METHODS OF KREĬN SPACE OPERATOR THEORY

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This paper is a survey of old and recent methods of Kreĭn space operator theory centering around Julia operators, extension problems for contraction operators, Hermitian kernels, and uniqueness questions. Examples related to coefficient problems for univalent functions are briefly discussed.

1. Introduction

The author was originally led to Kreĭn space operator theory by a problem of L. de Branges concerning the coefficients of univalent functions. The particular question was resolved in the negative, but the operator methods used to show this are related to other areas which remain currently active, such as the study of generalized Schur and Nevanlinna functions. The methods are of a general nature and based on familiar Hilbert space concepts, including contraction operators, their dilations, and reproducing kernel spaces. Today the Kreĭn space counterparts of many of these ideas are complete to a high degree. As always, there are difficulties and new issues in the indefinite theory. For example, it turns out that uniqueness questions play a more important role in the indefinite theory than in the definite case. In this paper we survey some old and recent results in these areas, with an aim to show that tools which have found wide applicability in Hilbert space problems are also available in Kreĭn space operator theory.

In outline, the contents are as follows:

§2. *Examples from function theory*

Generalizations of the Dirichlet space yield interesting examples, including contraction operators on indefinite inner product spaces defined by substitution by normalized univalent functions. Multiplication by the independent variable on similar spaces gives examples of indefinite two-isometries as studied by Agler, Richter, and others.

§3. *Definitions and basic notions*

Basic ideas are discussed here in order to make the paper self-contained.

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§4. *Three useful tools of Kreĭn space operator theory*

Our goal is to adapt Hilbert space methods to Kreĭn space operators, but some elementary constructions break down when positivity is abandoned. Here we show that there are simple replacements in the indefinite theory. For example, the replacement for the Hilbert space construction of a nonnegative square root of a nonnegative operator is a factorization of any selfadjoint operator C on a Kreĭn space $\mathfrak{H}$ in the form $C = AA^*$ where $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$ for some Kreĭn space $\mathfrak{A}$ and $\ker A = \{0\}$. Factorizations of this type are one of the main themes of this survey. Though elementary, they are extremely useful.

§5. *Julia operators and extension problems*

In §5.1 and §5.2, we discuss Julia operators and the most basic kinds of row, column, and matrix completions. In §5.3, we contrast several forms of commutant lifting in the indefinite setting.

§6. *Uniqueness questions*

A selfadjoint operator $C \in \mathfrak{L}(\mathfrak{H})$ is said to have the unique factorization property if the representation $C = AA^*$, $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$, described above can only be changed by replacing the Kreĭn space $\mathfrak{A}$ by an isomorphic copy. We give necessary and sufficient conditions for uniqueness and identify situations in which uniqueness is automatic.

§7. *Kolmogorov decompositions of Hermitian kernels*

L. Schwarz introduced a number of elegant ideas into Kreĭn space operator theory in a 1964 paper, but they have become mainstream only more recently. Here we present the ideas in the form of the theory of Hermitian kernels. Particular cases include finite and infinite block operator matrices and reproducing kernels. A Hermitian kernel is a collection of Kreĭn space operators $K_{ij} = K_{ji}^* \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{H}_i)$, $i, j \in J$. A Kolmogorov decomposition is a representation in the form

$$K_{ij} = V_i^* V_j, \qquad i, j \in J,$$

where $V_j \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{K})$, $j \in J$, for some Kreĭn space $\mathfrak{K}$ such that $\mathfrak{K} = \bigvee_{j \in J} V_j \mathfrak{H}_j$. The general theory is concerned with criteria for existence and uniqueness. Our account is expository and follows recent work of Constantinescu and Gheondea.

§8. *Examples of Hermitian kernels*

The theory of Kolmogorov decompositions is illustrated with reproducing kernel spaces and holomorphic kernels. Another special case yields criteria for existence and uniqueness of completions of pre-Kreĭn spaces, which behave differently from pre-Hilbert spaces.

§9. *The contractive substitution property*

We return to the coefficient problems discussed in §2 and show, by numerical evidence, that the contractive substitution property, while not sufficient to characterize coefficients, nevertheless does an excellent job constraining low order coefficients. Some open questions are stated.

Related topics appear in the six lectures of Dritschel and Rovnyak [37]. Definitive accounts of the general theory of operators on indefinite inner product spaces, along with authoritative literature notes, are given in the books by Azizov and Iokhvidov [12], Bognár [14], and Iokhvidov, Kreĭn, and Langer [47]. Azizov, Ginzburg, and Langer [11] discuss M. G. Kreĭn's vision and contributions in this area. These and other sources should be consulted to see the great diversity of Kreĭn space operator theory and something of the many topics that are omitted here.

2. Examples from function theory

We give some examples which arise from coefficient problems for univalent functions. For the author personally, these examples were a compelling reason to undertake learning the indefinite theory. A deeper understanding of them is a long-range goal and challenge for the subject.

A holomorphic function $f(z)$ is **univalent** if it takes distinct values at distinct points. Coefficient problems play a central role in the theory of univalent functions which are defined on the unit disk $\mathbf{D} = \{z : |z| < 1\}$. A highlight of the theory is de Branges' proof [16] of the Bieberbach conjecture: *Let $f(z)$ be univalent on $\mathbf{D}$ and normalized so that $f(0) = 0$ and $f'(0) > 0$. If $f(z) = a_1 z + a_2 z^2 + \cdots$, then $|a_n| \le n a_1$ for all $n \ge 2$.* The inequality, however, is satisfied by many functions which are not univalent. Ideally we would like to find stronger conditions which are more characteristic of univalent functions. We restrict attention to the subclass of functions which are bounded by one in $\mathbf{D}$. The following problem is classical.

Coefficient Interpolation Problem: *For any positive integer r, characterize all complex numbers $B_1, \dots, B_r$ $(B_1 > 0)$ such that there exists a univalent and normalized function $B(z)$ satisfying $|B(z)| \le 1$ on $\mathbf{D}$ and such that $B(z) = B_1 z + \cdots + B_r z^r + \mathcal{O}(z^{r+1})$.*

Necessary conditions follow from a generalized form of the area theorem. Assume that such a function $B(z)$ exists for given numbers $B_1, \dots, B_r$ $(B_1 > 0)$. For any real number ν, consider an arbitrary generalized power series

$$h(z) = a_1 z^{\nu+1} + a_2 z^{\nu+2} + \cdots \tag{2.1}$$

with complex coefficients (constants terms, which arise when ν is a negative integer, are identified to zero). Define $h(B(z)) = b_1 z^{\nu+1} + b_2 z^{\nu+2} + \cdots$ by formal substitution. Then

$$\sum_{n=1}^{r} (\nu+n)|b_n|^2 \le \sum_{n=1}^{r} (\nu+n)|a_n|^2.$$

Equivalently,

$$\langle h(B(z)), h(B(z)) \rangle_{\mathfrak{D}_r^\nu} \le \langle h(z), h(z) \rangle_{\mathfrak{D}_r^\nu}, \tag{2.2}$$

where $\mathfrak{D}_r^\nu$ is the linear space of series (2.1) in the inner product

$$\langle h(z), h(z) \rangle_{\mathfrak{D}_r^\nu} = \sum_{n=1}^{r} (\nu+n)|a_n|^2.$$

The inequality (2.2) is proved by de Branges [17] when $\nu \ge -r-1$, and the restriction on ν is removed by Li and Rovnyak [51]; for a proof, see [65, Section 7.5]. Nikolskii and Vasyunin [59, 60] give another view of these inequalities and explain their connection with subordination (see Section P45, p. 1202, in the English translation of [60]); see also Ghosechowdhury [43, 44] and Rovnyak [67]. The conditions (2.2) depend only on $B_1, \dots, B_r$ and are thus necessary conditions on these numbers for the existence of an interpolating function $B(z)$. It is natural to ask if the necessary conditions are sufficient:

Problem (de Branges [17, 19])**.** *Let $B_1, \dots, B_r$ be complex numbers with $B_1 > 0$ such that (2.2) holds for all real numbers ν and all generalized power series (2.1). Does it follow that $B(z) = B_1 z + \cdots + B_r z^r + \mathcal{O}(z^{r+1})$ where $B(z)$ is univalent and $|B(z)| \le 1$ on $\mathbf{D}$?*

The simple answer is negative (see §9).

The main point here, however, is that we obtain a large class of examples of contraction operators. Namely, by (2.2) the operator

$$T : h(z) \to h(B(z)) \tag{2.3}$$

is a contraction on the space $\mathfrak{D}_r^\nu$ for any positive integer r, any real number ν, and any function $B(z)$ which is univalent, normalized, and bounded by one in $\mathbf{D}$. The space $\mathfrak{D}_r^\nu$ is indefinite when $\nu < -1$. In the same way, (2.3) acts as a contraction in the infinite-dimensional space $\mathfrak{D}^\nu$ of series (2.1) such that $\sum_{n=1}^\infty (\nu+n)|a_n|^2 < \infty$ in the inner product

$$\langle h(z), h(z)\rangle_{\mathfrak{D}^\nu} = \sum_{n=1}^\infty (\nu+n)|a_n|^2,$$

and this inner product is indefinite when $\nu < -1$. Another interesting example in $\mathfrak{D}^\nu$ is multiplication by z:

$$S\colon\, f(z) \to zf(z).$$

In the classical case ($\nu = 0$), this is the Dirichlet shift. In general, S is a two-isometry in the sense that $S^{*2}S^2 - 2S^*S + 1 = 0$, or in terms of inner products,

$$\left\langle z^2 f(z), z^2 f(z)\right\rangle_{\mathfrak{D}^\nu} - 2\langle zf(z), zf(z)\rangle_{\mathfrak{D}^\nu} + \langle f(z), f(z)\rangle_{\mathfrak{D}^\nu} = 0$$

for all $f(z)$ in $\mathfrak{D}^\nu$. Two-isometries and more general operators on Hilbert spaces are studied by Agler and Stankus [1]. A two-isometry is called analytic if the intersection of the ranges of its powers is zero. Richter [63] constructed a model theory for cyclic analytic two-isometries on a Hilbert space, the Dirichlet shift being the motivating example [62, 64]. The beginnings of an indefinite theory have been made by Chris Hellings [46]. See also McCullough and Rodman [54, 55], who earlier proposed to extend Agler's ideas into the indefinite domain.

Such examples suggest the need for an approach that emphasizes the analogies with the Hilbert space case, and our purpose here is to outline such a viewpoint.

3. Definitions and basic notions

Inner products are assumed to be linear and symmetric. The antispace of an inner product space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ is $(\mathfrak{H}, -\langle\cdot,\cdot\rangle)$.

As we use the term, a **Kreĭn space** is an inner product space which is expressible as an orthogonal direct sum $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ of a Hilbert space $\mathfrak{H}_+$ and the antispace $\mathfrak{H}_-$ of a Hilbert space (for simplicity, Hilbert spaces are assumed to be separable). Any such representation is a **fundamental decomposition**. The induced Hilbert space topology is the **strong topology** of $\mathfrak{H}$. The dimensions of $\mathfrak{H}_\pm$ are the **indices** of $\mathfrak{H}$. A Kreĭn space is also called a **Pontryagin space** if it has finite negative index. These definitions do not depend on the choice of fundamental decomposition. When nothing is said, underlying spaces are assumed to be Kreĭn spaces (which might be Pontryagin spaces or finite-dimensional).

Spaces $\mathfrak{L}(\mathfrak{H})$ and $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ of continuous operators and adjoint operators are defined for Kreĭn spaces in the same way as for Hilbert spaces. Thus if $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, then $A^* \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ and $\langle Af, g\rangle = \langle f, A^*g\rangle$ for all f in $\mathfrak{H}$ and g in $\mathfrak{K}$. An operator $A \in \mathfrak{L}(\mathfrak{H})$ is

selfadjoint if $A^* = A$,
a **projection** if A is selfadjoint and $A^2 = A$, and
nonnegative if $\langle Af, f\rangle \ge 0$ for every $f \in \mathfrak{H}$.

If $A \in \mathfrak{L}(\mathfrak{H})$ is selfadjoint, let $\operatorname{ind}_+ A$ ($\operatorname{ind}_- A$) be the supremum of all r such that there exists an r-dimensonal subspace of $\mathfrak{H}$ which is a Hilbert space (antispace of a Hilbert space) in the inner product $\langle f, g \rangle_A = \langle Af, g \rangle$, $f, g \in \mathfrak{H}$. An operator $B \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is

> **isometric** if $B^*B = 1_{\mathfrak{H}}$,
> **partially isometric** if $BB^*B = B$,
> **unitary** if both B and B^* are isometric,
> a **contraction** if $B^*B \le 1_{\mathfrak{H}}$, and
> a **bicontraction** if both B and B^* are contractions.

An **isomorphism** of inner product spaces is a one-to-one and onto linear mapping which preserves inner products. As in the Hilbert space case, the class of isomorphisms between two Kreĭn spaces $\mathfrak{H}$ and $\mathfrak{K}$ coincides with the set of unitary operators between the spaces.

Orthogonality is defined for Kreĭn spaces as for Hilbert spaces. The relation $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ is not always true for all closed subspaces $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$, however. It is true for an important subclass of subspaces. A linear subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$ is a **Kreĭn subspace**, or a **regular subspace**, if $\mathfrak{M}$ is closed and a Kreĭn space in the inner product of $\mathfrak{H}$. If $\mathfrak{M}$ is a linear subspace of $\mathfrak{H}$, the following assertions are equivalent:

(1) $\mathfrak{M}$ is a Kreĭn subspace;
(2) $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$;
(3) $\mathfrak{M} = \operatorname{ran} P$, where $P \in \mathfrak{L}(\mathfrak{H})$ is a projection operator.

In this case, restriction of the strong topology of $\mathfrak{H}$ to $\mathfrak{M}$ coincides with the strong topology of $\mathfrak{M}$ as a Kreĭn space. For details and other basic notions, see [12, 14, 36, 47].

4. Three useful tools of Kreĭn space operator theory

Kreĭn space operator theory is much like the Hilbert space special case despite failure of some of the most basic notions in the indefinite situation. The explanation is that there are effective substitutes for the missing Hilbert space results.

Tool #1: a factorization theorem for selfadjoint operators.

One of the cornerstones of Hilbert space operator theory is that every nonnegative operator has a nonnegative square root. The Kreĭn space counterpart is a factorization theorem for any selfadjoint operator. The result is old, but its systematic use is more recent [26, 37, 36].

Theorem 4.1. *Every selfadjoint operator $C \in \mathfrak{L}(\mathfrak{H})$, $\mathfrak{H}$ a Kreĭn space, can be written $C = AA^*$ where $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$ for some Kreĭn space $\mathfrak{A}$ and* $\ker A = \{0\}$.

The first step in the proof, reduction to the Hilbert space case, is worth separate notice:

> *Every selfadjoint operator on a Kreĭn space is congruent to a selfadjoint operator on a Hilbert space.*

That is, if $\mathfrak{H}$ is a Kreĭn space and $C \in \mathfrak{L}(\mathfrak{H})$ is a selfadjoint operator, there is a Hilbert space $\mathfrak{K}$, a selfadjoint operator $B \in \mathfrak{L}(\mathfrak{K})$, and an invertible operator $X \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that

$$C = X^*BX.$$

In fact, let X be any invertible operator from $\mathfrak{H}$ onto any Hilbert space $\mathfrak{K}$, and take $B = X^{*-1}CX^{-1}$.

Proof of Theorem 4.1. It is sufficient to prove the theorem when $\mathfrak{H}$ is a Hilbert space. In this case, we can decompose $\mathfrak{H}$ into spectral subspaces for C for the sets $(0,\infty)$, $\{0\}$, $(-\infty,0)$, say $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_-$. Define $\mathfrak{A} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ in the inner product

$$\langle f, g\rangle_{\mathfrak{A}} = \pm\langle f, g\rangle_{\mathfrak{H}}, \qquad f, g \in \mathfrak{H}_\pm.$$

We easily check that the operator A defined by $Af = |C|^{1/2} f$, $f \in \mathfrak{A}$, has the required properties. ☐

Tool #2: extension theorems for densely defined operators.

A different factorization occurs in Hilbert space operator theory. In a typical situation, we are given Hilbert space operators $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{A})$ and $B \in \mathfrak{L}(\mathfrak{H}, \mathfrak{B})$ with $B^*B \le A^*A$. If A has dense range, then the partially defined operator

$$C_0 : Af \to Bf, \qquad f \in \mathfrak{H},$$

has a contractive (hence continuous) extension $C \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ such that $B = CA$. When the underlying spaces are Kreĭn spaces, C_0 may not be well defined (that is, $Af_1 = Af_2$ and $Bf_1 \ne Bf_2$ for some $f_1, f_2 \in \mathfrak{H}$), and even if it is it may not have a continuous extension. See [6, p. 429] for examples.

What is needed is a means to define continuous contraction operators by specifying their action on dense sets. An index condition resolves the difficulties. A **linear relation** from $\mathfrak{H}$ to $\mathfrak{K}$ is a linear subspace $\mathbf{R}$ of $\mathfrak{H} \times \mathfrak{K}$. The domain of $\mathbf{R}$ is the set of all first elements f of the pairs (f, g) in $\mathbf{R}$.

Theorem 4.2. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Pontryagin spaces such that* $\operatorname{ind}_- \mathfrak{H} = \operatorname{ind}_- \mathfrak{K}$. *Let $\mathbf{R}$ be a linear relation such that*

(1) $\mathbf{R}$ *has dense domain,*
(2) $\langle g, g\rangle_{\mathfrak{K}} \le \langle f, f\rangle_{\mathfrak{H}}$ *for all* $(f, g) \in \mathbf{R}$.

Then the closure of $\mathbf{R}$ is the graph of a contraction $C \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.

See [5, 6] for two different proofs ofTheorem 4.2. The known Kreĭn space generalizations of Theorem 4.2 require strong hypotheses which are difficult to verify in applications (Shmul'yan [70], Dritschel and Rovnyak [37, Theorem 1.4.4] and [36, Supplement]). An exception here is the following nice result which is given in Constantinescu and Gheondea [25, Lemma 2.3].

Theorem 4.3. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Kreĭn spaces. Let $\mathbf{R}$ be a linear relation such that*

(1) $\mathbf{R}$ *has dense domain and dense range,*
(2) $\langle g, g\rangle_{\mathfrak{K}} = \langle f, f\rangle_{\mathfrak{H}}$ *for all* $(f, g) \in \mathbf{R}$,
(3) *the domain of* $\mathbf{R}$ *contains one of the subspaces* $\mathfrak{H}_\pm$ *in some fundamental decomposition* $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

Then the closure of $\mathbf{R}$ is the graph of a unitary operator $U \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.

A finite-dimensional example in [6, p. 429] shows that Theorem 4.2 is not valid if $\operatorname{ind}_- \mathfrak{H} \ne \operatorname{ind}_- \mathfrak{K}$. The same example shows that the conclusion of Theorem 4.3 can fail if all conditions are met except the range of $\mathbf{R}$ is not dense.

Typical applications of Theorems 4.2 and 4.3 arise from inequalities $B^*B \le A^*A$, where $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{A})$ and $B \in \mathfrak{L}(\mathfrak{H}, \mathfrak{B})$ are Kreĭn space operators. Under suitable conditions, the linear relation

$$\mathbf{R} = \{(Af, Bf) : f \in \mathfrak{H}\}.$$

satisfies the hypotheses of the theorems. Then we obtain a factorization $B = CA$ with $C \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ a contraction operator or unitary operator, as in the Hilbert space case.

Tool #3: continuous isometries and partial isometries.

Recall that a partial isometry is defined as a Kreĭn space operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that $AA^*A = A$. Such operators have properties much the same as in the Hilbert space case.

Theorem 4.4. *If $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, $\mathfrak{H}$ and $\mathfrak{K}$ Kreĭn spaces, the following assertions are equivalent:*

(1) *A is a partial isometry;*
(2) *A^*A is a projection operator and* $\ker A^*A = \ker A$*;*
(3) *AA^* is a projection operator and* $\ker AA^* = \ker A^*$*;*
(4) *there exist Kreĭn subspaces $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{N}$ of $\mathfrak{K}$ such that A maps $\mathfrak{M}$ in a one-to-one way onto $\mathfrak{N}$ with $\langle Af, Ag\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}}$ for all $f, g \in \mathfrak{M}$, and $Af = 0$ for all $f \in \mathfrak{M}^\perp$.*

In this case, A^*A and AA^* are the projections onto $\mathfrak{M}$ and $\mathfrak{N}$. If, in fact, A is an isometry, then in addition

(5) *A maps closed subspaces of $\mathfrak{H}$ onto closed subspaces of $\mathfrak{K}$;*
(6) *A maps Kreĭn subspaces of $\mathfrak{H}$ onto Kreĭn subspaces of $\mathfrak{K}$.*

In particular, the range of an isometry $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a Kreĭn subspace of $\mathfrak{K}$.

The conditions on kernels in parts (2) and (3) of Theorem 4.4 do not appear in the Hilbert space case because they hold automatically when $\mathfrak{H}$ and $\mathfrak{K}$ are Hilbert spaces. For proofs of the assertions in Theorem 4.4, see the *Supplement and errata* cited in [36, pp. 156–57].

Theorem 4.4 plays a greater role in the indefinite theory than in the special case of Hilbert spaces. It can only be appreciated in the light of pathological examples of "isometries" on Kreĭn spaces: if $\mathfrak{H}$ is an infinite-dimensional Hilbert space and $\mathfrak{K}$ is an infinite dimensional Pontryagin space with $\operatorname{ind}_- \mathfrak{K} = 1$, there exists an everywhere defined linear transformation V on $\mathfrak{H}$ into $\mathfrak{K}$ such that $\langle Vf, Vg\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}}$ for all f and g in $\mathfrak{H}$, yet V is not continuous with respect to the strong topologies of $\mathfrak{H}$ and $\mathfrak{K}$ (for example, see [36, Supplement]). Obviously all manner of bad behavior is to be expected in such a situation, and the point of Theorem 4.4 is that order is restored with the hypothesis of continuity. While our definition of an "isometry" presumes continuity, this practice is not universal, and in other sources the meaning of the term should be verified.

5. Julia operators and extension problems

5.1 Defect and Julia operators

Much of the theory of contraction operators on Hilbert spaces in Sz.-Nagy and Foias [72] carries over to the indefinite setting. Dilation properties and model theory are discussed in Davis [28], Davis and Foias [29] and McEnnis [56, 57, 58]. We focus on more recent developments in the Kreĭn space theory that include notions of defect and Julia operators, matrix extension theorems, and the commutant lifting theorem. In the definite case, the history of results in this area is long and complex and closely connected with interpolation theory; for example, see Foias and Frazho [40]; a recent sequel to this standard source is given in Foias, Frazho, Kaashoek, and Gohberg [41]. The indefinite theory for these areas originates with Constantinescu and Gheondea [22, 24] and Dritschel [32].

Defect and Julia operators play an even greater role in Kreĭn space operator theory than in the Hilbert space case. The first constructions are due to Arsene, Constantinescu, and Gheondea [10]. Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces. By a **defect operator** for T we mean any operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$, where $\tilde{\mathfrak{D}}$ is a Kreĭn space, such that $\ker \tilde{D} = \{0\}$ and the operator

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}) \tag{5.1}$$

is an isometry, that is, $T^*T + \tilde{D}\tilde{D}^* = 1$. A **Julia operator** for T is any unitary operator

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}), \tag{5.2}$$

where $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ are Kreĭn spaces, such that the operators $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$ and $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ have zero kernels. Julia operators are also called **elementary rotations** in the literature.

The preceding definitions of defect and Julia operators apply to any operator $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, and they do not presume that T is a contraction operator. So even when $\mathfrak{H}$ and $\mathfrak{K}$ are Hilbert spaces, the definitions are more general than the standard definitions which are given in the Hilbert space case.

Theorem 5.1. *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces.*

(1) *A defect operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ for T exists, and for any such operator*

$$\operatorname{ind}_\pm \tilde{\mathfrak{D}} = \operatorname{ind}_\pm (1 - T^*T).$$

(2) *A Julia operator $U \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$ for T exists, and for any such operator*

$$\operatorname{ind}_\pm \mathfrak{D} = \operatorname{ind}_\pm (1 - TT^*) \qquad \text{and} \qquad \operatorname{ind}_\pm \tilde{\mathfrak{D}} = \operatorname{ind}_\pm (1 - T^*T).$$

Proof. We obtain (1) by applying Theorem 4.1 to $C = 1 - T^*T$. To prove (2), apply Theorem 4.1 a second time to $C = 1 - VV^*$, where V is given by (5.1). For details, see Dritschel and Rovnyak [36, Theorem 2.3]. □

We give an elementary illustration how Theorem 5.1, combined with the good behavior of isometric and unitary operators, can be used to obtain information about general operators. The result itself is old and has a simple direct proof [10].

Theorem 5.2. *If $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ for any Kreĭn spaces $\mathfrak{H}$ and $\mathfrak{K}$, then*

$$\operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm (1 - TT^*) = \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm (1 - T^*T).$$

In particular, if $\operatorname{ind}_- \mathfrak{H} = \operatorname{ind}_- \mathfrak{K} < \infty$, then $T^*T \le 1$ implies $TT^* \le 1$.

Proof. Choose a Julia operator (5.2) for T. By the unitarity of U and Theorem 5.1(2), $\operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm (1 - TT^*) = \operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm \mathfrak{D} = \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm \tilde{\mathfrak{D}} = \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm (1 - T^*T)$. □

Another basic problem is describe all contractive row, column, and matrix extensions

$$\begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K}),$$
$$\begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G}),$$
$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G}),$$

of a given contraction operator $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces. The problem has several variants, such as dropping the hypothesis that T is a contraction. We can alternatively consider operators T such that $\operatorname{ind}_- (1 - T^*T) < \infty$ and ask for contractive extensions or extensions which also satisfy index conditions.

5.2 Basic extension theorems

Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces. Choose a Julia operator

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T) \tag{5.3}$$

for T. This is, of course, a particular extension of T. When $\mathfrak{H}, \mathfrak{K}, \mathfrak{F}, \mathfrak{G}$ are Hilbert spaces and T is a contraction, it is a well-known result that all contractive row, column, and matrix extensions are given by

$$\begin{pmatrix} T & D_T X \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K}), \tag{5.4}$$

$$\begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G}), \tag{5.5}$$

and

$$\begin{pmatrix} T & D_T X \\ Y^* \tilde{D}_T^* & -Y^* L_T^* X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G}), \tag{5.6}$$

where X, Y, Z are contraction operators on appropriate spaces as required to make the formulas meaningful and $\tilde{D}_X$ and $\tilde{D}_Y$ are defect operators for X and Y.

The next result describes the situation when T is a contraction.

Theorem 5.3. *Assume that $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a contraction, $\mathfrak{H}, \mathfrak{K}, \mathfrak{F}$ are Kreĭn spaces, and $\mathfrak{G}$ is a Hilbert space. Then all contractive row, column, and matrix extensions of T are given by* (5.4), (5.5), *and* (5.6) *again where X, Y, Z are contraction operators on appropriate spaces as required to make the formulas meaningful and $\tilde{D}_X$ and $\tilde{D}_Y$ are defect operators for X and Y.*

The asymmetry in Theorem 5.3 is due to the fact that the adjoint of a contraction operator on Kreĭn spaces is not necessarily a contraction. Thus, for example, the row extension theorem cannot be deduced by applying the column extension to T^*; the row and column extensions need separate proofs. When $\mathfrak{G}$ is a Kreĭn space, the conclusions can fail [36, p. 172]. Nevertheless, a more general result holds and provides another illustration of the role played by index conditions in Kreĭn space operator theory.

When T is not necessarily a contraction, or $\mathfrak{G}$ is not a Hilbert space, similar conclusions hold but with other hypotheses in the form of index conditions.

Theorem 5.4 (Row extensions)**.** *Assume that $\mathfrak{H}, \mathfrak{K}, \mathfrak{F}$ are Kreĭn spaces. Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ (not necessarily a contraction), and let $D_T \in \mathfrak{L}(\mathfrak{D}_T, \mathfrak{H})$ be a defect operator for T^*. Let $R = \begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K})$ be a row extension of T satisfying at least one of the conditions*

$$\operatorname{ind}_- (1 - RR^*) + \operatorname{ind}_- \mathfrak{F} = \operatorname{ind}_- (1 - TT^*) < \infty, \tag{5.7}$$

$$\operatorname{ind}_- (1 - R^*R) = \operatorname{ind}_- (1 - T^*T) < \infty. \tag{5.8}$$

Then R has the form (5.4), where $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction. Conversely, every such operator (5.4) satisfies both of the equalities in (5.7) and (5.8) (with possibly infinite values).

Theorem 5.5 (Column extensions)**.** *Let $\mathfrak{H}, \mathfrak{K}, \mathfrak{G}$ be Kreĭn spaces. Assume that $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and that $\tilde{D}_T \in \mathfrak{L}(\tilde{\mathfrak{D}}_T, \mathfrak{H})$ is a defect operator for T. Let*

$$C = \begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

be a column extension of T satisfying at least one of the conditions

$$\operatorname{ind}_- (1 - C^*C) + \operatorname{ind}_- \mathfrak{G} = \operatorname{ind}_- (1 - T^*T) < \infty, \tag{5.9}$$

$$\operatorname{ind}_- (1 - CC^*) = \operatorname{ind}_- (1 - TT^*) < \infty. \tag{5.10}$$

Then C has the form (5.5), where $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction. Conversely, every such operator (5.5) satisfies both of the equalities in (5.9) and (5.10) (with possibly infinite values).

A similar result holds for matrix extensions of the form (5.6). Dritschel [33] has given a beautiful method of proof of such theorems. The results are first proved in the special case when the given operators are isometries; in this simple case we are able to use what are essentially Hilbert space methods, and these methods work for Kreĭn spaces because by Theorem 4.4 the properties of continuous partial isometries on Kreĭn spaces are much the same as in the Hilbert space case. The second step is to reduce the general results to the case of isometries by means of extensions using defect and Julia operators. It is only necessary to prove Theorems 5.4 and 5.5 and the counterpart for (5.6), as these imply Theorem 5.3; for example, the row and column statements in Theorem 5.3 follow when the equalities in (5.8) and (5.9) hold with the value zero. Full details are given in [36, Lecture 3].

5.3 Commutant lifting

Commutant lifting provides operator extensions with additional properties. Already in the definite case, the commutant lifting theorem has a number of formulations, but the different versions have essentially the same content. In the case of Kreĭn space operators, there are several natural extensions of the commutant lifting theorem. While obviously related, however, they are not easily compared. A survey of this area by itself would be a sizable undertaking, and we limit this discussion to several results and some citations to other sources.

One result simply says that the theorem of Sz.-Nagy and Foias [72] remains true if Hilbert spaces are replaced by Kreĭn spaces. If $\mathfrak{K}$ is a Kreĭn space with Kreĭn subspace $\mathfrak{H}$, let $P_{\mathfrak{H}}$ be the projection operator on $\mathfrak{K}$ with range $\mathfrak{H}$. A **minimal isometric dilation** of an operator $A \in \mathfrak{L}(\mathfrak{H})$, $\mathfrak{H}$ a Kreĭn space, is an isometric operator $U \in \mathfrak{L}(\mathfrak{K})$, where $\mathfrak{K}$ is a Kreĭn space containing $\mathfrak{H}$ as a Kreĭn subspace, such that $A^n = P_{\mathfrak{H}} U^n|_{\mathfrak{H}}$ for all $n = 1, 2, \ldots,$ and $\bigvee_{n=0}^{\infty} U^n \mathfrak{H} = \mathfrak{K}$. A minimal isometric dilation exists for any Kreĭn space operator $A \in \mathfrak{L}(\mathfrak{H})$; if A is a contraction, it is essentially unique as in the Hilbert space case [37].

Commutant Lifting Theorem I (Dritschel [32]). *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Kreĭn spaces, and let $T \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ be a contraction operator such that $TA_1 = A_2T$ for some contraction operators $A_1 \in \mathfrak{L}(\mathfrak{H}_1)$ and $A_2 \in \mathfrak{L}(\mathfrak{H}_2)$. Let $U_1 \in \mathfrak{L}(\mathfrak{K}_1)$ and $U_2 \in \mathfrak{L}(\mathfrak{K}_2)$ be minimal isometric dilations of A_1 and A_2. Then there is a contraction $\hat{T} \in \mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ such that $U_2\hat{T} = \hat{T}U_1$ and $P_{\mathfrak{H}_2}\hat{T} = TP_{\mathfrak{H}_1}$.*

The proof is an application of Theorems 5.4 and 5.5. It is simplified in Dritschel and Rovnyak [37]. For different proofs, see Dijksma, Dritschel, Marcantognini, and de Snoo [30], and Marcantognini [52]. A module formulation has been given by Dritschel [35]. Earlier results in the same direction were obtained by Constantinescu and Gheondea; see [22, 24].

Another version of the commutant lifting theorem also starts with the Sz.-Nagy and Foias theorem and weakens the hypothesis that the intertwining operator T is a contraction. In its original form, the underlying spaces are again Hilbert spaces.

Commutant Lifting Theorem II (Ball and Helton [13]). *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces, $A_1 \in \mathfrak{L}(\mathfrak{H}_1)$ and $A_2 \in \mathfrak{L}(\mathfrak{H}_2)$ contractions with minimal isometric dilations $U_1 \in \mathfrak{L}(\mathfrak{K}_1)$ and $U_2 \in \mathfrak{L}(\mathfrak{K}_2)$. Assume that $T \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is a contraction operator such that*

$$\operatorname{ind}_- (1 - T^*T) \leq \kappa$$

for some nonnegative integer κ. Then there is a U_1-invariant subspace $\hat{\mathfrak{K}}_1$ of $\mathfrak{K}_1$ of codimension at most κ and a contraction operator $\hat{T}\colon \hat{\mathfrak{K}}_1 \to \mathfrak{K}_2$ such that $U_2\hat{T} = \hat{T}U_1|_{\hat{\mathfrak{K}}_1}$ and $P_{\mathfrak{H}_2}\hat{T} = TP_{\mathfrak{H}_1}|_{\hat{\mathfrak{K}}_1}$.

Independently, Gheondea [42] and Arocena, Azizov, Dijksma, and Marcantognini [7, 8] have extended the Ball and Helton theorem to allow $\mathfrak{H}_1$ and $\mathfrak{H}_2$ to be Kreĭn spaces. In the generalization, the subspace $\hat{\mathfrak{K}}_1$ is not necessarily a Kreĭn subspace, but with a natural interpretation of contraction operator the statement is otherwise identical. The formulation in [8] is more general in another direction, namely, a broader notion of isometric dilation is adopted.

Canonical models also provide a setting for commutant lifting [72]. In Alpay [3] and de Branges [20], generalizations of the commutant lifting theorem in canonical model spaces are constructed. Concerning canonical models in Kreĭn spaces, see also Yang [74].

6. Uniqueness questions

6.1 General results

While factorizations as in Theorem 4.1 always exist, they are not in general unique even up to appropriate notions of isomorphism. Indices of the underlying Kreĭn space, at least, are unique [37, Theorem 1.2.1]:

Theorem 6.1. *Let $C \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator on a Kreĭn space $\mathfrak{H}$. In any way, factor C in the form $C = AA^*$ where $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$ for some Kreĭn space $\mathfrak{A}$ and $\ker A = \{0\}$ as in Theorem 4.1. Then*

$$\operatorname{ind}_\pm \mathfrak{A} = \operatorname{ind}_\pm C.$$

In particular, the indices $\operatorname{ind}_\pm \mathfrak{A}$ do not depend on the choice of factorization.

We turn to conditions which imply that a factorization $C = AA^*$, $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$, $\ker A = \{0\}$ is unique up to replacement of $\mathfrak{A}$ by an isomorphic copy. Examples show that this is not always the case (see [34, p. 217] and [38, p. 891]). Such a notion of uniqueness is of interest in its own right and also because some applications use special properties of the particular factorization which is constructed in the proof of Theorem 4.1; see Dritschel and Rovnyak [38, Lecture 6].

Definition 6.2. *A selfadjoint operator $C \in \mathfrak{L}(\mathfrak{H})$ is said to have the* **unique factorization property** *if for any two factorizations*

$$C = A_j A_j^*, \qquad A_j \in \mathfrak{L}(\mathfrak{A}_j, \mathfrak{H}), \quad \ker A_j = \{0\}, \quad j = 1, 2, \tag{6.1}$$

there is an isomorphism $U \in \mathfrak{L}(\mathfrak{A}_1, \mathfrak{A}_2)$ such that $A_1 = A_2 U$.

This property holds in many naturally occurring situations. In fact, it is possible to completely characterize when the property holds.

Theorem 6.3. *Let $\mathfrak{H}$ be a Kreĭn space, and let $C \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator. The following conditions are equivalent:*

(1) *C has the unique factorization property;*
(2) *for some Hilbert space selfadjoint operator B congruent to C, $\sigma(B)$ omits an interval of the form $(-\epsilon, 0)$ or $(0, \epsilon)$ with $\epsilon > 0$;*
(3) *for some factorization $C = AA^*$ as in Theorem 4.1, $\operatorname{ran} A^*$ contains one of the subspaces $\mathfrak{A}_+$ or $\mathfrak{A}_-$ in some fundamental decomposition $\mathfrak{A} = \mathfrak{A}_+ \oplus \mathfrak{A}_-$.*

In this case, (2) holds for any selfadjoint operator congruent to C, and (3) holds for any factorization of C as in Theorem 4.1.

For a proof see [26, Theorem 2.8]. Condition (2) in Theorem 6.3 was given by Constantinescu and Gheondea [23, 25], Ćurgus and Langer [27], and Hara [45]. Condition (3) is given in a different form in Dritschel [34] and Dritschel and Rovnyak [38].

Theorem 6.4. *Let $\mathfrak{H}$ be a Kreĭn space, and let $C \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator. Each of the following conditions is sufficient for C to have the unique factorization property:*

(1) *$C \geq 0$;*
(2) *one of the indices $\operatorname{ind}_\pm C$ is finite;*
(3) *$C^2 \leq C$.*

Sketch of proof. (1), (2) Assume that $\operatorname{ind}_- C < \infty$. We check condition (2) in Theorem 6.3. Suppose that B is a selfadjoint operator on a Hilbert space $\mathfrak{K}$ which is congruent to C. Then $\sigma(B) \cap (-\infty, 0)$ is a finite set, and so (2) holds. We obtain (1) as a special case of (2).

(3) We deduce this from Theorems 6.5 and 6.6 below. Assume that $C^2 \leq C$. Suppose that we have two factorizations $C = A_j A_j^*$, $A_j \in \mathfrak{L}(\mathfrak{A}_j, \mathfrak{H})$, $\ker A_j = \{0\}$, $j = 1, 2$. For $j = 1, 2$, let $\mathfrak{G}_j$ be the range of A_j in the inner product that makes A_j an isomorphism from $\mathfrak{A}_j$ onto $\mathfrak{G}_j$. Then $\mathfrak{G}_j$ is a Kreĭn space which is contained continuously in $\mathfrak{H}$, and $C = E_j E_j^*$, where $E_j : \mathfrak{G}_j \to \mathfrak{H}$ is the inclusion mapping. The inequality $C^2 \leq C$ implies that the inclusion operators E_j are contractions. Applying Theorems 6.5 and 6.6 with $P = C$, we see that C has the unique factorization property. $\square$

Alternatively, to prove Theorem 6.4(2) we can verify condition (3) in Theorem 6.3 with the aid of

Pontryagin's Theorem: *Let $\mathfrak{D}$ be a dense linear subspace of a Pontryagin space $\mathfrak{G}$. Then $\mathfrak{D}$ contains the negative subspace $\mathfrak{G}_-$ in some fundamental decomposition $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$.*

Suppose again that $\mathrm{ind}_- C < \infty$, and let $C = AA^*$ be any factorization as in Theorem 4.1. By Theorem 6.1, $\mathrm{ind}_- \mathfrak{A} = \mathrm{ind}_- C < \infty$. Since $\ker A = \{0\}$, $\mathrm{ran}\, A^*$ is dense in $\mathfrak{A}$, and so (3) follows from Pontryagin's theorem.

6.2 Examples of uniqueness results

(i) **Continuous inclusion of Kreĭn spaces and complementation in the sense of de Branges**. The simplest case here comes from a Kreĭn subspace $\mathfrak{G}$ of a Kreĭn space $\mathfrak{H}$. The inclusion mapping $E : \mathfrak{G} \to \mathfrak{H}$ is a continuous isometry in this case. The operator $P = EE^*$ is the projection on $\mathfrak{H}$ with range $\mathfrak{G}$. These notions have far-reaching generalizations in the work of de Branges [18]. We follow the operator range view in [38] in which P can be any selfadjoint operator on a Kreĭn space.

A Kreĭn space $\mathfrak{G}$ is said to be **contained continuously** in a Kreĭn space $\mathfrak{H}$ if $\mathfrak{G}$ is a linear subspace of $\mathfrak{H}$ and the inclusion mapping $E : \mathfrak{G} \to \mathfrak{H}$ is continuous. In this situation $P = EE^*$ is a selfadjoint operator on $\mathfrak{H}$. It is not hard to see that the range of P is contained in $\mathfrak{G}$ as a dense subspace, and

$$\langle Pf, Pg \rangle_{\mathfrak{G}} = \langle Pf, g \rangle_{\mathfrak{H}}, \qquad f, g \in \mathfrak{H}. \tag{6.2}$$

We call P the **generalized projection operator** for the inclusion of $\mathfrak{G}$ in $\mathfrak{H}$.

It is easy to see that every selfadjoint operator $P \in \mathfrak{L}(\mathfrak{H})$ arises as a generalized projection operator. In fact, if $P \in \mathfrak{L}(\mathfrak{H})$ is a given selfadjoint operator, write $P = AA^*$, $A \in \mathfrak{L}(\mathfrak{A}, \mathfrak{H})$, $\ker A = \{0\}$, as in Theorem 4.1. Let $\mathfrak{G}$ be the range of A in the inner product which makes A an isomorphism. It is not hard to see that $\mathfrak{G}$ is a Kreĭn space which is contained continuously in $\mathfrak{H}$, and $P = EE^*$ where E is the inclusion mapping.

Uniqueness questions arise. In the preceding situation, the indices $\mathrm{ind}_\pm \mathfrak{G}$ are determined by P. However, $\mathfrak{G}$ itself is not necessarily determined by P: it may occur that P is the generalized projection operator for distinct Kreĭn spaces $\mathfrak{G}_1$ and $\mathfrak{G}_2$ which are contained continuously in $\mathfrak{H}$; that is, $P = E_1E_1^* = E_2E_2^*$, where $E_1 : \mathfrak{G}_1 \to \mathfrak{H}$ and $E_2 : \mathfrak{G}_2 \to \mathfrak{H}$ are the inclusion mappings.

Theorem 6.5. *Let $\mathfrak{H}$ be a Kreĭn space, and let $P \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator. The following conditions are equivalent:*

1. *P is the generalized projection operator for a unique Kreĭn space which is contained continuously in $\mathfrak{H}$;*
2. *P has the unique factorization property.*

Uniqueness is automatic in some cases. Suppose that $\mathfrak{G}$ is contained continuously in $\mathfrak{H}$. We say that the inclusion is **contractive** if

$$\langle g, g \rangle_{\mathfrak{H}} \le \langle g, g \rangle_{\mathfrak{G}}, \qquad g \in \mathfrak{G},$$

that is, the inclusion mapping is contractive; by (6.2), this occurs if and only if the associated generalized projection operator P satisfies $P^2 \le P$. The notion of an **isometric** inclusion is defined similarly but with equality in the preceding inequalities.

Theorem 6.6. *Conditions* (1) *and* (2) *in Theorem* 6.5 *are satisfied if P is the generalized projection operator for some Kreĭn space $\mathfrak{G}$ which is contained continuously and contractively in $\mathfrak{H}$. In particular, such a space $\mathfrak{G}$ is unique.*

Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Kreĭn spaces which are contained continuously and contractively in a Kreĭn space $\mathfrak{H}$. We say that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are **complementary in the sense of de Branges** or simply **complementary** if the mapping $(h_1, h_2) \to h_1 + h_2$ is a contractive partial isometry from $\mathfrak{H}_1 \times \mathfrak{H}_2$ onto $\mathfrak{H}$. In this case, for every $h \in \mathfrak{H}$,

$$\langle h, h \rangle_{\mathfrak{H}} = \min_{h = h_1 + h_2} \left(\langle h_1, h_1 \rangle_{\mathfrak{H}_1} + \langle h_2, h_2 \rangle_{\mathfrak{H}_2} \right),$$

and $\operatorname{ind}_- \mathfrak{H} = \operatorname{ind}_- \mathfrak{H}_1 + \operatorname{ind}_- \mathfrak{H}_2$. Examples appear in the theory of reproducing kernel spaces (see §8). The general theory is given in [5, 18, 38].

(ii) **Defect and Julia operators**. Defect and Julia operators can be changed by replacing the underlying Kreĭn spaces by isomorphic spaces. It is of interest to know if any two defect or Julia operators for a given operator $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ are related in this way.

Definition 6.7. *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces.*

1. *We say that T has an* **essentially unique defect operator** *if any two defect operators $\tilde{D}_j \in \mathfrak{L}(\tilde{\mathfrak{D}}_j, \mathfrak{H})$, $j = 1, 2$, are related by $\tilde{D}_1 = \tilde{D}_2 \tilde{V}$, where $\tilde{V}$ is an isomorphism from $\tilde{\mathfrak{D}}_1$ onto $\tilde{\mathfrak{D}}_2$.*
2. *We say that T has an* **essentially unique Julia operator** *if any two Julia operators*

$$\begin{pmatrix} T & D_j \\ \tilde{D}_j^* & -L_j^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_j, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_j), \qquad j = 1, 2,$$

are related by

$$\begin{pmatrix} T & D_1 \\ \tilde{D}_1^* & -L_1^* \end{pmatrix} = \begin{pmatrix} 1_{\mathfrak{K}} & 0 \\ 0 & \tilde{V}^* \end{pmatrix} \begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & -L_2^* \end{pmatrix} \begin{pmatrix} 1_{\mathfrak{H}} & 0 \\ 0 & V \end{pmatrix}$$

where $\tilde{V}$ is an isomorphism from $\tilde{\mathfrak{D}}_1$ onto $\tilde{\mathfrak{D}}_2$ and V is an isomorphism from $\mathfrak{D}_1$ onto $\mathfrak{D}_2$

A complete analysis of these conditions is given in Dritschel [34]. The following result probably covers the most important special cases

Theorem 6.8. *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces. Each of the following conditions is sufficient for T to have essentially unique defect and Julia operators:*

1. *T is a contraction;*
2. *T^* is a contraction;*
3. *one of the four indices* $\operatorname{ind}_\pm (1 - T^*T)$, $\operatorname{ind}_\pm (1 - TT^*)$ *is finite.*

Conditions (1) and (2) in Theorem 6.8 are included for emphasis, but they are special cases of (3). In the case of Julia operators, Theorem 6.8 is given in Dritschel and Rovnyak [37, p. 298]. The result for defect operators can be deduced from this and the fact that a Julia operator (5.2) can be constructed with any prescribed defect operator $\tilde{D}$ for T.

7. Kolmogorov decompositions of Hermitian kernels

The theory of Hermitian kernels provides a unified environment for common constructions that appear in a number of areas including the study of reproducing kernels, inner products, and selfadjoint operator matrices. The indefinite theory originates with Schwartz [69]. We follow the approach of Constantinescu and Gheondea [26]. The form of the uniqueness result in Theorem 7.3 is implicit in [26] and was communicated privately by the authors.

A **(Hermitian) kernel** is an indexed collection

$$K = \{K_{ij}\}_{i,j\in J}, \qquad K_{ij} \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{H}_i), \tag{7.1}$$

of operators satisfying $K_{ij} = K_{ji}^*$ for all $i, j \in J$. Here J is an index set, and the underlying spaces $\mathfrak{H}_j$, $j \in J$, are Kreĭn spaces. We say that K has a **Kolmogorov decomposition** if there exist a Kreĭn space $\mathfrak{K}$ and operators $V_j \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{K})$, $j \in J$, such that

$$K_{ij} = V_i^* V_j, \qquad i, j \in J, \tag{7.2}$$

and $\mathfrak{K} = \bigvee_{j\in J} V_j \mathfrak{H}_j$. The term "Kolmogorov decomposition" is derived from a theorem of Kolmogorov [48] as it appears, for example, in Martin and Putinar [53, p. 34]. Two Kolmogorov decompositions with operators $V_{1j} \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{K}_1)$ and $V_{2j} \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{K}_2)$, $j \in J$, are called **equivalent** if there is an isomorphism $W \in \mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ such that $V_{2j} = W V_{1j}$ for all $j \in J$. If any two Kolmogorov decompositions are equivalent, we say that K has an **essentially unique Kolmogorov decomposition**.

Sums and differences of kernels are defined in the obvious way when the underlying spaces are the same, and the set of such kernels has the structure of a linear space. Given a Hermitian kernel (7.1), let $\mathfrak{F}$ be the linear space of all finitely nonzero indexed sets $f = \{f_j\}_{j\in J}$ of vectors $f_j \in \mathfrak{H}_j$, $j \in J$. Define a K**-inner product** on $\mathfrak{F}$ by

$$\langle f, g\rangle_K = \sum_{i,j\in J} \langle K_{ij} f_j, g_i\rangle_{\mathfrak{H}_i}, \qquad f, g \in \mathfrak{F}.$$

We call K **nonnegative** and write $K \geq 0$ if the K-inner product (7.3) is nonnegative. The inequality $K_1 \leq K_2$ for two Hermitian kernels means that $K_2 - K_1 \geq 0$

A **nonnegative majorant** for a Hermitian kernel K is a Hermitian kernel L having the same underlying spaces such that $L \geq 0$ and $-L \leq K \leq L$. In this situation, we associate a Hilbert space $\mathfrak{H}_L$ with L by a standard construction. A dense set in $\mathfrak{H}_L$ is the quotient space $\mathfrak{F}/\mathfrak{N}_L$, where $\mathfrak{F}$ is as above and $\mathfrak{N}_L$ the subspace of elements which are orthogonal to all of $\mathfrak{F}$ in the L-inner product. If $f \in \mathfrak{F}$, let $[f]$ be the corresponding coset in $\mathfrak{F}/\mathfrak{N}_L$. The inner product in $\mathfrak{H}_L$ is given on the dense set by

$$\langle [f], [g]\rangle_{\mathfrak{H}_L} = \langle f, g\rangle_L, \qquad f, g \in \mathfrak{F}.$$

Arguments in [26, p. 929] show that there is a unique operator $G \in \mathfrak{L}(\mathfrak{H}_L)$ such that

$$\langle G[f], [g]\rangle_{\mathfrak{H}_L} = \langle f, g\rangle_K, \qquad f, g \in \mathfrak{F}.$$

The operator G is selfadjoint and satisfies $\|G\| \leq 1$. It is called the **Gram operator** of the kernel K for the majorant L.

To avoid repetitive statements, throughout §7 the underlying spaces for Hermitian kernels are assumed to be as in (7.1), and $\mathfrak{F}$ has the same meaning as above. We likewise use the same notation $\mathfrak{F}/\mathfrak{N}_L$ for the dense set in the Hilbert space $\mathfrak{H}_L$ as in the definition of a Gram operator.

Theorem 7.1. *If K is a Hermitian kernel, the following assertions are equivalent:*

(1) *K has a Kolmogorov decomposition;*
(2) *K has a nonnegative majorant;*
(3) *$K = K_+ - K_-$ for some Hermitian kernels $K_+ \geq 0$ and $K_- \geq 0$.*

In this case, the decomposition in (3) *can be chosen such that the only Hermitian kernel M such that $0 \leq M \leq K_\pm$ is $M = 0$.*

Proof. (1) $\Leftrightarrow$ (2) Assume that a Kolmogorov decomposition (7.2) exists. In any way construct a Hilbert space $\mathfrak{M}$ and an invertible operator $X \in \mathfrak{L}(\mathfrak{K}, \mathfrak{M})$ such that

$$|\langle k, k\rangle_{\mathfrak{K}}| \leq \langle Xk, Xk\rangle_{\mathfrak{M}}, \qquad k \in \mathfrak{K}.$$

Define $L = \{L_{ij}\}_{i,j\in J}$ by $L_{ij} = V_i^* X^* X V_j \in \mathfrak{L}(\mathfrak{H}_j, \mathfrak{H}_i)$, $i, j \in J$. It is easy to see that L is a nonnegative majorant for K.

Conversely, let K have a nonnegative majorant L, and let $G \in \mathfrak{L}(\mathfrak{H}_L)$ be the associated Gram operator. Using Theorem 4.1, factor $G = AA^*$ with $A \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H}_L)$ and $\ker A = \{0\}$. For each $j \in J$, there is a natural continuous embedding operator E_j from $\mathfrak{H}_j$ into $\mathfrak{H}_L$, namely, $E_j u = [f_u]$, where f_u is the element of $\mathfrak{F}$ whose j-th component is u and all other components are zero. Then $V_j = A^* E_j$, $j \in J$, defines a Kolmogorov decomposition.

(2) $\Leftrightarrow$ (3) If $K = K_+ - K_-$ as in (3), $L = K_+ + K_-$ is a nonnegative majorant for K.

Conversely, suppose that K has a nonnegative majorant L, and let G be the corresponding Gram operator. In terms of the embedding operators E_j, $j \in J$, defined above, we have

$$K_{ij} = E_i^* G E_j, \qquad i, j \in J.$$

Let $P_\pm, P_0$ be the spectral projections for $(0, \infty)$, $(-\infty, 0)$, $\{0\}$ for G. Then the formula

$$K_{\pm ij} = E_i^* (\pm P_\pm) G E_j, \qquad i, j \in J.$$

defines kernels $K_\pm$ such that $K_\pm \geq 0$ and $K = K_+ - K_-$.

The kernels $K_\pm$ constructed in this way have the property in the last statement of the theorem. For assume that $0 \leq M \leq K_\pm$. Since $\|G\| \leq 1$, $K^\pm \leq L$, and thus $0 \leq M \leq L$. The Gram operator $H \in \mathfrak{L}(\mathfrak{H}_L)$ of M relative to L satisfies

$$0 \leq \langle H[f], [f]\rangle_{\mathfrak{H}_L} \leq \langle \pm P_\pm G[f], [f]\rangle_{\mathfrak{H}_L}, \qquad f \in \mathfrak{F}.$$

Since P_+G and P_-G are supported on orthogonal subspaces of $\mathfrak{H}_L$, $H = 0$. Hence $\langle f, g\rangle_M = \langle H[f], [g]\rangle_{\mathfrak{H}_L} = 0$ for all $f, g \in \mathfrak{F}$, and so $M = 0$. ☐

Let K be a Hermitian kernel with nonnegative majorant L. A Kolmogorov decomposition (7.2) for K is said to be L**-continuous** if the mapping $[f]$ into $\sum_{j\in J} V_j f_j$ on $\mathfrak{F}/\mathfrak{N}_L$ extends to a continuous operator from on $\mathfrak{H}_L$ into $\mathfrak{K}$.

Lemma 7.2. *Let K be a Hermitian kernel.*

(1) *If K has a Kolmogorov decomposition* (7.2)*, the decomposition is L-continuous with respect to the nonnegative majorant L constructed in the proof of Theorem* 7.1*,* (1) *implies* (2)*.*

(2) *If K has a nonnegative majorant L, the Kolmogorov decomposition of K constructed in the proof of Theorem* 7.1*,* (2) *implies* (1)*, is L-continuous.*

Proof. (1) In the notation of Theorem 7.1, (1) implies (2),

$$\langle [f],[g]\rangle_{\mathfrak{H}_L} = \left\langle X\sum_{j\in J} V_j f_j, X\sum_{i\in J} V_i g_i\right\rangle_{\mathfrak{M}}, \qquad f,g\in\mathfrak{F}.$$

Thus the mapping $[f]$ into $X\sum_{j\in J} V_j f_j$ is a Hilbert space isometry from $\mathfrak{H}_L$ into $\mathfrak{M}$; the mapping $[f]$ into $\sum_{j\in J} V_j f_j$ on $\mathfrak{F}/\mathfrak{N}_L$ into $\mathfrak{K}$ is the composite of this isometry and X^{-1} and hence is continuous.

(2) We wish to show that, in the proof of Theorem 7.1, (2) implies (1), the mapping $[f]$ into $\sum_{j\in J} V_j f_j$ on $\mathfrak{F}/\mathfrak{N}_L$ extends to a continuous operator from $\mathfrak{H}_L$ into $\mathfrak{K}$. In fact, we show that the mapping is A^*. Since $\mathfrak{K}$ is the closed span of the ranges of the operators V_j, it is sufficient to show that for any $f,g\in\mathfrak{F}$,

$$\left\langle [f], A\sum_{i\in J} V_i g_i\right\rangle_{\mathfrak{H}_L} = \left\langle \sum_{j\in J} V_j f_j, \sum_{i\in J} V_i g_i\right\rangle_{\mathfrak{K}}.$$

Since $V_i = A^*E_i$ for each $i\in J$ and $\sum_{i\in J} E_i g_i = [g]$, it is the same thing to show that

$$\langle [f], AA^*[g]\rangle_{\mathfrak{H}_L} = \left\langle \sum_{j\in J} V_j f_j, \sum_{i\in J} V_i g_i\right\rangle_{\mathfrak{K}}.$$

This holds because $AA^* = G$, and so both sides are equal to $\langle f,g\rangle_K$. ☐

Thus L-continuous Kolmogorov decompositions always exist. Uniqueness depends on the Gram operator.

Theorem 7.3. *Let K be a Hermitian kernel with nonnegative majorant L and Gram operator G. Any two L-continuous Kolmogorov decompositions are equivalent if and only if G has the unique factorization property.*

Proof. Assume that G has the unique factorization property. Let

$$K_{ij} = V_{1i}^* V_{1j}, \quad V_{1j}\in\mathfrak{L}(\mathfrak{H}_j,\mathfrak{K}_1), \quad i,j\in J, \tag{7.3}$$

$$K_{ij} = V_{2i}^* V_{2j}, \quad V_{2j}\in\mathfrak{L}(\mathfrak{H}_j,\mathfrak{K}_2), \quad i,j\in J, \tag{7.4}$$

be two L-continuous Kolmogorov decompositions. By hypothesis, the mapping $[f]$ into $\sum_{j\in J} V_{1j} f_j$ on $\mathfrak{F}/\mathfrak{N}_L$ extends to a continuous operator from $\mathfrak{H}_L$ into $\mathfrak{K}_1$. Denote its adjoint $A_1\in\mathfrak{L}(\mathfrak{K}_1,\mathfrak{H}_L)$. Since $\mathfrak{K}_1 = \bigvee_{j\in J} V_{1j}\mathfrak{H}_j$, $\ker A_1 = \{0\}$. For all $f,g\in\mathfrak{F}$,

$$\left\langle [f], A_1\sum_{i\in J} V_{1i} g_i\right\rangle_{\mathfrak{H}_L} = \left\langle \sum_{j\in J} V_{1j} f_j, \sum_{i\in J} V_{1i} g_i\right\rangle_{\mathfrak{K}_1}.$$

Thus

$$\langle G[f],[g]\rangle_{\mathfrak{H}_L} = \langle f,g\rangle_K = \left\langle \sum_{j\in J} V_{1j} f_j, \sum_{i\in J} V_{1i} g_i\right\rangle_{\mathfrak{K}_1} = \langle [f], A_1A_1^*[g]\rangle_{\mathfrak{H}_L},$$

and so $G = A_1A_1^*$. Construct a factorization $G = A_2A_2^*$ in a similar way from (7.4). Since G has the unique factorization property, there is a unitary operator $W\in\mathfrak{L}(\mathfrak{K}_1,\mathfrak{K}_2)$ such that $A_1 = A_2W$. For all $j\in J$,

$$V_{2j} = A_2^*E_j = WA_1^*E_j = WV_{1j},$$

and thus the two Kolmogorov decompositions are equivalent.

Assume that any two L-continuous Kolmogorov decompositions of K are equivalent. Let

$$G = A_1A_1^* = A_2A_2^*$$

with $A_1\in\mathfrak{L}(\mathfrak{K}_1,\mathfrak{H}_L)$, $A_2\in\mathfrak{L}(\mathfrak{K}_2,\mathfrak{H}_L)$, $\ker A_1 = \{0\}$, and $\ker A_2 = \{0\}$. By Lemma 7.2, we can construct L-continuous Kolmogorov decompositions (7.3) and (7.4) by setting $V_{1j} =$

$A_1^* E_j$ and $V_{2j} = A_2^* E_j$ for all $j \in J$. By hypothesis, there is a unitary operator $W \in \mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$ such that $V_{2j} = W V_{1j}$ for all $j \in J$. Using the properties $\mathfrak{K}_1 = \bigvee_{j \in J} V_{1j} \mathfrak{H}_j$ and $\mathfrak{K}_2 = \bigvee_{j \in J} V_{2j} \mathfrak{H}_j$ of the Kolmogorov decompositions, we obtain $W A_1^* = A_2^*$ and hence $A_1 = A_2 W$, and thus G has the unique factorization property. □

A stronger uniqueness result holds with a stronger hypothesis.

Theorem 7.4. *A Hermitian kernel K has an essentially unique Kolmogorov decomposition if and only if the Gram operators for all nonnegative majorants have the unique factorization property.*

Proof. If some Gram operator does not have the unique factorization property, Theorem 7.3 implies that there exist nonequivalent Kolmogorov decompositions, which proves necessity.

Conversely, assume that every Gram operator has the essential uniqueness property. Let (7.3) and (7.4) be any two Kolmogorov decompositions of K. By Lemma 7.2, the decompositions are continuous relative to some nonnegative majorants L_1 and L_2 for K. Then $L = L_1 + L_2$ is a nonnegative majorant for K. Since $L_1 \leq L$, the "identity mapping" on $\mathfrak{F}/\mathfrak{N}_L$ to $\mathfrak{F}/\mathfrak{N}_{L_1}$ is a densely defined contraction from $\mathfrak{H}_L$ into $\mathfrak{H}_{L_1}$. Since these are Hilbert spaces, the mapping $[f]$ into $\sum_{j \in J} V_{1j} f_j$ on $\mathfrak{F}/\mathfrak{N}_L$ into $\mathfrak{K}_1$ is a composition of continuous operators, and so (7.3) is L-continuous. Similarly, (7.4) is L-continuous. By Theorem 7.3 and our hypothesis on Gram operators, the two Kolmogorov decompositions are equivalent. □

The following sufficient condition for essential uniqueness is given in [26, Theorem 4.3].

Theorem 7.5. *Let K be a Hermitian kernel, and assume that there exists a Kolmogorov decomposition* (7.2) *such that the linear span of the subspaces $V_j \mathfrak{H}_j$, $j \in J$, contains contains one of the subspaces $\mathfrak{K}_\pm$ in some fundamental decomposition $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$. Then K has an essentially unique Kolmogorov decomposition.*

Proof. Suppose that the given Kolmogorov decomposition is relabeled as (7.3), and let (7.4) be any second Kolmogorov decomposition. Define a linear relation $\mathbf{R}$ from $\mathfrak{K}_1$ into $\mathfrak{K}_2$ by

$$\mathbf{R} = \left\{ \left(\sum_{j \in J} V_{1j} f_j, \sum_{j \in J} V_{2j} f_j \right) : f \in \mathfrak{F} \right\}.$$

By the definition of a Kolmogorov decomposition, $\mathbf{R}$ has dense domain and dense range. For all $f \in \mathfrak{F}$,

$$\left\langle \sum_{j \in J} V_{1j} f_j, \sum_{i \in J} V_{1i} f_i \right\rangle_{\mathfrak{K}_1} = \langle f, f \rangle_K = \left\langle \sum_{j \in J} V_{2j} f_j, \sum_{i \in J} V_{2i} f_i \right\rangle_{\mathfrak{K}_2}.$$

By hypothesis, the domain of $\mathbf{R}$ contains one of the subspaces $\mathfrak{K}_{1\pm}$ in some fundamental decomposition $\mathfrak{K}_1 = \mathfrak{K}_{1+} \oplus \mathfrak{K}_{1-}$. Hence by Theorem 4.3 the closure of $\mathbf{R}$ is the graph of a unitary operator $W \in \mathfrak{L}(\mathfrak{K}_1, \mathfrak{K}_2)$. By construction, $V_{2j} = W V_{1j}$ for all $j \in J$, and so the two Kolmogorov decompositions are equivalent. □

Corollary 7.6. *If a Hermitian kernel K has a Kolmogorov decomposition* (7.2) *such that $\mathfrak{K}$ is either a Pontryagin space or the antispace of a Pontryagin space, then K has an essentially unique Kolmogorov decomposition.*

Proof. Since $\mathfrak{K}$ is a Pontryagin space for the given Kolmogorov decomposition, the hypotheses of Theorem 7.5 are satisfied by Pontryagin's theorem (see §6). □

8. Examples of Hermitian kernels

8.1 Reproducing kernel Kreĭn spaces

The definite theory is classical and has many applications. In addition to the standard source of Aronszajn [9], see also, for example, Dym [39] and Saitoh [68]. The indefinite theory is due to Schwartz [69] and Sorjonen [71] and also also owes much to a series of papers in the 1970's by Kreĭn and Langer including [49, 50] and the thesis of Alpay [2]. The theory of §7 allows a quick derivation of the main results.

Consider a Hermitian kernel $K(s,t)$, $s,t \in \Omega$, with values in $\mathfrak{L}(\mathfrak{F})$ for some fixed Kreĭn space $\mathfrak{F}$ and nonempty set Ω. We call $K(s,t)$ a **reproducing kernel** for a Kreĭn space $\mathfrak{H}_K$ of $\mathfrak{F}$-valued functions on Ω if

(1) for each $s \in \Omega$ and $f \in \mathfrak{F}$, $K(s,\cdot)f$ belongs to $\mathfrak{H}_K$, and
(2) $\langle h(\cdot), K(s,\cdot)f\rangle_{\mathfrak{H}_K} = \langle h(s), f\rangle_{\mathfrak{F}}$ for every $h(\cdot)$ in $\mathfrak{H}_K$.

These conditions are equivalent to the existence of a Kolmogorov decomposition with a Kreĭn space $\mathfrak{H}_K$ and operators $V_s \colon \mathfrak{F} \to \mathfrak{H}_K$ such that

$$V_s f = K(s,\cdot)f, \qquad f \in \mathfrak{F}, \tag{8.1}$$

for all $s \in \Omega$. In other words, $V_s = E(s)^*$, where $E(s)\colon h(\cdot) \to h(s)$ is evaluation at any point $s \in \Omega$ (the evaluation mappings are continuous by the closed graph theorem).

Conversely, one can start with a Kreĭn space of functions:

Theorem 8.1. *Let $\mathfrak{H}$ be a Kreĭn space of functions on a set Ω with values in a Kreĭn space $\mathfrak{F}$. Then $\mathfrak{H}$ has a reproducing kernel if and only if all evaluation mappings $E(s)$, $s \in \Omega$, belong to $\mathfrak{L}(\mathfrak{H},\mathfrak{F})$. The reproducing kernel is uniquely determined by the space and given by $K(s,t) = E(t)E(s)^*$, $s,t \in \Omega$.*

Notions of nonnegative kernels and nonnegative majorants have the same meaning as in the general case. The definite case is well known: a nonnegative kernel $L(s,t)$ is the reproducing kernel for a unique Hilbert space $\mathfrak{H}_L$ (this also follows from the results of §7).

Existence and uniqueness are separate issues in the indefinite case. A reproducing kernel for a Kreĭn space, when it exists, is uniquely determined by the space. However, unlike the Hilbert space case, two distinct Kreĭn spaces can have the same reproducing kernel. The uniqueness of a Kreĭn space with given reproducing kernel can be restored in a restricted sense if suitable conditions are met.

Theorem 8.2. *If $K(s,t)$, $s,t \in \Omega$, is a Hermitian kernel with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$, the following assertions are equivalent:*

(1) *$K(s,t)$ is the reproducing kernel for some Kreĭn space $\mathfrak{H}_K$ of functions on Ω;*
(2) *$K(s,t)$ has a nonnegative majorant $L(s,t)$ on $\Omega \times \Omega$;*
(3) *$K(s,t) = K_+(s,t) - K_-(s,t)$ for some nonnegative kernels $K_\pm(s,t)$ on $\Omega \times \Omega$.*

When these conditions hold, then moreover:

(4) *For a given nonnegative majorant $L(s,t)$ for $K(s,t)$, there is a Kreĭn space $\mathfrak{H}_K$ with reproducing kernel $K(s,t)$ which is contained continuously in the Hilbert space $\mathfrak{H}_L$ with reproducing kernel $L(s,t)$.*

(5) *In the same situation, there is a continuous selfadjoint operator G on $\mathfrak{H}_L$ such that $G\colon L(s,\cdot)f \to K(s,\cdot)f$, $s \in \Omega$, $f \in \mathfrak{F}$. The space $\mathfrak{H}_K$ in (4) is unique if and only if G has the unique factorization property.*

When the equivalent conditions in Theorem 8.2 hold, then for any space $\mathfrak{H}_K$ as in (1), the decomposition in (3) can be chosen so that $\pm K_\pm(s,t)$ are reproducing kernels for the spaces $\mathfrak{H}_K^\pm$ in a fundamental decomposition $\mathfrak{H}_K = \mathfrak{H}_K^+ \oplus \mathfrak{H}_K^-$. In fact, we need only choose $\pm K_\pm(s,t)$ to be the reproducing kernels for the spaces in a fundamental decomposition.

Proof. The equivalence of (1)–(3) follows from Theorem 7.1.

(4) We use $L(s,t)$ to construct a reproducing kernel Kreĭn space $\mathfrak{H}_K$ for $K(s,t)$ as in the proof of Theorem 7.1. The reproducing kernel Hilbert space $\mathfrak{H}_L$ is naturally identified with the abstract space denoted in the same way in §7, and the associated Gram operator G has the action described in (5). By Lemma 7.2(2) there is a continuous operator A^* on $\mathfrak{H}_L$ into $\mathfrak{H}_K$ such that

$$A^*\colon L(s,\cdot)f \to K(s,\cdot)f, \qquad s \in \Omega,\ f \in \mathfrak{F}.$$

The adjoint of this operator is the inclusion mapping A from $\mathfrak{H}_K$ into $\mathfrak{H}_L$. Thus $G = AA^*$ and $\mathfrak{H}_K$ is contained continuously in $\mathfrak{H}_L$.

(5) Assume that G has the unique factorization property, and let $\mathfrak{H}_K'$ and $\mathfrak{H}_K''$ be two Kreĭn spaces with reproducing kernel $K(s,t)$ which are contained continuously in $\mathfrak{H}_L$. The two Kolmogorov decompositions induced as in (8.1) are equivalent by Theorem 7.3. It follows that the identity mapping on the linear span $\mathfrak{H}_0$ of all functions $K(s,\cdot)f$, $s \in \Omega$, and $f \in \mathfrak{F}$, extends to a unitary operator from $\mathfrak{H}_K'$ onto $\mathfrak{H}_K''$. The continuity of evaluation mappings for any reproducing kernel Kreĭn space implies that whenever $W\colon h_1(\cdot) \to h_1(\cdot)$, then $h(1(s) = h_2(s)$ for all $s \in \Omega$. Thus $\mathfrak{H}_K'$ and $\mathfrak{H}_K''$ are identical.

In the other direction, the existence of distinct Kreĭn spaces $\mathfrak{H}_K'$ and $\mathfrak{H}_K''$ with reproducing kernel $K(s,t)$ contained continuously in $\mathfrak{H}_L$ implies the existence of two nonequivalent L-continuous Kolmogorov decompositions. By Theorem 7.3, the Gram operator G does not have the essential uniqueness property in this situation. □

Suppose that Ω is an open set in the complex plane. A Hermitian kernel $K(w,z)$ on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$ is **holomorphic** if it is a holomorphic function of z and $\bar{w}$.

Theorem 8.3. *Let Ω be an open set in the complex plane, and let $K(w,z)$, $w,z \in \Omega$, be a holomorphic Hermitian kernel with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$. The following assertions are equivalent:*

(1) *$K(w,z)$ is the reproducing kernel for some Kreĭn space $\mathfrak{H}_K$ of holomorphic functions on Ω;*
(2) *$K(w,z)$ has a nonnegative holomorphic majorant $L(w,z)$ on $\Omega \times \Omega$;*
(3) *$K(w,z) = K_+(w,z) - K_-(w,z)$ for some nonnegative holomorphic kernels $K_\pm(w,z)$ on $\Omega \times \Omega$.*

When these conditions hold, then moreover:

(4) *For a given nonnegative holomorphic majorant $L(w,z)$ for $K(w,z)$, there is a Kreĭn space $\mathfrak{H}_K$ of holomorphic functions with reproducing kernel $K(w,z)$ which is contained continuously in the Hilbert space $\mathfrak{H}_L$ with reproducing kernel $L(w,z)$.*

(5) *In the same situation, there is a continuous selfadjoint operator G on $\mathfrak{H}_L$ such that $G\colon L(w,\cdot)f \to K(w,\cdot)f$, $w \in \Omega$, $f \in \mathfrak{F}$. The space $\mathfrak{H}_K$ in* (4) *is unique if and only if G has the unique factorization property.*

Proof. It is well known that the reproducing kernel Hilbert space associated with a nonnegative holomorphic Hermitian kernel consists of holomorphic functions, and conversely. Given this, we proceed as in the proof of Theorem 8.2 to obtain the result. □

The conditions for existence of a reproducing kernel Kreĭn space are automatically satisfied in many cases of interest. Suppose that Ω is an open set in the complex plane.

Theorem 8.4 (Alpay [4])**.** *Let $K(w,z)$ be a holomorphic Hermitian kernel on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$. Assume that Ω is a disk or half-plane, and that $K(w,z)$ is bounded relative to some and hence any norm which determines the strong topology of $\mathfrak{F}$. Then there exist nonnegative holomorphic Hermitian kernels $K_\pm(w,z)$ such that*

$$K(w,z) = K_+(w,z) - K_-(w,z), \qquad w,z \in \Omega.$$

In particular, $K(w,z)$ satisfies the equivalent conditions (1)–(3) *of Theorem* 8.2.

Proof. Without loss of generality, take $\Omega = \mathbf{D}$. It is also sufficient to prove the result when $\mathfrak{F}$ is a Hilbert space, since otherwise we need only consider $X^*K(w,z)X$, where X is an invertible continuous operator on $\mathfrak{F}$ onto a Hilbert space. Let $H^2_{\mathfrak{F}}$ be the Hardy space of holomorphic $\mathfrak{F}$-valued functions on $\mathbf{D}$.

For any polynomial $p(z)$ with coefficients in $\mathfrak{F}$ and $0 < r < 1$, the formula

$$h_p(z) = \frac{1}{2\pi}\int_0^{2\pi} K(re^{it},z)p(r^{-1}e^{it})\,dt$$

defines a holomorphic function on the disk $|z| < r$. This function is independent of r. It is enough to show this for a monomial $p(z) = fz^m$, $f \in \mathfrak{F}$. If $K(w,z) = \sum_0^\infty A_n(z)\bar{w}^n$, then in this case

$$h_p(z) = \frac{1}{2\pi}\int_0^{2\pi} \sum_0^\infty A_n(z)r^n e^{-int} r^{-m} e^{imt} f\,dt = A_m(z)f,$$

which is independent of r.

Let $p(z)$ be any polynomial with coefficients in $\mathfrak{F}$. We show that the function h_p belongs to $H^2_{\mathfrak{F}}$ and $\|h_p\|_{H^2_{\mathfrak{F}}} \le M\,\|p\|_{H^2_{\mathfrak{F}}}$, where $M > 0$ is a constant. Given any $\rho \in (0,1)$ and $\rho < r < 1$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \left\|h_p(\rho e^{i\theta})\right\|^2_{\mathfrak{F}}\,d\theta &\le \frac{1}{2\pi}\int_0^{2\pi} \left\|\frac{1}{2\pi}\int_0^{2\pi} K(re^{it},\rho e^{i\theta})p(r^{-1}e^{it})\,dt\right\|^2_{\mathfrak{F}}\,d\theta \\ &\le \frac{M^2}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\left\|p(r^{-1}e^{it})\right\|_{\mathfrak{F}}\right]^2\,d\theta \\ &\le \frac{M^2}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi} 1\cdot\,dt\right]\left[\frac{1}{2\pi}\int_0^{2\pi}\left\|p(r^{-1}e^{it})\right\|^2_{\mathfrak{F}}\,dt\right]\,d\theta \\ &= \frac{M^2}{2\pi}\int_0^{2\pi}\left\|p(r^{-1}e^{it})\right\|^2_{\mathfrak{F}}\,dt.\end{aligned}$$

The constant M in the estimate is any bound for $\|K(w,z)\|_{\mathfrak{L}(\mathfrak{F})}$ on $\mathbf{D}\times\mathbf{D}$. The assertion follows on letting $r\uparrow 1$ and using the arbitrariness of ρ. It follows that there is a bounded operator P on $H^2_{\mathfrak{F}}$ such that $P\colon p\to h_p$ on polynomials. The operator P is selfadjoint by the symmetry of the kernel $K(w,z)$. Applying P to monomials of the form $f\bar{w}^n z^n$ and summing over $n\geq 0$, we find that

$$P\colon (1-\bar{w}z)^{-1}f \to K(w,z)f, \qquad w\in\mathbf{D},\ f\in\mathfrak{F},$$

and so for any $f,g\in\mathfrak{F}$ and $\alpha,\beta\in\mathbf{D}$,

$$\langle K(\alpha,\beta)f,g\rangle_{\mathfrak{F}} = \left\langle P\{(1-\bar{\alpha}z)^{-1}f\},(1-\bar{\beta}z)^{-1}g\right\rangle_{H^2_{\mathfrak{F}}}.$$

In any way write $P=P_+-P_-$, where $P_\pm$ are nonnegative selfadjoint operators on $H^2_{\mathfrak{F}}$. Then kernels $K_\pm(w,z)$ of the required type are defined by requiring

$$\langle K_\pm(\alpha,\beta)f,g\rangle_{\mathfrak{F}} = \left\langle P_\pm\{(1-\bar{\alpha}z)^{-1}f\},(1-\bar{\beta}z)^{-1}g\right\rangle_{H^2_{\mathfrak{F}}}.$$

for all $f,g\in\mathfrak{F}$ and $\alpha,\beta\in\mathbf{D}$. □

Theorem 8.4 can be used for unbounded as well as bounded kernels.

Corollary 8.5. *Let $S(z)$ be a holomorphic function on $\Omega=\mathbf{D}$, with possibly an isolated set of points Z omitted. Assume that there is a bounded scalar-valued function $u(z)$ on $\mathbf{D}$ which is nonvanishing on Ω and such that $u(z)S(z)$ is bounded relative to some and hence any norm which determines the strong topology of $\mathfrak{F}$. Then the kernels*

$$\frac{1_{\mathfrak{F}}-S(z)S(w)^*}{1-z\bar{w}} \qquad \textit{and} \qquad \frac{S(z)S(w)^*}{1-z\bar{w}} \tag{8.2}$$

satisfy the equivalent conditions (1)–(3) of Theorem 8.2 on $\Omega\times\Omega$. In particular, they are reproducing kernels for Kreĭn spaces $\mathfrak{H}(S)$ and $\mathfrak{M}(S)$ of functions on Ω.

Proof. Each of the kernels has the form $K(w,z)=L(w,z)/(1-\bar{w}z)$, where $u(z)L(w,z)\overline{u(w)}$ is a bounded holomorphic Hermitian kernel on $\mathbf{D}\times\mathbf{D}$ (any removable singularities of $u(z)S(z)$ are presumed to be removed). By Theorem 8.4, $u(z)L(w,z)\overline{u(w)}=M_+(w,z)-M_-(w,z)$, where $M_\pm(w,z)$ are nonnegative holomorphic kernels on $\mathbf{D}\times\mathbf{D}$. Thus

$$K(w,z)=\frac{L(w,z)}{1-\bar{w}z}=\sum_{n=0}^{\infty}\frac{z^n}{u(z)}M_+(w,z)\left(\frac{w^n}{u(w)}\right)^{-}-\sum_{n=0}^{\infty}\frac{z^n}{u(z)}M_-(w,z)\left(\frac{w^n}{u(w)}\right)^{-}.$$

The two sums on the right define nonnegative holomorphic kernels $K_\pm(w,z)$ such that $K(w,z)=K_+(w,z)-K_-(w,z)$ on $\Omega\times\Omega$. This verifies condition (3) in Theorem 8.2, and the each of the kernels (8.2) is the reproducing kernel for some Kreĭn space of functions on Ω. □

8.2 Reproducing kernel Pontryagin spaces

Again consider a Hermitian kernel $K(s,t)$, $s,t\in\Omega$, with values in $\mathfrak{L}(\mathfrak{F})$ for some fixed Kreĭn space $\mathfrak{F}$ and nonempty set Ω. Stronger results than those above hold when $K(s,t)$ has κ **negative squares**, that is, the maximum number of negative eigenvalues of all matrices

$$\left(\langle K(s_j,s_i)f_j,f_i\rangle_{\mathfrak{F}}\right)_{i,j=1}^n, \qquad s_1,\dots,s_n\in\Omega,\ f_1,\dots,f_n\in\mathfrak{F},\ n\geq 1,$$

is a nonnegative integer κ. In this case, we write $\operatorname{sq}_- K=\kappa$. An associated reproducing kernel space $\mathfrak{H}_K$ automatically exists. It is unique and a Pontryagin space of negative index κ.

Conversely, the reproducing kernel of any given reproducing kernel Pontryagin space is a Hermitian kernel which has κ negative squares [71]. The classical Aronszajn theory of sums and differences of kernels has a natural generalization in the present setting.

Theorem 8.6. *Let $K(s,t), K_1(s,t), K_2(s,t)$ be Hermitian kernels on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$. If $K(s,t) = K_1(s,t) + K_2(s,t)$, then*

$$\operatorname{sq}_- K \leq \operatorname{sq}_- K_1 + \operatorname{sq}_- K_2.$$

Suppose these numbers are finite, and let $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}$ be the associated reproducing kernel Pontryagin spaces. Then the following conditions are equivalent:

(1) $\operatorname{sq}_- K = \operatorname{sq}_- K_1 + \operatorname{sq}_- K_2$;
(2) *$\mathfrak{H}_1$ and $\mathfrak{H}_2$ are contained continuously and contractively as complementary spaces in $\mathfrak{H}$;*
(3) *the intersection $\mathfrak{R} = \mathfrak{H}_1 \cap \mathfrak{H}_2$ is a Hilbert space in the inner product*

$$\langle h,k\rangle_{\mathfrak{R}} = \langle h,k\rangle_{\mathfrak{H}_1} + \langle h,k\rangle_{\mathfrak{H}_2}, \qquad h,k \in \mathfrak{R}.$$

Theorem 8.7. *Let $\mathfrak{H}, \mathfrak{H}_1$ be Pontryagin spaces of functions defined on a set Ω with values in a Kreĭn space $\mathfrak{F}$ and such that $\mathfrak{H}_1$ is contained continuously and contractively in $\mathfrak{H}$. If the spaces have reproducing kernels $K(s,t), K_1(s,t)$, then*

$$K_2(s,t) = K(s,t) - K_1(s,t)$$

defines a Hermitian kernel such that $\operatorname{sq}_- K_2 = \operatorname{sq}_- K - \operatorname{sq}_- K_1$. The associated reproducing kernel Pontryagin space $\mathfrak{H}_2$ is also contained continuously and contractively in $\mathfrak{H}$, and $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are complementary spaces in $\mathfrak{H}$.

Theorem 8.8. *Let $K(w,z)$ be a holomorphic kernel on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$ and region Ω in the complex plane. Let Ω_0 be a subregion of Ω, and assume that the restriction of $K(w,z)$ to $\Omega_0 \times \Omega_0$ has κ negative squares. Then $K(w,z)$ has κ negative squares on $\Omega \times \Omega$.*

Proofs. See [5, Theorems 1.1.4, 1.5.5, 1.5.6]. □

8.3 On pre-Kreĭn spaces

Another special case of the general theory of Kolmogorov decompositions gives results on completions of inner product spaces. We again follow [26].

If an inner product space $\mathfrak{H}_0$ is nonnegative, a standard quotient-completion construction produces an essentially unique Hilbert space. More generally, let $\mathfrak{H}_0$ be any linear and symmetric inner product space. Define a quotient space $\mathfrak{H}_0/\mathfrak{N}$, where $\mathfrak{N}$ is the set of elements of $\mathfrak{H}_0$ which are orthogonal to the full space. If $[f]$ is the coset determined by any $f \in \mathfrak{H}_0$, we obtain an inner product on $\mathfrak{H}_0/\mathfrak{N}$ by writing

$$\langle [f],[g]\rangle_{\mathfrak{H}_0/\mathfrak{N}} = \langle f,g\rangle_{\mathfrak{H}_0}, \qquad f,g \in \mathfrak{H}_0.$$

The quotient space is **nondegenerate**: the only element which is orthogonal to the full space is the zero element. In the nonnegative case, this means that the inner product is strictly positive, and therefore $\mathfrak{H}_0$ has an essentially unique completion to a Hilbert space. In general, we are interested to know, under what conditions does a nondegenerate inner product space have a "completion" to a Kreĭn space, and when is such a completion unique?

We first give formal definitions. Let $\mathfrak{H}_0$ be a nondegenerate inner product space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}_0}$. A **completion** of $\mathfrak{H}_0$ is a Kreĭn space $\mathfrak{H}$ which contains $\mathfrak{H}_0$ isometrically as a dense subspace (that is, $\mathfrak{H}_0$ is a dense linear subspace of $\mathfrak{H}$ and $\langle f, g \rangle_{\mathfrak{H}} = \langle f, g \rangle_{\mathfrak{H}_0}$ for all $f, g \in \mathfrak{H}_0$). By a **nonnegative majorant** for $\langle \cdot, \cdot \rangle_{\mathfrak{H}_0}$ we mean a nonnegative inner product $\langle \cdot, \cdot \rangle_+$ on $\mathfrak{H}_0$ such that

$$-\langle f, f \rangle_+ \leq \langle f, f \rangle_{\mathfrak{H}_0} \leq \langle f, f \rangle_+ , \qquad f \in \mathfrak{H}_0.$$

Since we assume that $\mathfrak{H}_0$ is nondegenerate, such a majorant is strictly positive [26, p.929]; thus $\mathfrak{H}_0$ has a completion to a Hilbert space $\mathfrak{H}_+$ relative to $\langle \cdot, \cdot \rangle_+$, and there is a **Gram operator** $G \in \mathfrak{L}(\mathfrak{H}_+)$ such that

$$\langle f, g \rangle_{\mathfrak{H}_0} = \langle Gf, g \rangle_+, \qquad f, g \in \mathfrak{H}_0.$$

The Gram operator G is selfadjoint and satisfies $\|G\| \leq 1$. Two completions $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of $\mathfrak{H}_0$ are **equivalent** if the identity mapping on $\mathfrak{H}_0$ extends to an isomorphism from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. We call $\mathfrak{H}_0$ a **pre-Kreĭn** space if it has a completion and any two completions are equivalent.

These notions correspond to their counterparts in §7 for an associated Hermitian kernel. Namely, given a nondegenerate inner product space $\mathfrak{H}_0$ as above, we define a Hermitian kernel by setting

$$K_{gf} = \langle f, g \rangle_{\mathfrak{H}_0} , \qquad f, g \in \mathfrak{H}_0.$$

The index set for the kernel is $\mathfrak{H}_0$ itself, and the underlying spaces are all chosen to be $\mathbf{C}$, the complex numbers in the Euclidean metric. It is immediate from the definitions that $\mathfrak{H}_0$ has a completion if and only if the Hermitian kernel has a Kolmogorov decomposition. The definitions of nonnegative majorant and Gram operator for the inner product correspond to the same notions for the Hermitian kernel.

Theorem 8.9. *If $\mathfrak{H}_0$ is a nondegenerate inner product space, the following are equivalent:*

(1) *$\mathfrak{H}_0$ has a completion to a Kreĭn space;*
(2) *the inner product of $\mathfrak{H}_0$ has a nonnegative majorant $\langle \cdot, \cdot \rangle_+$;*
(3) *the inner product of $\mathfrak{H}_0$ is a difference of nonnegative inner products.*

If these conditions hold and $\langle \cdot, \cdot \rangle_+$ is a nonnegative majorant as in (2), *there is a completion $\mathfrak{H}$ of $\mathfrak{H}_0$ which is contained continuously in the Hilbert space completion $\mathfrak{H}_+$ of $\mathfrak{H}_0$ in the* (*necessarily strictly positive*) *inner product $\langle \cdot, \cdot \rangle_+$. Any two such completions are equivalent if and only if the associated Gram operator for the nonnegative majorant has the unique factorization property.*

Theorem 8.9 is a special case of Theorems 7.1 and 7.3. It can also be proved directly by repeating arguments in this special case. In a similar way, Theorem 7.4 yields:

Theorem 8.10. *A nondegenerate inner product space $\mathfrak{H}_0$ is a pre-Kreĭn space if and only if* (i) *it satisfies the conditions* (1) – (3) *of Theorem* 8.9, *and* (ii) *the Gram operator for every nonnegative majorant has the unique factorization property.*

9. The contractive substitution property

We return to the problem of de Branges on the coefficients of univalent functions (see §2): do the conditions (2.2) characterize initial segments $B_1, \ldots, B_r$ of the coefficients of a normalized univalent function which is bounded by one in the unit disk? The answer, as already indicated, is negative, but there is more to the story. It is not hard to see that the conditions are sufficient for $r = 1, 2$ [51, Theorem 3.4], and they are also sufficient in the limit as $r \to \infty$ [15]:

Theorem 9.1 (de Branges)**.** *Let $B(z) = B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ be a formal power series such that $B_1 > 0$ and* (2.2) *holds for all $r = 1, 2, \ldots,$ every $\nu = -1, -2, \ldots,$ and every generalized power series* (2.1)*. Then $B(z)$ represents a univalent function which is bounded by one in the unit disk.*

See [66] for an account of the original proof by de Branges; a different proof is due to Nikolskii and Vasyunin in their work on coefficient problems and functional analytic aspects of the proof of the Bieberbach conjecture [59, 60] (see Theorem D180, p. 1219, and the remark D270, p. 1225, in the English translation of [60]).

The conditions (2.2) are reduced a procedure which is analogous to the Schur algorithm in Christner, Li, and Rovnyak [21]. The classical Schur algorithm does not make sense in the present context, but the operator transcription in Foias and Frazho [40] can be adapted to the indefinite situation using properties of Julia operators as discussed in §5. The outcome is that if $B_1, \ldots, B_r$, $B_1 > 0$, are given numbers satisfying (2.2) for all real numbers ν, then the set of numbers B_{r+1} such that $B_1, \ldots, B_r, B_{r+1}$, satisfy the same conditions with r replaced by $r + 1$ is the intersection of a family of closed disks $\Delta_r(\nu)$, $-\infty < \nu < \infty$. The centers and radii of the disks $\Delta_r(\nu)$ are functions of $B_1, \ldots, B_r$ which are given by recursive formulas. The formulas were implemented in a *Mathematica* program by an undergraduate student, A. Pitsillides [61]. A typical run is shown in the Figures 1–5 below. In each case, the white oval region inside the system of circles shows the possible values of B_{r+1} for the given numbers $B_1, \ldots, B_r$. The same formulas produced a counterexample when $r = 3$ in another undergraduate project by D. Dreibelbis [31]: the numbers

$$B_1 = \frac{1}{6}, \qquad B_2 = \frac{1}{4}, \qquad B_3 = \frac{4}{15} + \frac{i}{18} \tag{9.1}$$

satisfy (2.2) with $r = 3$ for all real ν, but there is no univalent function which has the form

$$B(z) = \frac{1}{6}\, z + \frac{1}{4}\, z^2 + \left(\frac{4}{15} + \frac{i}{18}\right) z^3 + \mathcal{O}(z^4)$$

and is bounded by one in the unit disk.

Nevertheless, there is numerical evidence in favor of something like (2.2). The most basic example of a normalized univalent function which is bounded by one in the unit disk is a **bounded Koebe mapping**, by which we mean a solution $B(z) = B_{b,a,u}(z)$ of the functional equation $f_{a,u}(z) = f_{b,u}(B(z))$, where $0 < a \leq b < \infty$, $|u| = 1$, and

$$f_{t,u}(z) = tz/(1 - uz)^2, \qquad 0 < t < \infty,$$

is a Koebe function [65, §8.1]; that is,

$$B_{b,a,u}(z) = f_{b,u}^{-1}(f_{a,u}(z)). \tag{9.2}$$

Compositions of bounded Koebe mappings provide many data sets for numerical experiments. It appears that counterexamples such as (9.1) are only possible when numbers are chosen very close to the boundaries of the regions predicted by (2.2). In private discussions, M. A. Dritschel and the author have considered possible variations of (2.2), including:

Problem. *Let B_1, B_2, B_3, B_4 be given numbers with $B_1 > 0$. If* (2.2) *holds for $r = 4$, all real numbers ν, and all generalized power series* (2.1)*, are B_1, B_2, B_3 the coefficients of a normalized univalent function which is bounded by one in the unit disk?*

The numbers (9.1) are not a counterexample because there is no way to choose B_4 to meet the conditions. More generally, if (2.2) holds for some numbers $B_1, \ldots, B_r$ ($B_1 > 0$), are some of these numbers the coefficients of a normalized univalent function which is bounded by one in the unit disk? A related problem asks if, in some sense, the bounded Koebe mappings (9.2) play a role analogous to single Blaschke factors.

Problem. *Let $B_1, \ldots, B_r$ be numbers with $B_1 > 0$. If there exists a univalent and normalized function $B(z)$ satisfying $|B(z)| \leq 1$ on $\mathbf{D}$ and such that $B(z) = B_1 z + \cdots + B_r z^r + \mathcal{O}(z^{r+1})$, can $B(z)$ be chosen to be a composition of r bounded Koebe mappings?*

The answer is affirmative for $r = 2$, and numerical evidence for $r = 3$ seems strong.

The coefficients of univalent functions, and in particular bounded univalent functions, are extensively studied in the literature on classical function theory. The first four coefficients are investigated by [73]; connections with the problems stated above may be present but are not transparent to the author.

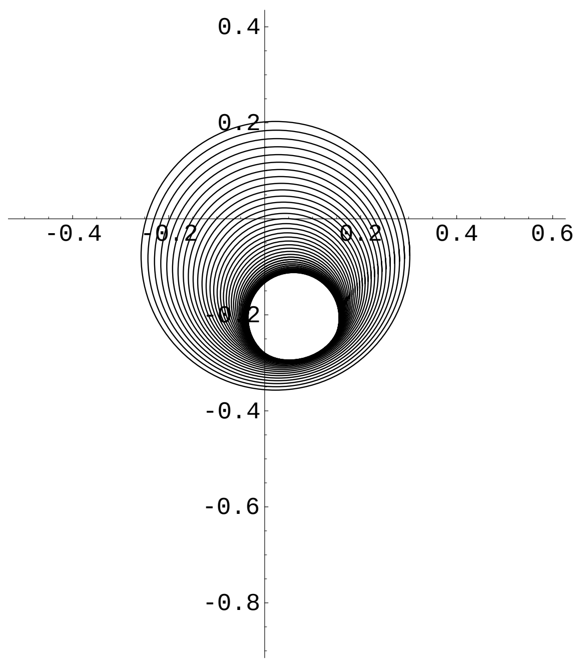

FIGURE 1. Possible values of B_3 if $B_1 = 0.2$ and $B_2 = -0.2 + 0.15\,i$

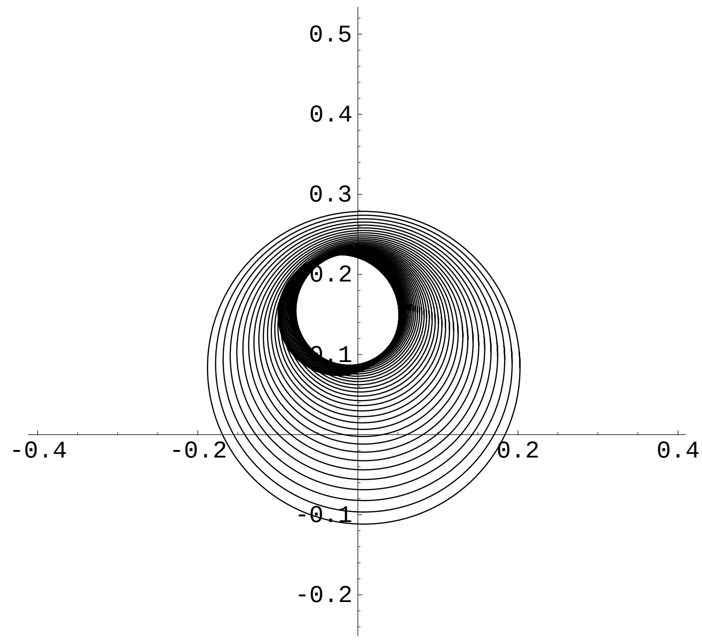

FIGURE 2. Possible values of B_4 if $B_3 = 0.1 - 0.2\,i$

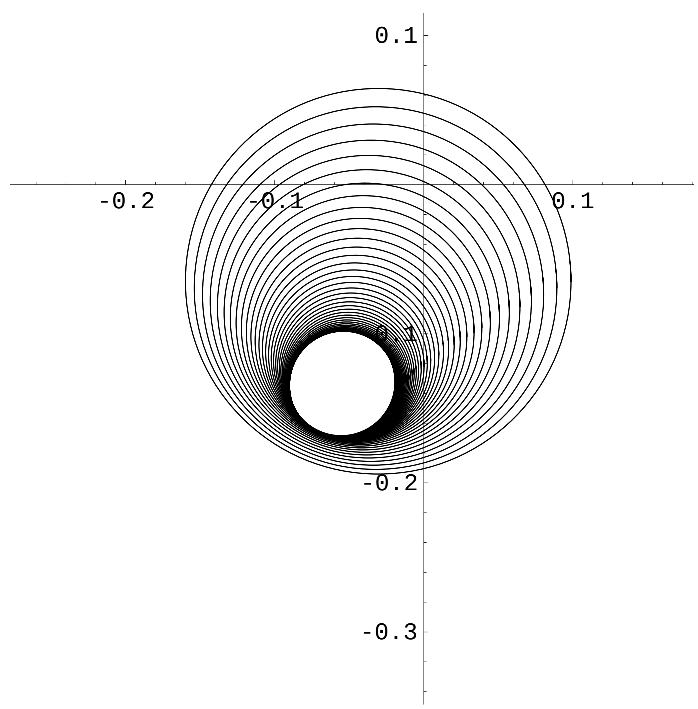

FIGURE 3. Possible values of B_5 if $B_4 = -0.02 + 0.2\,i$

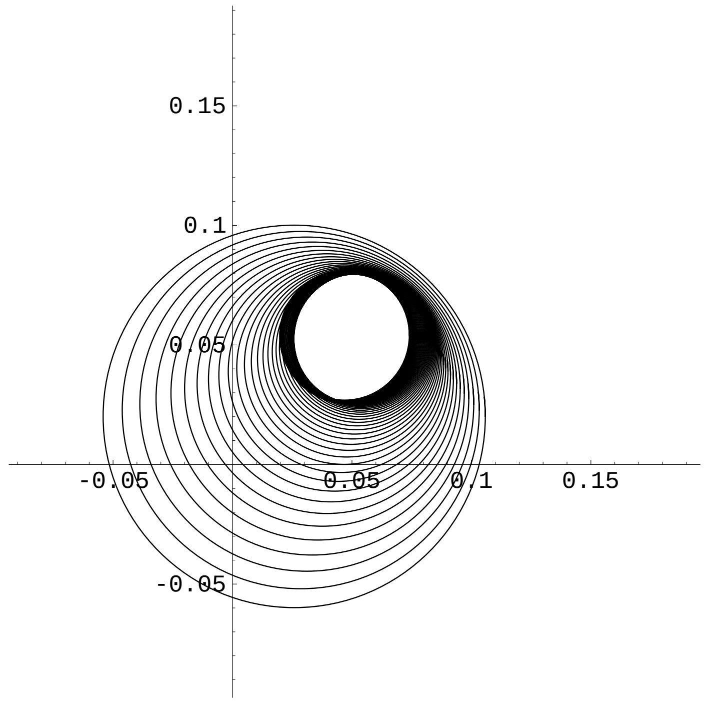

FIGURE 4. Possible values of B_6 if $B_5 = -0.05 - 0.15\,i$

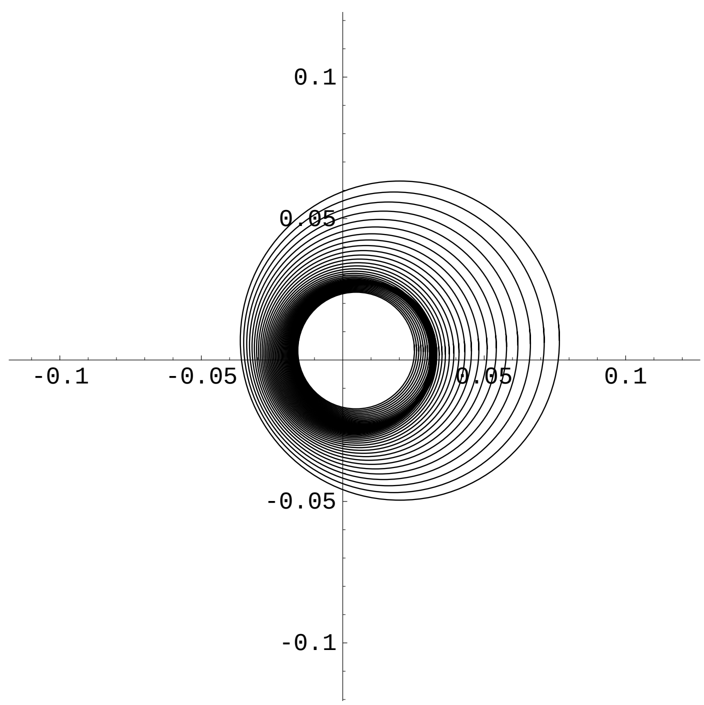

FIGURE 5. Possible values of B_7 if $B_6 = 0.06 + 0.05\,i$

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