SOME EXTENSIONS OF LOEWNER'S THEORY OF MONOTONE OPERATOR FUNCTIONS

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To Marvin Rosenblum, with best wishes on the occasion of his retirement.

Abstract. Several extensions of Loewner's theory of monotone operator functions are given. These include a theorem on boundary interpolation for matrix-valued functions in the generalized Nevanlinna class. The theory of monotone operator functions is generalized from scalar- to matrix-valued functions of an operator argument. A notion of κ-monotonicity is introduced and characterized in terms of classical Nevanlinna functions with removable singularities on a real interval. Corresponding results for Stieltjes functions are presented.

1. Introduction and preliminary results

Loewner [18] characterized the class of real-valued differentiable functions f on a real interval (a, b) such that $f(A) \leq f(B)$ whenever A and B are selfadjoint matrices whose eigenvalues lie in (a, b) and $A \leq B$. The class coincides with the set of all real-valued differentiable functions f on (a, b) such that the Hermitian kernel which is defined by $K_f(x, y) = [f(x) - f(y)]/(x - y)$ when $x \neq y$, and $K_f(x, x) = f'(x)$, is nonnegative on $(a, b) \times (a, b)$. In turn, such a kernel $K_f(x, y)$ is nonnegative if and only if f is the restriction of a Nevanlinna function which has an analytic continuation across (a, b). There are many related results and generalizations; for example, see [6, 11, 13, 20].

In this paper, we consider similar problems for matrix-valued functions F in place of scalar-valued functions f. A natural functional calculus is used to define $F(A)$ when F is a matrix-valued function and A is a Hilbert space operator. In the definite case, we show that, relative to this functional calculus, the Loewner theory has much the same character in the matrix case as in the scalar case. The interpolation part of the Loewner theory is carried out more generally for functions in the generalized Nevanlinna class $\mathcal{N}_\kappa$, extending a scalar result of Alpay and Rovnyak [5]. A different result on boundary interpolation in the indefinite case is given in Ball and Helton [7, Theorem 3.9]. We also consider functions which satisfy a notion of κ-monotonicity. The answer here is not what one might expect: κ-monotone functions do not arise from functions in $\mathcal{N}_\kappa$ but instead from classical Nevanlinna functions which are modified by adding removable singularities. We obtain analogous results for Stieltjes functions.

Let $\mathbf{C}$ be the complex numbers, $\mathbf{R}$ the real numbers, and $\mathbf{C}^{\pm}$ the open upper and lower half-planes in $\mathbf{C}$. Let $\mathbf{C}^p$ be Euclidean p-dimensional space, and let $\mathbf{C}^{p\times q}$ be the set of $p \times q$ matrices with complex entries, which we identify with operators on $\mathbf{C}^q$ into $\mathbf{C}^p$. The adjoint and norm of an operator or a matrix B are written B^* and $\|B\|$.

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We call a matrix-valued function K defined on a product set $\Omega \times \Omega$ a **kernel** if it is Hermitian: $K(t,s) = K(s,t)^*$ for all $s, t \in \Omega$. We say that a kernel K has κ negative squares on Ω and write $\mathrm{sq}_- K = \kappa$ if for every choice of a positive integer r and points $s_1, \dots, s_r \in \Omega$, the Hermitian matrix $(K(s_j, s_k))_{k,j=1}^r$ has at most κ negative eigenvalues, and at least one such matrix has exactly κ negative eigenvalues (counting multiplicities). The kernel K is called **nonnegative** if this property holds with $\kappa = 0$, that is, the matrix $(K(s_j, s_k))_{k,j=1}^r$ is nonnegative for every finite set of points $s_1, \dots, s_r \in \Omega$. We also write $\mathrm{sq}_\pm M$ for the numbers of positive and negative eigenvalues of a Hermitian matrix M.

We write $\rho(F)$ for the set of points of analyticity of a meromorphic function F.

Definition 1.1. *A function S with values in $\mathbf{C}^{m\times m}$ is in the generalized Nevanlinna class $\mathcal{N}_\kappa$ if S is meromorhic in $\mathbf{C} \setminus \mathbf{R}$, $\rho(S)$ is symmetric with respect to the real axis,*

$$(1.1) \qquad S(\bar z) = S(z)^*, \qquad z \in \mathbf{C}^+ \cap \rho(S),$$

and the kernel

$$(1.2) \qquad K_S(w,z) = \frac{S(z) - S(w)^*}{z - \bar w}$$

has κ negative squares on $\mathbf{C}^+ \cap \rho(S)$.

Definition 1.2. *An $m \times m$ matrix function S belongs to the generalized Stieltjes class $\mathcal{S}_\kappa^{\tilde\kappa}$ if $S(z) \in \mathcal{N}_\kappa$ and $zS(z) \in \mathcal{N}_{\tilde\kappa}$, that is, if S is meromorhic in $\mathbf{C} \setminus \mathbf{R}$, $\rho(S)$ is symmetric with respect to the real axis, S satisfies the symmetry relation* (1.1)*, and the kernels*

$$(1.3) \qquad K_S(w,z) = \frac{S(z) - S(w)^*}{z - \bar w}, \qquad \widetilde{K}_S(w,z) = \frac{zS(z) - \bar w S(w)^*}{z - \bar w}$$

have κ and $\widetilde\kappa$ negative squares on $\mathbf{C}^+ \cap \rho(S)$.

We write $\mathcal{N}_0$ and $\mathcal{S}_0^0$ more simply as $\mathcal{N}$ and $\mathcal{S}$. The generalized Nevanlinna classes appear in many places and are related to analogous classes of generalized Schur functions relative to the unit disk [4]. The Stieltjes class $\mathcal{S}_\kappa^0$ appears in [15, p. 206] and [16, p. 27]; the general classes appear in [10]. Nevanlinna-Pick interpolation problems for the generalized Stieltjes classes are studied in [3]. Classical sources in the scalar case are [15] and [17].

We note some alternative characterizations when $\kappa = 0$. The class $\mathcal{N}$ consists of all holomorphic functions S with $\mathbf{C} \setminus \mathbf{R} \subseteq \rho(\mathbf{S})$ which satisfy (1.1) and which have a nonnegative imaginary part in $\mathbf{C}^+$. It turns out that $\mathcal{S}$ coincides with the class of functions $S \in \mathcal{N}$ which are holomorphic across $(-\infty, 0)$ and there take nonnegative values (see [17, Theorem A5, p. 392] for the scalar case of this result). These properties are often taken as definitions of the classes.

Theorem 1.3. *A $\mathbf{C}^{m\times m}$-valued holomorphic function S with $\mathbf{C} \setminus \mathbf{R} \subseteq \rho(\mathbf{S})$ belongs to $\mathcal{N}$ if and only if it admits a representation*

$$(1.4) \qquad S(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\Sigma(\lambda),$$

where α and β are $m \times m$ matrices such that $\alpha = \alpha^*$ and $\beta \geq 0$, and $\Sigma(\lambda)$ is a nondecreasing $\mathbf{C}^{m\times m}$-valued function which satisfies

$$(1.5) \qquad \int_{-\infty}^{\infty} (\lambda^2+1)^{-1}\, d\langle \Sigma(\lambda)x,\, x\rangle < \infty, \qquad x \in \mathbf{C}^m.$$

With the normalization $\Sigma(0) = 0$ and $\Sigma(\lambda) = \frac{1}{2}(\Sigma(\lambda+0)+\Sigma(\lambda-0))$ for all λ, such a representation is unique, and $\Sigma(\lambda)$ is determined from S via the Stieltjes inversion formula

$$\Sigma(\lambda_2) - \Sigma(\lambda_1) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} S(x+i\varepsilon)\, dx, \qquad -\infty < \lambda_1 < \lambda_2 < \infty. \tag{1.6}$$

Theorem 1.4. *A $\mathbf{C}^{m\times m}$-valued holomorphic function S with $\mathbf{C} \setminus \mathbf{R} \subseteq \rho(\mathbf{S})$ belongs to $\mathcal{S}$ if and only if it admits a representation*

$$S(z) = \gamma + \int_0^\infty \frac{d\Sigma(\lambda)}{\lambda - z}, \tag{1.7}$$

where γ is an $m \times m$ matrix such that $\gamma \geq 0$, and $\Sigma(\lambda)$ is a nondecreasing $\mathbf{C}^{m\times m}$-valued function which satisfies

$$\int_0^\infty (\lambda+1)^{-1}\, d\langle \Sigma(\lambda)x,\, x\rangle < \infty, \qquad x \in \mathbf{C}^m. \tag{1.8}$$

With a suitable normalization as in Theorem 1.3, such a representation is uniquely determined.

These results are easily deduced from their scalar analogues [17, Appendix] (they are also true for operator–valued Stieltjes functions if the integrals in (1.4) and (1.7) are interpreted in the weak sense).

Section 2 is devoted to Loewner or boundary interpolation for the generalized Nevanlinna and Stieltjes classes. In Section 3 we characterize matrix-valued functions which have the monotone operator property in a suitable functional calculus. The notion of a κ-monotone operator function is defined and characterized in Section 4.

2. Matrix analogues of Loewner interpolation

We first extend the scalar result on Loewner interpolation in Alpay and Rovnyak [5] to matrix-valued functions. Let F be a continuously differentiable function on an open subset Δ of $\mathbf{R}$ whose values are selfadjoint matrices in $\mathbf{C}^{m\times m}$. Define kernels

$$K_F(x,t) = \begin{cases} \dfrac{F(x)-F(t)}{x-t}, & x \neq t, \\ F'(x), & x = t, \end{cases} \quad \text{and} \quad \widetilde{K}_F(x,t) = \begin{cases} \dfrac{xF(x)-tF(t)}{x-t}, & x \neq t, \\ xF'(x)+F(x), & x = t, \end{cases} \tag{2.1}$$

on $\Delta \times \Delta$.

Theorem 2.1. *Let F be a $\mathbf{C}^{m\times m}$-valued selfadjoint continuously differentiable function on an open subset $\Delta \subseteq \mathbf{R}$. Then*

(1) *There exists $S \in \mathcal{N}_\kappa$ which admits an analytic continuation across Δ and coincides with F on Δ if and only if the kernel K_F has κ negative squares on Δ.*
(2) *There exists $S \in \mathcal{S}_\kappa^{\tilde{\kappa}}$ which admits an analytic continuation across Δ and coincides with F on Δ if and only if the kernels K_F and $\widetilde{K}_F$ have, respectively, κ and $\widetilde{\kappa}$ negative squares on Δ.*

Thus for functions F which satisfy the hypotheses of the theorem, if K_F has κ negative squares on Δ, then K_F has κ negative squares on every open subset Δ' of Δ.

Proof. (1) *Necessity.* Assume that $F = S|_\Delta$ for some function $S \in \mathcal{N}_\kappa$ which admits an analytic continuation across Δ. The kernel K_S is holomorphic on an open set Ω that includes Δ, and it therefore has κ negative squares on Ω by [4, Theorem 1.1.4]. We view K_S as being defined on Ω and write $\mathcal{H}(K_S)$ for the associated reproducing kernel Pontryagin space of holomorphic functions on Ω. The space of restrictions of functions in $\mathcal{H}(K_S)$ to Δ is a Pontryagin space $\mathcal{H}(K_F)$ in the inner product which makes the restriction mapping an isometry, and its reproducing kernel is K_F, that is, the restriction of K_S to $\Delta \times \Delta$. Since $\mathcal{H}(K_F)$ then also has negative index κ, it follows that K_F has κ negative squares.

Sufficiency. We prove this using the known result in the scalar case [5, Theorem 3.1]. Assume that F is a continuously differentiable function on an open subset $\Delta \subseteq \mathbf{R}$ whose values are $\mathbf{C}^{m\times m}$ selfadjoint matrices such that K_F has κ negative squares on Δ. For any fixed vector $h \in \mathbf{C}^m$, we can apply the scalar result to the function $h^*F(x)h$ to obtain a scalar-valued meromorphic function $F_h \in \mathcal{N}_{\kappa_h}$ (with $\kappa_h \leq \kappa$), which is holomorphic across Δ and satisfies

$$F_h(x) = h^*F(x)h, \qquad x \in \Delta.$$

The singularities of $F_h(z)$ may depend on h, and we need to show that they all lie in some fixed finite set of points. For simplicity, suppose that $m = 2$. Define

$$W(z) = \begin{pmatrix} W_{11}(z) & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{pmatrix},$$

where

$$\begin{aligned}
W_{11}(z) &= F_{\binom{1}{0}}(z), \\
W_{12}(z) &= \frac{1}{2}\,F_{\binom{1}{1}}(z) - \frac{i}{2}\,F_{\binom{1}{i}}(z) - \frac{1-i}{2}\left[F_{\binom{1}{0}}(z) + F_{\binom{0}{1}}(z)\right], \\
W_{21}(z) &= \frac{1}{2}\,F_{\binom{1}{1}}(z) + \frac{i}{2}\,F_{\binom{1}{i}}(z) - \frac{1+i}{2}\left[F_{\binom{1}{0}}(z) + F_{\binom{0}{1}}(z)\right], \\
W_{22}(z) &= F_{\binom{0}{1}}(z).
\end{aligned}$$

These functions are meromorphic on $(\mathbf{C} \setminus \mathbf{R}) \cup \boldsymbol{\Delta}$ except for finitely many nonreal points; let $\Omega = \rho(W)$ be the region of holomorphy of W. A short calculation shows that

$$F(x) = \begin{pmatrix} W_{11}(x) & W_{12}(x) \\ W_{21}(x) & W_{22}(x) \end{pmatrix}$$

on Δ. It follows that for all $h \in \mathbf{C}^2$, $F_h(z)$ is holomorphic on Ω and

$$F_h(z) = h^*W(z)h, \qquad z \in \Omega.$$

For $m > 2$, a similar argument yields the same conclusion. Here we define the diagonal entries $W_{jj}(z)$ of $W(z)$ in the obvious way. Any off diagonal term of a matrix may be viewed as an element of a principal 2×2 submatrix. This allows us to define $W_{jk}(z)$ for $j \neq k$ as well by formulas as above.

It remains to show that $W \in \mathcal{N}_\kappa$. This function is meromorphic in $\mathbf{C} \setminus \mathbf{R}$ by construction, and the symmetry condition (1.1) is satisfied for $S = W$ because we assume that F has selfadjoint values on Δ. Since W and F coincide on Δ, K_W has at least κ negative squares on Ω; initially we cannot say, however, that K_W has a finite number of negative squares. Choose a disk D centered at a point of Δ whose closure is in the region of holomorphy of $W(z)$, and let $I = D \cap \Delta$. By a theorem of Alpay [2], the fact that K_W is a holomorphic kernel implies that the restriction $K_{W,D}$ of K_W to $D \times D$ is the reproducing kernel for some (not necessarily unique) Kreĭn space $\mathcal{K}_D$ of $\mathbf{C}^m$-valued holomorphic functions on D.

Since I is a set of uniqueness for the elements of $\mathcal{K}_D$, we may define a Kreĭn space $\mathcal{K}_I$ of functions on I such that the restriction mapping on $\mathcal{K}_D$ to $\mathcal{K}_I$ is a Kreĭn space isomorphism. Standard arguments show that $\mathcal{K}_I$ has reproducing kernel equal to the restriction $K_{W,I}$ of K_W to $I \times I$. Thus $\mathcal{K}_I$ is a Pontryagin space, because $K_{W,I}$ is the restriction of K_F to $I \times I$. Hence $\mathcal{K}_D$ is a Pontryagin space of the same negative index as $\mathcal{K}_I$. This negative index cannot exceed κ because of our hypotheses on F, and hence it is equal to κ. Since by [4, Theorem 1.1.4], the number of negative squares is independent of the region of holomorphy, we conclude that $W \in \mathcal{N}_\kappa$, as was to be shown.

(2) Let the kernels K_F and $\widetilde{K}_F$ have κ and $\widetilde{\kappa}$ negative squares, respectively. Applying the first part of the theorem to the functions $F(x)$ and $xF(x)$ we conclude that there exist two functions $S \in \mathcal{N}_\kappa$ and $\widetilde{S} \in \mathcal{N}_{\widetilde{\kappa}}$, which admit analytic continuations across Δ and coincide with $F(x)$ and $xF(x)$, respectively, on Δ. Since S and $\widetilde{S}$ are analytic at every point of $\mathbf{C}^+$ except for at most finitely many points and since $\widetilde{S}(x) = xS(x)$ at every point $x \in \Delta$, then by the uniqueness theorem, $\widetilde{S}(z) \equiv zS(z)$. Thus, $S(z) \in \mathcal{N}_\kappa$ and $zS(z) \in \mathcal{N}_{\widetilde{\kappa}}$ and therefore, $S \in \mathcal{S}_\kappa^{\widetilde{\kappa}}$. The converse implication is self-evident. □

3. Monotone operator-functions

In practice we are only concerned with functions defined on open intervals, but the definition of matrix- and operator-monotone functions can be stated more generally. Let F be a $\mathbf{C}^{m\times m}$-valued measurable function with selfadjoint values which is defined on a Borel set $\Delta \subseteq \mathbf{R}$. Write $F(x) = (f_{j\ell}(x))_{j,\ell=1}^m$, $x \in \Delta$, and assume that $\|F(x)\|$ is bounded on all compact subsets of Δ. Given a bounded selfadjoint operator A on a Hilbert space $\mathcal{H}$ with the spectrum $\sigma(A) \subseteq \Delta$ and the spectral representation

$$A = \int_{\sigma(A)} t\, dE(t), \tag{3.1}$$

let $F(A)$ denote the operator which acts on the space

$$\widehat{\mathcal{H}} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{\text{m times}} \tag{3.2}$$

and has the matrix representation

$$F(A) = \left(\int_{\sigma(A)} f_{j\ell}(t)\, dE(t) \right)_{j,\ell=1}^m \tag{3.3}$$

with respect to the orthogonal decomposition (3.2). Since the functions $f_{j\ell}$ are bounded on $\sigma(A)$, the operator $F(A)$ is bounded. It is easy to see that $F(A)$ is selfadjoint. If $\dim \mathcal{H} = n$, then $A = \sum_{k=1}^n \lambda_k E_k$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, $E_1, \dots, E_n$ are rank one orthogonal projections, and $F(A)$ is the $mn \times mn$ matrix

$$F(A) = \big(f_{j\ell}(A)\big)_{j,\ell=1}^m = \left(\sum_{k=1}^n f_{j\ell}(\lambda_k)\, E_k \right)_{j,\ell=1}^m = \sum_{k=1}^n F(\lambda_k) \otimes E_k. \tag{3.4}$$

Here $\otimes$ indicates a Kronecker or tensor product [13, p. 243]. Similar functional calculi are used, for example, in [1, p. 53] and [8]. We call F a **monotone operator function** if $F(A) \leq F(B)$ for every choice of a Hilbert space $\mathcal{H}$ and a pair of bounded selfadjoint operators A and B on $\mathcal{H}$ such that

$$\sigma(A) \subseteq \Delta, \quad \sigma(B) \subseteq \Delta, \quad \text{and} \quad A \leq B. \tag{3.5}$$

We say that F is a **monotone matrix function** if this condition holds under the additional restriction that $\dim \mathcal{H} < \infty$, that is, if for every choice of an integer n and a pair of Hermitian $n \times n$ matrices A and B satisfying (3.5), the $mn \times mn$ matrices $F(A)$ and $F(B)$ defined via (3.4) satisfy $F(A) \leq F(B)$.

Every monotone operator function is obviously a monotone matrix function. The converse assertion is an immediate consequence of Theorem 3.3 (see below). For scalar–valued functions the equivalence of matrix and operator monotonicities follows from Loewner's results [18] (for the proof see also [9, Lemma 2.2]).

The class of monotone matrix (operator) functions is closed under addition, multiplication by nonnegative scalars, and pointwise limits. By considering scalar multiples of the identity operator on a one-dimensional Hilbert space, we see that a monotone matrix function F satisfies $F(\lambda_1) \leq F(\lambda_2)$ whenever λ_1 and λ_2 are points in its domain such that $\lambda_1 < \lambda_2$, that is, F is nondecreasing in the usual sense. If the values of F are $m \times m$ matrices and h is any m-dimensional column vector, then, in the classical sense for scalar-valued functions, $h^* F h$ is matrix monotone. In particular, it follows from the polarization identity and scalar theory [20] that a monotone matrix function F on an interval is continuously differentiable.

We need some preliminary lemmas. The first is an analogue of the Schur theorem on the Hadamard product of matrices. The result is known (see [14, Theorem 3.1]), but we include a proof for the convenience of the reader.

Lemma 3.1. *Let $D = (d_{k\ell})_{k,\ell=1}^m$ be a nonnegative $m \times m$ matrix and let M be a bounded nonnegative operator on a Hilbert space $\widehat{\mathcal{H}}$ of the form* (3.2) *with the matrix representation $M = (M_{k\ell})_{k,\ell=1}^m$ with respect to the decomposition* (3.2)*. Then the operator*

$$D \circ M = (d_{k\ell} M_{k\ell})_{k,\ell=1}^m$$

is nonnegative.

Proof. Put $M^{1/2} = (\widetilde{M}_{k\ell})_{k,\ell=1}^m$ and $D^{1/2} = (\widetilde{d}_{k\ell})_{k,\ell=1}^m$. Then $d_{k\ell} = \sum_{p=1}^m \left(\widetilde{d}_{pk}\right)^- \widetilde{d}_{p\ell}$ and $M_{k\ell} = \sum_{q=1}^m \widetilde{M}_{qk}^* \widetilde{M}_{q\ell}$. Let h be any vector in $\widehat{\mathcal{H}}$ with components $h_1, \ldots, h_m$. Then

$$\begin{aligned}\langle (D \circ M)h,\ h\rangle_{\widehat{\mathcal{H}}} &= \sum_{k,\ell=1}^m \langle d_{k\ell} M_{k\ell} h_\ell, h_k\rangle_{\mathcal{H}} = \sum_{k,\ell=1}^m \sum_{p,q=1}^m \left\langle \left(\widetilde{d}_{pk}\right)^- \widetilde{d}_{p\ell} \widetilde{M}_{qk}^* \widetilde{M}_{q\ell} h_\ell, h_k \right\rangle_{\mathcal{H}} \\ &= \sum_{p,q=1}^m \left\langle \sum_{\ell=1}^m \widetilde{d}_{p\ell} \widetilde{M}_{q\ell} h_\ell, \sum_{k=1}^m \widetilde{d}_{pk} \widetilde{M}_{qk} h_k \right\rangle_{\mathcal{H}} \geq 0\end{aligned}$$

because every term in the sum is nonnegative. □

Lemma 3.2. *Let A and B be two Hermitian matrices of the same order with spectral decompositions $A = \sum_{k=1}^n \lambda_k E_k$ and $B = \sum_{i=1}^N \mu_i G_i$ corresponding to any labeling of eigenvalues.*

(1) *For any measurable* $\mathbf{C}^{m\times m}$*-function* $F = (f_{j\ell})_{j,\ell=1}^{m}$ *whose domain includes all of the eigenvalues of both* A *and* B*,*

$$F(B) - F(A) = \left(\sum_{k=1}^{n} \sum_{i=1}^{N}{}' \, \frac{f_{j\ell}(\lambda_k) - f_{j\ell}(\mu_i)}{\lambda_k - \mu_i} \, E_k(B-A)G_i \right)_{j,\ell=1}^{m}, \tag{3.6}$$

where the $'$ *indicates that when* $\lambda_k - \mu_i = 0$*, the difference quotient may be replaced by any number whatever, the choice being irrelevant because* $E_k(B-A)G_i = 0$*.*

(2) *If the eigenvalues of* A *are simple, then for all sufficiently small real* ε*, the eigenvalues of* $A^{(\varepsilon)} = A + \varepsilon B$ *are simple, and the spectral decomposition of* $A^{(\varepsilon)}$ *can be written in a form*

$$A^{(\varepsilon)} = \sum_{k=1}^{n} \lambda_k^{(\varepsilon)} E_k^{(\varepsilon)}$$

such that $\lim_{\varepsilon\to 0} \lambda_k^{(\varepsilon)} = \lambda_k$ *and* $\lim_{\varepsilon\to 0} E_k^{(\varepsilon)} = E_k$ *for all* $k = 1, \ldots, n$*.*

Lemma 3.2 follows from [20, Lemmas A and B on p. 42].

We note a connection between Lemma 3.2(2) and Geršgorin's theorem [12, Theorem 6.1.1]. Namely, Geršgorin's theorem immediately gives a part of this statement. For in the situation of the lemma, it implies that for all sufficiently small $\varepsilon > 0$ and a suitable labeling of eigenvalues, $|\lambda_k - \lambda_k^{(\varepsilon)}| \le \varepsilon(n-1)$ for all $k = 1, \ldots, n$, and thus $\lim_{\varepsilon\to 0} \lambda_k^{(\varepsilon)} = \lambda_k$.

Theorem 3.3. *Let* F *be a* $\mathbf{C}^{m\times m}$*-valued measurable function with selfadjoint values which is bounded on all compact subsets of the open interval* $\Delta \subseteq \mathbf{R}$*.*

(1) *If* F *is a monotone matrix function on* Δ*, then there exists a function* $S \in \mathcal{N}$ *which is analytic on* $\mathbf{C}^+ \cup \mathbf{C}^- \cup \Delta$ *and satisfies*

$$S(x) = F(x), \qquad x \in \Delta. \tag{3.7}$$

(2) *If* (3.7) *holds for some* $S \in \mathcal{N}$ *which is analytic on* $\mathbf{C}^+ \cup \mathbf{C}^- \cup \Delta$*, then* F *is a monotone operator function on* Δ*.*

Proof. (1) Let F be a monotone matrix function. As noted above, F is continuously differentiable, and so the kernel K_F defined in (2.1) is continuous on $\Delta \times \Delta$ (an alternative to using the scalar result to show that F is continuously differentiable is to reduce to this case by the smoothing method which is used in the scalar case [20, pp. 43–44]). Let n be any positive integer, and let $\lambda_1, \ldots, \lambda_n \in \Delta$ be any distinct points. Let

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} = \sum_{k=1}^{n} \lambda_k E_k, \tag{3.8}$$

where E_k is the matrix with the only nonzero entry $[E_k]_{kk} = 1$. Here and below we sometimes use the notation $[M]_{ij}$ for the entry in row i and column j of the matrix M. Let

$$B = A + \varepsilon Q, \qquad Q = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (1, \ldots, 1), \tag{3.9}$$

where ε is a positive number. By the second assertion of Lemma 3.2, B admits a spectral representation

$$B = \sum_{k=1}^{n} \lambda_k^{(\varepsilon)} E_k^{(\varepsilon)},$$

such that $\lambda_k^{(\varepsilon)} \to \lambda_k$ and $E_k^{(\varepsilon)} \to E_k$ as ε tends to zero. For all sufficiently small ε, $\sigma(B) \subseteq \Delta$. Since $B \geq A$ and F is a monotone matrix function, $F(B) \geq F(A)$, and by the first assertion of Lemma 3.2,

$$0 \leq F(B) - F(A) = \left(\sum_{k,i=1}^{n} \frac{f_{j\ell}(\lambda_k) - f_{j\ell}(\lambda_i^{(\varepsilon)})}{\lambda_k - \lambda_i^{(\varepsilon)}} E_k \varepsilon Q E_i^{(\varepsilon)} \right)_{j,\ell=1}^{m}.$$

Dividing by ε and then letting $\varepsilon \to 0$, we obtain, in view of (2.1),

$$\left(\sum_{k,i=1}^{n} [K_F(\lambda_k,\, \lambda_i)]_{j\ell}\, E_k Q E_i \right)_{j,\ell=1}^{m} \geq 0. \tag{3.10}$$

By the definitions of E_k and Q the matrix $E_k Q E_i$ has the only nonzero entry $[E_k Q E_i]_{ki} = 1$ and therefore,

$$\sum_{k,i=1}^{n} [K_F(\lambda_k,\, \lambda_i)]_{j\ell}\, E_k Q E_i = \Big([K_F(\lambda_k,\, \lambda_i)]_{j\ell} \Big)_{k,i=1}^{n}.$$

Thus, (3.10) takes the form

$$M = \left(\Big([K_F(\lambda_k,\, \lambda_i)]_{j\ell} \Big)_{k,i=1}^{n} \right)_{j,\ell=1}^{m} \geq 0. \tag{3.11}$$

Let $U \in \mathbf{C}^{mn \times mn}$ be the permutation matrix defined by

$$U_{kj} = 1 \quad \text{if and only if} \quad (k,\, j) = ((q-1)n + r,\ (r-1)m + q),$$

for any choice of $q \in \{1, \dots, m\}$ and $r \in \{1, \dots, n\}$. Multiplying M by U from on right and by U^* on the left we obtain

$$U^* M U = \left(\Big([K_F(\lambda_k,\, \lambda_i)]_{j\ell} \Big)_{j,\ell=1}^{m} \right)_{k,i=1}^{n} = (K_F(\lambda_k,\, \lambda_i))_{k,i=1}^{n}.$$

Since the points $\lambda_j \in \Delta$ are arbitrary, the kernel K_F is nonnegative[1] on $\Delta \times \Delta$. The existence of a function $S \in \mathcal{N}$ with the desired properties now follows from Theorem 2.1(1) in the case $\kappa = 0$ or [19, Theorem 3.5].

(2) Assume that a function $S \in \mathcal{N}$ exists as in the theorem. By the Stieltjes inversion formula (1.6), since $S(x)$ is selfadjoint for all $x \in \Delta$, the representation (1.4) has the form

$$F(x) = \alpha + \beta x + \int_{\mathbf{R} \setminus \boldsymbol{\Delta}} \left(\frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2 + 1} \right) d\Sigma(\lambda). \tag{3.12}$$

[1]Another way to see this is to choose column vectors $h_k = \operatorname{col}(h_1^{(k)}, \dots, h_m^{(k)}) \in \mathbf{C}^m$, $k = 1, \dots, n$, and define $\widetilde{h}_1, \dots, \widetilde{h}_m \in \mathbf{C}^n$ by $\widetilde{h}_j = \operatorname{col}(h_j^{(1)}, \dots, h_j^{(n)})$, $j = 1, \dots, m$. If $h = \operatorname{col}(\widetilde{h}_1, \dots, \widetilde{h}_m)$, then by (3.11),

$$0 \leq h^* M h = \sum_{j,\ell=1}^{m} \sum_{k,i=1}^{n} h_j^{(k)*} K_{f_{j\ell}}(\lambda_k,\, \lambda_i) h_\ell^{(i)} = \sum_{k,i=1}^{n} h_i^* K_F(\lambda_k,\, \lambda_i) h_k.$$

Since the set of monotone operator functions is closed under addition and pointwise limits, it is sufficient to show that the function $F_\beta(x) = \beta x$, $\beta = (\beta_{j\ell})_{j,\ell=1}^m \geq 0$, and all functions of the form

$$F_{\lambda,D}(x) = (\lambda - x)^{-1} D, \qquad \lambda \in \mathbf{R} \setminus \Delta, \;\; D = (d_{j\ell})_{j,\ell=1}^m \geq 0,$$

are monotone operator functions on Δ. To show the latter, we take two bounded selfadjoint operators A and B satisfying (3.5) and get

$$F_{\lambda,D}(A) = \Big(d_{j\ell}\,(\lambda I_{\mathcal{H}} - A)^{-1}\Big)_{j,\ell=1}^m \quad \text{and} \quad F_{\lambda,D}(B) = \Big(d_{j\ell}\,(\lambda I_{\mathcal{H}} - B)^{-1}\Big)_{j,\ell=1}^m .$$

Then $F_{\lambda,D}(B) - F_{\lambda,D}(A) = D \circ M$, where

$$M = \begin{pmatrix} I_{\mathcal{H}} \\ \vdots \\ I_{\mathcal{H}} \end{pmatrix} \Big((\lambda I_{\mathcal{H}} - B)^{-1} - (\lambda I_{\mathcal{H}} - A)^{-1} \Big) \left(I_{\mathcal{H}},\ \ldots,\ I_{\mathcal{H}}\right).$$

Since $A \leq B$, $(\lambda I_{\mathcal{H}} - A)^{-1} \leq (\lambda I_{\mathcal{H}} - B)^{-1}$. Hence M is nonnegative, and by Lemma 3.1, $F_{\lambda,D}(B) - F_{\lambda,D}(A) \geq 0$. Similarly, $F_\beta(B) - F_\beta(A) = \beta \circ M$, where

$$M = \begin{pmatrix} I_{\mathcal{H}} \\ \vdots \\ I_{\mathcal{H}} \end{pmatrix} (B - A) \left(I_{\mathcal{H}},\ \ldots,\ I_{\mathcal{H}}\right) \geq 0.$$

Another application of Lemma 3.1 gives $F_\beta(B) - F_\beta(A) \geq 0$. □

Let $\mathbf{R}_+$ be the set of positive real numbers.

Corollary 3.4. *Let F be a $\mathbf{C}^{m\times m}$-valued measurable selfadjoint function which is bounded on all compact subsets of the open interval $\Delta \in \mathbf{R}$.*

1. *If $F(x)$ and $xF(x)$ are monotone matrix functions on Δ, then there exists $S \in \mathcal{S}$ which is analytic on $(\mathbf{C}\setminus\mathbf{R}_+) \cup \boldsymbol{\Delta}$ and satisfies (3.7).*
2. *If (3.7) holds for some $S \in \mathcal{S}$ which is analytic on $(\mathbf{C} \setminus \mathbf{R}_+) \cup \boldsymbol{\Delta}$, then $F(x)$ and $xF(x)$ are monotone operator function on Δ.*

Proof. (1) Let $F(x)$ and $xF(x)$ be monotone matrix functions on Δ. By Theorem 3.3 there exist functions $S(z)$ and $\widetilde{S}(z)$ in $\mathcal{N}$ which are analytic across Δ and coincide with $F(x)$ and $xF(x)$ on Δ. Since $\widetilde{S}(x) = xS(x)$ on Δ, $\widetilde{S}(z) \equiv zS(z)$. Thus both $S(z)$ and $zS(z)$ belong to $\mathcal{N}$, and therefore $S(z)$ belongs to $\mathcal{S}$.

(2) Assume that (3.7) for some function $S(z)$ in $\mathcal{S}$ which is analytic on $(\mathbf{C} \setminus \mathbf{R}_+) \cup \boldsymbol{\Delta}$. Then both $S(z)$ and $\widetilde{S}(z)$ belong to $\mathcal{N}$ and are analytic across Δ. Applying Theorem 3.3 to $S(z)$ and $zS(z)$, we see that $F(x)$ and $xF(x)$ are both monotone operator functions. □

4. Indefinite analogues of monotone matrix functions

It is natural to ask if the notion of a monotone operator function F has a generalization in which, for some nonnegative integer κ,

$$\text{sq}_-(F(B) - F(A)) \leq \kappa \tag{4.1}$$

whenever $A \leq B$ and the expressions are defined. It turns out that if we further restrict A and B to have simple spectra, there is such a notion.

Let F be a measurable function which is defined on a Borel set $\Delta \subseteq \mathbf{R}$, whose values are $\mathbf{C}^{m\times m}$ selfadjoint matrices, and which is bounded on all compact subsets of Δ. We call F a **κ-monotone matrix function** if for every positive integer n and pair of Hermitian matrices A and B in $\mathbf{C}^{n\times n}$ satisfying (3.5) and having simple spectra, the $mn \times mn$ matrix $F(B) - F(A)$ has at most κ negative eigenvalues, and $F(B) - F(A)$ has exactly κ negative eigenvalues for at least one choice of A and B satisfying (3.5). We write $\mathcal{M}_{m,\kappa}(\Delta)$ for the set of all $\mathbf{C}^{m\times m}$-valued κ-monotone matrix functions on Δ.

Remark 4.1. *If Δ is an interval, the class $\mathcal{M}_{m,0}$ coincides with the set of monotone matrix functions as defined in Section* 3.

In one direction this is trivial: a monotone matrix function belongs to $\mathcal{M}_{m,0}$. Conversely, suppose that F is in $\mathcal{M}_{m,0}$. This means that $F(B) \leq F(A)$ whenever A and B are Hermitian matrices which have simple spectra and satisfy (3.5). To remove the condition on simple spectra, we observe that the proof of Theorem 3.3(1) goes through for any function F in $\mathcal{M}_{m,0}$. It then follows from Theorem 3.3(2) that, in fact, F is a monotone operator function.

Remark 4.2. *For $\kappa > 0$, if we were to drop the assumption that A and B have simple spectra in the definition of a κ-monotone matrix function, the class $\mathcal{M}_{m,\kappa}$ of all such functions would be empty.*

Indeed, if $\mathrm{sq}_-(F(B) - F(A)) = \kappa$ for some A and B which satisfy (3.5), then for every positive integer j,

$$\mathrm{sq}_-(F(B_j) - F(A_j)) = j\kappa,$$

where $A_j = \mathrm{diag}\{A, \ldots, A\}$ and $B_j = \mathrm{diag}\{B, \ldots, B\}$ with j diagonal entries in each case. Thus if $\kappa > 0$, no function can meet the condition $\mathrm{sq}_-(F(B) - F(A)) \leq \kappa$ if we allow A and B to have nonsimple spectra.

Theorem 3.3 characterizes $\mathcal{M}_{m,0}$ in terms of analytic continuations to Nevanlinna functions. We shall generalize this result to $\mathcal{M}_{m,\kappa}(\Delta)$ for $\kappa > 0$, but first we need some lemmas.

Lemma 4.3. *Let $M \in \mathbf{C}^{m\times m}$ and $N \in \mathbf{C}^{n\times n}$ be two Hermitian matrices. Then*

$$\mathrm{sq}_-(M \otimes N) = \mathrm{sq}_-(M)\mathrm{sq}_+(N) + \mathrm{sq}_+(M)\mathrm{sq}_-(N).$$

Proof. Let U be a unitary matrix such that

$$UMU^* = \widetilde{M} = \mathrm{diag}\{a_1, \ldots, a_m\}.$$

Then $\mathrm{sq}_\pm(M) = \mathrm{sq}_\pm(\widetilde{M})$ and it is easily seen that

$$\mathrm{sq}_-(\widetilde{M} \otimes N) = \mathrm{sq}_-(\mathrm{diag}\{a_1N, \ldots, a_mN\}) = \mathrm{sq}_-(\widetilde{M})\mathrm{sq}_+(N) + \mathrm{sq}_+(\widetilde{M})\mathrm{sq}_-(N).$$

Thus, it remains to show that $\mathrm{sq}_\pm(M \otimes N) = \mathrm{sq}_\pm(\widetilde{M} \otimes N)$. But this follows from the equality

$$\widetilde{M} \otimes N = (U \otimes I_n)(M \otimes N)(U \otimes I_n)^*,$$

since the matrix $U \otimes I_n$ is unitary (see [13, pp. 243–244] for the properties of Kronecker products used here). $\square$

Lemma 4.4. *Let $M \in \mathbf{C}^{n\times n}$ be Hermitian. Then there exists a $\nu > 0$ such that*

$$\mathrm{sq}_- M \leq \mathrm{sq}_- N \leq \mathrm{sq}_- M + \dim(\ker M), \tag{4.2}$$

for every Hermitian matrix $N \in \mathbf{C}^{n\times n}$ such that

$$\max_{i,j} |M_{ij} - N_{ij}| < \nu. \tag{4.3}$$

Proof. The eigenvalues of M are the roots of the characteristic polynomial $p(z)$ of M. Let γ_+ be a simple closed contour in the open right half-plane that encloses all of the positive eigenvalues of M, γ_- a simple closed contour in the open left half-plane that encloses all of the negative eigenvalues of M. Choose $\delta > 0$ such that $|p(z)| > \delta$ on $\gamma_+ \cup \gamma_-$. Choose $\nu > 0$ such that whenever $N \in \mathbf{C}^{n\times n}$ is a Hermitian matrix satisfying (4.3), then the characteristic polynomial $q(z)$ of N satisfies $|p(z) - q(z)| < \delta$ on $\gamma_+ \cup \gamma_-$. Then $|p(z) - q(z)| < \delta < |p(z)|$ on $\gamma_+ \cup \gamma_-$, and so by Rouché's theorem, $q(z)$ has the same number of zeros inside $\gamma_\pm$ as $p(z)$. In particular, N has at least $\mathrm{sq}_- M$ negative eigenvalues and at least $\mathrm{sq}_+ M$ positive eigenvalues, yielding (4.2). □

In the lemmas that follow we assume that Δ is an open interval, and that F is a measurable function on Δ whose values are $\mathbf{C}^{m\times m}$ selfadjoint matrices and which is bounded on all compact subsets of Δ.

Lemma 4.5. *Suppose that F is in $\mathcal{M}_{m,\kappa}(\Delta)$, and let Δ_1 and Δ_2 be disjoint open subintervals of Δ. Then the restrictions of F to Δ_1 and Δ_2 belong to the corresponding classes with integers κ_1 and κ_2 such that $\kappa \geq \kappa_1 + \kappa_2$.*

Proof. The fact that the restrictions belong to $\mathcal{M}_{m,\kappa_1}(\Delta_1)$ and $\mathcal{M}_{m,\kappa_2}(\Delta_2)$ for some $\kappa_1 \leq \kappa$ and $\kappa_2 \leq \kappa$ is immediate from the definition of the classes. To prove that $\kappa \geq \kappa_1 + \kappa_2$, for each $j = 1, 2$, choose matrices A_j, B_j of order n_j, having simple spectra in Δ_j, such that $A_j \leq B_j$ and

$$\mathrm{sq}_-(F(B_j) - F(A_j)) = \kappa_j.$$

Put $A = \mathrm{diag}\{A_1, A_2\}$ and $B = \mathrm{diag}\{B_1, B_2\}$. Then A and B are selfadjoint matrices such that $A \leq B$, and since Δ_1 and Δ_2 are disjoint subintervals of Δ, A and B have simple spectra in Δ. Moreover,

$$\mathrm{sq}_-(F(B) - F(A)) = \mathrm{sq}_- \mathrm{diag}\{F(B_1) - F(A_1), F(B_2) - F(A_2)\} = \kappa_1 + \kappa_2.$$

Since F belongs to $\mathcal{M}_{m,\kappa}(\Delta)$, $\kappa \geq \kappa_1 + \kappa_2$. □

Lemma 4.6. *Assume that F is in $\mathcal{M}_{m,\kappa}(\Delta)$, say $\Delta = (a, b)$ where $-\infty \leq a < b \leq \infty$. Then:*

1. *For each λ, $a < \lambda \leq b$, there is a $\lambda_- < \lambda$ such that $F|_{(\lambda_-,\lambda)} \in \mathcal{M}_{m,0}((\lambda_-, \lambda))$.*
2. *For each λ, $a \leq \lambda < b$, there is a $\lambda_+ > \lambda$ such that $F|_{(\lambda,\lambda_+)} \in \mathcal{M}_{m,0}((\lambda, \lambda_+))$.*
3. *If $a < \lambda < b$ and $\lambda_\pm$ are chosen as in (1) and (2), there is a function S in $\mathcal{N}$ which is holomorphic across (λ_-, λ_+) whose restriction to $(\lambda_-, \lambda) \cup (\lambda, \lambda_+)$ coincides with F.*

Proof. (1) The restriction of F to $\Delta_- = (a, \lambda)$ belongs to $\mathcal{M}_{m,\kappa_-}(\Delta_-)$ for some $\kappa_- \leq \kappa$, by Lemma 4.5. If $F \in \mathcal{M}_{m,0}((a, c))$ for every $c < \lambda$, then F belongs to $\mathcal{M}_{m,0}((a, \lambda))$, and we can choose $\lambda_- = a$. Otherwise, there is an interval (a, a_1) such that $F \in \mathcal{M}_{m,\widetilde{\kappa}}((a, a_1))$ for some $\widetilde{\kappa} > 0$. By Lemma 4.5, F belongs to $\mathcal{M}_{m,\kappa_1}((a_1, \lambda))$ for some $\kappa_1 < \kappa_-$. Repeating this argument ℓ ($\leq \kappa$) times, we get points $a = a_0 < a_1 < \ldots < a_\ell < \lambda$ and integers $\kappa_0 > \kappa_1 > \ldots > \kappa_\ell = 0$ such that

$$F \in \mathcal{M}_{m,\kappa_j}((a_j,\, \lambda)) \quad \text{for} \quad j = 0, \ldots, \ell.$$

In particular, $F \in \mathcal{M}_{m,0}((a_\ell,\, \lambda))$, which proves (1) with $\lambda_- = a_\ell$.

(2) This is similar.

(3) By Theorem 3.3 (see also Remark 4.1), F is a continuously differentiable function on the open set

$$\Delta_\lambda = (\lambda_-, \lambda) \cup (\lambda, \lambda_+).$$

Moreover, F belongs to $\mathcal{M}_{m,\kappa_\lambda}(\Delta_\lambda)$ for some $\kappa_\lambda \leq \kappa$, by Lemma 4.5. We show that $\kappa_\lambda = 0$. To this end, choose n arbitrary distinct points $\lambda_1, \ldots, \lambda_n \in \Delta_\lambda$, and consider the matrix M defined by (3.11). By Lemma 4.4, there is a number $\nu > 0$ such that (4.2) holds for every Hermitian matrix N which satisfies (4.3). Let A and Q be defined as in (3.8) and (3.9). It was shown in the proof of Theorem 3.3 that the matrix $M^{(\varepsilon)} = F(A + \varepsilon Q) - F(A)$ tends to M entrywise as $\varepsilon \to 0$. Choosing $\varepsilon > 0$ sufficiently small, we can arrange that

$$|M_{ij} - M_{ij}^{(\varepsilon)}| < \nu, \qquad i, j = 1, \ldots, n, \tag{4.4}$$

and therefore,

$$\mathrm{sq}_- M \leq \mathrm{sq}_- M^{(\varepsilon)} \leq \mathrm{sq}_- M + \dim\,(\ker M).$$

Since F is κ_λ-monotone on Δ_λ,

$$\mathrm{sq}_- M \leq \mathrm{sq}_- M^{(\varepsilon)} = \mathrm{sq}_-\,(F(A + \varepsilon Q) - F(A)) \leq \kappa_\lambda.$$

Since λ_j are arbitrary points on Δ_λ, the latter inequality means that the kernel $K_F(w, z)$ has at most κ_λ negative squares on $\Delta_\lambda \times \Delta_\lambda$. Let

$$\mathrm{sq}_-(K_F(w, z)) = \kappa_1 \leq \kappa_\lambda.$$

Then by Theorem 2.1, there exists $S \in \mathcal{N}_{\kappa_1}$ which admits an analytic continuation across Δ_λ and coincides with F on Δ_λ. Then the kernel $K_F(w, z)$ has κ_1 negative squares on any subinterval of Δ_λ. In view of Remark 4.1 and parts (1) and (2) of the lemma proved above, it follows from Theorem 3.3 that $\kappa_1 = 0$.

Thus F is the restriction to Δ_λ of a function $S \in \mathcal{N}$ which has an analytic continuation across Δ_λ. By the Stieltjes inverse formula (1.6), the measure $d\Sigma$ from the Herglotz representation (1.4) of S is supported by $\mathbf{R} \setminus \boldsymbol{\Delta}_\lambda$ and therefore

$$F(x) = \alpha + \beta x + \int_{\mathbf{R}\setminus(\lambda-\delta,\,\lambda+\delta)} \left(\frac{1}{t - x} - \frac{t}{t^2 + 1}\right) d\Sigma(t) + \frac{D}{\lambda - x}, \qquad x \in \Delta_\lambda, \tag{4.5}$$

for some nonnegative matrix D. Since F is defined and bounded on compact subsets of Δ, $D = 0$. Thus S is analytic across all of Δ. ☐

We are now ready to show that κ-monotone matrix functions can only arise by altering a classical monotone matrix function at a finite number of points.

Theorem 4.7. *Let F be a measurable function on an open interval $\Delta \subseteq \mathbf{R}$ whose values are $\mathbf{C}^{m\times m}$ selfadjoint matrices which is bounded on all compact subsets of Δ, and let κ be a nonnegative integer. Then F belongs to $\mathcal{M}_{m,\kappa}(\Delta)$ if and only if*

(1) *the discontinuity set of F consists of $\ell \leq \kappa$ points $\Lambda = \{\lambda_1, \ldots, \lambda_\ell\}$, and*

(2) *there is a function $S\in\mathcal{N}$ which is analytic on $\mathbf{C}^+\cup\mathbf{C}^-\cup\Delta$ such that F coincides with S on $\Delta\setminus\Lambda$ and*

$$\sum_{j=1}^{\ell}\operatorname{rank}\left(F(\lambda_j)-S(\lambda_j)\right)=\kappa. \tag{4.6}$$

Proof. When $\kappa=0$, this follows from Theorem 3.3, so in the proof we assume that $\kappa>0$.

Sufficiency. Assume that (1) and (2) hold. Let A and B be selfadjoint matrices of order n having simple spectra in Δ and spectral decompositions

$$A=\sum_{j=1}^{n}\nu_jE_j \quad\text{and}\quad B=\sum_{j=1}^{n}\mu_jG_j,$$

and satisfying $A\le B$. Then

$$\begin{aligned} F(B)-F(A) &= S(B)-S(A) \\ &\quad+\sum_{\mu_j\in\Lambda}(F(\mu_j)-S(\mu_j))\otimes G_j+\sum_{\nu_j\in\Lambda}(S(\nu_j)-F(\nu_j))\otimes E_j. \end{aligned}\tag{4.7}$$

By Theorem 3.3, $S(B)-S(A)\ge 0$. Rewriting (4.6) in the form

$$\sum_{j=1}^{\ell}\left\{\operatorname{sq}_-\left(F(\lambda_j)-S(\lambda_j)\right)+\operatorname{sq}_-\left(S(\lambda_j)-F(\lambda_j)\right)\right\}=\kappa \tag{4.8}$$

and taking into account that E_j and G_j are rank one projections, that is, that

$$\operatorname{sq}_-(E_j)=\operatorname{sq}_-(G_j)=0 \quad\text{and}\quad \operatorname{sq}_+(E_j)=\operatorname{sq}_+(G_j)=1,$$

we conclude by Lemma 4.3 that

$$\begin{aligned} &\operatorname{sq}_-\left(\sum_{\mu_j\in\Lambda}(F(\mu_j)-S(\mu_j))\otimes G_j+\sum_{\nu_j\in\Lambda}(S(\nu_j)-F(\nu_j))\otimes E_j\right) \\ &\qquad\le\sum_{\mu_j\in\Lambda}\operatorname{sq}_-\left((F(\mu_j)-S(\mu_j))\otimes G_j\right)+\sum_{\nu_j\in\Lambda}\operatorname{sq}_-\left((S(\nu_j)-F(\nu_j))\otimes E_j\right) \\ &\qquad=\sum_{\mu_j\in\Lambda}\operatorname{sq}_-\left(F(\mu_j)-S(\mu_j)\right)+\sum_{\nu_j\in\Lambda}\operatorname{sq}_-\left(S(\nu_j)-F(\nu_j)\right) \\ &\qquad\le\sum_{j=1}^{\ell}\left\{\operatorname{sq}_-\left(F(\lambda_j)-S(\lambda_j)\right)+\operatorname{sq}_-\left(S(\lambda_j)-F(\lambda_j)\right)\right\}=\kappa. \end{aligned}$$

Thus (4.1) holds by (4.7). On the other hand, take

$$\nu_{2j-1}=\lambda_j-\varepsilon,\quad \nu_{2j}=\mu_{2j-1}=\lambda_j,\quad\text{and}\quad \mu_{2j}=\lambda_j+\varepsilon \quad\text{for}\quad j=1,\dots,\ell, \tag{4.9}$$

where $\varepsilon>0$ is so small enough that $\nu_1<\nu_2<\ldots<\nu_\ell$ and $\mu_1<\mu_2<\ldots<\mu_\ell$, and let

$$A_0=\operatorname{diag}\{\nu_1,\dots,\nu_{2\ell}\} \quad\text{and}\quad B_0=\operatorname{diag}\{\mu_1,\dots,\mu_{2\ell}\}.$$

Then $A_0=\sum_{j=1}^{2\ell}\nu_jP_j$ and $B_0=\sum_{j=1}^{2\ell}\mu_jP_j$, where the P_j are $n\times n$ matrices, all the entries of which are equal to zero except for the jj-entry which is equal to one. By construction, $A_0\le B_0$, and according to (4.7),

$$F(B_0)-F(A_0)=S(B_0)-S(A_0)+Q,$$

where

$$Q=\sum_{j=1}^{\ell}(F(\mu_{2j-1})-S(\mu_{2j-1}))\otimes P_{2j-1}+\sum_{j=1}^{\ell}(S(\nu_{2j})-F(\nu_{2j}))\otimes P_{2j}.$$

In view of (4.9),

$$Q=\sum_{j=1}^{\ell}(F(\lambda_j)-S(\lambda_j))\otimes P_{2j-1}+\sum_{j=1}^{\ell}(S(\lambda_j)-F(\lambda_j))\otimes P_{2j}$$

and since P_j are mutually orthogonal rank one projections, we get by Lemma 4.3 and in view of (4.8), that

$$\begin{aligned}\operatorname{sq}_-(Q)&=\sum_{j=1}^{\ell}\operatorname{sq}_-((F(\lambda_j)-S(\lambda_j))\otimes P_{2j-1})+\sum_{j=1}^{\ell}\operatorname{sq}_-((S(\lambda_j)-F(\lambda_j))\otimes P_{2j})\\&=\sum_{j=1}^{\ell}\operatorname{sq}_-(F(\lambda_j)-S(\lambda_j))+\sum_{j=1}^{\ell}\operatorname{sq}_-(S(\lambda_j)-F(\lambda_j))=\kappa.\end{aligned}$$

Since S is continuous on Δ, the value of ε in (4.9) may be chosen that $\|S(B_0)-S(A_0)\|$ is arbitrarily small. Since we know that $\operatorname{sq}_-(F(B_0)-F(A_0))\leq\kappa$, it follows from Lemma 4.4 that for all sufficiently small ε,

$$\operatorname{sq}_-(F(B_0)-F(A_0))=\operatorname{sq}_-(S(B_0)-S(A_0)+Q)=\operatorname{sq}_-(Q)=\kappa, \tag{4.10}$$

and so $F\in\mathcal{M}_{m,\kappa}(\Delta)$.

Necessity. Assume that $F\in\mathcal{M}_{m,\kappa}(\Delta)$. By Lemma 4.6, for each λ in Δ we can choose $\lambda_-<\lambda$ and $\lambda_+>\lambda$ such that

(i) $F|_{(\lambda_-,\lambda)}\in\mathcal{M}_{m,0}((\lambda_-,\lambda))$,

(ii) $F|_{(\lambda,\lambda_+)}\in\mathcal{M}_{m,0}((\lambda,\lambda_+))$, and

(iii) there is a function S in $\mathcal{N}$ which is holomorphic across (λ_-,λ_+) whose restriction to $(\lambda_-,\lambda)\cup(\lambda,\lambda_+)$ coincides with F.

The same lemma also assures that if $\Delta=(a,b)$, $-\infty\leq a<b\leq\infty$, there are numbers a' and b', $a<a'<b'<b$, such that $F|_{(a,a')}\in\mathcal{M}_{m,0}((a,a'))$ and $F|_{(b',b)}\in\mathcal{M}_{m,0}((b',b))$. By a compactness argument, we can find points $\lambda_1,\dots,\lambda_n$ in Δ and associated $\lambda_{j\pm}$ such that (i)–(iii) hold with $\lambda=\lambda_j$ and $\lambda_\pm=\lambda_{j\pm}$ for all $j=1,\dots,n$, and in addition such that

(iv) $\lambda_1<\lambda_2<\cdots<\lambda_n$, and

(v) $\bigcup_{j=1}^{n}(\lambda_{j-},\lambda_{j+})=\Delta$.

Write S_j for the function in $\mathcal{N}$ provided by (iii), $j=1,\dots,n$. In fact, because consecutive intervals in the union in (v) overlap, $S=S_j$ is independent of $j=1,\dots,n$. In particular, the points of discontinuity of F occur among $\lambda_1,\dots,\lambda_n$, and we relabel these $\lambda_1,\dots,\lambda_\ell$. We obtain (1) and (2), except for the equality in (4.6). But this equality is then a consequence of the sufficiency part of the theorem, which has already been proved. □

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