

# A theorem of Beurling-Lax type for Hilbert spaces of functions analytic in the unit ball

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**Abstract.** Schur multipliers on the unit ball are operator-valued functions for which the  $N$ -variable Schwarz-Pick kernel is nonnegative. In this paper, the coefficient spaces are assumed to be Pontryagin spaces having the same negative index. The associated reproducing kernel Hilbert spaces are characterized in terms of generalized difference-quotient transformations. The connection between invariant subspaces and factorization is established.

**Mathematics Subject Classification (2000).** Primary 46E22. Secondary 32A37, 47A15, 47A48, 47B32.

## 1. Introduction

It is known that a number of constructions from single-variable operator theory extend to a setting in which a key role is played by the kernel

$$\frac{1}{1 - \langle z, w \rangle}, \quad \langle z, w \rangle = \sum_{j=1}^N z_j \bar{w}_j,$$

on the open unit ball  $B_N$  in  $\mathbf{C}^N$ , that is, the set of all  $z = (z_1, \dots, z_N)$  in  $\mathbf{C}^N$  such that  $\langle z, z \rangle < 1$ ; for example, see [1, 2, 7, 11, 15]. More generally, let  $\mathfrak{F}$  and  $\mathfrak{G}$  be Hilbert spaces, and let  $H_{\mathfrak{F}}(B_N)$  and  $H_{\mathfrak{G}}(B_N)$  be the Hilbert spaces of functions having reproducing kernels  $I_{\mathfrak{F}}/(1 - \langle z, w \rangle)$  and  $I_{\mathfrak{G}}/(1 - \langle z, w \rangle)$ . An analytic function  $S(z)$  on  $B_N$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  is said to be a Schur multiplier if multiplication

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Daniel Alpay was supported by the Israel Science Foundation (Grant no. 322/00). J. Rovnyak was supported by the National Science Foundation grant DMS-0100437, and he thanks University College London for its hospitality during a sabbatical visit when this work was completed.

by  $S(z)$  acts as a contraction from  $H_{\mathfrak{F}}(B_N)$  into  $H_{\mathfrak{G}}(B_N)$ , or equivalently if the  $N$ -variable Schwarz-Pick kernel

$$K_S(w, z) = \frac{I_{\mathfrak{G}} - S(z)S(w)^*}{1 - \langle z, w \rangle} \quad (1.1)$$

is nonnegative. When  $N = 1$  the class of Schur multipliers coincides with the set of holomorphic functions on the disk which are bounded by one, but for  $N > 1$  it is a smaller class (for example, see [8]).

We write  $\mathfrak{H}(S)$  for the Hilbert space with reproducing kernel  $K_S(w, z)$  whenever  $S(z)$  is a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ . In the special case that  $\mathfrak{H}(S)$  is contained isometrically in  $H_{\mathfrak{G}}(B_N)$ , the complementary space

$$\mathfrak{M}(S) = H_{\mathfrak{G}}(B_N) \ominus \mathfrak{H}(S)$$

is a closed subspace of  $H_{\mathfrak{G}}(B_N)$  which is invariant under multiplication by  $z_j$  for each  $j = 1, \dots, N$ . Beurling-Lax type theorems, that is, converse results, have been obtained in this context and in related settings by Arveson [9], McCullough and Trent [18], and Greene, Richter, and Sundberg [16]. A key difference with the classical Beurling-Lax representation [21] of an invariant subspace  $\mathfrak{M}$  in  $H_{\mathfrak{G}}(B_N)$  should be noted: when  $N = 1$  we can write  $\mathfrak{M} = \mathfrak{M}(S)$  where the multiplier  $S(z)$  has values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  with  $\mathfrak{F} = \mathfrak{G}$ , but for  $N > 1$  it may occur that  $\dim \mathfrak{F} > \dim \mathfrak{G}$ ; see Example 3 in Section 5. Other versions of the Beurling-Lax theorem are given by Bolotnikov and Rodman [12] and Muhly and Solel [19].

In this paper, we characterize the Hilbert spaces  $\mathfrak{H}(S)$  which are contractively (but not necessarily isometrically) contained in  $H_{\mathfrak{G}}(B_N)$ . No greater effort is involved and new examples are obtained by allowing the coefficient spaces  $\mathfrak{F}$  and  $\mathfrak{G}$  to be Pontryagin spaces having the same negative index (the Brune section examples in Remark 5.2 are only possible when  $\mathfrak{F}$  and  $\mathfrak{G}$  are indefinite). For matrix-valued functions, it is equivalent to consider nonnegative kernels of the form

$$\frac{J_1 - S(z)J_2S(w)^*}{1 - \langle z, w \rangle},$$

where  $J_1$  and  $J_2$  are signature matrices having the same negative indices (cf. Potapov [20]). Our main result, Theorem 3.2, was announced in [4]. When  $N = 1$ , this result reduces to a special case of [5, Theorem 3.1.2]:

**Theorem 1.1.** *Let  $\mathfrak{H}$  be a reproducing kernel Hilbert space whose elements are holomorphic functions with values in a Pontryagin space  $\mathfrak{G}$  on a subregion of the unit disk in  $\mathbf{C}$  containing the origin. Assume that  $[h(z) - h(0)]/z$  belongs to the space whenever  $h(z)$  is in the space, and that*

$$\left\| \frac{h(z) - h(0)}{z} \right\|_{\mathfrak{H}}^2 \leq \|h(z)\|_{\mathfrak{H}}^2 - \langle h(0), h(0) \rangle_{\mathfrak{G}}. \quad (1.2)$$

*Then  $\mathfrak{H}$  is isometrically equal to a space  $\mathfrak{H}(S)$  for a Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  for some Pontryagin space  $\mathfrak{F}$  having the same negative index as  $\mathfrak{G}$ .*

When  $N > 1$ , the role of the difference-quotient transformation

$$T: h(z) \rightarrow [h(z) - h(0)]/z$$

is played by  $N$  operators  $T_1, \dots, T_N$  of  $\mathfrak{H}(S)$  into itself. These operators, which appear in Alpay and Kaptanoğlu [8] in a different way, satisfy a generalization of the difference-quotient inequality (1.2). Our main tool is a coisometric realization of Schur multipliers, which we construct in Section 2 by means of an isometric linear relation as in [5, Theorem 2.2.1]. Unitary, isometric, and coisometric realizations have previously been constructed by the method of “lurking isometries” in [11, 15]. The main result is presented in Section 3. In Section 4 we show the connection between invariant subspaces and the factorization of Schur multipliers. Examples are given in Section 5.

It is natural to ask if similar results hold for kernels (1.1) which have a finite number of negative squares, say  $\kappa$ . That is, is there an  $N$ -variable analog of the generalized Schur class  $\mathbf{S}_\kappa$  of Kreĭn-Langer [17]? In this paper, we assume that  $\kappa = 0$ . The authors believe that, in general, there is an  $N$ -variable theory when the index condition  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$  in the case  $\kappa = 0$  is replaced by

$$\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G} + (N - 1) \kappa.$$

Notice that for  $N > 1$  and  $\kappa > 0$ , the new index condition can only be satisfied if  $\mathfrak{F}$  is a Pontryagin space. We hope to address these matters in another place.

## 2. Schur multipliers and their coisometric realizations

Let  $S(z)$  be an analytic function of  $z = (z_1, \dots, z_N)$  in a open connected set  $\Omega(S)$  in  $B_N$ . We assume that  $\Omega(S)$  contains the origin, and that the values of  $S(z)$  are operators in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ , where  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces having the same negative index.<sup>1</sup> We call  $S(z)$  a **Schur multiplier** if the kernel  $K_S(w, z)$  defined by (1.1) is nonnegative on  $\Omega(S) \times \Omega(S)$ . In this case,  $\mathfrak{H}(S)$  denotes the Hilbert space of  $\mathfrak{G}$ -valued functions on  $\Omega(S)$  with reproducing kernel  $K_S(w, z)$ .

If  $S(z)$  is a Schur multiplier, we write  $\mathfrak{H}_N(S)$  for the closed span in  $\mathfrak{H}(S)^N = \mathfrak{H}(S) \oplus \dots \oplus \mathfrak{H}(S)$  ( $N$  copies) of all tuples

$$w^* K_S(w, z)g = \begin{pmatrix} \bar{w}_1 K_S(w, z)g \\ \vdots \\ \bar{w}_N K_S(w, z)g \end{pmatrix},$$

<sup>1</sup>Notation and terminology generally follow [5]. In particular, a Pontryagin space is a Kreĭn space  $\mathfrak{F}$  for which the negative index  $\text{ind}_- \mathfrak{F}$ , that is, the dimension of  $\mathfrak{F}_-$  in any fundamental decomposition  $\mathfrak{F} = \mathfrak{F}_+ \oplus \mathfrak{F}_-$ , is finite.

where  $w = (w_1, \dots, w_N) \in \Omega(S)$  and  $g \in \mathfrak{G}$ . Equivalently,  $\mathfrak{H}_N(S) = \mathfrak{H}(S)^N \ominus \mathfrak{N}$ , where  $\mathfrak{N}$  is the set of elements

$$h(z) = \begin{pmatrix} h_1(z) \\ \vdots \\ h_N(z) \end{pmatrix}$$

in  $\mathfrak{H}(S)^N$  such that  $zh(z) = \sum_{j=1}^N z_j h_j(z) = 0$  for all  $z = (z_1, \dots, z_N) \in \Omega(S)$ .

We use coisometric realizations for which the colligations have the form

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{G} \end{pmatrix}, \quad (2.1)$$

where  $\mathfrak{H}$  is a Hilbert space,  $\mathfrak{K}$  is a Hilbert space which is contained isometrically in  $\mathfrak{H}^N = \mathfrak{H} \oplus \dots \oplus \mathfrak{H}$  ( $N$  copies), and  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces such that  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . We say that  $V$  is **coisometric** if  $VV^* = I$ . By a mild abuse of notation we may view  $T$  and  $F$  as acting into the larger space  $\mathfrak{H}^N$  and write

$$T = \begin{pmatrix} T_1 \\ \vdots \\ T_N \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ \vdots \\ F_N \end{pmatrix}, \quad (2.2)$$

where  $T_1, \dots, T_N \in \mathfrak{L}(\mathfrak{H})$  and  $F_1, \dots, F_N \in \mathfrak{L}(\mathfrak{F}, \mathfrak{H})$ . We can also view any  $z = (z_1, \dots, z_N) \in \mathbf{C}^N$  as an operator on  $N \times 1$  matrices of scalars, vectors, or operators. Thus

$$zT = \sum_{j=1}^N z_j T_j \quad \text{and} \quad zF = \sum_{j=1}^N z_j F_j, \quad (2.3)$$

and  $T^*z^* = (zT)^*$  and  $F^*z^* = (zF)^*$ . The **characteristic function** of a colligation (2.1) is the function

$$S_V(z) = H + G(I - zT)^{-1}(zF) \quad (2.4)$$

on the domain  $\Omega(S_V)$  that we take to be the component of the origin in the open subset of  $B_N$  where the inverse exists. The values of  $S_V(z)$  are therefore operators on  $\mathfrak{F}$  to  $\mathfrak{G}$ . We call  $V$  **closely outer connected** if

$$\mathfrak{H} = \overline{\text{span}} \left\{ (I - T^*w^*)^{-1}G^*g : w \in \Omega(S_V), g \in \mathfrak{G} \right\} \quad (2.5)$$

and

$$\mathfrak{K} = \overline{\text{span}} \left\{ w^*(I - T^*w^*)^{-1}G^*g : w \in \Omega(S_V), g \in \mathfrak{G} \right\}. \quad (2.6)$$

**Theorem 2.1.** *The characteristic function  $S_V(z)$  of a coisometric colligation (2.1) is a Schur multiplier, and*

$$K_{S_V}(w, z) = G(I - zT)^{-1}(I - T^*w^*)^{-1}G^*, \quad w, z \in \Omega(S_V). \quad (2.7)$$

**Lemma 2.2.** *The characteristic function (2.4) of any colligation (2.1) satisfies*

$$\begin{aligned} \frac{I_{\mathfrak{G}} - S_V(z)S_V(w)^*}{1 - \langle z, w \rangle} &= G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* \\ &\quad + (G(I - zT)^{-1}z \quad I_{\mathfrak{G}}) \frac{I - VV^*}{1 - \langle z, w \rangle} \begin{pmatrix} w^*(I - T^*w^*)^{-1}G^* \\ I_{\mathfrak{G}} \end{pmatrix} \end{aligned}$$

for all  $w, z \in \Omega(S_V)$ .

*Proof.* Set

$$V_z = \begin{pmatrix} zT & zF \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H} \\ \mathfrak{G} \end{pmatrix}.$$

Since  $zw^* = \langle z, w \rangle$ , the identity to be proved is equivalent to

$$\begin{aligned} \frac{I_{\mathfrak{G}} - S_V(z)S_V(w)^*}{1 - \langle z, w \rangle} &= G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* \\ &\quad + (G(I - zT)^{-1} \quad I_{\mathfrak{G}}) \frac{\begin{pmatrix} \langle z, w \rangle I_{\mathfrak{H}} & 0 \\ 0 & I_{\mathfrak{G}} \end{pmatrix} - V_z V_w^*}{1 - \langle z, w \rangle} \begin{pmatrix} (I - T^*w^*)^{-1}G^* \\ I_{\mathfrak{G}} \end{pmatrix}, \end{aligned}$$

that is,

$$\begin{aligned} I_{\mathfrak{G}} - S_V(z)S_V(w)^* &= G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* + I_{\mathfrak{G}} \\ &\quad - (G(I - zT)^{-1} \quad I_{\mathfrak{G}}) V_z V_w^* \begin{pmatrix} (I - T^*w^*)^{-1}G^* \\ I_{\mathfrak{G}} \end{pmatrix}. \end{aligned}$$

Simple algebraic manipulations with the resolvent operator  $(I - zT)^{-1}$  reduce the last term on the right to

$$\begin{aligned} &(G(I - zT)^{-1} \quad I_{\mathfrak{G}}) \begin{pmatrix} zT & zF \\ G & H \end{pmatrix} \begin{pmatrix} T^*w^* & G^* \\ F^*w^* & H^* \end{pmatrix} \begin{pmatrix} (I - T^*w^*)^{-1}G^* \\ I_{\mathfrak{G}} \end{pmatrix} \\ &= (G(I - zT)^{-1}(zT) + G \quad S_V(z)) \begin{pmatrix} (T^*w^*)(I - T^*w^*)^{-1}G^* + G^* \\ S_V(w)^* \end{pmatrix} \\ &= G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* + S_V(z)S_V(w)^*, \end{aligned}$$

yielding the result.  $\square$

*Proof of Theorem 2.1.* If the colligation (2.1) is coisometric, then  $VV^* = I$ , and we obtain (2.7) from Lemma 2.2. Since the intermediate space  $\mathfrak{H}$  in this factorization is a Hilbert space, the kernel  $K_{S_V}(w, z)$  is nonnegative, and hence  $S_V(z)$  is a Schur multiplier.  $\square$

The converse generalizes [5, Theorem 2.2.1].

**Theorem 2.3.** *Let  $S(z)$  be a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ , where  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces such that  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . There is a unique colligation*

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H}(S) \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H}_N(S) \\ \mathfrak{G} \end{pmatrix} \quad (2.8)$$

such that

$$\left. \begin{aligned} (zTh)(z) &= h(z) - h(0), \\ (zFf)(z) &= [S(z) - S(0)]f, \\ Gh &= h(0), \\ Hf &= S(0)f, \end{aligned} \right\} \quad (2.9)$$

for all  $h \in \mathfrak{H}(S)$ ,  $f \in \mathfrak{F}$ , and  $z \in \Omega(S)$ . This colligation is coisometric and closely outer connected, and its characteristic function coincides with  $S(z)$  on the set where both are defined, that is,  $S_V(z) = S(z)$  for all  $z \in \Omega(S_V) \cap \Omega(S)$ .

We call (2.8) the **canonical coisometric colligation** associated with  $S(z)$ .

*Proof.* We first construct a colligation having all of the properties in the theorem. Define a linear relation

$$\mathbf{R} \subseteq \begin{pmatrix} \mathfrak{H}_N(S) \\ \mathfrak{G} \end{pmatrix} \times \begin{pmatrix} \mathfrak{H}(S) \\ \mathfrak{F} \end{pmatrix} \quad (2.10)$$

as the span of all pairs

$$\left( \begin{pmatrix} \alpha^* K_S(\alpha, z) u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} [K_S(\alpha, z) - K_S(0, z)] u_1 + K_S(0, z) u_2 \\ [S(\alpha)^* - S(0)^*] u_1 + S(0)^* u_2 \end{pmatrix} \right)$$

where  $u_1, u_2 \in \mathfrak{G}$  and  $\alpha = (\alpha_1, \dots, \alpha_N) \in \Omega(S)$ . The domain of  $\mathbf{R}$  is dense by the definition of  $\mathfrak{H}_N(S)$ , and the domain and range spaces in (2.10) are Pontryagin spaces having the same negative index since  $\mathfrak{H}_N(S)$  and  $\mathfrak{H}(S)$  are Hilbert spaces and  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . It is not hard to see that  $\mathbf{R}$  is isometric, that is, for any  $u_1, u_2, v_1, v_2 \in \mathfrak{G}$  and  $\alpha, \beta \in \Omega(S)$ ,

$$\begin{aligned} & \left\langle \begin{pmatrix} \overline{\alpha}_1 K_S(\alpha, z) u_1 \\ \vdots \\ \overline{\alpha}_N K_S(\alpha, z) u_1 \end{pmatrix}, \begin{pmatrix} \overline{\beta}_1 K_S(\beta, z) v_1 \\ \vdots \\ \overline{\beta}_N K_S(\beta, z) v_1 \end{pmatrix} \right\rangle_{\mathfrak{H}_N(S)} + \langle u_2, v_2 \rangle_{\mathfrak{G}} \\ &= \left\langle [K_S(\alpha, z) - K_S(0, z)] u_1 + K_S(0, z) u_2, \right. \\ & \quad \left. [K_S(\beta, z) - K_S(0, z)] v_1 + K_S(0, z) v_2 \right\rangle_{\mathfrak{H}(S)} \\ &+ \left\langle [S(\alpha)^* - S(0)^*] u_1 + S(0)^* u_2, \right. \\ & \quad \left. [S(\beta)^* - S(0)^*] v_1 + S(0)^* v_2 \right\rangle_{\mathfrak{F}}. \end{aligned}$$

To check this, expand both sides, and then observe that all of the terms involving  $u_2$  or  $v_2$  cancel; the remaining terms are also dealt with in a routine way. By [5, Theorem 1.4.2] there is a coisometric colligation (2.8) such that the graph of  $V^*$  is the closure of  $\mathbf{R}$ . Thus for any  $w \in \Omega(S)$  and  $g \in \mathfrak{G}$ ,

$$\left. \begin{aligned} T^* : w^* K_S(w, z)g &\rightarrow [K_S(w, z) - K_S(0, z)]g, \\ F^* : w^* K_S(w, z)g &\rightarrow [S(w)^* - S(0)^*]g, \\ G^* : g &\rightarrow K_S(0, z)g, \\ H^* : g &\rightarrow S(0)g. \end{aligned} \right\} \quad (2.11)$$

The formulas for  $T, F, G, H$  in the theorem are simple consequences. We show that  $G(I - wT)^{-1}$  is evaluation at any point  $w$  in  $\Omega(S)$  such that the inverse exists. To see this, consider any  $h \in \mathfrak{H}(S)$  and set  $h = (I - wT)k$ . The relation

$$(wT k)(w) = k(w) - k(0)$$

says that the value of  $h = (I - wT)k$  at  $w$  is equal to  $k(0) = Gk$ . Hence

$$h(w) = Gk = G(I - wT)^{-1}h,$$

as was to be shown. It follows that the colligation (2.8) is closely outer connected. For by what we have just shown,

$$(I - T^*w^*)^{-1}G^* : g \rightarrow K_S(w, z)g$$

for all  $w$  in a neighborhood of the origin and any  $g \in \mathfrak{G}$ . Hence (2.5) and (2.6) hold with  $\mathfrak{H} = \mathfrak{H}(S)$  and  $\mathfrak{K} = \mathfrak{H}_N(S)$ .

We show that  $S_V(z) = S(z)$  on the set where both are defined. Let  $f \in \mathfrak{F}$  and  $w \in \Omega(S_V) \cap \Omega(S)$ . The formula for  $F$  in the theorem says that the value of  $(wF)f$  at  $w$  is  $S(w)f - S(0)f$ . Since evaluation at  $w$  is given by  $G(I - wT)^{-1}$ , we obtain

$$G(I - wT)^{-1}(wF)f = S(w)f - S(0)f,$$

which gives the result because  $S(0) = H$ .

To prove uniqueness, suppose that (2.8) is any colligation satisfying (2.9). Define  $T_1, \dots, T_N$  and  $F_1, \dots, F_N$  as in (2.2). From the first two equations in (2.9), for all  $w \in \Omega(S)$  and  $u \in \mathfrak{G}$ ,

$$\begin{aligned} \sum_{j=1}^N \bar{w}_j T_j^* : K_S(w, \cdot)u &\rightarrow [K_S(w, \cdot) - K_S(0, \cdot)]u, \\ \sum_{j=1}^N \bar{w}_j F_j^* : K_S(w, \cdot)u &\rightarrow [S(w)^* - S(0)^*]u. \end{aligned}$$

This implies the first two relations in (2.11), and hence the operators  $T$  and  $F$  in the given colligation coincide with those constructed via the isometric relation. The operators  $G$  and  $H$  are evidently the same also.  $\square$

We say that two colligations

$$\left. \begin{aligned} V &= \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K} \\ \mathfrak{G} \end{pmatrix}, \\ V' &= \begin{pmatrix} T' & F' \\ G' & H' \end{pmatrix} : \begin{pmatrix} \mathfrak{H}' \\ \mathfrak{F}' \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{K}' \\ \mathfrak{G}' \end{pmatrix} \end{aligned} \right\} \quad (2.12)$$

are **equivalent** if there exist isomorphisms  $W_1: \mathfrak{H} \rightarrow \mathfrak{H}'$  and  $W_2: \mathfrak{K} \rightarrow \mathfrak{K}'$  such that

$$W_1(\alpha T) = (\alpha T')W_1, \quad W_1(\alpha F) = \alpha F', \quad (2.13)$$

and

$$\left. \begin{aligned} W_2 T &= T' W_1, & W_2 F &= F', \\ G &= G' W_1, & H &= H'. \end{aligned} \right\} \quad (2.14)$$

Equivalent colligations  $V$  and  $V'$  have the same characteristic function:

$$\begin{aligned} S_{V'}(z) &= H' + G'(I - zT')^{-1}(zF') = H + GW_1^{-1}(I - zT')^{-1}W_1(zF) \\ &= H + G(I - zT)^{-1}(zF) = S_V(z). \end{aligned}$$

The converse is true if the colligations are coisometric and closely outer connected.

**Theorem 2.4.** *Let  $V$  and  $V'$  be two colligations of the form (2.12). If both are coisometric and closely outer connected, and if*

$$S_V(z) = S_{V'}(z) \quad (2.15)$$

*in a neighborhood of the origin, then  $V$  and  $V'$  are equivalent.*

*Proof.* The problem is to construct isomorphisms  $W_1: \mathfrak{H} \rightarrow \mathfrak{H}'$  and  $W_2: \mathfrak{K} \rightarrow \mathfrak{K}'$  such that the six relations (2.13) and (2.14) hold. Setting  $z = 0$  in (2.15), we obtain  $H = H'$ , and so in a neighborhood of the origin,

$$G(I - zT)^{-1}(zF) = G'(I - zT')^{-1}(zF').$$

Since both colligations are coisometric, by Theorem 2.1,

$$\begin{aligned} G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* &= K_{S_V}(w, z) \\ &= K_{S_{V'}}(w, z) = G'(I - zT')^{-1}(I - T'^*w'^*)^{-1}G'^*. \end{aligned}$$

In view of (2.5) and (2.6), this relation allows us to construct isomorphisms  $W_1$  and  $W_2$  such that

$$W_1: (I - T^*w^*)^{-1}G^*g \rightarrow (I - T'^*w'^*)^{-1}G'^*g$$

and

$$W_2: w^*(I - T^*w^*)^{-1}G^*g \rightarrow w'^*(I - T'^*w'^*)^{-1}G'^*g$$

for all  $w$  in a neighborhood of the origin and all  $g \in \mathfrak{G}$ . We omit the straightforward verifications that the relations (2.13) and (2.14) hold.  $\square$

The colligation  $V$  in Theorem 2.3 is coisometric and may have a nontrivial kernel. It is sometimes useful to know the form of  $\ker V$  (cf. [5, Theorem 3.2.3]).



**Theorem 2.5.** *The kernel of the colligation  $V$  constructed in Theorem 2.3 for any Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  coincides with the set of elements of  $\mathfrak{H}(S) \oplus \mathfrak{F}$  which have the form*

$$\begin{pmatrix} -S(z)f \\ f \end{pmatrix}$$

for some  $f \in \mathfrak{F}$ .

*Proof.* If  $\begin{pmatrix} h \\ f \end{pmatrix} \in \ker V$ , then

$$\begin{aligned} Th + Ff &= 0, \\ Gh + Hf &= 0. \end{aligned}$$

Hence also  $(zTh)(z) + (zFf)(z) \equiv 0$ . By Theorem 2.3,

$$\begin{aligned} [h(z) - h(0)] + [S(z) - S(0)]f &\equiv 0, \\ h(0) + S(0)f &= 0, \end{aligned}$$

and thus  $h(z) \equiv -S(z)f$ . Conversely, suppose

$$\begin{pmatrix} h(z) \\ f \end{pmatrix} = \begin{pmatrix} -S(z)f \\ f \end{pmatrix} \in \mathfrak{H}(S) \oplus \mathfrak{F}. \tag{2.16}$$

Reversing the preceding argument, we obtain

$$\begin{aligned} (zTh)(z) + (zFf)(z) &\equiv 0, \\ Gh + Hf &= 0. \end{aligned}$$

It follows that  $u = Th + Ff$  is an element of  $\mathfrak{H}_N(S)$  such that  $\sum_{j=1}^N z_j u_j(z) \equiv 0$ . By the definition of  $\mathfrak{H}_N(S)$ ,  $u = 0$ , that is,  $Th + Ff = 0$ . Hence the element (2.16) belongs to  $\ker V$ .  $\square$

### 3. Characterization of the spaces $\mathfrak{H}(S)$

We now ask, which functional Hilbert spaces have the form  $\mathfrak{H}(S)$  for some Schur multiplier  $S(z)$ ? Necessary conditions follow from Theorem 2.3.

**Theorem 3.1.** *Let  $S(z)$  be a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ , where  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces such that  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . Then there exist bounded operators  $T_1, \dots, T_N$  on  $\mathfrak{H}(S)$  into itself satisfying*

(1) for all  $h \in \mathfrak{H}(S)$  and  $z = (z_1, \dots, z_N) \in \Omega(S)$ ,

$$h(z) - h(0) = \sum_{j=1}^N z_j (T_j h)(z),$$

(2) for all  $h \in \mathfrak{H}(S)$ ,

$$\sum_{j=1}^N \|T_j h\|_{\mathfrak{H}(S)}^2 \leq \|h\|_{\mathfrak{H}(S)}^2 - \langle h(0), h(0) \rangle_{\mathfrak{G}}.$$

The operators  $T_1, \dots, T_N$  meeting these conditions are unique if the only  $h(z)$  in  $\mathfrak{H}(S)^N$  such that  $zh(z) \equiv 0$  is  $h(z) \equiv 0$ .

*Proof.* Let  $V$  be the canonical coisometric colligation (2.8), and write the operator  $T$  that appears there as in (2.2). The assertion (1) is a restatement of the first relation in (2.9). To prove (2), observe that  $V^*$  is an isometry from  $\mathfrak{H}_N(S) \oplus \mathfrak{G}$  into  $\mathfrak{H}(S) \oplus \mathfrak{F}$ , and the two Pontryagin spaces have the same negative index. Hence  $V^{**} = V$  is a contraction [14, Corollary 2.5]. Thus if  $h \in \mathfrak{H}(S)$ ,

$$\left\langle \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix}, \begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle_{\mathfrak{H}_N(S) \oplus \mathfrak{G}} \leq \left\langle \begin{pmatrix} h \\ 0 \end{pmatrix}, \begin{pmatrix} h \\ 0 \end{pmatrix} \right\rangle_{\mathfrak{H}(S) \oplus \mathfrak{F}},$$

which yields (2). We leave it to the reader to prove the uniqueness statement from (1).  $\square$

The conditions (1) and (2) in Theorem 3.1 state that Gleason's problem for the space  $\mathfrak{H}(S)$  can be solved in a certain way (see [8] and [23, p. 116]). The converse result generalizes the Hilbert space case of [5, Theorem 3.1.2].

**Theorem 3.2.** *Let  $\mathfrak{H}$  be a reproducing kernel Hilbert space whose elements are analytic functions defined on a common neighborhood of the origin  $\Omega$  in  $B_N$  with values in a Pontryagin space  $\mathfrak{G}$ . Assume that there exist bounded operators  $T_1, \dots, T_N$  on  $\mathfrak{H}$  into itself satisfying*

(1) for all  $h \in \mathfrak{H}$  and  $z = (z_1, \dots, z_N) \in \Omega$ ,

$$h(z) - h(0) = \sum_{j=1}^N z_j (T_j h)(z),$$

(2) for all  $h \in \mathfrak{H}$ ,

$$\sum_{j=1}^N \|T_j h\|_{\mathfrak{H}}^2 \leq \|h\|_{\mathfrak{H}}^2 - \langle h(0), h(0) \rangle_{\mathfrak{G}}.$$

Then there exists a Pontryagin space  $\mathfrak{F}$  satisfying  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$  and a Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  such that  $\mathfrak{H}$  is equal isometrically to  $\mathfrak{H}(S)$ . We can choose  $S(z)$  such that the only  $f$  in  $\mathfrak{F}$  such that  $S(z)f \equiv 0$  is  $f = 0$ .

More precisely,  $\mathfrak{H}$  and  $\mathfrak{H}(S)$  coincide when their elements are restricted to a common neighborhood of the origin.

The Schur multiplier in Theorem 3.2 is essentially unique. In fact, assume that for each  $j = 1, 2$ ,  $S_j(z)$  is a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}_j, \mathfrak{G})$ , where  $\mathfrak{F}_j$

and  $\mathfrak{G}$  are Pontryagin spaces having the same negative index, and that  $S_j(z)f_j \equiv 0$  only for  $f_j = 0$ . Then  $\mathfrak{H}(S_1)$  and  $\mathfrak{H}(S_2)$  are equal isometrically if and only if

$$S_1(z) \equiv S_2(z)W,$$

where  $W \in \mathfrak{L}(\mathfrak{F}_1, \mathfrak{F}_2)$  is unitary. The proof is the same as in [5, Theorem 3.1.3].

*Proof.* Define  $T$  as in (2.2), and let  $G: \mathfrak{H} \rightarrow \mathfrak{G}$  be evaluation at 0. Without loss of generality, we can assume that the range of  $T$  is contained in

$$\mathfrak{H}_0^N = \mathfrak{H}^N \ominus \mathfrak{N},$$

where  $\mathfrak{H}^N = \mathfrak{H} \oplus \dots \oplus \mathfrak{H}$  ( $N$  copies) and  $\mathfrak{N}$  is the set of elements

$$h(z) = \begin{pmatrix} h_1(z) \\ \vdots \\ h_N(z) \end{pmatrix}$$

of  $\mathfrak{H}^N$  such that  $zh(z) = \sum_{j=1}^N z_j h_j(z) \equiv 0$ . For if this is not the case we can replace  $T$  by  $PT$ , where  $P$  is the projection of  $\mathfrak{H}^N$  onto  $\mathfrak{H}_0^N$ , and conditions (1) and (2) hold for  $PT$  if they hold for  $T$ . By restricting functions in  $\mathfrak{H}$  to a smaller set if necessary, we can also assume that  $I - zT$  is invertible for all  $z$  in  $\Omega$ .

**Claim 1:** *The reproducing kernel for  $\mathfrak{H}$  is given by*

$$K(w, z) = G(I - zT)^{-1}(I - T^*w^*)^{-1}G^*, \quad w, z \in \Omega. \tag{3.1}$$

To see this, note that for any  $w$  in  $\Omega$ , the operator  $E(w) = G(I - wT)^{-1}$  is evaluation at  $w$  on  $\mathfrak{H}$ . For given  $h \in \mathfrak{H}$ , setting  $h = (I - wT)k$ ,  $k \in \mathfrak{H}$ , we obtain

$$\begin{aligned} h(w) &= k(w) - ((wT)k)(w) = k(w) - [k(w) - k(0)] \\ &= Gk = G(I - zT)^{-1}h = E(w)h, \end{aligned}$$

where the second equality is by (1). By the definition of a reproducing kernel,  $K(w, z) = E(z)E(w)^*$ , and so the assertion follows.

**Claim 2:** *There exist a Pontryagin space  $\mathfrak{F}$  having the same negative index as  $\mathfrak{G}$  and operators  $F \in \mathfrak{L}(\mathfrak{F}, \mathfrak{H}_0^N)$  and  $H \in \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  such that*

$$I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* = \begin{pmatrix} F \\ H \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}^* \tag{3.2}$$

and

$$\ker \begin{pmatrix} F \\ H \end{pmatrix} = \{0\}. \tag{3.3}$$

To prove this, write  $C = \begin{pmatrix} T \\ G \end{pmatrix}$ . Since  $C \in \mathfrak{L}(\mathfrak{H}, \mathfrak{H}_0^N \oplus \mathfrak{G})$ ,

$$\text{ind}_- (I - CC^*) + \text{ind}_- \mathfrak{H} = \text{ind}_- (I - C^*C) + \text{ind}_- (\mathfrak{H}_0^N \oplus \mathfrak{G})$$

by [14, Theorem 2.4]. Since  $\mathfrak{H}$  and  $\mathfrak{H}_0^N$  are Hilbert spaces,

$$\text{ind}_-(I - CC^*) = \text{ind}_-(I - C^*C) + \text{ind}_- \mathfrak{G}.$$

By condition (2),  $I - C^*C = I - T^*T - G^*G \geq 0$ , and therefore

$$\text{ind}_-(I - CC^*) = \text{ind}_- \mathfrak{G}.$$

The assertion follows by applying [14, Theorem 2.1] to the selfadjoint operator  $I - CC^*$  on  $\mathfrak{K} = \mathfrak{H}_0^N \oplus \mathfrak{G}$ .

Now to prove the theorem, use the operators constructed above to define a colligation

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} \mathfrak{H}_0^N \\ \mathfrak{G} \end{pmatrix}.$$

By (3.2), the colligation is coisometric:

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} = \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* + \begin{pmatrix} F \\ H \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}^* = I,$$

Hence by Theorem 2.1, the characteristic function  $S(z) = S_V(z)$  of the colligation is a Schur multiplier satisfying (2.7). By Claim 1,

$$K(w, z) = G(I - zT)^{-1}(I - T^*w^*)^{-1}G^* = K_S(w, z)$$

on  $\Omega \times \Omega$ , and so  $\mathfrak{H}$  is isometrically equal to  $\mathfrak{H}(S)$ .

The Schur multiplier  $S(z)$  constructed in this way has the additional property in the last statement. For suppose that  $f \in \mathfrak{F}$  and  $S(z)f = 0$  identically. By the definition of  $S(z) = S_V(z)$ ,

$$G(I - zT)^{-1}(zFf) = [S(z) - S(0)]f = 0$$

identically. Recall that  $G(I - zT)^{-1}$  is evaluation at  $z$ , so the last relation gives

$$(zFf)(z) = 0, \quad z \in \Omega.$$

Equivalently,  $Ff \in \mathfrak{N}$ . By construction,  $Ff$  is in  $\mathfrak{H}_0^N = \mathfrak{H}^N \ominus \mathfrak{N}$ , and hence  $Ff = 0$ . Since also  $Hf = S(0)f = 0$ ,  $f = 0$  by (3.3).  $\square$

*Remark 3.3.* The proof of Theorem 3.2 provides a constructive procedure to find a Schur multiplier  $S(z)$  having the required properties. We first find a Pontryagin space  $\mathfrak{F}$  having the same negative index as  $\mathfrak{G}$  and operators  $F \in \mathfrak{L}(\mathfrak{F}, \mathfrak{H}^N)$  and  $H \in \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  satisfying (3.2) and (3.3). In principle, this can always be done, and then the required Schur multiplier is given by

$$S(z) = H + G(I - zT)^{-1}(zF).$$

We remark that a factorization (3.2) which does not necessarily satisfy (3.3) is still useful; in this case  $S(z)$  fulfills all of the conditions in Theorem 3.2 except perhaps for the last statement.

#### 4. Invariant subspaces and factorization

In the classical Beurling-Lax theory, inclusions of invariant subspaces correspond to factorizations of inner functions. We derive a result of this type, Theorem 4.1, for the spaces  $\mathfrak{H}(S)$  associated with Schur multipliers  $S(z)$ . Particular cases are equivalent to results that appear, for example, in [15, 18]. The one-variable case is proved in [5, Theorem 4.1.3] by a different method.

**Theorem 4.1.** *Let  $S(z)$  be a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ , where  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces such that  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ , and let  $T_1, \dots, T_N$  be operators on  $\mathfrak{H}(S)$  satisfying the conditions (1) and (2) in Theorem 3.1. Let  $\mathfrak{H}_1$  be a closed subspace of  $\mathfrak{H}(S)$  which is invariant under each of the operators  $T_1, \dots, T_N$ .*

(1) *There is a Pontryagin space  $\mathfrak{K}$  which has the same negative index as  $\mathfrak{F}$  and  $\mathfrak{G}$  and a Schur multiplier  $S_1(z)$  with values in  $\mathfrak{L}(\mathfrak{K}, \mathfrak{G})$  such that  $\mathfrak{H}_1$  is isometrically equal to  $\mathfrak{H}(S_1)$ .*

(2) *There is a Schur multiplier  $S_2(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$  such that*

$$S(z) = S_1(z)S_2(z)$$

*in a neighborhood of the origin.*

We shall deduce Theorem 4.1 as a corollary of Theorem 3.2 and a factorization theorem of a different nature.

**Theorem 4.2.** *Let  $S(z)$  and  $S_1(z)$  be Schur multipliers with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  and  $\mathfrak{L}(\mathfrak{K}, \mathfrak{G})$ , respectively, where  $\mathfrak{F}$ ,  $\mathfrak{G}$ , and  $\mathfrak{K}$  are Pontryagin spaces having the same negative index. Then  $\mathfrak{H}(S_1)$  is contained contractively in  $\mathfrak{H}(S)$  if and only if there is a Schur multiplier  $S_2(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$  such that*

$$S(z) = S_1(z)S_2(z)$$

*in a neighborhood of the origin.*

The property that  $\mathfrak{H}(S_1)$  is contained contractively in  $\mathfrak{H}(S)$  is equivalent to the nonnegativity of the kernel

$$\frac{S_1(z)S_1(w)^* - S(z)S(w)^*}{1 - \langle z, w \rangle},$$

and thus Theorem 4.2 is a factorization theorem of Leech type (for example, see [5, §3.5 C]). Our proof of Theorem 4.2 generalizes [6, Theorem 8] and uses the method of lurking isometries [10].

*Proof of Theorem 4.2.* For notational simplicity, throughout the proof we write  $\Omega$  for a sufficiently small open connected set in  $B_N$  containing the origin in which all of the functions involved are defined.

*Sufficiency.* Given such a function  $S_2(z)$ , we can write

$$K_S(w, z) = K_{S_1}(w, z) + S_1(z)K_{S_2}(w, z)S_1(w)^*.$$

Both kernels on the right side are nonnegative, and therefore  $\mathfrak{H}(S_1)$  is contained contractively in  $\mathfrak{H}(S)$  by general properties of reproducing kernels.

*Necessity.* Conversely, if  $\mathfrak{H}(S_1)$  is contained contractively in  $\mathfrak{H}(S)$ , then

$$K_S(w, z) = K_{S_1}(w, z) + \tilde{K}(w, z),$$

where

$$\tilde{K}(w, z) = \frac{S_1(z)S_1(w)^* - S(z)S(w)^*}{1 - \langle z, w \rangle}$$

is a nonnegative kernel. Hence  $\tilde{K}(w, z)$  is the reproducing kernel for a Hilbert space  $\tilde{\mathfrak{H}}$  of functions on  $\Omega$ . The space  $\tilde{\mathfrak{H}}$  is also contained contractively in  $\mathfrak{H}(S)$ . Write

$$\tilde{K}(w, z) = E(z)E(w)^*,$$

where  $E(w): \tilde{\mathfrak{H}} \rightarrow \mathfrak{G}$  is the evaluation mapping for any  $w \in \Omega$ . Let

$$\mathfrak{M} = \overline{\text{span}} \left\{ \begin{pmatrix} E(w)^*g \\ S(w)^*g \end{pmatrix} : w \in \Omega, g \in \mathfrak{G} \right\}.$$

Then  $\mathfrak{M}$  is a closed subspace of  $\tilde{\mathfrak{H}} \oplus \mathfrak{F}$ .

**Claim:**  $\mathfrak{M}$  is regular and has same negative index as  $\mathfrak{F}$ .

To see this it is enough to show that the orthogonal complement  $\mathfrak{M}^\perp$  of  $\mathfrak{M}$  in  $\tilde{\mathfrak{H}} \oplus \mathfrak{F}$  is a Hilbert space. By the definition of  $\mathfrak{M}$ ,

$$\mathfrak{M}^\perp = \left\{ \begin{pmatrix} h \\ f \end{pmatrix} : h \in \tilde{\mathfrak{H}}, f \in \mathfrak{F}, h(z) + S(z)f \equiv 0 \right\}.$$

Let  $V$  be the coisometric colligation constructed in Theorem 2.3 for the Schur multiplier  $S(z)$ . Consider the operator  $\Phi = V\Delta$ :

$$\tilde{\mathfrak{H}} \oplus \mathfrak{F} \xrightarrow{\Delta} \mathfrak{H}(S) \oplus \mathfrak{F} \xrightarrow{V} \mathfrak{H}_N(S) \oplus \mathfrak{G},$$

where  $\Delta$  is the inclusion mapping. Then  $\Phi$  is a contraction because both  $\Delta$  and  $V$  are contractions. Since  $\tilde{\mathfrak{H}} \oplus \mathfrak{F}$  and  $\mathfrak{H}_N(S) \oplus \mathfrak{G}$  have the same negative index,  $\ker \Phi$  is a Hilbert subspace of  $\tilde{\mathfrak{H}} \oplus \mathfrak{F}$  [14, Theorem 2.7]. It follows from Theorem 2.5 that  $\mathfrak{M}^\perp \subseteq \ker \Phi$ . Hence  $\mathfrak{M}^\perp$  is a Hilbert space in the inner product of  $\tilde{\mathfrak{H}} \oplus \mathfrak{F}$ , and the claim follows.

By the claim, we may view  $\mathfrak{M}$  as a Pontryagin space in its own right and define a linear relation

$$\mathbf{R} \subseteq \mathfrak{M} \times \begin{pmatrix} \tilde{\mathfrak{H}}^N \\ \mathfrak{R} \end{pmatrix}$$

by

$$\mathbf{R} = \text{span} \left\{ \begin{pmatrix} E(w)^*g \\ S(w)^*g \end{pmatrix}, \begin{pmatrix} w^*E(w)^*g \\ S_1(w)^*g \end{pmatrix} : w \in \Omega, g \in \mathfrak{G} \right\},$$

where  $\tilde{\mathfrak{H}}^N = \tilde{\mathfrak{H}} \oplus \cdots \oplus \tilde{\mathfrak{H}}$  ( $N$  copies). The identity

$$E(z)E(w)^* + S(z)S(w)^* = \langle z, w \rangle E(z)E(w)^* + S_1(z)S_1(w)^*$$

shows that  $\mathbf{R}$  is isometric. Since the domain and range spaces for  $\mathbf{R}$  are Pontryagin spaces having the same negative index, by [5, Theorem 1.4.2] the closure of  $\mathbf{R}$  is the graph of a continuous isometry  $W_0$ . Let  $\mathfrak{N}$  be the range of  $W_0$ . We view  $W_0$  as an operator in  $\mathfrak{L}(\mathfrak{M}, \mathfrak{N})$ ,

$$\mathfrak{M} \subseteq \begin{pmatrix} \tilde{\mathfrak{H}} \\ \mathfrak{F} \end{pmatrix}, \quad \mathfrak{N} \subseteq \begin{pmatrix} \tilde{\mathfrak{H}}^N \\ \mathfrak{K} \end{pmatrix}, \tag{4.1}$$

and construct a unitary extension. To this end, define a Hilbert space  $\mathfrak{L}$  to be  $\{0\}$  if the orthogonal complements of  $\mathfrak{M}$  and  $\mathfrak{N}$  in the inclusions (4.1) have the same dimension, and otherwise let  $\mathfrak{L}$  be a separable Hilbert space of infinite dimension. Then

$$\mathfrak{M} \subseteq \begin{pmatrix} \tilde{\mathfrak{H}} \oplus \mathfrak{L} \\ \mathfrak{F} \end{pmatrix}, \quad \mathfrak{N} \subseteq \begin{pmatrix} (\tilde{\mathfrak{H}} \oplus \mathfrak{L})^N \\ \mathfrak{K} \end{pmatrix}, \tag{4.2}$$

where  $(\tilde{\mathfrak{H}} \oplus \mathfrak{L})^N$  is the direct sum of  $N$  copies of  $\tilde{\mathfrak{H}} \oplus \mathfrak{L}$ . The orthogonal complements of  $\mathfrak{M}$  and  $\mathfrak{N}$  in the inclusions (4.2) have the same dimension, and hence we can extend  $W_0$  to a unitary operator

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \tilde{\mathfrak{H}} \oplus \mathfrak{L} \\ \mathfrak{F} \end{pmatrix} \rightarrow \begin{pmatrix} (\tilde{\mathfrak{H}} \oplus \mathfrak{L})^N \\ \mathfrak{K} \end{pmatrix}.$$

For any  $g \in \mathfrak{G}$  and  $w \in \Omega$ ,

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} w^* E(w)^* g \\ S_1(w)^* g \end{pmatrix} = \begin{pmatrix} E(w)^* g \\ S(w)^* g \end{pmatrix}.$$

We obtain the relations

$$\begin{aligned} A^* w^* \{E(w)^* g\} + C^* \{S_1(w)^* g\} &= E(w)^* g, \\ B^* w^* \{E(w)^* g\} + D^* \{S_1(w)^* g\} &= S(w)^* g, \end{aligned}$$

and hence

$$S(w)^* = \left[ D^* + B^* w^* (I_{\tilde{\mathfrak{H}} \oplus \mathfrak{L}} - A^* w^*)^{-1} C^* \right] S_1(w)^*.$$

Therefore  $S(z) = S_1(z)S_W(z)$  in a neighborhood of the origin, where

$$S_W(z) = D + C(I_{\tilde{\mathfrak{H}} \oplus \mathfrak{L}} - zA)^{-1}(zB)$$

is the characteristic function of the colligation  $W$ . Since  $W$  is coisometric (and even unitary),  $S_2(z) = S_W(z)$  is a Schur multiplier with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$ . By construction,  $S(z) = S_1(z)S_2(z)$ .  $\square$

*Proof of Theorem 4.1.* (1) We easily verify that the space  $\mathfrak{H}_1$  together with the operators  $T_1|_{\mathfrak{H}_1}, \dots, T_N|_{\mathfrak{H}_1}$  satisfy the two conditions in Theorem 3.2. Thus (1) follows from Theorem 3.2.

(2) In view of (1), the hypotheses of Theorem 4.2 are met, and (2) follows from that result.  $\square$

The converse to Theorem 4.1 requires an additional assumption.

**Theorem 4.3.** *Let  $S(z)$  and  $S_1(z)$  be Schur multipliers with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  and  $\mathfrak{L}(\mathfrak{K}, \mathfrak{G})$ , respectively, where  $\mathfrak{F}$ ,  $\mathfrak{G}$ , and  $\mathfrak{K}$  are Pontryagin spaces having the same negative index. Assume that the only  $h(z)$  in  $\mathfrak{H}(S)^N$  such that  $zh(z) \equiv 0$  is the function  $h(z) \equiv 0$ , and hence the operators  $T_1, \dots, T_N$  associated with  $\mathfrak{H}(S)$  as in Theorem 3.1 are unique. If  $\mathfrak{H}(S_1)$  is contained isometrically in  $\mathfrak{H}(S)$ , then  $\mathfrak{H}(S_1)$  is a common invariant subspace for  $T_1, \dots, T_N$ .*

*Proof.* The operators  $T'_1, \dots, T'_N$  associated with  $\mathfrak{H}(S_1)$  as in Theorem 3.1 are also unique by the last statement in Theorem 3.1. For every  $h \in \mathfrak{H}(S_1)$ ,

$$\sum_{j=1}^N z_j(T_j h)(z) = h(z) - h(0) = \sum_{j=1}^N z_j(T'_j h)(z).$$

Therefore  $T_j h = T'_j h \in \mathfrak{H}(S_1)$  for all  $j = 1, \dots, N$ , which proves the assertion.  $\square$

Hilbert space coefficient spaces  $\mathfrak{F}$  and  $\mathfrak{G}$ , in one sense, are already sufficient to model arbitrary tuples of operators. We give one simple result that extends the notion of a universal model in the case  $N = 1$ . The idea of a universal model originates with Rota [22]. The case  $N = 1$  of the following result is given, for example, in [21, §1.5].

**Universal Model.** *Up to unitary equivalence and nonzero multiplicative constants, the class of operators  $T_1, \dots, T_N$  on a space  $\mathfrak{H}(S)$  which satisfy conditions (1) and (2) of Theorem 3.1, with Hilbert space coefficient spaces  $\mathfrak{F}$  and  $\mathfrak{G}$ , includes every tuple of  $N$  bounded linear operators on a Hilbert space.*

This is a corollary of Example 2 in Section 5 with  $K = I$ . Indeed, let  $A_1, \dots, A_N$  be any bounded operators on a Hilbert space  $\mathfrak{K}$ . By multiplying by scalars, we may assume that  $I - \sum_{j=1}^N A_j^* A_j$  is nonnegative and invertible. Hence there is a nonnegative invertible operator  $C$  on  $\mathfrak{K}$  such that

$$I - \sum_{j=1}^N A_j^* A_j = C^2.$$

The construction in Example 2 in Section 5 now yields a Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  for some Hilbert spaces  $\mathfrak{F}$  and  $\mathfrak{G}$ , and an isomorphism from  $\mathfrak{K}$  onto  $\mathfrak{H}(S)$  relative to which  $A_1, \dots, A_N$  are unitarily equivalent to operators  $T_1, \dots, T_N$  on  $\mathfrak{H}(S)$  which satisfy conditions (1) and (2) of Theorem 3.1. This verifies the assertion. We note in addition that the Schur multiplier  $S(z)$  constructed in this way has additional properties:

- (i) Equality holds in condition (2) in Theorem 3.1.
- (ii) The only  $h(z)$  in  $\mathfrak{H}(S)^N$  such that  $zh(z) \equiv 0$  is the function  $h(z) \equiv 0$ .
- (iii) The only  $f$  in  $\mathfrak{F}$  such that  $S(z)f \equiv 0$  is  $f = 0$ .

Recall that (ii) is required as a hypothesis in Theorem 4.3.



When  $N = 1$ , a version of the model with Pontryagin coefficient spaces  $\mathfrak{F}$  and  $\mathfrak{G}$  is given in [5, §4.5]. We shall not pursue more precise versions of the Universal Model here but hope to discuss this idea in another place.

### 5. Examples

**Example 1.** We illustrate Theorem 3.2 by determining all Schur multipliers  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ , where  $\mathfrak{F}$  and  $\mathfrak{G}$  are Pontryagin spaces having the same negative index, such that  $\dim \mathfrak{H}(S) = 1$ .

A one-dimensional space  $\mathfrak{H}$  satisfies the conditions of Theorem 3.2 if and only if it consists of all constant multiples of a function

$$h_0(z) = \frac{g_0}{1 - \langle z, a \rangle}, \tag{5.1}$$

where  $\|h_0\|_{\mathfrak{H}} = 1$ ,  $a \in \mathbf{C}^N$ ,  $g_0 \in \mathfrak{G}$  ( $g_0 \neq 0$ ), and

$$\langle g_0, g_0 \rangle_{\mathfrak{G}} \leq 1 - \langle a, a \rangle. \tag{5.2}$$

In fact, if  $T_j$  is multiplication by  $\bar{a}_j$ ,  $j = 1, \dots, N$ , conditions (1) and (2) of Theorem 3.2 are equivalent to (5.1) and (5.2) with  $g_0 = h_0(0)$ . For such a space  $\mathfrak{H}$ , the only tuple  $h(z)$  in  $\mathfrak{H}^N$  such that  $zh(z) \equiv 0$  is  $h(z) \equiv 0$ .

By Theorem 3.2,  $\mathfrak{H} = \mathfrak{H}(S)$  for some Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  where  $\mathfrak{F}$  is a Pontryagin space such that  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . We show how to find  $\mathfrak{F}$  and  $S(z)$  by the method in the proof of Theorem 3.2. We use the same notation as in that proof, except that, for simplicity, we identify  $\mathfrak{H}$  and  $\mathfrak{H}^N$  with  $\mathbf{C}$  and  $\mathbf{C}^N$  in the natural way. The main problem is to factor

$$I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* = \begin{pmatrix} I - TT^* & -TG^* \\ -GT^* & I - GG^* \end{pmatrix} : \begin{pmatrix} \mathbf{C}^N \\ \mathfrak{G} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{C}^N \\ \mathfrak{G} \end{pmatrix}$$

as in (3.2) and (3.3). Writing  $g_0^*$  for the linear functional on  $\mathfrak{G}$  such that  $g_0^*g = \langle g, g_0 \rangle_{\mathfrak{G}}$ ,  $g \in \mathfrak{G}$ , we obtain

$$I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* = \begin{pmatrix} I - a^*a & -a^*g_0^* \\ -g_0a & I - g_0g_0^* \end{pmatrix}. \tag{5.3}$$

**Case 1:**  $\langle a, a \rangle \neq 1$ .

When  $\langle a, a \rangle > 1$ ,  $I - a^*a$  has eigenvalue 1 with multiplicity  $N - 1$  and eigenvalue  $1 - \langle a, a \rangle < 0$  with multiplicity 1. By similarly examining the case  $\langle a, a \rangle < 1$ , we can write

$$I - a^*a = \beta\beta^*,$$

where  $\beta$  is an operator from a Pontryagin space  $\mathfrak{F}_1$  to  $\mathbf{C}^N$ ,  $\ker \beta = \{0\}$ , and

$$\text{ind}_- \mathfrak{F}_1 = \begin{cases} 0 & \text{if } \langle a, a \rangle < 1, \\ 1 & \text{if } \langle a, a \rangle > 1. \end{cases}$$

If  $\delta = 1 - \langle a, a \rangle$ , then  $\beta\beta^*a^* = (I - a^*a)a^* = a^*\delta$  and  $a\beta\beta^* = \delta a$ , and so by (5.3),

$$I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* = \begin{pmatrix} \beta & 0 \\ -\frac{g_0 a \beta}{\delta} & D \end{pmatrix} \begin{pmatrix} \beta^* & -\frac{\beta^* a^* g_0^*}{\delta} \\ 0 & D^* \end{pmatrix},$$

where  $D$  is an operator on a Pontryagin space  $\mathfrak{D}$  to  $\mathfrak{G}$  such that  $\ker D = \{0\}$  and

$$DD^* = I - \frac{g_0 g_0^*}{\delta}.$$

**Lemma 5.1.** *In this situation,*

$$\text{ind}_- \mathfrak{D} = \begin{cases} \text{ind}_- \mathfrak{G} & \text{if } \langle a, a \rangle < 1, \\ \text{ind}_- \mathfrak{G} - 1 & \text{if } \langle a, a \rangle > 1. \end{cases}$$

The proof is given below. Thus  $\mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{D}$  is a Pontryagin space satisfying  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . A factorization satisfying (3.2) and (3.3) is obtained with the operators on  $\mathfrak{F}$  defined by

$$F = \begin{pmatrix} \beta & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} -\frac{g_0 a \beta}{\delta} & D \end{pmatrix}.$$

Under the present assumptions,  $\dim \mathfrak{F}_1 = N$  and  $\beta$  is invertible. We obtain the Schur multiplier

$$\begin{aligned} S(z) &= H + G(I - zT)^{-1}(zF) \\ &= \begin{pmatrix} -\frac{g_0 a \beta}{\delta} & D \end{pmatrix} + g_0 \frac{1}{1 - \langle z, a \rangle} \begin{pmatrix} z\beta & 0 \end{pmatrix} \\ &= \begin{pmatrix} g_0 \left\{ -\frac{a}{\delta} + \frac{z}{1 - \langle z, a \rangle} \right\} \beta & D \end{pmatrix} \\ &= \begin{pmatrix} g_0 \frac{z - a}{1 - \langle z, a \rangle} \beta^{*-1} & D \end{pmatrix}. \end{aligned}$$

These functions reduce to Blaschke-Potapov factors in the case  $N = 1$ . We remark that in the case  $\langle a, a \rangle < 1$ , a different approach to these  $N$ -variable generalizations is given in [3, 8].

**Case 2:**  $\langle a, a \rangle = 1$ .

By (5.2),  $\langle g_0, g_0 \rangle_{\mathfrak{G}} \leq 0$ , and hence  $\tau = 1/(1 - \langle g_0, g_0 \rangle_{\mathfrak{G}})$  is a number in  $(0, 1]$ . Then  $I - \tau a^* a \geq 0$ , and hence we can write

$$I - \tau a a^* = \beta \beta^*,$$

where  $\beta$  is an operator from a Hilbert space  $\mathfrak{F}_\tau$  to  $\mathbf{C}^N$  with zero kernel. Let  $k$  be one of the (real) solutions of the quadratic equation

$$k^2 \langle g_0, g_0 \rangle_{\mathfrak{G}} - 2k + 1 = 0.$$

Then using Schur complements and (5.3) we find that

$$\begin{aligned} I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* &= \begin{pmatrix} I & -\tau a^* g_0^* \\ 0 & I \end{pmatrix} \begin{pmatrix} I - \tau a^* a & 0 \\ 0 & I - g_0 g_0^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -\tau g_0 a & I \end{pmatrix} \\ &= \begin{pmatrix} F \\ H \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}^*, \end{aligned}$$

where

$$\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} I & -\tau a^* g_0^* \\ 0 & I \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & I - k g_0 g_0^* \end{pmatrix} = \begin{pmatrix} \beta & ((k-1)\tau - k)a^* g_0^* \\ 0 & I - k g_0 g_0^* \end{pmatrix}$$

is an operator on the space  $\mathfrak{F} = \mathfrak{F}_\tau \oplus \mathfrak{G}$ ,  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ , satisfying (3.2) and (3.3). The required Schur multiplier is given by

$$\left( \begin{array}{c|c} \frac{g_0 z \beta}{1 - \langle z, a \rangle} & I - g_0 g_0^* \left\{ \frac{k + \tau(1-k)\langle z, a \rangle}{1 - \langle z, a \rangle} \right\} \end{array} \right).$$

*Remark 5.2.* The case  $\langle a, a \rangle = 1$  and  $\langle g_0, g_0 \rangle_{\mathfrak{G}} = 0$  is of special interest. Then  $\tau = 1$ ,  $k = \frac{1}{2}$ , and

$$S(z) = \left( \begin{array}{c|c} \frac{g_0 z \beta}{1 - \langle z, a \rangle} & I - \frac{g_0 g_0^*}{2} \frac{1 + \langle z, a \rangle}{1 - \langle z, a \rangle} \end{array} \right).$$

The  $N = 1$  case reduces to Brune sections. The multi-variable versions of Brune sections that appear here are used in a boundary interpolation problem in [7]. These examples are possible only when  $\mathfrak{G}$  is an indefinite space.

**Example 2.** The next class of examples is based on a generalized Stein equation. Similar examples are treated in [3] by another method.

Let  $K, A_1, \dots, A_N \in \mathfrak{L}(\mathfrak{K})$  and  $C \in \mathfrak{L}(\mathfrak{K}, \mathfrak{G})$ , where  $\mathfrak{K}$  is a Hilbert space and  $\mathfrak{G}$  is a Pontryagin space, and assume that

$$K - \sum_{j=1}^N A_j^* K A_j = C^* C. \tag{5.4}$$

Assume also that  $K$  is invertible, and the only  $k \in \mathfrak{K}$  such that  $C A_j^n k = 0$  for all  $j = 1, \dots, N$  and  $n \geq 0$  is  $k = 0$ . We can use more compact notation by writing

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}.$$

Thus, for example, (5.4) becomes  $K - A^* K A = C^* C$ , and  $zA = \sum_{j=1}^N z_j A_j$ .

The preceding assumptions allow us to define a Hilbert space  $\mathfrak{H}$  as the set of functions analytic on some common neighborhood of the origin of the form

$$h(z) = C(I - zA)^{-1} K^{-\frac{1}{2}} k, \quad k \in \mathfrak{K},$$

with  $\|h\|_{\mathfrak{H}} = \|k\|_{\mathfrak{K}}$ . For each  $j = 1, \dots, N$ , define an operator  $T_j$  on  $\mathfrak{H}$  by

$$T_j: C(I - zA)^{-1}K^{-\frac{1}{2}}k \rightarrow C(I - zA)^{-1}A_jK^{-\frac{1}{2}}k, \quad k \in \mathfrak{K}.$$

Conditions (1) and (2) of Theorem 3.2 are satisfied with equality in (2) by (5.4). To exhibit a factorization (3.2), it is convenient to identify  $\mathfrak{H}$  and  $\mathfrak{H}^N$  with  $\mathfrak{K}$  and  $\mathfrak{K}^N$  in the obvious way. With these identifications,

$$\begin{aligned} T &= K^{\frac{1}{2}}AK^{-\frac{1}{2}}, & T^* &= K^{-\frac{1}{2}}A^*K^{\frac{1}{2}}, \\ G &= CK^{-\frac{1}{2}}, & G^* &= K^{-\frac{1}{2}}C^*. \end{aligned}$$

We obtain

$$\begin{aligned} I - \begin{pmatrix} T \\ G \end{pmatrix} \begin{pmatrix} T \\ G \end{pmatrix}^* &= \begin{pmatrix} I - TT^* & -TG^* \\ -GT^* & I - GG^* \end{pmatrix} \\ &= \begin{pmatrix} I - K^{\frac{1}{2}}AK^{-1}A^*K^{\frac{1}{2}} & -K^{\frac{1}{2}}AK^{-1}C^* \\ -CK^{-1}A^*K^{\frac{1}{2}} & I - CK^{-1}C^* \end{pmatrix} \\ &= \begin{pmatrix} F \\ H \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}^*, \end{aligned}$$

where  $F$  and  $H$  act on  $\mathfrak{F} = \mathfrak{G} \oplus \mathfrak{K}^N$  and are given by

$$\begin{aligned} F &= (-K^{\frac{1}{2}}AK^{-1}C^*, \quad I - K^{\frac{1}{2}}AK^{-1}A^*K^{\frac{1}{2}}), \\ H &= (I - CK^{-1}C^*, \quad -CK^{-1}A^*K^{\frac{1}{2}}). \end{aligned}$$

The verifications here make repeated use of the identity  $K - A^*KA = C^*C$  and are straightforward. Clearly  $\text{ind}_- \mathfrak{F} = \text{ind}_- \mathfrak{G}$ . By Remark 3.3 a Schur multiplier  $S(z)$  with values in  $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$  such that  $\mathfrak{H}$  is equal isometrically to  $\mathfrak{H}(S)$  is given by

$$\begin{aligned} S(z) &= H + G(I - zT)^{-1}(zF) \\ &= (I - CK^{-1}C^*, \quad -CK^{-1}A^*K^{\frac{1}{2}}) \\ &\quad + CK^{-\frac{1}{2}}(I - zK^{\frac{1}{2}}AK^{-\frac{1}{2}})^{-1}z(-K^{\frac{1}{2}}AK^{-1}C^*, \quad I - K^{\frac{1}{2}}AK^{-1}A^*K^{\frac{1}{2}}) \\ &= (I, \quad 0) + C(I - zA)^{-1}K^{-1}(-C^*, \quad (z - A^*)K^{\frac{1}{2}}). \end{aligned}$$

We do not assert, however, that this choice of  $S(z)$  necessarily has the property in the last statement in Theorem 3.2, since we cannot say that the factorization (3.2) which is used in the construction satisfies (3.3).

A recent paper of Constantinescu, Sayed, and Kailath [13, pp. 166–168] uses equations of the form (5.4) in a Nevanlinna-Pick interpolation problem for several noncommuting variables.

**Example 3.** For  $N \geq 2$  there are no scalar-valued Schur multipliers  $S(z) \neq \text{const.}$  such that  $\mathcal{H}(S)$  is contained isometrically in  $H_{\mathbf{C}}(B_N)$ . For example, suppose that

$N = 2$  and that  $S(z)$  is a scalar-valued Schur multiplier such that  $\mathcal{H}(S)$  is contained isometrically in  $H_{\mathbf{C}}(B_N)$ . By the identity

$$\frac{1}{1 - \langle z, w \rangle} = \frac{1 - S(z)\overline{S(w)}}{1 - \langle z, w \rangle} + \frac{S(z)\overline{S(w)}}{1 - \langle z, w \rangle},$$

the Hilbert space with reproducing kernel  $S(z)\overline{S(w)}/(1 - \langle z, w \rangle)$  is also contained isometrically in  $H_{\mathbf{C}}(B_N)$ , or, what is the same thing, multiplication by  $S(z)$  maps  $H_{\mathbf{C}}(B_N)$  isometrically into itself. In particular,

$$\|S(z)z_1\|_{H_{\mathbf{C}}(B_N)} = \|z_1\|_{H_{\mathbf{C}}(B_N)}, \quad \|S(z)z_2\|_{H_{\mathbf{C}}(B_N)} = \|z_2\|_{H_{\mathbf{C}}(B_N)},$$

$$\|S(z)1\|_{H_{\mathbf{C}}(B_N)} = \|1\|_{H_{\mathbf{C}}(B_N)}.$$

Write  $S(z) = \sum_{m,n=0}^{\infty} S_{mn}z_1^m z_2^n$ . By a standard formula for the norm in  $H_{\mathbf{C}}(B_N)$  (for example, see [8]),

$$\begin{aligned} 1 &= \sum_{m,n=0}^{\infty} |S_{mn}|^2 \frac{m!n!}{(m+n)!}, \\ 1 &= \sum_{m,n=0}^{\infty} |S_{mn}|^2 \frac{(m+1)!n!}{(m+1+n)!}, \\ 1 &= \sum_{m,n=0}^{\infty} |S_{mn}|^2 \frac{m!(n+1)!}{(m+n+1)!}. \end{aligned}$$

Since

$$\frac{(m+1)!n!}{(m+1+n)!} < \frac{m!n!}{(m+n)!},$$

for  $n \neq 0$ , the first two equalities imply that  $S_{mn} = 0$  for  $n \neq 0$ . Similarly,  $S_{mn} = 0$  for  $m \neq 0$ . Thus  $S(z)$  is a constant of modulus one.

There exist nonscalar-valued Schur multipliers  $S(z) \not\equiv \text{const.}$  such that  $\mathfrak{H}(S)$  is contained isometrically in  $H_{\mathbf{C}}(B_N)$ : one-dimensional spaces  $\mathfrak{H}(S)$  which are contained isometrically in  $H_{\mathbf{C}}(B_N)$  can be constructed from Example 1 above.

*Proof of Lemma 5.1.* Set  $\Delta = I - \delta^{-1}g_0g_0^*$ . Thus  $\Delta \in \mathfrak{L}(\mathfrak{G})$  and  $\Delta = DD^*$ .

**Case 1:**  $\langle a, a \rangle > 1$ . In this case,  $\langle g_0, g_0 \rangle_{\mathfrak{G}} \leq \delta < 0$  by (5.2). The span  $\mathfrak{G}_0$  of  $g_0$  is a regular subspace of  $\mathfrak{G}$ , hence  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ , where  $\text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G} - 1$ . Write

$$\Delta = \begin{pmatrix} \Delta_0 & 0 \\ 0 & I_{\mathfrak{G}_1} \end{pmatrix}, \tag{5.5}$$

$$\langle \Delta_0(cg_0), cg_0 \rangle_{\mathfrak{G}} = [1 - \delta^{-1}\langle g_0, g_0 \rangle_{\mathfrak{G}}] |c|^2 \langle g_0, g_0 \rangle_{\mathfrak{G}}, \quad c \in \mathbf{C}. \tag{5.6}$$

Since  $1 - \delta^{-1}\langle g_0, g_0 \rangle_{\mathfrak{G}} \leq 0$  and  $\langle g_0, g_0 \rangle_{\mathfrak{G}} < 0$ , it follows that  $\Delta_0 \geq 0$ , and hence  $\text{ind}_- \mathfrak{D} = \text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G} - 1$ .

**Case 2:**  $\langle a, a \rangle < 1$ . Then  $\delta > 0$ . If  $\langle g_0, g_0 \rangle_{\mathfrak{G}} \neq 0$ , then  $g_0$  spans a regular subspace  $\mathfrak{G}_0$  of  $\mathfrak{G}$ , and we can write  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  and (5.5)–(5.6) in the same way.

If  $\langle g_0, g_0 \rangle_{\mathfrak{G}} < 0$ , then  $\text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G} - 1$  and  $1 - \delta^{-1} \langle g_0, g_0 \rangle_{\mathfrak{G}} > 0$ . Hence

$$\langle \Delta_0(cg_0), cg_0 \rangle_{\mathfrak{G}} \leq 0, \quad c \in \mathbf{C},$$

with strict inequality for  $c \neq 0$ . Therefore  $\text{ind}_- \mathfrak{D} = 1 + \text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G}$ .

If  $\langle g_0, g_0 \rangle_{\mathfrak{G}} > 0$ , then  $\text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G}$ . By (5.2),  $0 < \langle g_0, g_0 \rangle_{\mathfrak{G}} \leq \delta$ . In this case,  $\Delta_0 \geq 0$  and  $\text{ind}_- \mathfrak{D} = \text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G}$ .

For the case  $\langle g_0, g_0 \rangle_{\mathfrak{G}} = 0$ , we write  $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$  in a different way. Since  $g_0 \neq 0$ , we may suppose that  $\mathfrak{G}_0$  is  $\mathbf{C}^2$  in the inner product

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathfrak{G}_0} = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

for all  $u_1, v_1, u_2, v_2 \in \mathbf{C}$ , and

$$\frac{g_0}{\sqrt{\delta}} = \begin{pmatrix} \gamma \\ \gamma \end{pmatrix}, \quad \gamma \neq 0.$$

Thus  $\text{ind}_- \mathfrak{G}_0 = 1$  and  $\text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G} - 1$ . We obtain a decomposition (5.5), where for any  $x, y \in \mathbf{C}$ ,

$$\begin{aligned} \Delta_0 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \right\rangle_{\mathfrak{G}_0} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - |\gamma|^2 & |\gamma|^2 \\ -|\gamma|^2 & 1 + |\gamma|^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \left\langle \Delta_0 \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{\mathfrak{G}_0} &= \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 - |\gamma|^2 & |\gamma|^2 \\ -|\gamma|^2 & 1 + |\gamma|^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} 1 - |\gamma|^2 & |\gamma|^2 \\ |\gamma|^2 & -1 - |\gamma|^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Since

$$\det \begin{pmatrix} 1 - |\gamma|^2 & |\gamma|^2 \\ |\gamma|^2 & -1 - |\gamma|^2 \end{pmatrix} = -1,$$

the matrix has one positive and one negative eigenvalue. Therefore in this case, we obtain  $\text{ind}_- \mathfrak{D} = 1 + \text{ind}_- \mathfrak{G}_1 = \text{ind}_- \mathfrak{G}$ .  $\square$

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