qquad AF_1^* + BF_2^* + E_1G^* = 0,$$

which gives $A^*E_1 = -F_1^*G$ and $AF_1^* = -E_1G^*$. Consequently, for $k \geq 1$, from (9.1) and (9.2), we have

$$B^*A^{*k}E_1 = B^*(A^*)^{k-1}(A^*E_1) = -B^*(A^*)^{k-1}F_1^*G = -(F_1A^{k-1}B)^*G = 0,$$

and

$$CA^kF_1^* = CA^{k-1}(AF_1^*) = -CA^{k-1}E_1G^* = 0.$$

We have verified (9.3) for all $k \geq 0$, and the result follows. □

Corollary 9.5 *The class $\mathbf{C}^\varkappa_{00}$ coincides with the set of all simple passive systems Σ in $\mathbf{P}^\varkappa_{00}$ such that $\Theta_\Sigma(z) \in \mathbf{U}_\varkappa(\mathcal{U},\mathcal{Y})$.*

P r o o f. This follows from Theorem 9.2 and Theorem 9.4. □

10 Models for $\mathbf{C}_{00}^{\varkappa}$

In this section we briefly describe two models that further illustrate Kreĭn-Langer factorizations and the cascade synthesis of corresponding systems.

We first consider simple conservative systems in the class $\mathbf{C}_{00}^{\varkappa}$ and use a special case of the canonical model of de Branges and Rovnyak [9], as presented in an indefinite form in [1]. By [1, Theorem 2.2.1] every $\Theta(z) \in \mathbf{S}_{\varkappa}(\mathcal{U},\mathcal{Y})$ is the transfer function of an observable system $\Sigma_\Theta = (T,F,G,H;\mathcal{H}(\Theta),\mathcal{U},\mathcal{Y};\varkappa)$, where $\mathcal{H}(\Theta)$ is the Pontryagin space with reproducing kernel $[I-\Theta(z)\Theta(w)^*]/(1-z\bar{w})$, and

$$T\colon h(z) \to \frac{h(z)-h(0)}{z}, \qquad F\colon u \to \frac{\Theta(z)-\Theta(0)}{z}\,u,$$
$$G\colon h(z) \to h(0), \qquad H\colon u \to \Theta(0)u,$$

for all $h(z)$ in $\mathcal{H}(\Theta)$ and u in $\mathcal{U}$. The system operator V_{Σ_Θ} is coisometric, and thus Σ_Θ is passive.

Theorem 10.1 (i) *If $\Theta(z) \in \mathbf{U}_{\varkappa}(\mathcal{U},\mathcal{Y})$ and $\Theta(z) = \Theta_r(z)b_r(z)^{-1} = b_\ell(z)^{-1}\Theta_\ell(z)$ are right and left Kreĭn-Langer factorizations, then $\Sigma_\Theta \cong \Sigma_{\Theta_r} \circ \Sigma_{b_r^{-1}}$ and $\Sigma_\Theta \cong \Sigma_{b_\ell^{-1}} \circ \Sigma_{\Theta_\ell}$.*

(ii) *If $\Theta(z) \in \mathbf{U}_{\varkappa}(\mathcal{U},\mathcal{Y})$, Σ_Θ is a simple conservative system in $\mathbf{C}_{00}^{\varkappa}$. Conversely, if Σ is a simple conservative system in $\mathbf{C}_{00}^{\varkappa}$, then $\Sigma \cong \Sigma_\Theta$ where $\Theta(z) = \Theta_\Sigma(z)$.*

The relation $\cong$ of equivalence of systems is defined in Section 3.

P r o o f. (i) If $\Theta(z) \in \mathbf{U}_{\varkappa}(\mathcal{U},\mathcal{Y})$, we deduce that $\Sigma_{b_r^{-1}}, \Sigma_{b_\ell^{-1}}, \Sigma_{\Theta_r}, \Sigma_{\Theta_\ell}$ are simple conservative systems (see Lemma 7.5, [24, Proposition 3.5, p. 257] and [3, Proposition 2]). Write

$$\Sigma_{b_r^{-1}} = (A_1,B_1,C_1,D_1;\mathcal{H}(b_r^{-1}),\mathcal{U},\mathcal{Y};\varkappa),$$
$$\Sigma_{\Theta_r} = (A_2,B_2,C_2,D_2;\mathcal{H}(\Theta_r),\mathcal{U},\mathcal{Y};\varkappa).$$

We exhibit an equivalence between the systems

$$\Sigma_{\Theta_r} \circ \Sigma_{b_r^{-1}} = (A,B,C,D;\mathcal{H}(b_r^{-1}) \oplus \mathcal{H}(\Theta_r),\mathcal{U},\mathcal{Y};\varkappa),$$
$$\Sigma_\Theta = (T,F,G,H;\mathcal{H}(\Theta),\mathcal{U},\mathcal{Y};\varkappa).$$

By [1, Theorem 4.2.3(4)], $\mathcal{H}(\Theta) = \mathcal{H}(\Theta_r) \oplus \Theta_r\mathcal{H}(b_r^{-1})$. This decomposition allows us to define an operator $W\colon \mathcal{H}(b_r^{-1}) \oplus \mathcal{H}(\Theta_r) \to \mathcal{H}(\Theta)$ by

$$W\colon \begin{bmatrix} u(z) \\ f(z) \end{bmatrix} \to f(z) + \Theta_r(z)u(z)$$

for all $u(z)$ in $\mathcal{H}(b_r^{-1})$ and $f(z)$ in $\mathcal{H}(\Theta_r)$. Straightforward verifications show that $T = WAW^{-1}$, $F = WB$, $G = CW^{-1}$, and $H = D$. Hence $\Sigma_\Theta \cong \Sigma_{\Theta_r} \circ \Sigma_{b_r^{-1}}$. We obtain $\Sigma_\Theta \cong \Sigma_{b_\ell^{-1}} \circ \Sigma_{\Theta_\ell}$ by considering adjoints.

(ii) If $\Theta(z) \in \mathbf{U}_{\varkappa}(\mathcal{U},\mathcal{Y})$, Σ_Θ is conservative by (i) and Theorem 7.1. Since Σ_Θ is observable, it is simple. Thus Σ_Θ belongs to $\mathbf{C}_{00}^{\varkappa}$ by Theorem 9.2. Conversely, if Σ is any simple conservative system in $\mathbf{C}_{00}^{\varkappa}$ and $\Theta(z) = \Theta_\Sigma(z)$, then Σ and Σ_Θ are simple conservative systems having the same transfer function, and hence they are equivalent by Theorem 4.1. □

Theorem 10.2 *Every simple conservative system Σ in $\mathbf{C}_{00}^{\varkappa}$ is minimal.*

A different criterion for a simple conservative system to be minimal is given in [14, Proposition 3.5].

P r o o f. By Theorem 10.1, we can take $\Sigma = \Sigma_\Theta$ where $\Theta(z) \in \mathbf{U}_{\varkappa}(\mathcal{U},\mathcal{Y})$. Set $\widetilde{\Theta}(z) = \Theta(\bar{z})^*$. Then $\Sigma^* = \Sigma_{\widetilde{\Theta}}$ is a simple conservative system in $\mathbf{C}_{00}^{\varkappa}$ by Theorem 9.2. We know that $\Sigma = \Sigma_\Theta$ is observable. We prove that Σ is controllable using a criterion for controllability given in Theorem 3.4.1 of [1]. Define an operator $\Lambda\colon \mathcal{H}(\Theta) \to \mathcal{H}(\widetilde{\Theta})$ first on a fundamental set by

$$\Lambda\colon \frac{I-\Theta(z)\Theta(w)^*}{1-z\bar{w}}\,y \to \frac{\widetilde{\Theta}(z)-\widetilde{\Theta}(\bar{w})}{z-\bar{w}}\,y$$

for all w in the domain of holomorphy of $\Theta(z)$ in the unit disk and all y in $\mathcal{Y}$, and then by linearity and continuity on all of $\mathcal{H}(\Theta)$. By Theorem 3.4.1 of [1], $\Sigma = \Sigma_\Theta$ is controllable if and only if Λ has zero kernel. In fact, Λ is an isometry. For since the system operator for $\Sigma^* = \Sigma_{\widetilde{\Theta}}$ is unitary, Theorem 3.3.5 of [1] gives

$$\left\langle \frac{\widetilde{\Theta}(z) - \widetilde{\Theta}(\bar{\alpha})}{z - \bar{\alpha}} y_1, \frac{\widetilde{\Theta}(z) - \widetilde{\Theta}(\bar{\beta})}{z - \bar{\beta}} y_2 \right\rangle_{\mathcal{H}(\widetilde{\Theta})} = \left\langle \frac{I - \Theta(\beta)\Theta(\alpha)^*}{1 - \beta\bar{\alpha}} y_1, y_2 \right\rangle_{\mathcal{Y}}$$
$$= \left\langle \frac{I - \Theta(z)\Theta(\alpha)^*}{1 - z\bar{\alpha}} y_1, \frac{I - \Theta(z)\Theta(\beta)^*}{1 - z\bar{\beta}} y_2 \right\rangle_{\mathcal{H}(\Theta)}$$

for all α, β in the domain of holomorphy of $\Theta(z)$ in the unit disk and all $y_1, y_2 \in \mathcal{Y}$. The kernel of an everywhere defined and continuous isometry on a Pontryagin space is zero (see [16, Corollary 1.9]). Thus $\Sigma = \Sigma_\Theta$ is both observable and controllable and hence minimal. □

We next consider the more general case of functions $\Theta(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ and the model of Sz.-Nagy and Foias [24] and M.S. Brodskiĭ [10]. Write $L^2(\mathcal{U})$ and $H^2(\mathcal{U})$ for the usual Lebesgue and Hardy spaces of functions on the unit circle $|\zeta| = 1$ with values in a Hilbert space $\mathcal{U}$. Define $\Delta_S(\zeta)$ for any function $S(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ as in Section 8.

In the case $\varkappa = 0$, given a function $\Theta(z) \in \mathbf{S}(\mathcal{U}, \mathcal{Y})$, the model of Sz.-Nagy and Foias produces a system $\mathring{\Sigma} = (\mathring{A}, \mathring{B}, \mathring{C}, \mathring{D}; \mathring{\mathcal{X}}(\Theta), \mathcal{U}, \mathcal{Y}; 0)$ with a Hilbert state space

$$\mathring{\mathcal{X}}(\Theta) = \Big(H^2(\mathcal{Y}) \oplus \overline{\Delta_\Theta L^2(\mathcal{U})}\Big) \ominus \overline{\begin{bmatrix} \Theta(\zeta) \\ \Delta_\Theta(\zeta) \end{bmatrix} H^2(\mathcal{U})} \tag{10.1}$$

and operators defined by

$$\begin{cases} \mathring{A}\colon \begin{bmatrix} y(\zeta) \\ u(\zeta) \end{bmatrix} \to \bar{\zeta} \begin{bmatrix} y(\zeta) - y(0) \\ u(\zeta) \end{bmatrix}, & \mathring{B}\colon u_0 \to \bar{\zeta} \begin{bmatrix} (\Theta(\zeta) - \Theta(0))\, u_0 \\ \Delta_\Theta(\zeta) u_0 \end{bmatrix}, \\ \mathring{C}\colon \begin{bmatrix} y(\zeta) \\ u(\zeta) \end{bmatrix} \to y(0), & \mathring{D}\colon u_0 \to \Theta(0) u_0, \end{cases} \tag{10.2}$$

for all $y(\zeta) \oplus u(\zeta)$ in $\mathring{\mathcal{X}}(\Theta)$ and all u_0 in $\mathcal{U}$. The system $\mathring{\Sigma}$ is simple and conservative, and has transfer function $\Theta(z)$. In the special case of inner functions, $\Delta_\Theta(\zeta) = 0$ a.e., and therefore we have the simpler formulas

$$\mathring{\mathcal{X}}(\Theta) = H^2(\mathcal{Y}) \ominus \Theta(\zeta) H^2(\mathcal{U}) \tag{10.3}$$

and

$$\begin{cases} \mathring{A}\colon y(\zeta) \to \bar{\zeta}\,(y(\zeta) - y(0))\,, & \mathring{B}\colon u_0 \to \bar{\zeta}\,(\Theta(\zeta) - \Theta(0))\, u_0, \\ \mathring{C}\colon y(\zeta) \to y(0), & \mathring{D}\colon u_0 \to \Theta(0) u_0. \end{cases} \tag{10.4}$$

for all $y(\zeta)$ in $\mathring{\mathcal{X}}(\Theta)$ and all u in $\mathcal{U}$. We formulate an analogous model for a simple conservative system with transfer function $\Theta(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$.

Theorem 10.3 *Let $\Theta(z) \in \mathbf{S}_\varkappa(\mathcal{U}, \mathcal{Y})$ and let $\Theta(z) = \Theta_r(z) b_r(z)^{-1}$ be a right Kreĭn-Langer factorization. Define a system*

$$\mathring{\Sigma} = (\mathring{A}, \mathring{B}, \mathring{C}, \mathring{D}; \mathring{\mathcal{X}}(\Theta), \mathcal{U}, \mathcal{Y}; \varkappa), \qquad \mathring{\mathcal{X}}(\Theta) = \mathring{\mathcal{X}}(\Theta_r) \oplus \Big(-\mathring{\mathcal{X}}(b_r)\Big),$$

where

$$\begin{cases} \overset{\circ}{A}\colon \begin{bmatrix} y(\zeta) \\ u_1(\zeta) \\ u_2(\zeta) \end{bmatrix} \to \bar\zeta \begin{bmatrix} y(\zeta) - y(0) - \big(\Theta_r(\zeta) b_r(0)^{-1} - \Theta(0)\big)\, u_2(\zeta) \\ u_1(\zeta) - \Delta_{\Theta_r}(\zeta) b_r(0)^{-1} u_2(0) \\ u_2(\zeta) - u_2(0) \end{bmatrix}, \\ \qquad \overset{\circ}{B}\colon u_0 \to \bar\zeta \begin{bmatrix} \big(\Theta_r(\zeta) b_r(0)^{-1} - \Theta(0)\big)\, u_0 \\ \Delta_{\Theta_r}(\zeta) b_r(0)^{-1} u_0 \\ \big(b_r(\zeta) b_r(0)^{-1} - I\big)\, u_0 \end{bmatrix}, \\ \overset{\circ}{C}\colon \begin{bmatrix} y(\zeta) \\ u_1(\zeta) \\ u_2(\zeta) \end{bmatrix} \to y(0) - \Theta(0) u_2(0), \\ \qquad \overset{\circ}{D}\colon u_0 \to \Theta(0) u_0, \end{cases} \tag{10.5}$$

for all $y(\zeta) \oplus u_1(\zeta) \oplus u_2(\zeta)$ in $\overset{\circ}{\mathcal{X}}(\Theta)$ and all u_0 in $\mathcal{U}$. Then $\overset{\circ}{\Sigma}$ is a simple conservative system with transfer function $\Theta(z)$. Moreover, the identities

$$(I - z\,\overset{\circ}{A})^{-1}\,\overset{\circ}{B}\,u_0 = \frac{1}{e^{it} - z} \begin{bmatrix} \Theta_r(\zeta) - \Theta_r(z) \\ \Delta_{\Theta_r}(\zeta) \\ b_r(\zeta) - b_r(z) \end{bmatrix} b_r(z)^{-1} u_0 \tag{10.6}$$

$$(I - \overline{w}\,\overset{\circ}{A}{}^*)^{-1}\,\overset{\circ}{C}{}^*\,y_0 = \frac{1}{1 - \overline{w} e^{it}} \begin{bmatrix} I - \Theta_r(\zeta)\Theta_r^*(w) \\ -\Delta_{\Theta_r}(\zeta)\Theta_r^*(w) \\ b_r^{*-1}(w) - b_r(\zeta)\Theta_r^*(w) \end{bmatrix} y_0. \tag{10.7}$$

hold for all $u_0 \in \mathcal{U}$ and all $y_0 \in \mathcal{Y}$.

P r o o f. Construct a simple conservative system Σ_1 with transfer function $\Theta_r(z) \in \mathbf{S}(\mathcal{U},\mathcal{Y})$ using (10.1) and (10.2). A simple conservative system Σ_2 with inner transfer function $b_r(z) \in \mathbf{U}(\mathcal{U},\mathcal{U})$ can be constructed using (10.3) and (10.4). Starting with Σ_2 and using (5.3), we can construct a system Σ_2'' which, by Theorem 5.3, is also simple and conservative and satisfies $\Theta_{\Sigma_2}(z) = b_r^{-1}(z)$. By Theorem 7.3,

$$\overset{\circ}{\Sigma} = \Sigma_1 \circ \Sigma_2''$$

is a simple conservative system. Straightforward calculations verify (10.5), (10.6), and (10.7). □

If $\Theta(z) \in \mathbf{U}_\varkappa(\mathcal{U},\mathcal{Y})$, (10.5) –(10.7) have the simpler form

$$\overset{\circ}{A}\colon \begin{bmatrix} y(\zeta) \\ u_2(\zeta) \end{bmatrix} \to \bar\zeta \begin{bmatrix} y(\zeta) - y(0) - \big(\Theta_r(\zeta) b_r(0)^{-1} - \Theta(0)\big)\, u_2(\zeta) \\ u_2(\zeta) - u_2(0) \end{bmatrix},$$

$$\overset{\circ}{B}\colon u_0 \to \bar\zeta \begin{bmatrix} \big(\Theta_r(\zeta) b_r(0)^{-1} - \Theta(0)\big)\, u_0 \\ \big(b_r(\zeta) b_r(0)^{-1} - I\big)\, u_0 \end{bmatrix},$$

$$\overset{\circ}{C}\colon \begin{bmatrix} y(\zeta) \\ u_2(\zeta) \end{bmatrix} \to y(0) - \Theta(0) u_2(0),$$

$$\overset{\circ}{D}\colon u_0 \to \Theta(0) u_0,$$

and

$$(I - z\,\overset{\circ}{A})^{-1}\,\overset{\circ}{B}\,u_0 = \frac{1}{\zeta - z} \begin{bmatrix} \Theta_r(\zeta) - \Theta_r(z) \\ b_r(\zeta) - b_r(z) \end{bmatrix} b_r^{-1}(z) u_0,$$

$$(I - \overline{w}\,\overset{\circ}{A}{}^*)^{-1}\,\overset{\circ}{C}{}^*\,y_0 = \frac{1}{1 - \overline{w}\zeta} \begin{bmatrix} I - \Theta_r(\zeta)\Theta_r^*(w) \\ b_r^{*-1}(w) - b_r(\zeta)\Theta_r^*(w) \end{bmatrix} y_0.$$

Acknowledgements D. Z. Arov and S. M. Saprikin were partially supported by the joint grant UM1-2567-00 03 from the U. S. Civilian Research & Development Foundation (CRDF) and Ukrainian Government, and they thank the University of Virginia for hospitality during visits when this work was carried out. J. Rovnyak was supported by the National Science Foundation Grant DMS-0100437.

References

[1] D. Alpay, A. Dijksma, J. Rovnyak, and H. S. V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory Adv. Appl., vol. 96, Birkhäuser, Basel, 1997.
[2] D. Z. Arov, *Passive linear steady-state dynamical systems*, Sibirsk. Mat. Zh. **20** (1979), no. 2, 211–228, 457, English transl. Siberian Math. J. 20 (1979), no. 2, 149–162.
[3] ______, *Stable dissipative linear stationary dynamical scattering systems*, J. Operator Theory **2** (1979), no. 1, 95–126, English transl. with an appendix by D. Z. Arov and J. Rovnyak, Interpolation theory, systems theory, and related topics, Oper. Theory Adv. Appl. 134, Birkhäuser, Basel, 2002, pp. 99–136.
[4] D. Z. Arov and S. M. Saprikin, *Maximal solutions for the embedding problem for a generalized Schur function and optimal dissipative scattering systems with Pontryagin state spaces*, Methods Funct. Anal. Topology **7** (2001), no. 4, 69–80.
[5] T. Ya. Azizov and I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, "Nauka", Moscow, 1986, English transl., John Wiley & Sons Ltd., Chichester, 1989.
[6] J. A. Ball, *Models for noncontractions*, J. Math. Anal. Appl. **52** (1975), no. 2, 235–254.
[7] J. A. Ball and N. Cohen, *de Branges-Rovnyak operator models and systems theory: a survey*, Topics in matrix and operator theory (Rotterdam, 1989), Oper. Theory Adv. Appl., vol. 50, Birkhäuser, Basel, 1991, pp. 93–136.
[8] J. Bognár, *Indefinite inner product spaces*, Springer-Verlag, New York, 1974, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78.
[9] L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), Wiley, New York, 1966, pp. 295–392.
[10] M. S. Brodskiĭ, *Unitary operator colligations and their characteristic functions*, Uspekhi Mat. Nauk **33** (1978), no. 4(202), 141–168, 256, English transl. Russian Math. Surveys 33 (1978), no. 4, 159-191.
[11] V. M. Brodskiĭ and M. S. Brodskiĭ, *Factorization of the characteristic function and invariant subspaces of a contraction operator*, Funkcional. Anal. i Priložen. **8** (1974), no. 2, 63–64, English transl. Funct. Anal. Appl. 8 (1974), 145–147.
[12] V. M. Brodskiĭ and Ja. S. Švarcman, *Invariant subspaces of a contraction, and factorization of the characteristic function*, Teor. Funkciĭ Funkcional. Anal. i Priložen. **22** (1975), 15–35, 160.
[13] D. N. Clark, *On functional models for non-contraction operators*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23** (1975), no. 2, 159–163.
[14] B. Ćurgus, A. Dijksma, H. Langer, and H. S. V. de Snoo, *Characteristic functions of unitary colligations and of bounded operators in Kreĭn spaces*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), Oper. Theory Adv. Appl., vol. 41, Birkhäuser, Basel, 1989, pp. 125–152.
[15] A. Dijksma, H. Langer, and H. S. V. de Snoo, *Characteristic functions of unitary operator colligations in π_κ-spaces*, Operator theory and systems (Amsterdam, 1985), Oper. Theory Adv. Appl., vol. 19, Birkhäuser, Basel, 1986, pp. 125–194.
[16] M. A. Dritschel and J. Rovnyak, *Operators on indefinite inner product spaces*, Lectures on operator theory and its applications (Waterloo, ON, 1994), Amer. Math. Soc., Providence, RI, 1996, pp. 141–232.
[17] I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators*, Izdat. "Nauka", Moscow, 1965, English transl. American Mathematical Society, Providence, R.I., 1969.
[18] I. S. Iokhvidov, M. G. Kreĭn, and H. Langer, *Introduction to the spectral theory of operators in spaces with an indefinite metric*, Akademie-Verlag, Berlin, 1982.
[19] M. G. Kreĭn and H. Langer, *Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_κ*, Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), North-Holland, Amsterdam, 1972, pp. 353–399. Colloq. Math. Soc. János Bolyai, 5.
[20] A. V. Kuzhel′, *A characteristic operator function of an arbitrary bounded operator*, Dopovīdī Akad. Nauk Ukraïn. RSR Ser. A **1968** (1968), 233–236.
[21] S. A. Kuzhel′, *The abstract Lax-Phillips scattering scheme in Pontryagin spaces*, Mat. Sb. **187** (1996), no. 10, 87–108, English transl. Sb. Math. 187 (1996), no. 10, 1503–1523.
[22] B. W. McEnnis, *Models for operators with bounded characteristic function*, Acta Sci. Math. (Szeged) **43** (1981), no. 1-2, 71–90.
[23] S. M. Saprikin, *The theory of linear discrete time-invariant dissipative scattering systems with π_κ state spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **282** (2001), Issled. po Linein. Oper. i Teor. Funkts. 29, 192–215, 281, English transl. J. Math. Sci. (N. Y.) **120** (2004), no. 5, 1752–1765.
[24] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., Amsterdam, 1970.