OPERATORS ON INDEFINITE INNER PRODUCT SPACES

Michael A. Dritschel and James Rovnyak
Department of Mathematics
University of Virginia
Charlottesville, Virginia 22903-3199
U. S. A.

1991 *Mathematics Subject Classification.* Primary 47-02, 47B50, 47A20;
Secondary 46C20, 46E22, 47A15, 47A48, 47A57.

Contents

Preface

Kreĭn space operator theory has long been a principal area of research in eastern mathematics, and it is now enjoying a surge of interest in western countries as well. The present lectures survey some recent methods and results which center around factorization properties of operators. Topics include the geometry of Kreĭn spaces, Pontryagin spaces, isometric operators, contractions, dilation and extension problems, reproducing kernel spaces, operator colligations, definitizable operators, and invariant subspaces.

Factorization ideas lead to notions of defect and Julia operators, which are tools for reducing the study of general operators to the isometric and unitary cases, where Hilbert space methods can often be used. Connections with interpolation theory and reproducing kernel spaces are explored. Factorization of selfadjoint operators also has applications in spectral theory and invariant subspace problems.

It is not possible to represent the field in six lectures, and the selection of topics is based on a particular viewpoint. Roughly, the viewpoint is that Kreĭn spaces provide a natural setting for operator theory. We assume Hilbert spaces or Pontryagin spaces only when there is some reason for it. Insights into the Hilbert space case come out of this approach, which separates the effects of positivity of vectors and positivity of operators. Connections between operator and spatial positivity then arise in the form of index formulas. Hilbert space methods find new applications in this approach, and operator theory is enriched by concepts which originate from the indefiniteness of inner products.

The list of omitted topics is long. For example, we do not discuss mappings of operator spheres and domains of holomorphy, connections with univalent functions, commutant lifting, interpolation theory, or complementation theory for Kreĭn spaces. Our bibliography is likewise limited, and we regret omissions. On any topic, the bibliography needs to be supplemented from the usual sources. Authoritative literature notes may be found in the books of Azizov and Iokhvidov [1989], Bognár [1974], and Iokhvidov, Kreĭn and Langer [1982].

Two other lecture series in this work complement our account. Systems aspects are treated in the lectures of M. A. Kaashoek on *State-Space Theory of Rational Matrix Functions and Applications*. Applications to differential operators and related topics such as operator polynomials are covered in the lectures of Aad Dijksma and Heinz Langer on *Operator Theory and Ordinary Differential Operators*.

We thank Peter Lancaster and the Program Committee for inviting us to contribute this series of lectures. We also thank The Fields Institute for their support, efficient organization, and hospitality. Both authors are supported by the National Science Foundation.

Introduction:
Preliminaries and Notation

A. Prerequisites

A **Hilbert space** is a strictly positive inner product space $\mathfrak{H}$ over the complex numbers which is complete in its norm metric. We also include separability in the axioms for a Hilbert space. Some familiarity with Hilbert spaces and operator theory is presumed. Otherwise the material covered in a first year graduate course of analysis is sufficient background. For example, see Halmos [1951] and Rudin [1974].

B. Inner product spaces

As we use the term, an **inner product space** is a complex vector space $\mathfrak{H}$ together with a complex-valued function $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{H}}$ on $\mathfrak{H} \times \mathfrak{H}$ which satisfies these axioms:

(1) (linearity) $\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle$ for all $f, g, h \in \mathfrak{H}$ and $a, b \in \mathbf{C}$;
(2) (symmetry) $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in \mathfrak{H}$.

The function $\langle \cdot, \cdot \rangle$ is called an **inner product**. By the **antispace** of an inner product space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ we mean the inner product space $(\mathfrak{H}, -\langle \cdot, \cdot \rangle)$. Suppressing inner products, we sometimes write $-\mathfrak{H}$ for the antispace of an inner product space $\mathfrak{H}$. Thus $\langle \cdot, \cdot \rangle_{-\mathfrak{H}} = -\langle \cdot, \cdot \rangle_{\mathfrak{H}}$.

In any inner product space $\mathfrak{H}$, we use the notions of subspace, orthogonality, and direct sum in the same manner as for Hilbert spaces. For example, vectors f and g in an inner product space $\mathfrak{H}$ are called **orthogonal** if $\langle f, g \rangle = 0$. Elementary properties which follow as in the Hilbert space case are taken for granted.

An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{H}$ is said to be **nondegenerate** if the only vector f in $\mathfrak{H}$ such that $\langle f, g \rangle = 0$ for all vectors $g \in \mathfrak{H}$ is $f = 0$. In applications, this is usually a consequence of stronger assumptions. For example, Hilbert spaces, Pontryagin spaces, and Kreĭn spaces (defined below) are nondegenerate inner product spaces. An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{H}$ is said to be

(1) **positive** if $\langle f, f \rangle \geq 0$ for all $f \in \mathfrak{H}$,
(2) **negative** if $\langle f, f \rangle \leq 0$ for all $f \in \mathfrak{H}$,
(3) **definite** if it is either positive or negative,
(4) **indefinite** if it is neither positive nor negative, and
(5) **neutral** if it is both positive and negative (and hence the inner product is identically zero by the polarization identity).

A positive (negative) inner product is called **strictly positive (strictly negative)** if $\langle f, f \rangle = 0$ only for $f = 0$. These terms are applied to the space $\mathfrak{H}$ as well as the inner product. They are also applied to subspaces $\mathfrak{M}$ of $\mathfrak{H}$ if the restriction of the inner product has the corresponding property.

The term **operator** always refers to a linear transformation. A complex number λ is identified with the operator of multiplication by λ; the identity operator on a space $\mathfrak{H}$ is denoted 1 or $1_{\mathfrak{H}}$. We write dom A and ran A for the domain and range of an operator A.

Two inner product spaces $(\mathfrak{H}, \langle\cdot,\cdot\rangle_{\mathfrak{H}})$ and $(\mathfrak{K}, \langle\cdot,\cdot\rangle_{\mathfrak{K}})$ are called **isomorphic** if there is a one-to-one operator A from $\mathfrak{H}$ onto $\mathfrak{K}$ such that $\langle Af, Ag\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}}$ for all $f, g \in \mathfrak{H}$. Such an operator A is said to be an **isomorphism**.

C. Kreĭn spaces and fundamental decompositions

The definition of a Kreĭn space is made in such a way that the class of Kreĭn spaces includes all Hilbert spaces, antispaces of Hilbert spaces, and is closed under the formation of orthogonal direct sums. It is the smallest such class of inner product spaces.

Definition. (1) *By a **Kreĭn space** we mean an inner product space $\mathfrak{H}$ which can be expressed as an orthogonal direct sum*

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \tag{0.1}$$

*where $\mathfrak{H}_+$ is a Hilbert space and $\mathfrak{H}_-$ is the antispace of a Hilbert space. Any such representation is called a **fundamental decomposition** of the space.*

(2) *Given a Kreĭn space $\mathfrak{H}$ and fundamental decomposition* (0.1), *by $|\mathfrak{H}|$ we mean the Hilbert space obtained from* (0.1) *by replacing $\mathfrak{H}_-$ by its antispace $|\mathfrak{H}_-| = -\mathfrak{H}_-$:*

$$|\mathfrak{H}| = \mathfrak{H}_+ \oplus |\mathfrak{H}_-|. \tag{0.2}$$

*The Hilbert space norm $\|\cdot\| = \|\cdot\|_{|\mathfrak{H}|}$ is called a **norm** for the Kreĭn space $\mathfrak{H}$.*

In this situation, the operator $J_{\mathfrak{H}}$ defined on $\mathfrak{H}$ by

$$J_{\mathfrak{H}} f = f_+ - f_-$$

whenever $f = f_+ + f_-$ with $f_\pm \in \mathfrak{H}_\pm$ is called a **fundamental symmetry** or **signature operator** for the space. The relationship between the Kreĭn space

$$(\mathfrak{H}, \langle\cdot,\cdot\rangle_{\mathfrak{H}})$$

and the Hilbert space

$$(|\mathfrak{H}|, \langle\cdot,\cdot\rangle_{|\mathfrak{H}|})$$

is that $\mathfrak{H}$ and $|\mathfrak{H}|$ coincide as vector spaces and

$$\langle f, g\rangle_{|\mathfrak{H}|} = \langle J_{\mathfrak{H}} f, g\rangle_{\mathfrak{H}} \qquad \text{and} \qquad \langle f, g\rangle_{\mathfrak{H}} = \langle J_{\mathfrak{H}} f, g\rangle_{|\mathfrak{H}|}$$

for any vectors f and g.

Hilbert spaces and antispaces of Hilbert spaces are Kreĭn spaces. Excluding these cases, a Kreĭn space has infinitely many fundamental decompositions and therefore infinitely many associated Hilbert spaces $|\mathfrak{H}|$ and norms $\|\cdot\| = \|\cdot\|_{|\mathfrak{H}|}$. We compare these structures for two fundamental decompositions.

Theorem. *Let $\mathfrak{H}$ be a Kreĭn space with two fundamental decompositions*

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \qquad \text{and} \qquad \mathfrak{H} = \mathfrak{H}'_+ \oplus \mathfrak{H}'_-. \tag{0.3}$$

Define $X \in \mathfrak{L}(\mathfrak{H}_+, \mathfrak{H}'_+)$ by

$$Xf_+ = f'_+, \qquad f_+ \in \mathfrak{H}_+,$$

if $f_+ = f'_+ + f'_-$, $f'_\pm \in \mathfrak{H}'_\pm$. Define $Y \in \mathfrak{L}(|\mathfrak{H}_-|, |\mathfrak{H}'_-|)$ by

$$Yf_- = f'_-, \qquad f_- \in |\mathfrak{H}_-|,$$

if $f_- = f'_+ + f'_-$, $f'_\pm \in \mathfrak{H}'_\pm$. Then X and Y are one-to-one and onto.

Proof. Let us show that X has a closed graph. For any $z \in \mathfrak{H}$, the mapping $f \to \langle f, z\rangle_{\mathfrak{H}}$ is a continuous linear functional on $\mathfrak{H}_+$ and on $\mathfrak{H}'_+$. If $f_n = g_n + h_n$, where $f_n \in \mathfrak{H}_+$, $g_n \in \mathfrak{H}'_+$, and $h_n \in \mathfrak{H}'_-$ for all $n \geq 1$, and if $f_n \to f$ in $\mathfrak{H}_+$ and $g_n \to g$ in $\mathfrak{H}'_+$, then $f_n - g_n \perp \mathfrak{H}'_+$ in the inner product of $\mathfrak{H}$ for all $n \geq 1$, and so $f - g \perp \mathfrak{H}'_+$ in the inner product of $\mathfrak{H}$. In other words, $f = g + h$ with $h = f - g \in \mathfrak{H}'_-$. It follows that the graph of X is closed, and so X is continuous by the closed graph theorem.

Suppose that $f_+ \in \mathfrak{H}_+$ and $f_+ = f'_+ + f'_-$ with $f'_\pm \in \mathfrak{H}'_\pm$. Since

$$\langle f_+, f_+\rangle_{\mathfrak{H}_+} = \langle f'_+, f'_+\rangle_{\mathfrak{H}'_+} + \langle f'_-, f'_-\rangle_{\mathfrak{H}'_-} \leq \langle f'_+, f'_+\rangle_{\mathfrak{H}'_+},$$

$\|Xf_+\| \geq \|f_+\|$. Therefore X is one-to-one and has closed range.

The range of X is all of $\mathfrak{H}'_+$. For suppose that h belongs to $\mathfrak{H}'_+$ and is orthogonal to the range of X. Let $f_+ \in \mathfrak{H}_+$ and $f_+ = f'_+ + f'_-$ with $f'_\pm \in \mathfrak{H}'_\pm$. Then h is orthogonal to both $f'_+ = Xf_+$ and f'_-, and so h is orthogonal to f_+. Therefore $h \in \mathfrak{H}_-$ and $\langle h, h\rangle_{\mathfrak{H}'_+} = \langle h, h\rangle_{\mathfrak{H}_-} \leq 0$, and so $h = 0$. We have now shown that X is one-to-one and onto.

The assertions for Y may be deduced similarly or by applying what we just proved to the antispace of $\mathfrak{H}$. □

Corollary 1. *If $\mathfrak{H}$ is a Kreĭn space, the numbers*

$$\operatorname{ind}_\pm \mathfrak{H} = \dim\, \mathfrak{H}_\pm$$

do not depend on the choice of decomposition (0.1)*.*

We call $\operatorname{ind}_\pm \mathfrak{H}$ the **positive** and **negative** indices of the space. A **Pontryagin space** is a Kreĭn space $\mathfrak{H}$ with $\operatorname{ind}_- \mathfrak{H} < \infty$.

Corollary 2. *Let $\mathfrak{H}$ be a Kreĭn space with two fundamental decompositions* (0.3) *and associated Hilbert spaces*

$$|\mathfrak{H}| = \mathfrak{H}_+ \oplus |\mathfrak{H}_-| \qquad \text{and} \qquad |\mathfrak{H}|' = \mathfrak{H}'_+ \oplus |\mathfrak{H}'_-|. \tag{0.4}$$

Then $|\mathfrak{H}|$ and $|\mathfrak{H}|'$ have equivalent norms, that is, there exist $m > 0$ and $M > 0$ such that

$$m\, \|f\|_{|\mathfrak{H}|} \leq \|f\|_{|\mathfrak{H}|'} \leq M\, \|f\|_{|\mathfrak{H}|} \tag{0.5}$$

for all $f \in \mathfrak{H}$. Hence the norm topologies of the Hilbert spaces (0.4) *are identical.*

Proof. Represent an arbitrary vector f in $\mathfrak{H}$ in the form $f = f_+ + f_- = f'_+ + f'_-$, where $f_\pm \in \mathfrak{H}_\pm$ and $f'_\pm \in \mathfrak{H}'_\pm$. If X, Y are the operators of the theorem, then

$$\begin{aligned} f_+ &= Xf_+ + (1-X)f_+, \\ f_- &= (1-Y)f_- + Yf_-, \end{aligned}$$

where $Xf_+, (1-Y)f_- \in \mathfrak{H}'_+$ and $(1-X)f_+, Yf_- \in \mathfrak{H}'_-$. It follows that

$$\begin{aligned} f'_+ &= Xf_+ + (1-Y)f_-, \\ f'_- &= (1-X)f_+ + Yf_-. \end{aligned}$$

Similar relations hold with the roles of the fundamental decompositions reversed, and we obtain (0.5) for suitable constants m and M. □

Definition. *The **strong topology** of a Kreĭn space $\mathfrak{H}$ is the norm topology of any Hilbert space* (0.2) *associated with a fundamental decomposition* (0.1) *for $\mathfrak{H}$.*

Notions of convergence and continuity are understood to be with respect to the strong topology unless otherwise stated.

The closed graph theorem holds for Kreĭn spaces: an everywhere defined linear transformation on a Kreĭn space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ is continuous if it has a closed graph. The proof is simply the observation that the strong topology of a Kreĭn space is a Hilbert space topology, and the closed graph theorem is true for Hilbert spaces.

The Riesz representation theorem holds for any Kreĭn space $\mathfrak{H}$: every continuous linear functional L on $\mathfrak{H}$ has the form

$$Lf = \langle f, g \rangle, \qquad f \in \mathfrak{H},$$

for a unique $g \in \mathfrak{H}$.

D. Example

Define the **Minkowski space** $\mathbf{M}^{n+1}$ as the set of $(n+1)$-dimensional column vectors

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ \gamma \end{pmatrix}$$

with complex entries. Typographically, it is easier to write $c = (c_1, \dots, c_n, \gamma)^t$, using a superscript t to indicate a matrix transpose. Given two elements $a = (a_1, \dots, a_n, \alpha)^t$ and $b = (b_1, \dots, b_n, \beta)^t$ of the space, define their inner product by

$$\langle a, b \rangle_{\mathbf{M}^{n+1}} = a_1\bar{b}_1 + \cdots + a_n\bar{b}_n - \alpha\bar{\beta}.$$

Then $\mathbf{M}^{n+1}$ is a Kreĭn space with $\operatorname{ind}_+ \mathbf{M}^{n+1} = n$ and $\operatorname{ind}_- \mathbf{M}^{n+1} = 1$. If we represent operators on the space as matrices in the usual way, then a fundamental symmetry for the space is given by

$$J_{\mathbf{M}^{n+1}} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

where I_n is the identity matrix of order n. Then $|\mathbf{M}^{n+1}| = \mathbf{C}^{n+1}$ in the Euclidean metric.

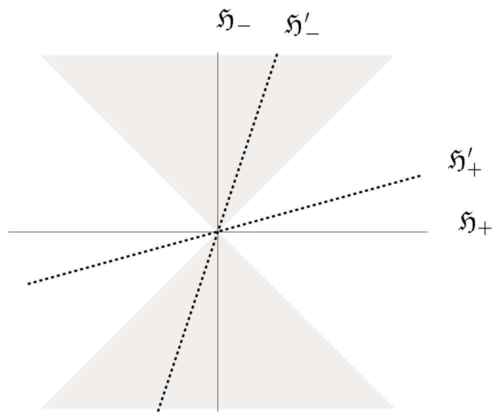

Figure 1

One fundamental decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ of the space $\mathbf{M}^2$ is obtained with $\mathfrak{H}_+$ equal to the span of $(1,0)^t$ and $\mathfrak{H}_-$ equal to the span of $(0,1)^t$. A second can be chosen, for example, with the spans of $(4,1)^t$ and $(1,4)^t$, as shown in figure 1. In figure 1, the shaded region represents all vectors f such that $\langle f, f\rangle \leq 0$ (negative cone), the light region all vectors f such that $\langle f, f\rangle \geq 0$ (positive cone).

Figure 1 is actually a 2-dimensional slice of a 4-dimensional space. A similar figure for $\mathbf{M}^3$ can also be visualized. In this case, the "negative" part of the space is a solid cone. This method of visualization is common in approaches that feature applications, such as Hassibi, Sayed, and Kailath [1994], and relativity texts.

E. Books and expositions

Standard sources on indefinite inner product spaces are Andô [1979], Azizov and Iokhvidov [1989], Bognár [1974], and Iokhvidov, Kreĭn, and Langer [1982]. The finite-dimensional case is treated in Gohberg, Lancaster, and Rodman [1983]. Dritschel and Rovnyak [1990] also include some introductory material.

Lecture 1: Kreĭn Spaces and Operators

Key ideas:

- Kreĭn spaces are treated like Hilbert spaces, with the same notation used for geometric concepts and classes of operators, including selfadjoint operators, projections, and isometries.
- The focus is on unique objects (strong topology, inner product, operators), and we de-emphasize nonunique ones (fundamental decompositions, norms).

The geometry of Kreĭn spaces has many similarities with the Hilbert space case but also some important differences such as the presence of negative subspaces. It is convenient to study Kreĭn space geometry concurrently with basic classes of operators, including selfadjoint operators, projections, and isometries. In this lecture we introduce the main classes of operators and show their relationship to the geometry of Kreĭn spaces.

Kreĭn spaces and their strong topologies have already been defined in §C of the Introduction. Underlying spaces throughout are assumed to be Kreĭn spaces if nothing further is stated. By $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ we mean the space of continuous operators on $\mathfrak{H}$ to $\mathfrak{K}$. Set $\mathfrak{L}(\mathfrak{H}) = \mathfrak{L}(\mathfrak{H}, \mathfrak{H})$. For every $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ there is a unique $A^* \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H})$ such that

$$\langle Af, g\rangle_{\mathfrak{K}} = \langle f, A^* g\rangle_{\mathfrak{H}}, \qquad f \in \mathfrak{H},\ g \in \mathfrak{K}.$$

We call A^* the **adjoint** of A. The existence of the adjoint follows from the Riesz representation theorem, and elementary properties are immediate as in the Hilbert space case. For example, when the operations are defined,

$$(A + B)^* = A^* + B^*, \qquad (AB)^* = B^* A^*.$$

The space $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ has the structure of a Banach space depending on choices of fundamental decompositions $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ and $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$ and associated fundamental symmetries $J_{\mathfrak{H}}$ and $J_{\mathfrak{K}}$. Let $\|\cdot\|_{|\mathfrak{H}|}$ and $\|\cdot\|_{|\mathfrak{K}|}$ be the norms for the corresponding Hilbert spaces $|\mathfrak{H}| = \mathfrak{H}_+ \oplus |\mathfrak{H}_-|$ and $|\mathfrak{K}| = \mathfrak{K}_+ \oplus |\mathfrak{K}_-|$. Our notation for fundamental symmetries and associated Hilbert spaces follows §C of the Introduction. Then $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ coincides with $\mathfrak{L}(|\mathfrak{H}|, |\mathfrak{K}|)$. We call the norm $\|\cdot\|$ of $\mathfrak{L}(|\mathfrak{H}|, |\mathfrak{K}|)$ an **operator norm** for $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. For any $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$,

$$|\langle Af, g\rangle_{\mathfrak{K}}| \le \|A\|\, \|f\|_{|\mathfrak{H}|}\, \|g\|_{|\mathfrak{K}|}, \qquad f \in \mathfrak{H},\ g \in \mathfrak{K}.$$

Any two operator norms for $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ are equivalent, and we thus obtain a unique **uniform topology** for $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. **Strong** and **weak operator topologies** for $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ are similarly defined from the corresponding Hilbert space notions.

The Kreĭn space and Hilbert space concepts of adjoint are related. Viewing $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ as an element of $\mathfrak{L}(|\mathfrak{H}|, |\mathfrak{K}|)$ as above, we write $A^\times \in \mathfrak{L}(|\mathfrak{K}|, |\mathfrak{H}|)$ for the Hilbert space adjoint. Then $A^* = J_\mathfrak{H} A^\times J_\mathfrak{K}$. For if $f \in \mathfrak{H}$ and $g \in \mathfrak{K}$, then

$$\langle Af, g \rangle_\mathfrak{K} = \langle J_\mathfrak{K} Af, g \rangle_{|\mathfrak{K}|} = \langle f, A^\times J_\mathfrak{K} g \rangle_{|\mathfrak{H}|} = \langle f, J_\mathfrak{H} A^\times J_\mathfrak{K} g \rangle_\mathfrak{H}.$$

In particular, $\|A^*\| = \|A\|$ for any $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and any operator norm on $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.

We say that

(1) $A \in \mathfrak{L}(\mathfrak{H})$ is **selfadjoint** if $A^* = A$;
(2) $A \in \mathfrak{L}(\mathfrak{H})$ is a **projection** if A is selfadjoint and $A^2 = A$;
(3) $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is **isometric** if $A^* A = 1_\mathfrak{H}$;
(4) $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is **partially isometric** if $AA^*A = A$;
(5) $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is **unitary** if both A and A^* are isometric;
(6) $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a **contraction** if $A^* A \leq 1_\mathfrak{H}$;
(7) $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a **bicontraction** if both A and A^* are contractions.

The definition of a contraction in part (6) uses the partial ordering of selfadjoint operators. The selfadjoint operators on $\mathfrak{H}$ are partially ordered by writing $A \geq 0$ if

$$\langle Af, f \rangle \geq 0, \qquad f \in \mathfrak{H}.$$

Then $A \leq B$ means that $B - A \geq 0$. A selfadjoint operator A is said to be **positive** or **negative** according as $A \geq 0$ or $A \leq 0$.

An operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a contraction if and only if

$$\langle Af, Af \rangle_\mathfrak{K} \leq \langle f, f \rangle_\mathfrak{H}, \qquad f \in \mathfrak{H}.$$

An operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is unitary if and only if it is an isomorphism between the Kreĭn spaces $\mathfrak{H}$ and $\mathfrak{K}$.

Theorem 1.1 (Bognár-Krámli Factorization) *Let $\mathfrak{H}$ be a Kreĭn space. Any selfadjoint operator $A \in \mathfrak{L}(\mathfrak{H})$ can be written in the form*

$$A = DD^*,$$

where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$ for some Kreĭn space $\mathfrak{D}$ and $\ker\, D = \{0\}$.

Proof. Let $J_\mathfrak{H}$ be a fundamental symmetry for $\mathfrak{H}$ and $|\mathfrak{H}|$ the associated Hilbert space. Since $A = A^* = J_\mathfrak{H} A^\times J_\mathfrak{H}$, $AJ_\mathfrak{H}$ is selfadjoint as an element of $\mathfrak{L}(|\mathfrak{H}|)$. Its polar decomposition has the form

$$AJ_\mathfrak{H} = RU = UR,$$

where R is a nonnegative operator in $\mathfrak{L}(|\mathfrak{H}|)$ and U is a partial isometry in $\mathfrak{L}(|\mathfrak{H}|)$ with initial and final space equal to $\overline{\text{ran}\, R}$. In fact, U is also selfadjoint, and

$$\mathfrak{D} = \overline{\text{ran}\, R}$$

is a Kreĭn space with signature operator $J_\mathfrak{D} = U\big|_\mathfrak{D}$. Define $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$ by

$$Df = R^{1/2} f, \qquad f \in \mathfrak{D}.$$

A short calculation shows that $D^* = UR^{1/2} J_\mathfrak{H}$, and so

$$DD^* = R^{1/2} U R^{1/2} J_\mathfrak{H} = AJ_\mathfrak{H} J_\mathfrak{H} = A.$$

By construction, D has zero kernel. $\square$

If $\mathfrak{M}$ is a subspace of a Kreĭn space $\mathfrak{H}$, its **orthogonal companion** is defined by

$$\mathfrak{M}^{\perp} = \{g \; : \; g \in \mathfrak{H} \text{ and } \langle f, g \rangle = 0 \text{ for all } f \in \mathfrak{M}\}.$$

For any subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$,

(1) $\mathfrak{M}^{\perp\perp} = \overline{\mathfrak{M}}$,

(2) $(\mathfrak{M} + \mathfrak{N})^{\perp} = \mathfrak{M}^{\perp} \cap \mathfrak{N}^{\perp}$,

(3) if $\mathfrak{M}$ and $\mathfrak{N}$ are closed, then $(\mathfrak{M} \cap \mathfrak{N})^{\perp} = \overline{\mathfrak{M}^{\perp} + \mathfrak{N}^{\perp}}$,

(4) $\overline{\mathfrak{M}} = \mathfrak{H}$ if and only if $\mathfrak{M}^{\perp} = \{0\}$.

The bars indicate closure. The assertions can be deduced from the Hilbert space case by choosing a fundamental symmetry $J_{\mathfrak{H}}$, and then noting that $\mathfrak{M}^{\perp}$ coincides with the orthogonal complement of $J_{\mathfrak{H}}\mathfrak{M}$ in the corresponding Hilbert space $|\mathfrak{H}|$.

The term "orthogonal complement" for $\mathfrak{M}^{\perp}$ is avoided, because the projection theorem fails in general for Kreĭn spaces. For a Hilbert space $\mathfrak{H}$, the projection theorem asserts that the relation

$$\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^{\perp} \tag{1.1}$$

holds for any closed subspace $\mathfrak{M}$ of $\mathfrak{H}$. An example where (1.1) fails is the span $\mathfrak{M}$ of the vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

in the Minkowski space $\mathbf{M}^2$ (Introduction, §D). If (1.1) holds for a closed subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$, we also call $\mathfrak{M}^{\perp}$ the **orthogonal complement** of $\mathfrak{M}$.

Every closed subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$ has a **closed complement**: by this we mean a closed subspace $\mathfrak{N}$ of $\mathfrak{H}$ such that

$$\mathfrak{H} = \mathfrak{M} + \mathfrak{N} \qquad \text{and} \qquad \mathfrak{M} \cap \mathfrak{N} = \{0\}.$$

This is proved by choosing a fundamental decomposition for $\mathfrak{H}$ and taking $\mathfrak{N}$ to be the Hilbert space orthogonal complement of $\mathfrak{M}$ in the associated space $|\mathfrak{H}|$. Then $\mathfrak{M}$ and $\mathfrak{N}$ are orthogonal relative to the Hilbert space inner product but generally not in the Kreĭn space inner product. A closed subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$ typically has infinitely many closed complements. It is an important problem to identify those subspaces which have orthogonal complements, that is, for which (1.1) holds.

Definition 1.2 *By a* ***Kreĭn subspace*** *of a Kreĭn space* $\mathfrak{H}$ *we mean a subspace* $\mathfrak{M}$ *of* $\mathfrak{H}$ *which is itself a Kreĭn space in the inner product of* $\mathfrak{H}$*, that is,*

$$\langle f, g \rangle_{\mathfrak{M}} = \langle f, g \rangle_{\mathfrak{H}}, \qquad f, g \in \mathfrak{M}.$$

In the literature, these are called regular subspaces. For an example, let $\mathfrak{M}$ be a finite-dimensional subspace of $\mathfrak{H}$ spanned by the linearly independent vectors $e_1, \dots, e_n$. Then $\mathfrak{M}$ is a Kreĭn subspace of $\mathfrak{H}$ if and only if the **Gram matrix**

$$G(e_1, \dots, e_n) = \left(\langle e_j, e_k \rangle_{\mathfrak{H}} \right)_{j,k=1}^{n}$$

is invertible. Hint for the proof: put the Gram matrix in diagonal form.

The next result shows that Kreĭn space projections are similar to Hilbert space projections, but the geometry applies only to Kreĭn subspaces.

Theorem 1.3 *If $\mathfrak{M}$ is a subspace of a Kreĭn space $\mathfrak{H}$, the following assertions are equivalent:*
(1) *$\mathfrak{M}$ is a Kreĭn subspace;*
(2) *$\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{M}^{\perp}$;*
(3) *$\mathfrak{M}$ is the range of a projection operator P.*
In this case $\mathfrak{M}$ is closed and the strong topology of $\mathfrak{M}$ viewed as a Kreĭn space in the inner product of $\mathfrak{H}$ coincides with the restriction of the strong topology of $\mathfrak{H}$ to $\mathfrak{M}$.

We often denote the projection of a Kreĭn space $\mathfrak{H}$ onto a Kreĭn subspace $\mathfrak{M}$ by $\mathrm{Pr}_{\mathfrak{M}}$.

Proof of Theorem 1.3. (3) $\Rightarrow$ (1) Let $\mathfrak{M} = \operatorname{ran} P$, where $P \in \mathfrak{L}(\mathfrak{H})$ is a projection. Use the Bognár-Krámli factorization to write $P = DD^*$, where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$ and $\ker D = \{0\}$. Since $P^2 = P$,

$$DD^*DD^* = DD^*.$$

Since D has zero kernel, D^* has dense range, and therefore $DD^*D = D$ and $D^*D = 1$. It follows that D preserves inner products and $\mathfrak{M} = \operatorname{ran} DD^* = \operatorname{ran} D$. Now (1) follows because by what we have shown, D is an isomorphism from the Kreĭn space $\mathfrak{D}$ onto $\mathfrak{M}$ in the inner product of $\mathfrak{H}$. Notice that $\mathfrak{M} = \operatorname{ran} D$ is a closed subspace of $\mathfrak{H}$ because D has a left inverse.

(2) $\Rightarrow$ (3) If (2) holds, then every $h \in \mathfrak{H}$ has a unique representation $h = f + g$ with $f \in \mathfrak{M}$ and $g \in \mathfrak{M}^{\perp}$. The relation $Ph = f$ defines an operator $P \in \mathfrak{L}(\mathfrak{H})$ by the closed graph theorem. Using (2), we verify that P is a projection. Clearly $\mathfrak{M} = \operatorname{ran} P$, and so (3) follows.

The proof will be completed later. □

Recall the classification of subspaces in §B of the Introduction: A subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$ is positive, negative, definite, indefinite, neutral, strictly positive, or strictly negative if the restriction of the inner product of $\mathfrak{H}$ to $\mathfrak{M}$ has the corresponding property. These notions are illustrated for the space $\mathbf{M}^2$ in figure 2. Subspaces $\mathfrak{M}_0$ in the boundary of the shaded region in figure 2 are neutral. Subspaces $\mathfrak{M}_-$ in the shaded region are negative (strictly negative when they are not on the boundary). Subspaces $\mathfrak{M}_+$ in the unshaded region are positive (strictly positive when they are not on the boundary).

Definition 1.4 *Let $\mathfrak{H}$ be a Kreĭn space with norm $\|\cdot\|$. A subspace $\mathfrak{M}$ of $\mathfrak{H}$ is called **uniformly positive** if there is a number $\delta > 0$ such that*

$$\langle f, f \rangle \geq \delta \|f\|^2, \qquad f \in \mathfrak{M}.$$

*We say that $\mathfrak{M}$ is **uniformly negative** if there is a number $\delta > 0$ such that*

$$\langle f, f \rangle \leq -\delta \|f\|^2, \qquad f \in \mathfrak{M}.$$

These notions are independent of the choice of norm.

A closed subspace $\mathfrak{M}$ of a Kreĭn space $\mathfrak{H}$ is uniformly positive if and only if it is a Hilbert space in the inner product of $\mathfrak{H}$. The necessity part of this assertion is clear. Conversely, suppose that $\mathfrak{M}$ is a Hilbert space in the inner product of $\mathfrak{H}$. Denote this Hilbert space $\mathfrak{M}_1$. Let $\mathfrak{M}_2$ be $\mathfrak{M}$ in any norm of $\mathfrak{H}$. The identity mapping from

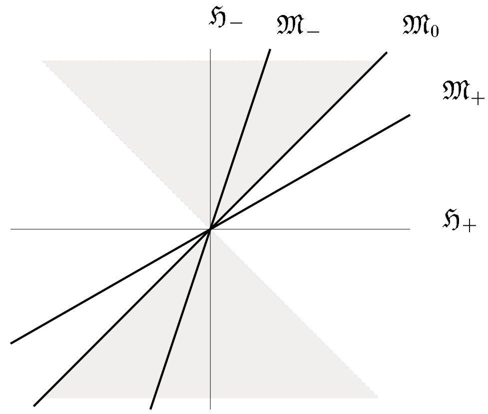

Figure 2

$\mathfrak{M}_2$ to $\mathfrak{M}_1$ is bounded and hence continuous. By the open mapping theorem, the inverse mapping is also bounded, and therefore $\mathfrak{M}$ is uniformly positive. Similar remarks apply to uniformly negative subspaces.

A subspace is said to be **maximal** with respect to some property if it has the property and is not properly contained in another subspace which has the property. For example, a maximal positive subspace is a positive subspace $\mathfrak{M}$ of $\mathfrak{H}$ which is not properly contained in another positive subspace of $\mathfrak{H}$. For "positive" we can substitute "negative," "uniformly positive," and so forth.

Definition 1.5 (Graph representation of a definite subspace) *Let $\mathfrak{H}$ be a Kreĭn space with fundamental decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, and let $\mathfrak{M}$ be a negative subspace of $\mathfrak{H}$. We identify $\mathfrak{H}$ as a vector space with $\mathfrak{H}_+ \times |\mathfrak{H}_-|$, so $\mathfrak{H}$ may alternatively be viewed as a space of pairs*

$$(g, f)^t = \begin{pmatrix} g \\ f \end{pmatrix}$$

*with $g \in \mathfrak{H}_+$ and $f \in |\mathfrak{H}_-|$. The **angle operator** for $\mathfrak{M}$ is the Hilbert space contraction operator K with $\operatorname{dom} K \subseteq |\mathfrak{H}_-|$ and $\operatorname{ran} K \subseteq \mathfrak{H}_+$ such that as a subset of $\mathfrak{H}_+ \times |\mathfrak{H}_-|$,*

$$\mathfrak{M} = \text{Graph } (K) = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} \; : \; f \in \operatorname{dom} K \right\}.$$

The graph representation of a positive subspace is defined analogously.

To see that such an operator K exists, observe that $\mathfrak{M}$ can contain no nonzero element of the form $(g, 0)^t$ because $\mathfrak{M}$ is negative. This is the condition that $\mathfrak{M} = \text{Graph } (K)$ for some operator K with $\operatorname{dom} K \subseteq |\mathfrak{H}_-|$ and $\operatorname{ran} K \subseteq \mathfrak{H}_+$. Since $\mathfrak{M}$ is negative, for each $f \in \operatorname{dom} K$,

$$\langle Kf + f, Kf + f \rangle_{\mathfrak{H}} \leq 0$$

and so

$$\|Kf\|_{\mathfrak{H}_+} \leq \|f\|_{|\mathfrak{H}_-|}.$$

Thus $\|K\| \leq 1$. This proves the existence of the angle operator.

Let $\mathfrak{M}$ be a negative subspace of a Kreĭn space $\mathfrak{H}$ represented as a graph as in Definition 1.5. The following assertions are straightforward:

(1) $\mathfrak{M}$ is closed if and only if $\operatorname{dom} K$ is closed in $|\mathfrak{H}_-|$;

(2) $\mathfrak{M}$ is uniformly negative if and only if $\|K\| < 1$;

(3) $\mathfrak{M}$ is maximal negative if and only if $\operatorname{dom} K = |\mathfrak{H}_-|$, that is, the projection of $\mathfrak{M}$ onto $\mathfrak{H}_-$ is all of $\mathfrak{H}_-$;

(4) $\mathfrak{M}$ is maximal uniformly negative if and only if $\operatorname{dom} K = |\mathfrak{H}_-|$ and $\|K\| < 1$.

It follows that a maximal uniformly negative subspace is maximal negative. Every maximal negative subspace is closed.

Property (3) has these frequently used consequences. Let $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{G}$, where $\mathfrak{H}$ is a Kreĭn space and $\mathfrak{G}$ is a Hilbert space. Every maximal negative subspace of $\mathfrak{H}$ is a maximal negative subspace of $\mathfrak{K}$. The projection of any maximal negative subspace of $\mathfrak{K}$ into $\mathfrak{H}$ is maximal negative in $\mathfrak{H}$.

Theorem 1.6 *Let $\mathfrak{H}$ be a Kreĭn space.*

(1) *A subspace $\mathfrak{M}$ of $\mathfrak{H}$ is maximal negative if and only if $\mathfrak{M}^\perp$ is maximal positive in $\mathfrak{H}$.*

(2) *If $\mathfrak{M}_+$ is a maximal uniformly positive subspace of $\mathfrak{H}$, then $\mathfrak{M}_- = \mathfrak{M}_+^\perp$ is maximal uniformly negative. Moreover $\mathfrak{H} = \mathfrak{M}_+ \oplus \mathfrak{M}_-$ is a fundamental decomposition.*

(3) *Every negative subspace of $\mathfrak{H}$ is contained in a maximal negative subspace of $\mathfrak{H}$.*

(4) *Every uniformly negative subspace of $\mathfrak{H}$ is contained in a maximal uniformly negative subspace of $\mathfrak{H}$.*

(5) *Assume that $\mathfrak{M}_\pm$ are uniformly positive/negative subspaces of $\mathfrak{H}$ such that $\mathfrak{M}_+ \perp \mathfrak{M}_-$. Then there exists a fundamental decomposition $\mathfrak{H} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$ such that $\mathfrak{M}_\pm \subseteq \mathfrak{K}_\pm$.*

A similar result holds with the words "positive" and "negative" interchanged. Part (5) is related to Phillips' theorem in Lecture 6. We prove parts (2), (4), and (5). See, for example, Andô [1979] and Azizov and Iokhvidov [1989] for further details and historical notes on the graph method.

Proof of Theorem 1.6, parts (2), (4) and (5). We use the notation for fundamental decompositions in Definition 1.5.

(2) By the discussion above, $\mathfrak{M}_+$ has the form

$$\mathfrak{M}_+ = \left\{ \begin{pmatrix} u \\ Ku \end{pmatrix} \ : \ u \in \mathfrak{H}_+ \right\},$$

where $K \in \mathfrak{L}(\mathfrak{H}_+, |\mathfrak{H}_-|)$ is a uniform contraction: $\|K\| < 1$. Then

$$\mathfrak{M}_- = \left\{ \begin{pmatrix} K^\times v \\ v \end{pmatrix} \ : \ v \in |\mathfrak{H}_-| \right\}.$$

Since $K^\times \in \mathfrak{L}(|\mathfrak{H}_-|, \mathfrak{H}_+)$ and $\|K^\times\| < 1$, $\mathfrak{M}_-$ is maximal uniformly negative. By the remarks following Definition 1.4, $\mathfrak{M}_+$ and $-\mathfrak{M}_-$ are Hilbert spaces in the inner product of $\mathfrak{H}$. By elementary algebra, every element of $\mathfrak{H}_+ \oplus |\mathfrak{H}_-|$ has a unique representation

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u \\ Ku \end{pmatrix} + \begin{pmatrix} K^\times v \\ v \end{pmatrix},$$

with $u \in \mathfrak{H}_+$ and $v \in |\mathfrak{H}_-|$. It follows that $\mathfrak{H} = \mathfrak{M}_+ \oplus \mathfrak{M}_-$ is a fundamental decomposition.

(4) Let $\mathfrak{M}$ be a uniformly negative subspace of $\mathfrak{H}$, and represent $\mathfrak{M}$ as a graph as in Definition 1.5. Then $\|K\| < 1$. Extend K in any way to an operator in $\mathfrak{L}(|\mathfrak{H}_-|, \mathfrak{H}_+)$ with norm less than 1. The graph of the extension is a maximal uniformly negative subspace of $\mathfrak{H}$ which contains $\mathfrak{M}$.

(5) By (4), we can extend $\mathfrak{M}_+$ to a maximal uniformly positive subspace $\tilde{\mathfrak{M}}_+$ of $\mathfrak{H}$ (strictly speaking, we use the version of (4) for positive subspaces). By the remarks following Definition 1.4, $\mathfrak{M}_+$ and $\tilde{\mathfrak{M}}_+$ are Hilbert spaces in the inner product of $\mathfrak{H}$. By the projection theorem for Hilbert spaces,

$$\tilde{\mathfrak{M}}_+ = \mathfrak{M}_+ \oplus \mathfrak{N}_+,$$

where $\mathfrak{N}_+$ is a subspace of $\mathfrak{H}$ which is a Hilbert space in the inner product of $\mathfrak{H}$. By (2), $\tilde{\mathfrak{M}}_- = \tilde{\mathfrak{M}}_+^\perp$ is maximal uniformly negative and $\mathfrak{H} = \tilde{\mathfrak{M}}_+ \oplus \tilde{\mathfrak{M}}_-$ is a fundamental decomposition. Therefore

$$\mathfrak{H} = (\mathfrak{M}_+ \oplus \mathfrak{N}_+) \oplus \tilde{\mathfrak{M}}_- = \mathfrak{M}_+ \oplus (\mathfrak{N}_+ \oplus \tilde{\mathfrak{M}}_-).$$

One sees easily out of this that $\mathfrak{N}_+ \oplus \tilde{\mathfrak{M}}_- = \mathfrak{M}_+^\perp$ is a Kreĭn space in the inner product of $\mathfrak{H}$. Since $\mathfrak{M}_-$ is now a uniformly negative subspace of $\mathfrak{M}_+^\perp$, we can apply a similar argument to show that

$$\mathfrak{M}_+^\perp = \mathfrak{M}_- \oplus (\mathfrak{P}_+ \oplus \mathfrak{P}_-),$$

where $\mathfrak{P}_+$ and $-\mathfrak{P}_-$ are subspaces of $\mathfrak{H}$ which are Hilbert spaces in the inner product of $\mathfrak{H}$. Therefore

$$\mathfrak{H} = \mathfrak{M}_+ \oplus \mathfrak{M}_+^\perp = (\mathfrak{M}_+ \oplus \mathfrak{P}_+) \oplus (\mathfrak{M}_- \oplus \mathfrak{P}_-).$$

The required fundamental decomposition is obtained with $\mathfrak{K}_\pm = \mathfrak{M}_\pm \oplus \mathfrak{P}_\pm$. □

The proof of Theorem 1.3 can now be completed.

Proof of Theorem 1.3, remaining parts. (1) ⇒ (2) Assume that $\mathfrak{M}$ is a Kreĭn space in the inner product of $\mathfrak{H}$. Then $\mathfrak{M} = \mathfrak{M}_+ \oplus \mathfrak{M}_-$, where $\mathfrak{M}_+$ is a Hilbert space and $\mathfrak{M}_-$ is the antispace of a Hilbert space in the inner product of $\mathfrak{H}$. By Theorem 1.6(5), $\mathfrak{M}_\pm \subseteq \mathfrak{H}_\pm$ for some fundamental decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ of $\mathfrak{H}$. We easily check that $\mathfrak{M}^\perp$ is the direct sum of $\mathfrak{H}_+ \ominus \mathfrak{M}_+$ and $\mathfrak{H}_- \ominus \mathfrak{M}_-$, and so (2) holds.

The equivalence of the three conditions has been established. Recall that we showed $\mathfrak{M}$ closed in the implication (3) ⇒ (1). The remaining topological properties at the end of the theorem are clear from the relationship between the fundamental decompositions of $\mathfrak{M}$ and $\mathfrak{H}$ in the implication (1) ⇒ (2). □

Partial isometries on Kreĭn spaces have much the same behavior as in the Hilbert space case. The schematic diagram in figure 3 shows the behavior of a partial isometry $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ with respect to its initial and final subspaces $\mathfrak{M}$ and $\mathfrak{N}$. Details are in Theorems 1.7 and 1.8.

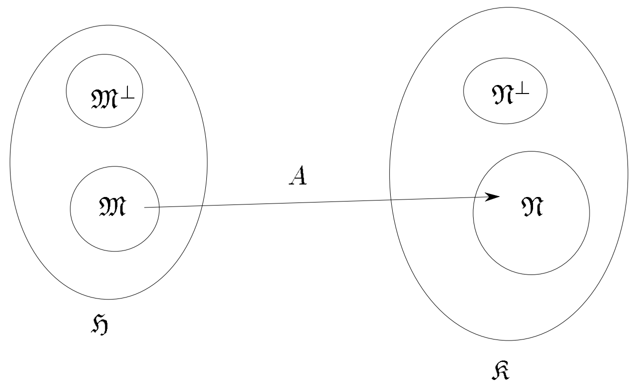

Figure 3

Theorem 1.7 *If $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ for some Kreĭn spaces $\mathfrak{H}$ and $\mathfrak{K}$, then A is a partial isometry if and only if there exist Kreĭn subspaces $\mathfrak{M}$ of $\mathfrak{H}$ and $\mathfrak{N}$ of $\mathfrak{K}$ such that*

(1) *A maps $\mathfrak{M}$ in a one-to-one way onto $\mathfrak{N}$, and*

$$\langle Af, Ag \rangle_{\mathfrak{K}} = \langle f, g \rangle_{\mathfrak{H}}, \qquad f, g \in \mathfrak{M},$$

(2) *$Af = 0$ for all $f \in \mathfrak{M}^\perp$.*

In this case, we call $\mathfrak{M}$ and $\mathfrak{N}$ the ***initial*** *and* ***final subspaces*** *for A. Moreover A^* is a partial isometry with initial space $\mathfrak{N}$ and final space $\mathfrak{M}$, and*

$$\begin{aligned} A^*A &= \mathrm{Pr}_{\mathfrak{M}}\,, \qquad & \ker\, A &= \ker\, \mathrm{Pr}_{\mathfrak{M}}\,, \\ AA^* &= \mathrm{Pr}_{\mathfrak{N}}\,, \qquad & \ker\, A^* &= \ker\, \mathrm{Pr}_{\mathfrak{N}}\,. \end{aligned}$$

Theorem 1.8 *Let $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, where $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces. The following assertions are equivalent:*

(1) *A is a partial isometry;*

(2) *A^*A is a projection operator and $\ker A^*A = \ker A$;*

(3) *AA^* is a projection operator and $\ker AA^* = \ker A^*$.*

Proofs of Theorems 1.7 and 1.8 follow basically Hilbert space methods. Where positivity of inner products is apparently required, the conditions on kernels provide what is needed. See Dritschel and Rovnyak [1990], pp. 8–10.

Notice that in Theorem 1.8, the condition $\ker A^*A = \ker A$ in (2) holds automatically if $\mathfrak{K}$ is a Hilbert space, and the condition $\ker AA^* = \ker A^*$ in (3) holds automatically if $\mathfrak{H}$ is a Hilbert space. An example of an operator A such that A^*A is a projection but $\ker A^*A \neq \ker A$ is any nonzero operator whose range is a neutral subspace, since then $A^*A = 0$. Such operators appear later in Lecture 3. We can even construct an example such that both A^*A and AA^* are projections but A is not a partial isometry. To do this, let A be a Hilbert space projection onto a nonzero closed neutral subspace of a Kreĭn space $\mathfrak{H}$ relative to some fundamental symmetry J. As before, $A^*A = 0$. This means that $JAJA = 0$, and hence $AA^* = AJAJ = 0$. The operator A is not a partial isometry because $AA^*A \neq A$.

The results on partial isometries apply in particular to isometries. Using this information and some elementary reasoning, we deduce:

Corollary 1.9 *An isometry* $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ *is a one-to-one operator whose range is a Kreĭn subspace of* $\mathfrak{K}$*. Therefore* A *is a Kreĭn-space isomorphism from* $\mathfrak{H}$ *onto* ran A *viewed as a Kreĭn space in the inner product of* $\mathfrak{K}$*.*

In particular, an isometry $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ maps closed subspaces of $\mathfrak{H}$ onto closed subspaces of $\mathfrak{K}$ and negative (positive) subspaces of $\mathfrak{H}$ onto negative (positive) subspaces of $\mathfrak{K}$. It also maps Kreĭn subspaces onto Kreĭn subspaces.

The structure of Kreĭn space isometries is more complicated than in the Hilbert space case. The Wold decomposition has no completely satisfactory generalization for Kreĭn spaces. Even when one exists, the shift component may not have the same behavior as in the Hilbert space situation. Those Kreĭn space isometries which have a structure analogous to Hilbert space shifts have been characterized by McEnnis [1990].

Lecture 2:
Julia Operators and Contractions

Key ideas:

- Basic tools of Kreĭn space operator theory include defect and Julia operators, Shmul′yan's theorem on extending densely defined operators, and Pontryagin's theorem on dense sets in a Pontryagin space.
- Contractions are an important special case, and they are simpler because their defect spaces are Hilbert spaces.

It will be seen that an arbitrary Kreĭn space operator $A \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ has extensions to isometric, coisometric, and unitary operators. The construction is an application of the Bognár-Krámli factorization. The extensions provide a very useful tool for the study of general operators.

A sharper form of the Bognár-Krámli factorization is needed. It uses a notion of index for selfadjoint operators. If $A \in \mathfrak{L}(\mathfrak{H})$ is a selfadjoint operator, let $\operatorname{ind}_- (A)$ be the supremum of all positive integers r such that there exists a nonpositive and invertible matrix of the form $\left(\langle Af_j, f_k\rangle_{\mathfrak{H}}\right)_{j,k=1}^{r}$; if no such r exists, set $\operatorname{ind}_- (A) = 0$. Put $\operatorname{ind}_+ (A) = \operatorname{ind}_- (-A)$. We call $\operatorname{ind}_\pm (A)$ the **indices** of A. They measure how much A behaves like a positive or negative operator. For example, $A \geq 0$ if and only if $\operatorname{ind}_- (A) = 0$.

Theorem 2.1 *Let $\mathfrak{H}$ be a Kreĭn space. Any selfadjoint operator $A \in \mathfrak{L}(\mathfrak{H})$ can be written in the form*

$$A = DD^*,$$

where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$ for some Kreĭn space $\mathfrak{D}$ and $\ker\ D = \{0\}$. *For any such factorization,* $\operatorname{ind}_\pm \mathfrak{D} = \operatorname{ind}_\pm (A)$.

The first statement in Theorem 2.1 is Theorem 1.1. The second is proved in Dritschel and Rovnyak [1990], Th. 1.2.1.

Definition 2.2 *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.*

(i) *By a **defect operator** for T we mean any operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ having zero kernel such that*

$$\begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}) \tag{2.1}$$

*is an isometry, that is, $T^*T + \tilde{D}\tilde{D}^* = 1_{\mathfrak{H}}$. We then call $\tilde{\mathfrak{D}}$ a **defect space** for T.*

(ii) *By a **Julia operator** for T we mean any unitary operator*

$$\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}) \tag{2.2}$$

such that $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ and $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ have zero kernels.

The notation in (2.2) is a change from Dritschel and Rovnyak [1990], [1991]: the 22-entry is now written $-L^*$ instead of L. This is closer to the notation in Sz.-Nagy and Foiaş [1970] for the Hilbert space case, and it makes some later formulas more symmetric.

If U is the matrix (2.2), the relations

$$U^*U = 1_{\mathfrak{H}\oplus\mathfrak{D}} \qquad \text{and} \qquad UU^* = 1_{\mathfrak{K}\oplus\tilde{\mathfrak{D}}}$$

are equivalent to the six identities

$$\left.\begin{aligned} T^*T + \tilde{D}\tilde{D}^* &= 1_{\mathfrak{H}}, \\ D^*D + LL^* &= 1_{\mathfrak{D}}, \\ TT^* + DD^* &= 1_{\mathfrak{K}}, \\ \tilde{D}^*\tilde{D} + L^*L &= 1_{\tilde{\mathfrak{D}}}, \end{aligned}\right\} \tag{2.3a,b,c,d}$$

and

$$\left.\begin{aligned} T^*D &= \tilde{D}L^*, \\ T\tilde{D} &= DL, \end{aligned}\right\} \tag{2.4a,b}$$

In particular, $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is a defect operator for T and $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ is a defect operator for T^*. The operator U^* is a Julia operator for T^*.

We use the Bognár-Krámli factorization to show that Julia operators exist. This argument is similar to the method used by Davis [1970] to construct unitary dilations of general operators on a Hilbert space: the method of Davis works with minor adaptations for Kreĭn space operators. An existence proof based on a lemma of Kreĭn was given by Arsene, Constantinescu, and Gheondea [1987]. A generalization is given in Theorem 6.2.

Theorem 2.3 *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.*
(i) *A defect operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ for T exists, and for any such operator,*

$$\operatorname{ind}_\pm \tilde{\mathfrak{D}} = \operatorname{ind}_\pm (1 - T^*T). \tag{2.5}$$

(ii) *If $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is any defect operator for T, there exists a Julia operator for T of the form*

$$\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}). \tag{2.6}$$

For any such operator,

$$\left.\begin{aligned} \operatorname{ind}_\pm \mathfrak{D} &= \operatorname{ind}_\pm (1 - TT^*), \\ \operatorname{ind}_\pm \tilde{\mathfrak{D}} &= \operatorname{ind}_\pm (1 - T^*T). \end{aligned}\right\} \tag{2.7}$$

Proof. (i) Apply Theorem 2.1 to the selfadjoint operator $1 - T^*T$.
(ii) If $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is a defect operator for T, then

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$$

is an isometry. By Theorem 1.7, VV^* is the projection onto ran V, and therefore $1 - VV^*$ is the projection onto $(\operatorname{ran} V)^\perp = \ker V^*$. Factor

$$1 - VV^* = BB^*,$$

where $B \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$ has zero kernel. Then B has the form

$$B = \begin{pmatrix} D \\ -L^* \end{pmatrix},$$

where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ and $L \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{D})$. We verify that with this choice of D and L, (2.6) is a Julia operator.

By construction, BB^* is a projection and $\ker BB^* = \ker B^*$ (because B has zero kernel). It follows from Theorem 1.8 that B is a partial isometry. Thus B is an isometry with $\operatorname{ran} B = \ker V^*$. Therefore $V^*B = 0$ and $B^*B = 1$. These identities imply (2.3b) and (2.4a). The relation (2.3a) follows from the assumption that $\tilde{D}$ is a defect operator for T. The identities (2.3c,d) and (2.4b) follow from the relation $1 - VV^* = BB^*$.

The operator $\tilde{D}$ has zero kernel because it is a defect operator by hypothesis. It remains to show that D has zero kernel. If $Df = 0$ for some $f \in \mathfrak{D}$, then $L^*f = 0$ by (2.4a). Hence $Bf = 0$, and so $f = 0$. Therefore D has zero kernel, and we have shown that (2.6) is a Julia operator.

The relations (2.7) follow from part (i) and the fact that $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is a defect operator for T and $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ is a defect operator for T^*. □

In the proof of Theorem 2.3(ii), when $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is an isometry, $\tilde{\mathfrak{D}}$ is the zero space and so $T = V$. It follows that an isometry $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ has Julia operator

$$\begin{pmatrix} V & D \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K}),$$

where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ is a defect operator for V^*.

A characterization of contractions and bicontractions follows from Theorem 2.3. Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ have Julia operator (2.6). By (2.7), T is a contraction if and only if $\tilde{\mathfrak{D}}$ is a Hilbert space, and T is a bicontraction if and only if both $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ are Hilbert spaces.

Theorem 2.4 *If $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, then*

$$\operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm (1 - TT^*) = \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm (1 - T^*T). \tag{2.8}$$

In particular, if $\mathfrak{H}$ and $\mathfrak{K}$ are Pontryagin spaces with the same negative index, then

$$\operatorname{ind}_- (1 - TT^*) = \operatorname{ind}_- (1 - T^*T). \tag{2.9}$$

Proof. Let (2.6) be a Julia operator for T. This establishes an isomorphism between the Kreĭn spaces $\mathfrak{H} \oplus \mathfrak{D}$ and $\mathfrak{K} \oplus \tilde{\mathfrak{D}}$. Hence by (2.7),

$$\begin{aligned} \operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm (1 - TT^*) &= \operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm \mathfrak{D} \\ &= \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm \tilde{\mathfrak{D}} \\ &= \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm (1 - T^*T). \end{aligned}$$

The second statement follows immediately from the first. □

Corollary 2.5 *If $\mathfrak{H}$ and $\mathfrak{K}$ are Pontryagin spaces with the same negative index, then every contraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a bicontraction.*

Proof. By (2.9), $1 - TT^* \geq 0$ if and only if $1 - T^*T \geq 0$. □

Two Julia operators

$$\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$$

and

$$\begin{pmatrix} T & D' \\ \tilde{D}'^* & -L'^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}', \mathfrak{K} \oplus \tilde{\mathfrak{D}}')$$

for a given $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ are abstractly indistinguishable if there exist Kreĭn space isomorphisms $\varphi : \mathfrak{D} \to \mathfrak{D}'$ and $\psi : \tilde{\mathfrak{D}} \to \tilde{\mathfrak{D}}'$ such that

$$D = D'\varphi, \qquad \tilde{D} = \tilde{D}'\psi, \qquad \varphi L = L'\psi,$$

since this has the effect merely to replace the defect spaces $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ by isomorphic copies. We say that T has **essentially unique Julia operator** if any two Julia operators for T are so related.

Theorem 2.6 *An operator $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ has essentially unique Julia operator if either* $\operatorname{ind}_-(1_{\mathfrak{H}} - T^*T) < \infty$ *or* $\operatorname{ind}_-(1_{\mathfrak{K}} - TT^*) < \infty$. *In particular, T has essentially unique Julia operator if either T or T^* is a contraction.*

For a proof, see Dritschel and Rovnyak [1990], Th. B4.

The basic properties of contraction operators on a Kreĭn space were worked out by Potapov, Ginzburg, Kreĭn, and Shmul′yan in the 1950's and 60's (references are given in Dritschel and Rovnyak [1990], p. 33). The key to these properties is the behavior of a contraction operator on negative subspaces.

Theorem 2.7 *If $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a contraction, then*

(1) *T maps every closed negative subspace $\mathfrak{M}$ of $\mathfrak{H}$ in a one-to-one way onto a closed negative subspace of $\mathfrak{K}$;*

(2) *T maps every closed uniformly negative subspace $\mathfrak{M}$ of $\mathfrak{H}$ in a one-to-one way onto a closed uniformly negative subspace of $\mathfrak{K}$;*

(3) *the kernel of T is a Hilbert space in the inner product of $\mathfrak{H}$.*

Proof. Fix norms for $\mathfrak{H}$ and $\mathfrak{K}$. Choose a defect operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ for T, so

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix}$$

is an isometry.

(1) Since T is a contraction, $\tilde{\mathfrak{D}}$ is a Hilbert space, and

$$\begin{aligned} \left\|\tilde{D}^* f\right\|^2 &= \langle (1 - T^*T)f, f\rangle_{\mathfrak{H}} \\ &= \langle f, f\rangle_{\mathfrak{H}} - \langle Tf, Tf\rangle_{\mathfrak{K}} \\ &\le -\langle Tf, Tf\rangle_{\mathfrak{K}} \\ &\le \left\|Tf\right\|^2 \end{aligned} \tag{2.10}$$

whenever $f \in \mathfrak{H}$ and $\langle f, f\rangle_{\mathfrak{H}} \le 0$. In particular, if $\langle f, f\rangle_{\mathfrak{H}} \le 0$ and $Tf = 0$, then $Vf = 0$ and $f = 0$ by Corollary 1.9. Thus T is one-to-one on any negative subspace.

Suppose that $\mathfrak{M}$ is a closed negative subspace of $\mathfrak{H}$. Clearly $T\mathfrak{M}$ is negative, because T is a contraction. Suppose that $\{f_n\}_1^\infty$ is a sequence in $\mathfrak{M}$ and $Tf_n \to g$

in the strong topology of $\mathfrak{K}$. By (2.10), $\tilde{D}^* f_n \to h$ in the strong topology of $\tilde{\mathfrak{D}}$ for some $h \in \tilde{\mathfrak{D}}$. Then

$$V f_n \to \begin{pmatrix} g \\ h \end{pmatrix}.$$

Since $V\mathfrak{M}$ is closed,

$$\begin{pmatrix} g \\ h \end{pmatrix} = Vf$$

for some $f \in \mathfrak{M}$, so $g = Tf \in T\mathfrak{M}$. This shows that $T\mathfrak{M}$ is closed.

(2) Let $\mathfrak{M}$ be a closed uniformly negative subspace of $\mathfrak{H}$. Since $\tilde{\mathfrak{D}}$ is a Hilbert space and $V\mathfrak{M}$ is uniformly negative, there exists a $\delta > 0$ such that for all $f \in \mathfrak{M}$,

$$\begin{aligned} \langle Tf, Tf \rangle_{\mathfrak{K}} &\le \langle Vf, Vf \rangle_{\mathfrak{K} \oplus \tilde{\mathfrak{D}}} \\ &\le -\delta \|Vf\|^2 \\ &= -\delta \left(\|Tf\|^2 + \|\tilde{D}^* f\|^2 \right) \\ &\le -\delta \|Tf\|^2. \end{aligned} \tag{2.11}$$

It follows that $T\mathfrak{M}$ is uniformly negative. Since $T\mathfrak{M}$ is closed by (1), (2) follows.

(3) Let $\mathfrak{N} = \ker T$. Then $V\mathfrak{N} = \{0\} \oplus \tilde{D}^*\mathfrak{N}$ is closed and uniformly positive. Since V^* acts as an isometry on $\operatorname{ran} V$ into $\mathfrak{H}$, V^* maps closed uniformly positive subspaces of $\operatorname{ran} V$ onto closed uniformly positive subspaces of $\mathfrak{H}$. Thus $V^*V\mathfrak{N} = \mathfrak{N}$ is a Hilbert space in the inner product of $\mathfrak{H}$. □

By Theorem 2.7(2), a contraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ maps a closed negative subspace $\mathfrak{M}$ of $\mathfrak{H}$ onto a closed negative subspace $T\,\mathfrak{M}$ of $\mathfrak{K}$. If $\mathfrak{M}$ is maximal negative in $\mathfrak{H}$, it is closed and $T\,\mathfrak{M}$ is therefore closed and negative in $\mathfrak{K}$; however, $T\,\mathfrak{M}$ is not necessarily maximal negative in $\mathfrak{K}$. For an example, consider the operator $T = 0$ on a Hilbert space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ which is not a Hilbert space. Notice that in this example, T is a contraction but not a bicontraction.

Theorem 2.8 (Ginzburg) *If $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a contraction, the following are equivalent:*

(1) *T is a bicontraction;*
(2) *T maps some maximal negative subspace of $\mathfrak{H}$ onto a maximal negative subspace of $\mathfrak{K}$;*
(3) *T maps every maximal negative subspace of $\mathfrak{H}$ onto a maximal negative subspace of $\mathfrak{K}$.*

In this case, T maps every maximal uniformly negative subspace of $\mathfrak{H}$ onto a maximal uniformly negative subspace of $\mathfrak{K}$.

Other equivalent conditions for a contraction to be a bicontraction are given in an addendum at the end of this lecture.

Proof. (1) ⇒ (3) Assume that T is a bicontraction. Then in any Julia operator

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$$

for T, the defect spaces $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ are Hilbert spaces. Let $\mathfrak{M}$ be any maximal negative subspace of $\mathfrak{H}$. Then $\mathfrak{M}$ is a maximal negative subspace of $\mathfrak{H} \oplus \mathfrak{D}$. Therefore $U\,\mathfrak{M}$

is a maximal negative subspace of $\mathfrak{K} \oplus \tilde{\mathfrak{D}}$. It follows that

$$T\,\mathfrak{M} = \mathrm{Pr}_{\mathfrak{K}}\ U\,\mathfrak{M}$$

is a maximal negative subspace of $\mathfrak{K}$, by remarks preceding Theorem 1.6. Here $\mathrm{Pr}_{\mathfrak{K}}$ denotes projection onto $\mathfrak{K}$. Thus (3) holds.

(3) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (1) Assume that T maps the maximal negative subspace $\mathfrak{M}$ of $\mathfrak{H}$ onto the maximal negative subspace $\mathfrak{N}$ of $\mathfrak{K}$. Since T is a contraction by hypothesis, $\tilde{\mathfrak{D}}$ is a Hilbert space. Therefore the operator

$$T' = \mathrm{Pr}_{\mathfrak{K}}\ U \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K})$$

is a contraction: for any $f \in \mathfrak{H}$ and $g \in \mathfrak{D}$,

$$\begin{aligned} \left\langle T'\begin{pmatrix} f \\ g \end{pmatrix}, T'\begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathfrak{K}} &= \left\langle \mathrm{Pr}_{\mathfrak{K}}\ U\begin{pmatrix} f \\ g \end{pmatrix}, \mathrm{Pr}_{\mathfrak{K}}\ U\begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathfrak{K}} \\ &\le \left\langle U\begin{pmatrix} f \\ g \end{pmatrix}, U\begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathfrak{K}\oplus\tilde{\mathfrak{D}}} = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathfrak{H}\oplus\mathfrak{D}}. \end{aligned}$$

If $\mathfrak{D}_-$ is any closed negative subspace of $\mathfrak{D}$, then $\mathfrak{M}' = \mathfrak{M} \oplus \mathfrak{D}_-$ is a closed negative subspace of $\mathfrak{H} \oplus \mathfrak{D}$. By Theorem 2.7(1), T' maps $\mathfrak{M}'$ in a one-to-one way onto a negative subspace $\mathfrak{N}'$ of $\mathfrak{K}$. Since

$$T'\,\mathfrak{M} = \mathrm{Pr}_{\mathfrak{K}}\ U\,\mathfrak{M} = T\,\mathfrak{M} = \mathfrak{N},$$

$\mathfrak{N}' \supseteq \mathfrak{N}$, and the inclusion is proper if $\mathfrak{D}_- \neq \{0\}$. By the maximality of $\mathfrak{N}$, $\mathfrak{D}_- = \{0\}$ and so $\mathfrak{D}$ is a Hilbert space. Therefore T^* is a contraction, which proves (1).

The last assertion follows from Theorem 2.7(2) and the graph representation of negative subspaces. □

In Hilbert space operator theory, a densely defined contraction can automatically be extended by continuity to an everywhere defined contraction. In contrast, there exist discontinuous densely defined operators T_0 on a Kreĭn space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ such that

$$\langle T_0 f, T_0 g\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}}, \qquad f, g \in \mathrm{dom}\ T_0. \tag{2.12}$$

Example:

There exists a densely defined operator V_0 on a Hilbert space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ such that

$$\langle V_0 f, V_0 g\rangle_{\mathfrak{K}} = \langle f, g\rangle_{\mathfrak{H}}, \qquad f, g \in \mathrm{dom}\ V_0,$$

but V_0 has no extension to an operator $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. Let $\mathfrak{H}$ have orthonormal basis $\varphi_1, \varphi_2, \dots$. Let $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$ be a fundamental decomposition, and assume that $\mathfrak{K}_\pm$ has orthonormal basis $e_1^\pm, e_2^\pm, \dots$. Let dom V_0 be the span of $\varphi_1, \varphi_2, \dots$, and define V_0 on this domain so as to be linear and satisfy

$$V_0\varphi_n = a_n e_n^+ + b_n e_n^-,$$

where the constants are chosen such that $|a_n|^2 - |b_n|^2 = 1$ for all $n \ge 1$. In figure 4 we take $\mathfrak{K}$ to be the direct sum of a sequence of copies of $\mathbf{M}^2$, and we choose $V_0\varphi_n$

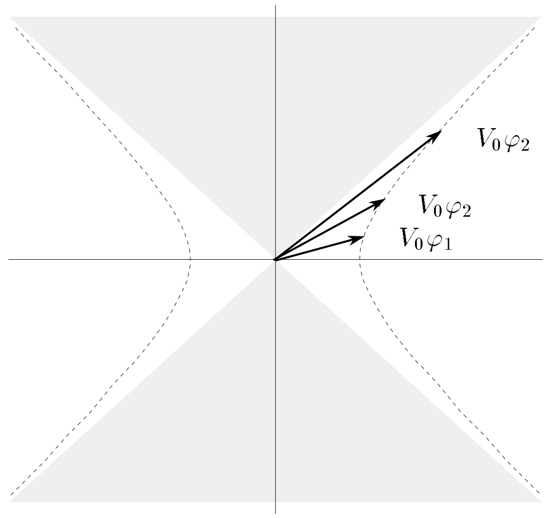

Figure 4

in the n-th copy. The figure is obtained from superimposing the copies. It is easy to see that

$$\langle V_0\varphi_m, V_0\varphi_n\rangle_{\mathfrak{K}} = \delta_{mn}, \qquad n \geq 1,$$

and so V_0 preserves inner products on its domain. If the constants are unbounded, V_0 can have no extension to an operator $V \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.

A theorem of Shmul′yan [1967] shows how such behavior may be excluded.

Theorem 2.9 (Shmul′yan) *Let T_0 be a densely defined operator on a Kreĭn space $\mathfrak{H}$ into a Kreĭn space $\mathfrak{K}$ satisfying*

$$\langle T_0 f, T_0 f\rangle_{\mathfrak{K}} \leq \langle f, f\rangle_{\mathfrak{H}}, \qquad f \in \operatorname{dom} T_0. \tag{2.13}$$

Assume that dom T_0 *contains a maximal uniformly negative subspace $\mathfrak{M}$ and that $T_0\,\mathfrak{M}$ is maximal uniformly negative in $\mathfrak{K}$. Then T_0 has an extension to a bicontraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.*

The proof in Shmul′yan [1967] is attributed to M. L. Brodskiĭ. Special cases were apparently known earlier, but we do not know references. See also Dritschel and Rovnyak [1990], Theorem 1.4.4.

In the case of Pontryagin spaces, the following result is helpful in checking the hypotheses on maximal uniformly negative subspaces in Theorem 2.9. For different proofs, see Iokhvidov, Kreĭn, and Langer [1982], Lemma 2.1 on p. 18, or Bognár [1974], Theorem 1.4 on p. 185.

Pontryagin's Theorem. *If $\mathfrak{M}$ is a dense subspace of a Pontryagin space $\mathfrak{H}$, there is a fundamental decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ such that $\mathfrak{H}_- \subseteq \mathfrak{M}$.*

Proof. The argument is based on this evident property: If a dense subspace of a Kreĭn space $\mathfrak{K}$ contains no nonzero vector f such that $\langle f, f\rangle < 0$, then $\mathfrak{K}$ is a Hilbert space.

Suppose that $\mathfrak{H}$ has negative index n and that $\mathfrak{M}$ is a dense subspace of $\mathfrak{H}$. If $n = 0$, the conclusion is trivial. If $n > 0$, by the property, there is a vector f_1 in $\mathfrak{M}$ such that $\langle f_1, f_1 \rangle < 0$. Let $\mathfrak{N}_1$ be the span of f_1. Then $\mathfrak{N}_1$ is a one-dimensional Kreĭn subspace of $\mathfrak{H}$. It is easy to see that

$$\mathfrak{M} = \mathfrak{N}_1 \oplus (\mathfrak{M} \ominus \mathfrak{N}_1)$$

and that $\mathfrak{M}_1 = \mathfrak{M} \ominus \mathfrak{N}_1$ is dense in $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{N}_1$. Here $\mathfrak{H}_1$ is a Pontryagin space with negative index $n-1$. If $n > 1$, we can repeat the argument and find a vector f_2 in $\mathfrak{M}_1$ such that $\langle f_2, f_2 \rangle < 0$. The span $\mathfrak{N}_2$ of f_2 is a Kreĭn subspace of $\mathfrak{H}_1$, and $\mathfrak{M}_2 = \mathfrak{M}_1 \ominus \mathfrak{N}_2$ is a dense subspace of the Pontryagin space $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{N}_2$. The process will terminate after n stages, and then $\mathfrak{H}_- = \mathfrak{N}_1 \oplus \cdots \oplus \mathfrak{N}_n$ is a subspace of $\mathfrak{H}$ with the required properties. □

Using Pontryagin's theorem, it is not hard to deduce the following version of Shmul′yan's theorem.

Theorem 2.10 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Pontryagin spaces with the same negative index, and let T_0 be a densely defined linear operator from $\mathfrak{H}$ into $\mathfrak{K}$ satisfying* (2.13). *Then T_0 has a continuous extension to a bicontraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$.*

The version for Pontryagin spaces is simpler to prove than Theorem 2.9. A nice proof by T. Ya. Azizov appears in Alpay, Dijksma, and de Snoo [1994]. The following version, also taken from the same source, is particularly useful in applications. Details will appear in Alpay, Dijksma, Rovnyak, and de Snoo [in preparation].

Theorem 2.10 (Alternative Form) *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Pontryagin spaces with the same negative index. Let $\mathbf{R}$ be a linear subspace of $\mathfrak{H} \times \mathfrak{K}$ such that*
(1) *$\langle g, g \rangle_{\mathfrak{K}} \le \langle f, f \rangle_{\mathfrak{H}}$ for all $(f, g) \in \mathbf{R}$, and*
(2) *the set of all $f \in \mathfrak{H}$ such that $(f, g) \in \mathbf{R}$ for some $g \in \mathfrak{K}$ is dense in $\mathfrak{H}$.*
Then there is a bicontraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that $g = Tf$ for all $(f, g) \in \mathbf{R}$.

We conclude this lecture with some formal properties of Julia operators which are very useful in the study of extension properties of operators. Assumptions are only made on the form of the extensions, and otherwise the theorems hold for arbitrary operators $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. All underlying spaces in Theorems 2.11, 2.12, and 2.13 are Kreĭn spaces.

Theorem 2.11 *Assume $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, and let*

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T)$$

be a Julia operator. Assume $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$, and let

$$\begin{pmatrix} X & D_X \\ \tilde{D}_X^* & -L_X^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{F} \oplus \mathfrak{D}_X, \mathfrak{D}_T \oplus \tilde{\mathfrak{D}}_X)$$

be a Julia operator. Then a Julia operator for

$$R = \begin{pmatrix} T & D_T X \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K})$$

is given by

$$\begin{pmatrix} R & D_R \\ \tilde{D}_R^* & -L_R^* \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} T & D_T X \end{pmatrix} & \begin{pmatrix} D_T D_X \end{pmatrix} \\ \begin{pmatrix} \tilde{D}_T^* & -L_T^* X \\ 0 & \tilde{D}_X^* \end{pmatrix} & -\begin{pmatrix} L_T^* D_X \\ L_X^* \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} T & D_T & 0 \\ \tilde{D}_T^* & -L_T^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & X & D_X \\ 0 & \tilde{D}_X^* & -L_X^* \end{pmatrix} \tag{2.14}$$

as an operator on $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F} \oplus \mathfrak{D}_X, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T \oplus \tilde{\mathfrak{D}}_X)$. *Thus defect spaces for* R *and* R^* *are given by*

$$\tilde{\mathfrak{D}}_R = \tilde{\mathfrak{D}}_T \oplus \tilde{\mathfrak{D}}_X \qquad \textit{and} \qquad \mathfrak{D}_R = \mathfrak{D}_X.$$

As in Theorem 2.4, the unitarity of the various operators in Theorem 2.11 gives a sequence of useful index formulas. Repeating (2.8), these read:

$$\left.\begin{aligned} \operatorname{ind}_\pm (1 - R^*R) &= \operatorname{ind}_\pm (1 - T^*T) + \operatorname{ind}_\pm (1 - X^*X), \\ \operatorname{ind}_\pm (1 - RR^*) &= \operatorname{ind}_\pm (1 - XX^*), \\ \operatorname{ind}_\pm \mathfrak{H} + \operatorname{ind}_\pm (1 - TT^*) &= \operatorname{ind}_\pm \mathfrak{K} + \operatorname{ind}_\pm (1 - T^*T), \\ \operatorname{ind}_\pm \mathfrak{F} + \operatorname{ind}_\pm (1 - XX^*) &= \operatorname{ind}_\pm (1 - TT^*) + \operatorname{ind}_\pm (1 - X^*X). \end{aligned}\right\} \tag{2.15}$$

Theorem 2.12 *Assume* $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, *and let*

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T)$$

be a Julia operator. Assume $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$, *and let*

$$\begin{pmatrix} Y & D_Y \\ \tilde{D}_Y^* & -L_Y^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{G} \oplus \mathfrak{D}_Y, \tilde{\mathfrak{D}}_T \oplus \tilde{\mathfrak{D}}_Y)$$

be a Julia operator. Then a Julia operator for

$$C = \begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

is given by

$$\begin{pmatrix} C & D_C \\ \tilde{D}_C^* & -L_C^* \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix} & \begin{pmatrix} D_T & 0 \\ -Y^* L_T^* & \tilde{D}_Y \end{pmatrix} \\ \begin{pmatrix} D_Y^* \tilde{D}_T^* \end{pmatrix} & -\begin{pmatrix} D_Y^* L_T^* & L_Y \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & Y^* & \tilde{D}_Y \\ 0 & D_Y^* & -L_Y \end{pmatrix} \begin{pmatrix} T & D_T & 0 \\ \tilde{D}_T^* & -L_T^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

as an operator on $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T \oplus \tilde{\mathfrak{D}}_Y, \mathfrak{K} \oplus \mathfrak{G} \oplus \mathfrak{D}_Y)$. *Thus defect spaces for* C *and* C^* *are given by*

$$\tilde{\mathfrak{D}}_C = \mathfrak{D}_Y \qquad \textit{and} \qquad \mathfrak{D}_C = \mathfrak{D}_T \oplus \tilde{\mathfrak{D}}_Y.$$

The corresponding index formulas are:

$$\left.\begin{aligned} \operatorname{ind}_\pm(1-C^*C) &= \operatorname{ind}_\pm(1-YY^*), \\ \operatorname{ind}_\pm(1-CC^*) &= \operatorname{ind}_\pm(1-TT^*)+\operatorname{ind}_\pm(1-Y^*Y), \\ \operatorname{ind}_\pm\mathfrak{H}+\operatorname{ind}_\pm(1-TT^*) &= \operatorname{ind}_\pm\mathfrak{K}+\operatorname{ind}_\pm(1-T^*T), \\ \operatorname{ind}_\pm\mathfrak{G}+\operatorname{ind}_\pm(1-YY^*) &= \operatorname{ind}_\pm(1-T^*T)+\operatorname{ind}_\pm(1-Y^*Y). \end{aligned}\right\} \tag{2.17}$$

Theorem 2.13 *Assume $T \in \mathfrak{L}(\mathfrak{H},\mathfrak{K})$, and let*

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}\oplus\mathfrak{D}_T, \mathfrak{K}\oplus\tilde{\mathfrak{D}}_T)$$

be a Julia operator. Assume $X \in \mathfrak{L}(\mathfrak{F},\mathfrak{D}_T)$, $Y \in \mathfrak{L}(\mathfrak{G},\tilde{\mathfrak{D}}_T)$, $Z \in \mathfrak{L}(\tilde{\mathfrak{D}}_X,\tilde{\mathfrak{D}}_Y)$, and let

$$\begin{aligned} &\begin{pmatrix} X & D_X \\ \tilde{D}_X^* & -L_X^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{F}\oplus\mathfrak{D}_X, \mathfrak{D}_T\oplus\tilde{\mathfrak{D}}_X), \\ &\begin{pmatrix} Y & D_Y \\ \tilde{D}_Y^* & -L_Y^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{G}\oplus\mathfrak{D}_Y, \tilde{\mathfrak{D}}_T\oplus\tilde{\mathfrak{D}}_Y), \\ &\begin{pmatrix} Z & D_Z \\ \tilde{D}_Z^* & -L_Z^* \end{pmatrix} \in \mathfrak{L}(\tilde{\mathfrak{D}}_X\oplus\mathfrak{D}_Z, \tilde{\mathfrak{D}}_Y\oplus\tilde{\mathfrak{D}}_Z), \end{aligned}$$

be Julia operators. Then a Julia operator for

$$A = \begin{pmatrix} T & D_T X \\ Y^*\tilde{D}_T^* & -Y^*L_T^*X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}\oplus\mathfrak{F}, \mathfrak{K}\oplus\mathfrak{G})$$

is given by

$$\begin{pmatrix} A & D_A \\ \tilde{D}_A^* & -L_A^* \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} T & D_T X \\ Y^*\tilde{D}_T^* & -Y^*L_T^*X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix} & \begin{pmatrix} D_T D_X & 0 \\ -Y^*L_T^*D_X - \tilde{D}_Y Z L_X^* & \tilde{D}_Y D_Z \end{pmatrix} \\ \begin{pmatrix} D_Y^*\tilde{D}_T^* & -D_Y^*L_T^*X - L_Y Z \tilde{D}_X^* \\ 0 & \tilde{D}_Z^*\tilde{D}_X^* \end{pmatrix} & -\begin{pmatrix} D_Y^*L_T^*D_X - L_Y Z L_X^* & L_Y D_Z \\ \tilde{D}_Z^* L_X^* & L_Z^* \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Y^* & \tilde{D}_Y & 0 \\ 0 & D_Y^* & -L_Y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T & D_T & 0 & 0 \\ \tilde{D}_T^* & -L_T^* & 0 & 0 \\ 0 & 0 & Z & D_Z \\ 0 & 0 & \tilde{D}_Z^* & -L_Z^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X & D_X & 0 \\ 0 & \tilde{D}_X^* & -L_X^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

as an operator on $\mathfrak{L}(\mathfrak{H}\oplus\mathfrak{F}\oplus\mathfrak{D}_X\oplus\mathfrak{D}_Z, \mathfrak{K}\oplus\mathfrak{G}\oplus\mathfrak{D}_Y\oplus\tilde{\mathfrak{D}}_Z)$. Thus defect spaces for A and A^* are given by

$$\tilde{\mathfrak{D}}_A = \mathfrak{D}_Y \oplus \tilde{\mathfrak{D}}_Z \qquad \text{and} \qquad \mathfrak{D}_A = \mathfrak{D}_X \oplus \mathfrak{D}_Z.$$

Now the index identities read:

$$\left.\begin{aligned} \operatorname{ind}_{\pm}(1-A^*A) &= \operatorname{ind}_{\pm}(1-YY^*)+\operatorname{ind}_{\pm}(1-Z^*Z), \\ \operatorname{ind}_{\pm}(1-AA^*) &= \operatorname{ind}_{\pm}(1-XX^*)+\operatorname{ind}_{\pm}(1-ZZ^*), \\ \operatorname{ind}_{\pm}\mathfrak{H}+\operatorname{ind}_{\pm}(1-TT^*) &= \operatorname{ind}_{\pm}\mathfrak{K}+\operatorname{ind}_{\pm}(1-T^*T), \\ \operatorname{ind}_{\pm}\mathfrak{F}+\operatorname{ind}_{\pm}(1-XX^*) &= \operatorname{ind}_{\pm}(1-TT^*)+\operatorname{ind}_{\pm}(1-X^*X), \\ \operatorname{ind}_{\pm}\mathfrak{G}+\operatorname{ind}_{\pm}(1-YY^*) &= \operatorname{ind}_{\pm}(1-T^*T)+\operatorname{ind}_{\pm}(1-Y^*Y), \\ \operatorname{ind}_{\pm}(1-X^*X)+\operatorname{ind}_{\pm}(1-ZZ^*) &= \operatorname{ind}_{\pm}(1-Y^*Y)+\operatorname{ind}_{\pm}(1-Z^*Z). \end{aligned}\right\} \tag{2.19}$$

Proofs of Theorems 2.11, 2.12, and 2.13. The matrices (2.14), (2.16), and (2.18) are unitary because each is the product of unitary operators. By inspection the 12-entries and adjoints of the 21-entries have zero kernels, and thus the conditions for a Julia operator are met. □

Addendum: Adjoints of contractions

Adjoints of contractions also have special mapping properties with respect to negative subspaces (Dritschel and Rovnyak [1990]), and these too can be studied by the method of isometric extensions. We illustrate the idea in an elaboration of Theorem 2.8 that includes a condition on adjoints. We first prove a part of this result (Theorem 2.8′ below) for the special case of isometries.

Lemma 2.14 *Let $V \in \mathfrak{L}(\mathfrak{H},\mathfrak{K})$ be an isometry. If V^* maps some maximal negative subspace $\mathfrak{N}$ in a one-to-one way into a negative subspace, then V is a bicontraction.*

Proof. By Theorem 1.7, V^* is isometric on the range of V. To prove that V is a bicontraction, it is sufficient to show that $(\operatorname{ran} V)^{\perp} = \ker V^*$ is a Hilbert space.

Since $\ker V^*$ is a Kreĭn subspace of $\mathfrak{K}$, it has a fundamental decomposition $\ker V^* = \mathfrak{E}_+ \oplus \mathfrak{E}_-$. Put

$$\begin{aligned} \mathfrak{K}' &= \operatorname{ran} V \oplus \mathfrak{E}_-, \\ \mathfrak{N}' &= \operatorname{Pr}_{\mathfrak{K}'} \mathfrak{N}, \\ V' &= V|_{\mathfrak{K}'} \in \mathfrak{L}(\mathfrak{H},\mathfrak{K}'). \end{aligned}$$

It is easy to see that the hypotheses of the lemma are satisfied with $V, N, \mathfrak{N}$ replaced by $V', N', \mathfrak{N}'$ (the space $\mathfrak{N}'$ is maximal negative by the comments preceding Theorem 1.6). Since $\ker V'^* = \mathfrak{E}_-$, by a change of notation we may assume in the original situation that $\ker V^*$ is uniformly negative.

We show that under the assumption that $\ker V^*$ is uniformly negative, it contains no nonzero element. Let $u \in \ker V^*$. For any $g \in \mathfrak{N}$ and any scalar c,

$$0 \geq \langle V^*g, V^*g\rangle_{\mathfrak{H}} = \langle VV^*(cu+g), VV^*(cu+g)\rangle_{\mathfrak{K}} \geq \langle cu+g, cu+g\rangle_{\mathfrak{K}},$$

where the second inequality results from the assumption that $\ker V^*$ is uniformly negative. Since $\mathfrak{N}$ is maximal negative, $u \in \mathfrak{N}$. Since V^* is one-to-one on $\mathfrak{N}$ and $V^*u = 0$, we obtain $u = 0$. Thus $\ker V^* = \{0\}$, and the result follows. □

Theorem 2.8′ *If $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is a contraction, the following are equivalent:*
(1) *T is a bicontraction;*
(2) *αT^* is a contraction for some scalar $\alpha \neq 0$;*
(3) *T maps some maximal negative subspace of $\mathfrak{H}$ onto a maximal negative subspace of $\mathfrak{K}$;*
(4) *T maps every maximal negative subspace of $\mathfrak{H}$ onto a maximal negative subspace of $\mathfrak{K}$;*
(5) *T^* maps some maximal negative subspace of $\mathfrak{K}$ in a one-to-one way into a negative subspace of $\mathfrak{H}$;*
(6) *if $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is a defect operator for T, the isometry*

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix}$$

is a bicontraction.

In this case, T maps every maximal uniformly negative subspace of $\mathfrak{H}$ onto a maximal uniformly negative subspace of $\mathfrak{K}$.

Proof. The equivalence of (1), (3), (4) and last assertion are shown in Theorem 2.8. The proof is completed by first establishing (1) $\Leftrightarrow$ (6). Then (2) is added to the list of equivalences by showing (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4). Finally, (5) is added by showing (2) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6).

(1) $\Leftrightarrow$ (6) By Theorem 2.3, we may choose a Julia operator

$$\begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$$

for T such that $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ is the defect operator which is given in (6). Writing

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \qquad \text{and} \qquad D_V = \begin{pmatrix} D \\ -L^* \end{pmatrix},$$

we obtain a Julia operator $\begin{pmatrix} V & D_V \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}, (\mathfrak{K} \oplus \tilde{\mathfrak{D}}) \oplus \{0\})$ for V with defect spaces $\tilde{\mathfrak{D}}_V = \{0\}$ and $\mathfrak{D}_V = \mathfrak{D}$. Thus V is a bicontraction if and only if $\mathfrak{D}$ is a Hilbert space, that is, T^* is a contraction. Since we assume that T is a contraction, this proves the equivalence of (1) and (6).

(1) $\Rightarrow$ (2) Take $\alpha = 1$.

(2) $\Rightarrow$ (4) Assume that αT^* is a contraction for some $\alpha \neq 0$. Let $D_\alpha \in \mathfrak{L}(\mathfrak{D}_\alpha, \mathfrak{K})$ be a defect operator for αT^*, so

$$V_\alpha = \begin{pmatrix} \alpha T^* \\ D_\alpha^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H} \oplus \mathfrak{D}_\alpha)$$

is an isometry. Since αT^* is a contraction, $\mathfrak{D}_\alpha$ is a Hilbert space. Therefore any maximal negative subspace $\mathfrak{M}$ of $\mathfrak{H}$ is maximal negative in $\mathfrak{H} \oplus \mathfrak{D}_\alpha$. Now

$$V_\alpha^*|_{\mathfrak{M}} = \alpha T|_{\mathfrak{M}},$$

and since T is a contraction, αT maps $\mathfrak{M}$ in a one-to-one way into a negative subspace by Theorem 2.7. By Lemma 2.14, V_α is a bicontraction. It follows that $T\mathfrak{M} = V_\alpha^* \mathfrak{M}$ is a maximal negative subspace of $\mathfrak{K}$, proving (4).

(2) $\Rightarrow$ (5) By Theorem 2.7, if αT^* is a contraction, then αT^* maps any maximal negative subspace of $\mathfrak{K}$ in a one-to-one way onto a negative subspace of $\mathfrak{H}$. Since $\alpha \neq 0$, T^* has the same property.

(5) $\Rightarrow$ (6) By assumption, there is a maximal negative subspace $\mathfrak{N}$ of $\mathfrak{K}$ which T^* maps in a one-to-one way into a negative subspace of $\mathfrak{H}$. Let

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}})$$

as in (6) for some defect operator $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ for T. Since T is a contraction, $\tilde{\mathfrak{D}}$ is a Hilbert space, and therefore $\mathfrak{N}$ is a maximal negative subspace of $\mathfrak{K} \oplus \tilde{\mathfrak{D}}$. Therefore $V^*|_{\mathfrak{N}} = T^*|_{\mathfrak{N}}$ maps $\mathfrak{N}$ in a one-to-one way into a negative subspace. By Lemma 2.14, V is a bicontraction. $\square$

Lecture 3: Extension and Completion Problems

Key ideas:

- Extension and completion problems for operators on Kreĭn spaces often have the same form as in the Hilbert space case, even in very general situations. The theory is particularly simple for contractions.
- The problems are reduced to the isometric case via defect and Julia operators, and this reduction allows us to apply Hilbert space methods.

Kreĭn space operator theory comes very close to matching the completeness of the Hilbert space case in some areas, notably the extension properties of operators. This is the first of several lectures which will explore such problems.

Extension and completion problems come in numerous variations, leading to commutant lifting theorems, the Schur algorithm, and interpolation theory, with their infinite possibilities (see Foiaş and Frazho [1990]). Here we focus on a single but basic type of problem, which in many ways serves as a prototype.

Given a fixed contraction $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, we wish to determine all contractive operators of the form

$$\begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K}), \tag{3.1}$$

$$\begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G}), \tag{3.2}$$

and

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G}). \tag{3.3}$$

Variations of the problem replace contractions T by operators such that

$$\operatorname{ind}_-(1 - T^*T) < \infty \qquad \text{or} \qquad \operatorname{ind}_-(1 - TT^*) < \infty. \tag{3.4}$$

Then we seek a description of row, column, and two-by-two extensions which have similar properties.

Particular extensions have already been encountered, namely,

$$\begin{pmatrix} T & D_T \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K}),$$

$$\begin{pmatrix} T \\ \tilde{D}_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T),$$

and

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T),$$

where the last operator is a Julia operator for T.

In the Hilbert space case, contractive row, column, and two-by-two completions take the form

$$\begin{pmatrix} T & D_T X \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K}), \tag{3.5}$$

$$\begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G}), \tag{3.6}$$

$$\begin{pmatrix} T & D_T X \\ Y^* \tilde{D}_T^* & -Y^* L_T^* X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G}), \tag{3.7}$$

where X, Y, Z are contractions on appropriate spaces. The Hilbert space theory originates in work of Davis [1976] and Parrott [1978] and has been much studied (see Dritschel and Rovnyak [1990] for references).

An example shows some of the problems that can arise with indefinite inner product spaces.

Example:

Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ be an isometry on a Hilbert space $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$. Let

$$C = \begin{pmatrix} T \\ N \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

be a column extension of T such that $N \in \mathfrak{L}(\mathfrak{H}, \mathfrak{G})$ is a **neutral operator**, which we define as an operator such that $N^* N = 0$. Such an operator is constructed simply by arranging that ran N is a neutral subspace of $\mathfrak{G}$. For example, we can find a neutral operator $N \neq 0$ if $\mathfrak{G} = \mathbf{M}^2$. Then

$$C^* C = T^* T + N^* N = T^* T = 1.$$

Thus C is a contractive (even isometric) column extension of T. It does not have the form (3.6) because $1 - T^* T = 0$ and hence $\tilde{\mathfrak{D}}_T = \{0\}$ and $\tilde{D}_T = 0$.

Nevertheless, it is desirable to try to derive positive results in which the forms (3.5)–(3.7) are obtained. For example, when these representations hold, the Julia operators for the extensions are given by the matrix formulas in Theorems 2.11–2.13. All of the index formulas stated there are valid. We can easily read off conditions on the parameters X, Y, Z that the operators are isometries or contractions, and so forth. This information is useful for more structured problems, such as those involving commutant lifting or the Schur algorithm (Lecture 4).

We follow a simplified method given in Dritschel [1993c]. The row extension problem will first be solved for isometries.

Theorem 3.1 *Let*

$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K}) \tag{3.8}$$

be an isometry, and let $D_{V_1} \in \mathfrak{L}(\mathfrak{D}_{V_1}, \mathfrak{K})$ *be a defect operator for* V_1^*. *Then*

$$V_2 = D_{V_1} W, \tag{3.9}$$

where $W = D_{V_1}^* V_2 \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_{V_1})$ *is an isometry. If* V *is unitary and* V_2 *is written in any way in the form (3.9) with* $W \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_{V_1})$ *an isometry, then* W *is unitary.*

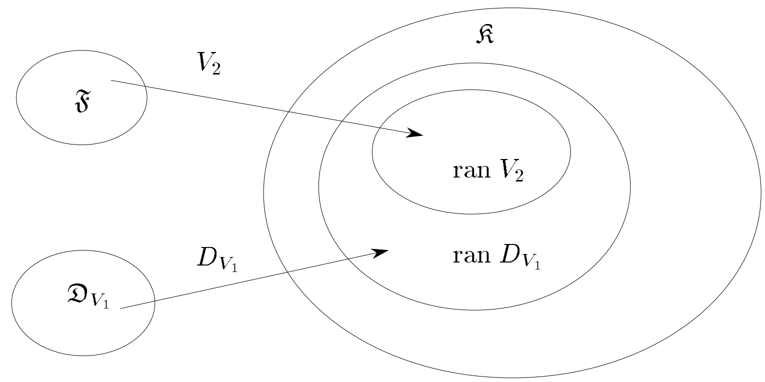

Figure 5

Proof. Since V is isometric, so are its restrictions $V_1 = V|_{\mathfrak{H}}$ and $V_2 = V|_{\mathfrak{F}}$. The identity

$$0 = 1 - V^*V = \begin{pmatrix} * & -V_1^*V_2 \\ * & * \end{pmatrix}$$

implies that $V_1^*V_2 = 0$, and so $\operatorname{ran} V_2 \subseteq \ker V_1^*$. Since V_1 is an isometry,

$$D_{V_1}D_{V_1}^* = 1 - V_1V_1^*$$

is a projection with range $(\operatorname{ran} V_1)^\perp = \ker V_1^*$. Thus

$$D_{V_1}D_{V_1}^*D_{V_1}D_{V_1}^* = D_{V_1}D_{V_1}^*,$$

and hence $D_{V_1}^*D_{V_1} = 1$, because D_{V_1} has zero kernel and $D_{V_1}^*$ has dense range. Therefore D_{V_1} is an isometry with $\operatorname{ran} D_{V_1} = \ker V_1^* \supseteq \operatorname{ran} V_2$ (see figure 5). Hence $W = D_{V_1}^*V_2$ defines an isometry from $\mathfrak{F}$ into $\mathfrak{D}_{V_1}$. Since $D_{V_1}D_{V_1}^*$ acts like the identity on $\operatorname{ran} D_{V_1}$,

$$D_{V_1}W = D_{V_1}D_{V_1}^*V_2 = V_2,$$

as was to be shown.

Assume V is unitary and V_2 is written in the form (3.9) with $W \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_{V_1})$ isometric. Then

$$0 = 1 - VV^* = 1 - V_1V_1^* - D_{V_1}WW^*D_{V_1}^* = D_{V_1}(1 - WW^*)D_{V_1}^*,$$

and $1 - WW^* = 0$ since D_{V_1} has zero kernel and $D_{V_1}^*$ has dense range. Since $W^*W = 1$ by assumption, W is unitary. $\square$

We shall first state the main results in their simplest form — for contractions. In Theorems 3.2–3.4, we make these assumptions on underlying spaces:

(1) The main spaces $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces.

(2) The extension space $\mathfrak{F}$ is a Kreĭn space, but $\mathfrak{G}$ is a Hilbert space.

Notice that the example above is excluded then from Theorem 3.3 because the space $\mathfrak{G}$ in the example is not a Hilbert space.

Theorem 3.2 *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction, and let $D_T \in \mathfrak{L}(\mathfrak{D}_T, \mathfrak{K})$ be a defect operator for T^*. Every contractive row extension*

$$R = \begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K})$$

of T has the form

$$R = \begin{pmatrix} T & D_T X \end{pmatrix}, \tag{3.10}$$

where $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction. Conversely, if $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction, then the operator R defined by (3.10) *is a contraction.*

Theorem 3.3 *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction, and let $\tilde{D}_T \in \mathfrak{L}(\tilde{\mathfrak{D}}_T, \mathfrak{H})$ be a defect operator for T. Every contractive column extension*

$$C = \begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

of T with $\mathfrak{G}$ a Hilbert space has the form

$$C = \begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix}, \tag{3.11}$$

where $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction. Conversely, if $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction, then the operator C defined by (3.11) *is a contraction.*

Theorem 3.4 *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ be a contraction with Julia operator*

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T).$$

Every contractive extension

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G})$$

of T with $\mathfrak{G}$ a Hilbert space has the form

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} = \begin{pmatrix} T & D_T X \\ Y^* \tilde{D}_T^* & -Y^* L_T^* X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix}, \tag{3.12}$$

where
(i) *$X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction with defect operator $\tilde{D}_X \in \mathfrak{L}(\tilde{\mathfrak{D}}_X, \mathfrak{F})$,*
(ii) *$Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction with defect operator $\tilde{D}_Y \in \mathfrak{L}(\tilde{\mathfrak{D}}_Y, \mathfrak{G})$,*
(iii) *$Z \in \mathfrak{L}(\tilde{\mathfrak{D}}_X, \tilde{\mathfrak{D}}_Y)$ is a contraction.*
Conversely, given X, Y, Z satisfying (i), (ii), (iii), then the operator defined by (3.12) is a contraction.

The method of proof of Theorems 3.2–3.4 applies with no extra input to operators satisfying (3.4). We now generalize each of these results, with no restriction on underlying spaces:
(1) The main spaces $\mathfrak{H}$ and $\mathfrak{K}$ are Kreĭn spaces.
(2) The extension spaces $\mathfrak{F}$ and $\mathfrak{G}$ are Kreĭn spaces.

Theorem 3.2 (Generalized Form) *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, and let $D_T \in \mathfrak{L}(\mathfrak{D}_T, \mathfrak{K})$ be a defect operator for T^*. Let*

$$R = \begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K})$$

be a row extension of T satisfying at least one of the conditions

$$\begin{aligned} \operatorname{ind}_-(1 - RR^*) + \operatorname{ind}_- \mathfrak{F} &= \operatorname{ind}_-(1 - TT^*) < \infty, \\ \operatorname{ind}_-(1 - R^*R) &= \operatorname{ind}_-(1 - T^*T) < \infty. \end{aligned}$$

Then

$$R = \begin{pmatrix} T & D_T X \end{pmatrix}, \tag{3.13}$$

where $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction. Conversely, if R has the form (3.13) where $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction, then the operator R defined by (3.13) satisfies both of the identities

$$\begin{aligned} \operatorname{ind}_-(1 - RR^*) + \operatorname{ind}_- \mathfrak{F} &= \operatorname{ind}_-(1 - TT^*), \\ \operatorname{ind}_-(1 - R^*R) &= \operatorname{ind}_-(1 - T^*T). \end{aligned}$$

Theorem 3.3 (Generalized Form) *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$, and let $\tilde{D}_T \in \mathfrak{L}(\tilde{\mathfrak{D}}_T, \mathfrak{H})$ be a defect operator for T. Let*

$$C = \begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

be a column extension of T satisfying at least one of the conditions

$$\begin{aligned} \operatorname{ind}_-(1 - C^*C) + \operatorname{ind}_- \mathfrak{G} &= \operatorname{ind}_-(1 - T^*T) < \infty, \\ \operatorname{ind}_-(1 - CC^*) &= \operatorname{ind}_-(1 - TT^*) < \infty. \end{aligned}$$

Then

$$C = \begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix}, \tag{3.14}$$

where $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction. Conversely, if C has the form (3.14) where $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ is a contraction, then the operator C defined by (3.14) satisfies both of the identities

$$\begin{aligned} \operatorname{ind}_-(1 - C^*C) + \operatorname{ind}_- \mathfrak{G} &= \operatorname{ind}_-(1 - T^*T), \\ \operatorname{ind}_-(1 - CC^*) &= \operatorname{ind}_-(1 - TT^*). \end{aligned}$$

Theorem 3.4 (Generalized Form) *Let $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ have Julia operator*

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T),$$

and let

$$A = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G})$$

be a 2×2 matrix extension of T satisfying at least one of the conditions

$$\begin{aligned} \operatorname{ind}_-(1 - A^*A) + \operatorname{ind}_- \mathfrak{G} &= \operatorname{ind}_-(1 - T^*T) < \infty, \\ \operatorname{ind}_-(1 - AA^*) + \operatorname{ind}_- \mathfrak{F} &= \operatorname{ind}_-(1 - TT^*) < \infty. \end{aligned}$$

Then

$$A = \begin{pmatrix} T & D_T X \\ Y^* \tilde{D}_T^* & -Y^* L_T^* X + \tilde{D}_Y Z \tilde{D}_X^* \end{pmatrix}, \tag{3.15}$$

where

(i) $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ *is a contraction with defect operator* $\tilde{D}_X \in \mathfrak{L}(\tilde{\mathfrak{D}}_X, \mathfrak{F})$,
(ii) $Y \in \mathfrak{L}(\mathfrak{G}, \tilde{\mathfrak{D}}_T)$ *is a contraction with defect operator* $\tilde{D}_Y \in \mathfrak{L}(\tilde{\mathfrak{D}}_Y, \mathfrak{G})$,
(iii) $Z \in \mathfrak{L}(\tilde{\mathfrak{D}}_X, \mathfrak{D}_Y)$ *is a contraction.*

Conversely, given X, Y, Z *satisfying* (i), (ii), (iii), *then the operator defined by* (3.15) *satisfies both of the identities*

$$\begin{aligned} \operatorname{ind}_-(1 - A^*A) + \operatorname{ind}_- \mathfrak{G} &= \operatorname{ind}_-(1 - T^*T), \\ \operatorname{ind}_-(1 - AA^*) + \operatorname{ind}_- \mathfrak{F} &= \operatorname{ind}_-(1 - TT^*). \end{aligned}$$

It is possible to give the complete proofs in just a few pages, and they are included here. It is sufficient to prove the generalized forms of the theorems.

Proof of Theorem 3.2 (Generalized Form). Assume that

$$R = \begin{pmatrix} T & F \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K})$$

and that one of the two conditions holds, say the first:

$$\operatorname{ind}_-(1 - RR^*) + \operatorname{ind}_- \mathfrak{F} = \operatorname{ind}_-(1 - TT^*) < \infty.$$

Choose a defect operator $D_R \in \mathfrak{L}(\mathfrak{D}_R, \mathfrak{K})$ for R^*. Then

$$\begin{pmatrix} R^* \\ D_R^* \end{pmatrix} = \begin{pmatrix} T^* \\ F^* \\ D_R^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H} \oplus \mathfrak{F} \oplus \mathfrak{D}_R)$$

is an isometry. The operator

$$\begin{pmatrix} T^* \\ D_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{K}, \mathfrak{H} \oplus \mathfrak{D}_T)$$

is also an isometry. Thus

$$\left\langle \begin{pmatrix} T^* \\ F^* \\ D_R^* \end{pmatrix} f, \begin{pmatrix} T^* \\ F^* \\ D_R^* \end{pmatrix} g \right\rangle_{\mathfrak{H} \oplus \mathfrak{F} \oplus \mathfrak{D}_R} = \left\langle \begin{pmatrix} T^* \\ D_T^* \end{pmatrix} f, \begin{pmatrix} T^* \\ D_T^* \end{pmatrix} g \right\rangle_{\mathfrak{H} \oplus \mathfrak{D}_T}$$

for all $f, g \in \mathfrak{K}$. Therefore

$$\left\langle \begin{pmatrix} F^* \\ D_R^* \end{pmatrix} f, \begin{pmatrix} F^* \\ D_R^* \end{pmatrix} g \right\rangle_{\mathfrak{F} \oplus \mathfrak{D}_R} = \langle D_T^* f, D_T^* g \rangle_{\mathfrak{D}_T}$$

for all $f, g \in \mathfrak{K}$. Since D_T has zero kernel, ran D_T^* is dense in $\mathfrak{D}_T$. Our hypotheses imply that

$$\operatorname{ind}_- \mathfrak{D}_T = \operatorname{ind}_-(\mathfrak{F} \oplus \mathfrak{D}_R) < \infty,$$

and hence $\mathfrak{F} \oplus \mathfrak{D}_R$ and $\mathfrak{D}_T$ are Pontryagin spaces having the same negative index. Hence by Shmul′yan's theorem (see Theorem 2.10 (Alternative Form)), there is an isometry $V \in \mathfrak{L}(\mathfrak{D}_T, \mathfrak{F} \oplus \mathfrak{D}_R)$ such that

$$\begin{pmatrix} F^* \\ D_R^* \end{pmatrix} = V D_T^*.$$

Define $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ by

$$X^* = \mathrm{Pr}_{\mathfrak{F}}\, V \in \mathfrak{L}(\mathfrak{D}_T, \mathfrak{F}).$$

By Corollary 2.5, V is a bicontraction. The operator $X = V^*|_{\mathfrak{F}}$ is thus a contraction. By construction, $F = D_T X$, and we have obtained the conclusion in case the first of the two conditions holds.

Suppose that the second condition holds, that is,

$$\mathrm{ind}_-\,(1 - R^*R) = \mathrm{ind}_-\,(1 - T^*T) < \infty,$$

or equivalently $\mathrm{ind}_- \tilde{\mathfrak{D}}_R = \mathrm{ind}_- \tilde{\mathfrak{D}}_T < \infty$. Let $\tilde{D}_R \in \mathfrak{L}(\tilde{\mathfrak{D}}_R, \mathfrak{H} \oplus \mathfrak{F})$ be a defect operator for R, and choose a Julia operator

$$\begin{pmatrix} T & D_T \\ \tilde{D}_T^* & -L_T^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{D}_T, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_T).$$

The operator

$$V = \begin{pmatrix} R \\ \tilde{D}_R^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_R)$$

is an isometry and may be written in the form

$$V = \begin{pmatrix} T & F \\ \tilde{D}_1^* & \tilde{D}_2^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_R) \tag{3.16}$$

where $\tilde{D}_R^* = \begin{pmatrix} \tilde{D}_1^* & \tilde{D}_2^* \end{pmatrix}$. The first column in (3.16),

$$V_1 = \begin{pmatrix} T \\ \tilde{D}_1^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_R),$$

is an isometric extension of T, implying that $\mathrm{ind}_-\,(1 - V_1^* V_1) = 0$. It is easy to see that the row extension

$$V_1^* = \begin{pmatrix} T^* & \tilde{D}_1 \end{pmatrix} \in \mathfrak{L}(\mathfrak{K} \oplus \tilde{\mathfrak{D}}_R, \mathfrak{H})$$

of T^* satisfies the first of the two conditions of the theorem (the index formula that is needed reduces to $\mathrm{ind}_- \tilde{\mathfrak{D}}_R = \mathrm{ind}_- \tilde{\mathfrak{D}}_T < \infty$), and the result has already been established in this case. Therefore

$$\tilde{D}_1^* = Y^* \tilde{D}_T^*,$$

where $Y \in \mathfrak{L}(\tilde{\mathfrak{D}}_R, \tilde{\mathfrak{D}}_T)$ is a contraction. Now apply Theorem 3.1 to the isometry

$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \left(\begin{pmatrix} T \\ \tilde{D}_1^* \end{pmatrix} \begin{pmatrix} F \\ \tilde{D}_2^* \end{pmatrix} \right) = \left(\begin{pmatrix} T \\ Y^* \tilde{D}_T^* \end{pmatrix} \begin{pmatrix} F \\ \tilde{D}_2^* \end{pmatrix} \right).$$

By Theorem 2.12, a defect operator for V_1^* is given by

$$D_{V_1} = \begin{pmatrix} D_T & 0 \\ -Y^* L_T^* & \tilde{D}_Y \end{pmatrix} \in \mathfrak{L}(\mathfrak{D}_T \oplus \tilde{\mathfrak{D}}_Y, \mathfrak{K} \oplus \tilde{\mathfrak{D}}_R).$$

Therefore by Theorem 3.1,

$$\begin{pmatrix} F \\ \tilde{D}_2^* \end{pmatrix} = \begin{pmatrix} D_T & 0 \\ -Y^* L_T^* & \tilde{D}_Y \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix},$$

where

$$W = \begin{pmatrix} X \\ X' \end{pmatrix} \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T \oplus \tilde{\mathfrak{D}}_Y)$$

is an isometry. This yields

$$F = D_T X,$$

where $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$. Since $W^*W = 1$,

$$1 - X^*X = X'^*X' \geq 0,$$

because $X' \in \mathfrak{L}(\mathfrak{F}, \tilde{\mathfrak{D}}_Y)$ and $\tilde{\mathfrak{D}}_Y$ is a Hilbert space (since Y is a contraction). Thus X is a contraction, and we have obtained the conclusion in case the second of the two conditions holds.

Conversely, assume that $X \in \mathfrak{L}(\mathfrak{F}, \mathfrak{D}_T)$ is a contraction, and define R by (3.13). Choose Julia operators for T and X as in Theorem 2.11. By that result, R has defect spaces

$$\begin{aligned} \tilde{\mathfrak{D}}_R &= \tilde{\mathfrak{D}}_T \oplus \tilde{\mathfrak{D}}_X, \\ \mathfrak{D}_R &= \mathfrak{D}_X. \end{aligned}$$

Since $\tilde{\mathfrak{D}}_X$ is a Hilbert space, $\operatorname{ind}_-(1 - X^*X) = 0$, and by the index formulas (2.15),

$$\begin{aligned} \operatorname{ind}_-(1 - R^*R) &= \operatorname{ind}_-(1 - T^*T), \\ \operatorname{ind}_-(1 - RR^*) &= \operatorname{ind}_-(1 - XX^*), \\ \operatorname{ind}_- \mathfrak{F} + \operatorname{ind}_-(1 - XX^*) &= \operatorname{ind}_-(1 - TT^*). \end{aligned}$$

The last two identities in the statement of the theorem follow. □

Proof of Theorem 3.3 (Generalized Form). On considering adjoints, this follows from Theorem 3.2. □

Proof of Theorem 3.4 (Generalized Form). It is enough to treat the case

$$\operatorname{ind}_-(1 - A^*A) + \operatorname{ind}_- \mathfrak{G} = \operatorname{ind}_-(1 - T^*T) < \infty, \tag{3.17}$$

since if the other condition holds we can apply this result to the adjoint of A. Form an isometric column extension

$$\begin{pmatrix} A \\ \tilde{D}_A \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{K} \oplus \mathfrak{G} \oplus \tilde{\mathfrak{D}}_A)$$

using some choice of a defect operator $\tilde{D}_A \in \mathfrak{L}(\tilde{\mathfrak{D}}_A, \mathfrak{H} \oplus \mathfrak{F})$ for A. Write

$$\begin{pmatrix} A \\ \tilde{D}_A \end{pmatrix} = \begin{pmatrix} C & C' \\ \tilde{D}_1^* & \tilde{D}_2^* \end{pmatrix}, \tag{3.18}$$

where

$$C = \begin{pmatrix} T \\ G \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G})$$

is the first column of A and C' is the second column of A. The first column in (3.18), that is,

$$K = \begin{pmatrix} C \\ \tilde{D}_1^* \end{pmatrix} = \begin{pmatrix} T \\ \begin{pmatrix} G \\ \tilde{D}_1^* \end{pmatrix} \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G} \oplus \tilde{\mathfrak{D}}_A),$$

is an isometry because it is the restriction of an isometry. Viewing K as a column extension of T, we may apply Theorem 3.3 (Generalized Form) because (3.17) insures that the first of the two conditions required in that result is met. In fact, since K is an isometry, $\operatorname{ind}_-(1-K^*K)=0$, and so

$$\begin{aligned}\operatorname{ind}_-(1-K^*K) &+ \operatorname{ind}_-(\mathfrak{G}\oplus\tilde{\mathfrak{D}}_A)\\ &= 0+\big(\operatorname{ind}_-\mathfrak{G}+\operatorname{ind}_-(1-A^*A)\big)=\operatorname{ind}_-(1-T^*T)<\infty.\end{aligned}$$

Hence

$$\begin{pmatrix}G\\ \tilde{D}_1^*\end{pmatrix}=V^*\tilde{D}_T^*,$$

where $V\in\mathfrak{L}(\mathfrak{G}\oplus\tilde{\mathfrak{D}}_A,\tilde{\mathfrak{D}}_T)$ is a contraction. This implies that

$$C=\begin{pmatrix}T\\ G\end{pmatrix}=\begin{pmatrix}T\\ Y^*\tilde{D}_T^*\end{pmatrix},$$

where $Y^*=\operatorname{Pr}_{\mathfrak{G}}V^*$. Here $Y=V|_{\mathfrak{G}}\in\mathfrak{L}(\mathfrak{G},\tilde{\mathfrak{D}}_T)$ is the restriction of a contraction, and so Y is a contraction. Choose a defect operator $\tilde{D}_Y\in\mathfrak{L}(\tilde{\mathfrak{D}}_Y,\mathfrak{G})$ for Y.

Additional information is obtained by viewing A as a row extension of C:

$$A=\begin{pmatrix}T & F\\ G & H\end{pmatrix}=\left(\begin{pmatrix}T\\ G\end{pmatrix}\ \begin{pmatrix}F\\ H\end{pmatrix}\right)=\left(C\ \begin{pmatrix}F\\ H\end{pmatrix}\right).$$

By (3.17) and Theorem 3.3 (Generalized Form),

$$\operatorname{ind}_-(1-A^*A)=\operatorname{ind}_-(1-T^*T)-\operatorname{ind}_-\mathfrak{G}=\operatorname{ind}_-(1-C^*C)<\infty.$$

The hypotheses of Theorem 3.2 (Generalized Form) are met, and therefore

$$A=\begin{pmatrix}C & D_CX_C\end{pmatrix},$$

where $X_C\in\mathfrak{L}(\mathfrak{F},\mathfrak{D}_C)$ is a contraction. Here we choose $D_C\in\mathfrak{L}(\mathfrak{D}_C,\mathfrak{K}\oplus\mathfrak{G})$ as in Theorem 2.12, that is, $\mathfrak{D}_C=\mathfrak{D}_T\oplus\tilde{\mathfrak{D}}_Y$ and

$$D_C=\begin{pmatrix}D_T & 0\\ -Y^*L_T^* & \tilde{D}_Y\end{pmatrix}\in\mathfrak{L}(\mathfrak{D}_T\oplus\tilde{\mathfrak{D}}_Y,\mathfrak{K}\oplus\mathfrak{G}).$$

Then X_C has the form

$$X_C=\begin{pmatrix}X\\ X'\end{pmatrix},$$

where $X=\operatorname{Pr}_{\mathfrak{D}_T}X_C\in\mathfrak{L}(\mathfrak{F},\mathfrak{D}_T)$ is a contraction because $\tilde{\mathfrak{D}}_Y$ is a Hilbert space. Let $\tilde{D}_X\in\mathfrak{L}(\tilde{\mathfrak{D}}_X,\mathfrak{F})$ be a defect operator for X. Applying Theorem 3.3 (in either its simple form or generalized form) to X_C, we can write, finally,

$$X'=Z\tilde{D}_X^*,$$

where $Z\in\mathfrak{L}(\tilde{\mathfrak{D}}_X,\tilde{\mathfrak{D}}_Y)$ is a Hilbert space contraction. With these choices of X,Y,Z,

$$D_CX_C=\begin{pmatrix}D_T & 0\\ -Y^*L_T^* & \tilde{D}_Y\end{pmatrix}\begin{pmatrix}X\\ Z\tilde{D}_X^*\end{pmatrix}=\begin{pmatrix}D_TX\\ -Y^*L_T^*X+\tilde{D}_YZD_X^*\end{pmatrix},$$

and therefore A has the required form.

The other direction follows from Theorem 2.13 and the index formulas (2.15), (2.17), and (2.19). □

Example:

Let us return to the example at the beginning of the lecture:

$$C = \begin{pmatrix} T \\ N \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K} \oplus \mathfrak{G}),$$

where $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ is an isometry with $\mathfrak{H}$ and $\mathfrak{K}$ Hilbert spaces and N is a nonzero operator satisfying $N^*N = 0$. We showed that C is a contractive column extension of T but not of the form given in Theorem 3.3 (Generalized Form). It is not hard to see why this result is not applicable. The hypotheses state that at least one of the conditions

$$\begin{aligned} \operatorname{ind}_-(1 - C^*C) + \operatorname{ind}_- \mathfrak{G} &= \operatorname{ind}_-(1 - T^*T) < \infty, \\ \operatorname{ind}_-(1 - CC^*) &= \operatorname{ind}_-(1 - TT^*) < \infty, \end{aligned}$$

holds. However, in this example,

$$\begin{aligned} \operatorname{ind}_-(1 - C^*C) + \operatorname{ind}_- \mathfrak{G} &> \operatorname{ind}_-(1 - T^*T), \\ \operatorname{ind}_-(1 - CC^*) &> \operatorname{ind}_-(1 - TT^*). \end{aligned}$$

The first relation holds because C and T are contractions and $\mathfrak{G}$ is not a Hilbert space (a nonzero operator N with the required properties cannot exist if $\mathfrak{G}$ is a Hilbert space). To verify the second relation, notice that $\operatorname{ind}_-(1 - TT^*) = 0$ because T^* is a contraction. It suffices to show that C^* is not a contraction. One way to do this to argue that $\ker C^*$ is not a Hilbert space (see Theorem 2.7). In fact, $\ker C^* = \ker \begin{pmatrix} T^* & N^* \end{pmatrix}$ contains $\{0\} \oplus \operatorname{ran} N$, which is a neutral subspace, and so $\ker C^*$ is not a Hilbert space.

There is a more general theory which covers the preceding example and many more situations (see Dritschel [1993a]). Neutral operators, that is, operators N such that $N^*N = 0$, appear systematically in these more general results. It can be shown that Theorems 3.2–3.4 (Generalized Forms) give exactly the cases in which the neutral operators are not present.

As indicated at the outset, we have treated just one kind of extension problem. For related material on isometric dilations, commutant lifting, as well as other methods of approach, see, for example, Arov and Grossman [1983], [1992], Constantinescu and Gheondea [1989], [1992], [1993], Dijksma, Marcantognini, and de Snoo [1994], Dijksma, Dritschel, Marcantognini, and de Snoo [1993], Dritschel and Rovnyak [1990], and Marcantognini [1992]. In the Hilbert space case, such extension and completion problems are closely associated with interpolation theory. These problems are more complex in the indefinite case. They have been successfully treated by invariant subspace methods in a series of papers by J. A. Ball and J. W. Helton, including Ball and Helton [1983].

Lecture 4: The Schur Algorithm

Key ideas:

- Given Kreĭn spaces $\mathfrak{A}$ and $\mathfrak{B}$, a Schur class $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ is defined whose elements are holomorphic functions with values in $\mathfrak{L}(\mathfrak{A},\mathfrak{B})$ such that the Schur algorithm and Carathéodory-Fejér problem have generalizations.
- The approach uses a formal calculus of Julia operators. The calculus is equivalent to the choice sequence method of describing extensions of operators.

In complex function theory, the **Schur class S** is defined as the set of complex-valued analytic functions $s(z) = \sum_{n=0}^{\infty} s_n z^n$ which are defined and bounded by one in the unit disk $D = \{z : |z| < 1\}$ in the complex plane.

Classical Schur algorithm: Given $s(z)$ in $\mathbf{S}$, define $s_0(z), s_1(z), s_2(z), \dots$ in $\mathbf{S}$ by writing $s_0(z) = s(z)$ and

$$\begin{aligned} s_1(z) &= \frac{1}{z}\,\frac{s_0(z)-s_0(0)}{1-\bar{s}_0(0)s_0(z)}, \\ s_2(z) &= \frac{1}{z}\,\frac{s_1(z)-s_1(0)}{1-\bar{s}_1(0)s_1(z)}, \\ &\cdots \end{aligned}$$

The sequence is well defined. For by assumption, $s_0(z)$ belongs to $\mathbf{S}$. If $s_n(z)$ belongs to $\mathbf{S}$ for some $n \geq 0$, then $s_{n+1}(z)$ is in $\mathbf{S}$ by Schwarz's lemma. Here $s_n(0)$ has modulus at most one because $s_n(z)$ is in $\mathbf{S}$; if $|s_n(0)| = 1$, we take $s_k(z) \equiv 0$ for all $k > n$. The numbers $\alpha_n = s_n(0)$, $n \geq 0$, are called the **Schur parameters** of $s(z)$. They form a sequence $\{\alpha_n\}_{n=0}^{\infty}$ in $\bar{D}$ such that if $|\alpha_r| = 1$ for some r, then $\alpha_k = 0$ for all $k > r$. Conversely, every such sequence occurs as the Schur parameters of a unique $s(z)$ in $\mathbf{S}$. The Schur algorithm thus labels $\mathbf{S}$ by the set of sequences $\{\alpha_n\}_{n=0}^{\infty}$ of the above type.

The Schur algorithm was introduced by Schur [1917] to study the Carathéodory-Fejér problem, and this is still one of its most striking applications. Assume that numbers $s_0, \dots, s_n$ are given. The Carathéodory-Fejér problem asks for conditions that there exists a function $s(z)$ in $\mathbf{S}$ such that the first $n+1$ coefficients in the Taylor series for $s(z)$ are the given numbers. Schur's theorem is that such a function exists if and only if the lower triangular matrix

$$\begin{pmatrix} s_0 & & & \\ s_1 & s_0 & & 0 \\ & \ddots & \ddots & \\ s_n & & s_1 & s_0 \end{pmatrix}$$

acts as a contraction on $\mathbf{C}^{n+1}$. The Schur algorithm is also closely related to Szegő's theory of orthogonal polynomials on the unit circle and interpolation theory.

Hilbert space generalizations of these classical theories have been known for a long time, and they play an important role in operator theory and its applications. If $\mathfrak{A}$ and $\mathfrak{B}$ are Hilbert spaces, the Schur class $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ is defined to be the set of analytic functions $S(z)$ on D whose values are contractions in $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$. The scalar theory can be written in such a way that large portions carry over essentially intact to the Hilbert space case. This definition of the Schur class is not suitable for Kreĭn spaces, however. For example, it is not natural to require that functions be analytic in the full unit disk in the indefinite case.

Problem. *How should the theory of the Schur class be formulated for functions whose values are Kreĭn space operators?*

The generalized Schur class is defined in Definition 4.6, after suitable preparation. Our approach is essentially the choice sequence method of Bakonyi and Constantinescu [1992], with the main constructions encoded in a formal calculus of Julia operators. In one key place, we use the extension theory from Lecture 3. Full details may be found in Christner and Rovnyak [to appear].

The Schur algorithm was studied in an indefinite setting by Ivanchenko [1979]. An approach by reproducing kernel spaces is given in Alpay and Dym [1986], [1992]. Our interest in indefinite generalizations of the Schur algorithm was sparked, in part, from an appearance of a variant of the algorithm in a coefficient problem for univalent functions (Christner, Li, and Rovnyak [1994]); indefinite inner products are central here also, but there are differences with the present situation, such as the use of composition operators. For the Hilbert space generalization of the Schur algorithm, see the books of Bakonyi and Constantinescu [1992] and Dubovoj, Fritzsche, and Kirstein [1992]. Additional references may be found in the historical accounts in Herglotz, Schur, et al. [1991] and the book review Rovnyak [1994].

To begin, assume that we are given Kreĭn spaces $\mathfrak{A}_0, \mathfrak{A}_1, \dots$ and $\mathfrak{B}_0, \mathfrak{B}_1, \dots$ and a sequence of Julia operators

$$\left.\begin{aligned}
&\begin{pmatrix}\alpha_0 & \beta_0\\ \gamma_0 & \delta_0\end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_0 \oplus \mathfrak{B}_1, \mathfrak{B}_0 \oplus \mathfrak{A}_1),\\
&\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_1 \oplus \mathfrak{B}_2, \mathfrak{B}_1 \oplus \mathfrak{A}_2),\\
&\qquad\cdots\\
&\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_j \oplus \mathfrak{B}_{j+1}, \mathfrak{B}_j \oplus \mathfrak{A}_{j+1}),\\
&\qquad\cdots
\end{aligned}\right\} \tag{4.1}$$

Put

$$\mathfrak{A} = \mathfrak{A}_0 \qquad \text{and} \qquad \mathfrak{B} = \mathfrak{B}_0.$$

No other assumptions are made at this point: the key hypothesis is the particular way in which the domains of the Julia operators are linked.

The sequence (4.1) is used in Theorems 4.1–4.4. For simplicity, we assume that the sequence is infinite. However, the main constructions are recursive and depend only on segments, so essentially everything that we say applies to sequences of the form (4.1) which terminate. Define a Julia operator

$$\begin{pmatrix} R_k & D_{R_k} \\ \tilde{D}^*_{R_k} & -L^*_{R_k} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \gamma_1 & \delta_1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \alpha_2 & \beta_2 & & 0 \\ & \gamma_2 & \delta_2 & & \\ & & & 1 & \\ & 0 & & & \ddots \\ & & & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & \alpha_k & \beta_k \\ & & & & \gamma_k & \delta_k \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1\alpha_2 & \beta_1\beta_2\alpha_3 & \ldots & \beta_1\beta_2\cdots\beta_{k-1}\alpha_k \end{pmatrix} & \begin{pmatrix} \beta_1\beta_2\cdots\beta_k \end{pmatrix} \\ \begin{pmatrix} \gamma_1 & \delta_1\alpha_2 & \delta_1\beta_2\alpha_3 & \ldots & \delta_1\beta_2\cdots\beta_{k-1}\alpha_k \\ 0 & \gamma_2 & \delta_2\alpha_3 & \ldots & \delta_2\beta_3\cdots\beta_{k-1}\alpha_k \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_k \end{pmatrix} & \begin{pmatrix} \delta_1\beta_2\cdots\beta_k \\ \delta_2\beta_3\cdots\beta_k \\ \vdots \\ \delta_k \end{pmatrix} \end{pmatrix}$$

with domain $(\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_k) \oplus \mathfrak{D}_{R_k} = (\mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_k) \oplus \mathfrak{B}_{k+1}$ and range

$$\mathfrak{B}_1 \oplus \tilde{\mathfrak{D}}_{R_k} = \mathfrak{B}_1 \oplus (\mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_{k+1}) \,.$$

Similarly,

$$\begin{pmatrix} C_k & D_{C_k} \\ \tilde{D}^*_{C_k} & -L^*_{C_k} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2\gamma_1 \\ \alpha_3\gamma_2\gamma_1 \\ \vdots \\ \alpha_k\gamma_{k-1}\cdots\gamma_1 \end{pmatrix} & \begin{pmatrix} \beta_1 & 0 & \ldots & 0 \\ \alpha_2\delta_1 & \beta_2 & \ldots & 0 \\ \alpha_3\gamma_2\delta_1 & \alpha_3\delta_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_k\gamma_{k-1}\cdots\gamma_2\delta_1 & \alpha_k\gamma_{k-1}\cdots\gamma_3\delta_2 & \ldots & \beta_k \end{pmatrix} \\ \begin{pmatrix} \gamma_k\gamma_{k-1}\cdots\gamma_1 \end{pmatrix} & \begin{pmatrix} \gamma_k\gamma_{k-1}\cdots\gamma_2\delta_1 & \gamma_k\gamma_{k-1}\cdots\gamma_3\delta_2 & \ldots & \delta_k \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & \alpha_k & \beta_k \\ & & & & \gamma_k & \delta_k \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & \\ & \alpha_2 & \beta_2 & & 0 \\ & \gamma_2 & \delta_2 & & \\ & & & 1 & \\ & 0 & & & \ddots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \gamma_1 & \delta_1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

is a Julia operator with domain $\mathfrak{A}_1 \oplus \mathfrak{D}_{C_k} = \mathfrak{A}_1 \oplus (\mathfrak{B}_2 \oplus \cdots \oplus \mathfrak{B}_{k+1})$ and range

$$(\mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_k) \oplus \tilde{\mathfrak{D}}_{C_k} = (\mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_k) \oplus \mathfrak{A}_{k+1}.$$

Theorem 4.1 *In the situation just described, construct Julia operators*

$$\begin{pmatrix} T_k & D_{T_k} \\ \tilde{D}^*_{T_k} & -L^*_{T_k} \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}^{k+1} \oplus \mathfrak{D}_{T_k}, \mathfrak{B}^{k+1} \oplus \tilde{\mathfrak{D}}_{T_k}),$$

$$\tilde{\mathfrak{D}}_{T_k} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_{k+1},$$
$$\mathfrak{D}_{T_k} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_{k+1},$$

for $0 \leq k \leq r$ *by setting*

$$\begin{pmatrix} T_0 & D_{T_0} \\ \tilde{D}^*_{T_0} & -L^*_{T_0} \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$$

and

$$\begin{pmatrix} T_k & D_{T_k} \\ \tilde{D}^*_{T_k} & -L^*_{T_k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_0 & \beta_0 & 0 \\ 0 & \gamma_0 & \delta_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R_k & D_{R_k} \\ 0 & 0 & \tilde{D}^*_{R_k} & -L^*_{R_k} \end{pmatrix} \begin{pmatrix} T_{k-1} & 0 & D_{T_{k-1}} & 0 \\ 0 & 1 & 0 & 0 \\ \tilde{D}^*_{T_{k-1}} & 0 & -L^*_{T_{k-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $1 \leq k \leq r$*. Then for* $1 \leq k \leq r$*,*

$$\begin{pmatrix} T_k & D_{T_k} \\ \tilde{D}^*_{T_k} & -L^*_{T_k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & T_{k-1} & D_{T_{k-1}} & 0 \\ 0 & \tilde{D}^*_{T_{k-1}} & -L^*_{T_{k-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C_k & D_{C_k} \\ 0 & 0 & \tilde{D}^*_{C_k} & -L^*_{C_k} \end{pmatrix} \begin{pmatrix} \alpha_0 & 0 & \beta_0 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_0 & 0 & \delta_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The theorem is proved by an inductive argument (Christner and Rovnyak [to appear]). It would be interesting to have a more conceptual proof. For example, is there a systems explanation?

On expanding the matrix products, we obtain

$$\begin{pmatrix} T_k & D_{T_k} \\ \tilde{D}^*_{T_k} & -L^*_{T_k} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} T_{k-1} & 0 \\ \beta_0 R_k \tilde{D}^*_{T_{k-1}} & \alpha_0 \end{pmatrix} & \begin{pmatrix} D_{T_{k-1}} & 0 \\ -\beta_0 R_k L^*_{T_{k-1}} & \beta_0 D_{R_k} \end{pmatrix} \\ \begin{pmatrix} \delta_0 R_k \tilde{D}^*_{T_{k-1}} & \gamma_0 \\ \tilde{D}^*_{R_k} \tilde{D}^*_{T_{k-1}} & 0 \end{pmatrix} & \begin{pmatrix} -\delta_0 R_k L^*_{T_{k-1}} & \delta_0 D_{R_k} \\ -\tilde{D}^*_{R_k} L^*_{T_{k-1}} & L_{R_k} \end{pmatrix} \end{pmatrix}$$

and

$$\begin{pmatrix} T_k & D_{T_k} \\ \tilde{D}^*_{T_k} & -L^*_{T_k} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \alpha_0 & 0 \\ D_{T_{k-1}} C_k \gamma_0 & T_{k-1} \end{pmatrix} & \begin{pmatrix} \beta_0 & 0 \\ D_{T_{k-1}} C_k \delta_0 & D_{T_{k-1}} D_{C_k} \end{pmatrix} \\ \begin{pmatrix} -L^*_{T_{k-1}} C_k \gamma_0 & \tilde{D}^*_{T_{k-1}} \\ \tilde{D}^*_{C_k} \gamma_0 & 0 \end{pmatrix} & \begin{pmatrix} -L^*_{T_{k-1}} C_k \delta_0 & -L^*_{T_{k-1}} D_{C_k} \\ \tilde{D}^*_{C_k} \delta_0 & -L^*_{C_k} \end{pmatrix} \end{pmatrix}.$$

These relations concisely summarize many useful identities. For example, they give recursive formulas for the operators T_k, D_{T_k}, and $\tilde{D}_{T_k}$. They give two different sets of formulas for these operators, and the fact that the sets are identical gives other relations.

It is not apparent in Theorem 4.1 why the operators $T_0, T_1, T_2, \ldots$ should be of interest. The reason is that they have a Toeplitz structure.

Theorem 4.2 *In Theorem* 4.1*, there exist operators* $S_0, \ldots, S_r$ *in* $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ *such that*

$$T_k = \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_k & & S_1 & S_0 \end{pmatrix}$$

for each $k = 0, \ldots, r$.

This is also proved by induction. The recursive formulas for T_k, D_{T_k}, and $\tilde{D}_{T_k}$ described above lead to similar formulas for the operators $S_0, S_1, \ldots$. Closed form expressions for $S_0, S_1, \ldots$ can be given. Define formal matrices

$$\begin{pmatrix} \widehat{\alpha}_0 & \widehat{\beta}_0 \\ \widehat{\gamma}_0 & \widehat{\delta}_0 \end{pmatrix} = \begin{pmatrix} (\alpha_0) & (\beta_0 \;\; 0 \;\; 0 \;\; \ldots\;) \\ \begin{pmatrix} \gamma_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} & \begin{pmatrix} \delta_0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ & \ldots & & \ddots \end{pmatrix} \end{pmatrix}$$

and

$$X = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 & \\ \gamma_1 & \delta_1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & \alpha_2 & \beta_2 & & 0 & \\ & \gamma_2 & \delta_2 & & & \\ & & & 1 & & \\ & & 0 & & 1 & \\ & & & & & \ddots \end{pmatrix} \cdots .$$

The definition of X is meaningful because each term in the expansion of the product involves only a finite number of algebraic expressions.

Theorem 4.3 *Let* $S(z) = S_0 + S_1 z + S_2 z^2 + \cdots$*, where* $S_0, S_1, S_2, \ldots$ *are the operators in Theorem* 4.2*. Then*

$$S(z) = \widehat{\alpha}_0 + z\widehat{\beta}_0 X \left(1 - z\widehat{\delta}_0 X\right)^{-1} \widehat{\gamma}_0$$

in the sense of formal power series.

The theorem asserts that

$$S_0 = \widehat{\alpha}_0$$

and

$$S_n = \widehat{\beta}_0 X \left(\widehat{\delta}_0 X\right)^{n-1} \widehat{\gamma}_0, \qquad n \geq 1.$$

The proof is mainly a matter of algebra proceeding from the recursive construction of the coefficients.

The structure of the coefficients $S_0, S_1, \dots$ in $S(z)$ described in Theorem 4.3 leads to a linear fractional representation.

Theorem 4.4 *Define $S(z)$ from the Julia operators*

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}, \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}, \dots$$

in the manner described in Theorem 4.3. In a similar way, define $S'(z)$ from the shifted sequence

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}, \dots .$$

Then $S(z)$ and $S'(z)$ are related by

$$S(z) = \alpha_0 + z\beta_0 S'(z) \left(1 - z\delta_0 S'(z)\right)^{-1} \gamma_0.$$

Theorems 4.1–4.4 are much more general than we need for the application to the Schur algorithm. In (4.1) we assume that $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ and $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots$ are Kreĭn spaces. An interesting special case arises when $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ and $\mathfrak{B}_1, \mathfrak{B}_2, \dots$ are Hilbert spaces. In this situation, $\mathfrak{A} = \mathfrak{A}_0$ and $\mathfrak{B} = \mathfrak{B}_0$ are Kreĭn spaces having the same negative index. In fact, by the unitarity of the first Julia operator in (4.1),

$$\operatorname{ind}_- \mathfrak{A}_0 = \operatorname{ind}_- (\mathfrak{A}_0 \oplus \mathfrak{B}_1) = \operatorname{ind}_- (\mathfrak{A}_0 \oplus \mathfrak{B}_1) = \operatorname{ind}_- \mathfrak{B}_0$$

because $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are Hilbert spaces

Theorem 4.5 *Let $\mathfrak{A} = \mathfrak{A}_0$ and $\mathfrak{B} = \mathfrak{B}_0$ be Kreĭn spaces having the same negative index, and let $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ and $\mathfrak{B}_1, \mathfrak{B}_2, \dots$ be Hilbert spaces. Let (4.1) be a sequence of Julia operators using these spaces, and define*

$$S(z) = S_0 + S_1 z + S_2 z^2 + \cdots$$

as in Theorems 4.1–4.4. Then for each $k = 0, 1, 2, \dots,$

$$T_k = \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_k & & S_1 & S_0 \end{pmatrix}$$

is a bicontraction on $\mathfrak{A}^{k+1}$ to $\mathfrak{B}^{k+1}$.

In the statement of the theorem, $\mathfrak{A}^{k+1}$ and $\mathfrak{B}^{k+1}$ denote $(k+1)$-fold Cartesian products of $\mathfrak{A}$ and $\mathfrak{B}$. The proof of the theorem is very easy.

Proof. Let k be any nonnegative integer. By Theorem 4.1, we may choose a Julia operator for T_k with defect spaces given by

$$\begin{aligned} \tilde{\mathfrak{D}}_{T_k} &= \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_{k+1}, \\ \mathfrak{D}_{T_k} &= \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_{k+1}. \end{aligned}$$

Our hypotheses imply that these are Hilbert spaces, and so T_k is a bicontraction by Theorem 2.3. □

In view of these results, we are now able to propose a definition of the Schur class $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ when $\mathfrak{A}$ and $\mathfrak{B}$ are Kreĭn spaces:

Definition 4.6 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kreĭn spaces having the same negative index. By the* ***Schur class*** $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ *we mean the set of all formal power series*

$$S(z) = \sum_{n=0}^{\infty} S_n z^n$$

with coefficients in $\mathfrak{L}(\mathfrak{A},\mathfrak{B})$ such that for each $k = 0, 1, 2, \dots$, the matrix

$$T_k = \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_k & & S_1 & S_0 \end{pmatrix}$$

is a bicontraction on $\mathfrak{A}^{k+1}$ to $\mathfrak{B}^{k+1}$.

The condition that $\mathfrak{A}$ and $\mathfrak{B}$ have the same negative index is necessary to insure that the Schur class $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ is nontrivial. For if $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ contains a single element $S(z) = \sum_{n=0}^{\infty} S_n z^n$, then

$$\operatorname{ind}_- \mathfrak{A} = \operatorname{ind}_- \mathfrak{B}$$

by Theorem 2.8, because $T_0 = S_0$ is a bicontraction on $\mathfrak{A}$ to $\mathfrak{B}$. When the condition is satisfied, Theorem 4.5 provides a rich class of examples of elements $S(z)$ of $\mathbf{S}(\mathfrak{A},\mathfrak{B})$.

A number of questions arise.

Question (1): Does every function $S(z)$ in $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ have the form described in Theorem 4.5? The answer is affirmative:

Every $S(z)$ in $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ is obtained as in Theorem 4.5 for some sequence of Julia operators

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_n \oplus \mathfrak{B}_{n+1}, \mathfrak{B}_n \oplus \mathfrak{A}_{n+1}), \qquad n \geq 0, \tag{4.2}$$

such that $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ and $\mathfrak{B}_1, \mathfrak{B}_2, \dots$ are Hilbert spaces.

The proof uses an extension theorem of the type discussed in Lecture 3: Let $A_1 \in \mathfrak{L}(\mathfrak{H}_1)$ and $A_2 \in \mathfrak{L}(\mathfrak{H}_2)$ be bicontractions, and choose defect operators $\tilde{D}_{A_1} \in \mathfrak{L}(\tilde{\mathfrak{D}}_{A_1}, \mathfrak{H}_1)$ and $D_{A_2} \in \mathfrak{L}(\mathfrak{D}_{A_2}, \mathfrak{H}_2)$ for A_1^* and A_2. An operator

$$\begin{pmatrix} A_1 & 0 \\ Q & A_2 \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}_1 \oplus \mathfrak{H}_2, \mathfrak{K}_1 \oplus \mathfrak{K}_2)$$

is a bicontraction if and only if

$$Q = D_{A_2} Y \tilde{D}_{A_1}^*,$$

where $Y \in \mathfrak{L}(\tilde{\mathfrak{D}}_{A_1}, \mathfrak{D}_{A_2})$ is a bicontraction. This result is an easy consequence of theorems in Lecture 3.

Now suppose that $S(z) = S_0 + S_1 z + S_2 z^2 + \cdots$. We choose $\alpha_0 = S_0$ and take the first term of the sequence (4.2) to be any Julia operator

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_0 \oplus \mathfrak{B}_1, \mathfrak{B}_0 \oplus \mathfrak{A}_1)$$

for α_0. Since α_0 is bicontraction by assumption, $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are Hilbert spaces. The next Julia operator is obtained by applying the extension theorem to the triangular matrix

$$\begin{pmatrix} A_1 & 0 \\ Q & A_2 \end{pmatrix} = \begin{pmatrix} S_0 & 0 \\ S_1 & S_0 \end{pmatrix}.$$

The extension theorem gives a factorization

$$S_1 = \beta_0 \alpha_1 \gamma_0,$$

where $\alpha_1 \in \mathfrak{L}(\mathfrak{A}_1, \mathfrak{B}_1)$ is a bicontraction (in fact, a Hilbert space contraction). For the second term of the sequence (4.2), we choose any Julia operator

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_1 \oplus \mathfrak{B}_2, \mathfrak{B}_1 \oplus \mathfrak{A}_2)$$

for α_1. The new spaces $\mathfrak{A}_2$ and $\mathfrak{B}_2$ are Hilbert spaces because α_1 is a Hilbert space contraction.

The construction proceeds recursively. In the general step, we apply the extension theorem to the triangular matrix

$$\begin{pmatrix} A_1 & 0 \\ Q & A_2 \end{pmatrix} = \begin{pmatrix} S_0 & 0 \\ \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_{k+1} \end{pmatrix} & \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_k & & S_1 & S_0 \end{pmatrix} \end{pmatrix}.$$

A certain amount of algebra is needed to see that the pattern persists (see Christner and Rovnyak [to appear], Th. 4.3).

Question (2): Can two essentially different sequences of Julia operators lead to the same element $S(z)$ of $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ in Theorem 4.5? The answer here is negative. An element $S(z)$ of $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ determines the sequence of Julia operators up to isomorphisms between the defect spaces. See Christner and Rovnyak [to appear], Th. 4.4.

Question (3): Does the definition of $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ agree with the previous one when $\mathfrak{A}$ and $\mathfrak{B}$ are Hilbert spaces? Here the answer is affirmative and a consequence of known Hilbert space theorems as well as results given below.

Generalized Schur algorithm:

One can ask more broadly if the Schur class $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ has properties analogous to the scalar and Hilbert space cases. The Schur algorithm is one test. We shall confirm that this test is passed.

First note that Theorems 4.1–4.4 apply in particular to the Schur class $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$. The information can be put into this form.

Theorem 4.7 (Christner [1993]) *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be Kreĭn spaces having the same negative index. Let* $S_0 \in \mathfrak{L}(\mathfrak{A}, \mathfrak{B})$ *be a bicontraction with Julia operator*

$$\begin{pmatrix} S_0 & D_0 \\ \tilde{D}_0^* & -L_0^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{A} \oplus \mathfrak{D}_0, \mathfrak{B} \oplus \tilde{\mathfrak{D}}_0).$$

(1) *If* $S'(z)$ *belongs to* $\mathbf{S}(\tilde{\mathfrak{D}}_0, \mathfrak{D}_0)$, *then*

$$S(z) = S_0 + zD_0 S'(z) \left(1 + zL_0^* S'(z)\right)^{-1} \tilde{D}_0^*$$

defines an element of $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ *such that* $S(0) = S_0$. (2) *Every* $S(z)$ *in* $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ *with* $S(0) = S_0$ *has this form, and the choice of* $S'(z)$ *in* $\mathbf{S}(\tilde{\mathfrak{D}}_0, \mathfrak{D}_0)$ *is unique when the Julia operator for* S_0 *is fixed.*

One consequence is that the elements $S(z)$ of the Schur class $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ represent analytic functions in a neighborhood of the origin. For since $\tilde{\mathfrak{D}}_0$ and $\mathfrak{D}_0$ are Hilbert spaces, $\mathbf{S}(\tilde{\mathfrak{D}}_0, \mathfrak{D}_0)$ is the set of analytic functions on the unit disk whose values are contraction operators in $\mathfrak{L}(\tilde{\mathfrak{D}}_0, \mathfrak{D}_0)$. Analyticity in a neighborhood of the origin therefore follows from the linear fractional representation in part (1) of the theorem.

The Schur algorithm in the Kreĭn space setting exactly follows the Hilbert space case, which is given in Bakonyi and Constantinescu [1992] and Dubovoj, Fritzsche, and Kirstein [1992]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Kreĭn spaces having the same negative index, and assume that $S(z)$ belongs to $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$. Set $\mathfrak{A}_0 = \mathfrak{A}$ and $\mathfrak{B}_0 = \mathfrak{B}$. Define $S_0(z) \in \mathbf{S}(\mathfrak{A}_0, \mathfrak{B}_0)$ and $\alpha_0 \in \mathfrak{L}(\mathfrak{A}_0, \mathfrak{B}_0)$ by

$$S_0(z) = S(z) \qquad \text{and} \qquad \alpha_0 = S_0(0),$$

and choose a Julia operator

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_0 \oplus \mathfrak{B}_1, \mathfrak{B}_0 \oplus \mathfrak{A}_1).$$

The defect spaces $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are Hilbert spaces because α_0 is a bicontraction. By Theorem 4.7, there is a unique solution $S_1(z) \in \mathbf{S}(\mathfrak{A}_1, \mathfrak{B}_1)$ of the functional equation

$$S_0(z) = \alpha_0 + z\beta_0 S_1(z) \left(1 - z\delta_0 S_1(z)\right)^{-1} \gamma_0.$$

Set $\alpha_1 = S_1(0)$, and then choose a Julia operator for α_1 with defect Hilbert spaces $\mathfrak{A}_2$ and $\mathfrak{B}_2$ and repeat the construction. For the inductive step, suppose that for each $k = 0, \dots, n$, we have chosen $S_k(z) \in \mathbf{S}(\mathfrak{A}_k, \mathfrak{B}_k)$ and a Julia operator

$$\begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_k \oplus \mathfrak{B}_{k+1}, \mathfrak{B}_k \oplus \mathfrak{A}_{k+1})$$

such that $\alpha_k = S_k(0)$ and $\mathfrak{A}_{k+1}$ and $\mathfrak{B}_{k+1}$ are Hilbert spaces. Use Theorem 4.7 to choose $S_{k+1}(z) \in \mathbf{S}(\mathfrak{A}_{k+1}, \mathfrak{B}_{k+1})$ such that

$$S_k(z) = \alpha_k + z\beta_k S_{k+1}(z) \left(1 - z\delta_k S_{k+1}(z)\right)^{-1} \gamma_k.$$

Set $\alpha_{k+1} = S_{k+1}(0)$ and choose a Julia operator for α_{k+1} with new defect Hilbert spaces $\mathfrak{A}_{k+2}$ and $\mathfrak{B}_{k+2}$ to complete the inductive construction. The construction produces a sequence

$$S_n(z) \in \mathbf{S}(\mathfrak{A}_n, \mathfrak{B}_n), \qquad n \geq 0.$$

The operators

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_n \oplus \mathfrak{B}_{n+1}, \mathfrak{B}_n \oplus \mathfrak{A}_{n+1}), \qquad n = 0, 1, 2, \dots ,$$

which arise in the construction are called the **Schur parameters** of $S(z)$. For a parallel with the classical case, we also call

$$\alpha_0, \alpha_1, \alpha_2, \dots$$

the Schur parameters of $S(z)$ with a choice of Julia operators understood.

The Schur parameters of an element $S(z)$ of $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ are defined up to replacement of domain spaces by isomorphic copies. If we identify such sequences, then our results show that there is a one-to-one correspondence between the Schur class $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ and the set of sequences of Julia operators

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}_n \oplus \mathfrak{B}_{n+1}, \mathfrak{B}_n \oplus \mathfrak{A}_{n+1}), \qquad n = 0, 1, 2, \dots ,$$

such that $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{B}_0 = \mathfrak{B}$, and $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ and $\mathfrak{B}_1, \mathfrak{B}_2, \dots$ are Hilbert spaces.

This version of the Schur algorithm can be used as in the scalar and Hilbert space cases. A great number of familiar constructions can be extended to the indefinite setting. For example, we obtain a Kreĭn space generalization of Schur's theorem.

Theorem 4.8 *Let $S_0, \dots, S_r$ be given operators in $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Kreĭn spaces having the same negative index. The operators occur as the initial segment of coefficients of an element $S(z)$ in $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ if and only if*

$$T_r = \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_r & & S_1 & S_0 \end{pmatrix}$$

is a bicontraction on $\mathfrak{A}^{r+1}$ to $\mathfrak{B}^{r+1}$.

It is instructive to see how the classical Schur algorithm is recovered from the generalized Schur algorithm. As before, write $\mathbf{S}$ for $\mathbf{S}(\mathfrak{A}, \mathfrak{B})$ when $\mathfrak{A}$ and $\mathfrak{B}$ coincide with the complex numbers in the Euclidean metric. Start with an element $s_0(z) = s(z)$ of $\mathbf{S}$. Then $\alpha_0 = s_0(0)$ is a number of modulus at most one. Choose the Julia operator

$$\begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \bar{\beta}_0 & -\bar{\alpha}_0 \end{pmatrix},$$

where β_0 is any complex number such that $|\beta_0|^2 = 1 - |\alpha_0|^2$. Then $s_1(z)$ is the solution of the equation

$$\begin{aligned} s_0(z) &= \alpha_0 + z\beta_0 s_1(z) \left(1 - z\delta_0 s_1(z)\right)^{-1} \gamma_0 \\ &= \alpha_0 + z|\beta_0|^2 s_1(z) \left(1 + z\bar{\alpha}_0 s_1(z)\right)^{-1}, \end{aligned}$$

and therefore

$$\frac{s_0(z) - \alpha_0}{z} = \frac{|\beta_0|^2 s_1(z)}{1 + z\bar{\alpha}_0 s_1(z)}.$$

Solving, we get

$$s_1(z) = \frac{1}{z}\,\frac{s_0(z) - s_0(0)}{1 - \bar{s}_0(0)s_0(z)},$$

in agreement with the construction at the beginning of this lecture. The Kreĭn space Schur algorithm thus reduces to the classical Schur algorithm in the scalar case.

In the Hilbert space case, the main ideas in this lecture also appear in the context of commutant lifting. For example, see Foiaş and Frazho [1990]. A Kreĭn space generalization of the commutant lifting theorem is known (see, for example, Dritschel and Rovnyak [1990]). We believe that commutant lifting has the potential to provide a viable approach to the Schur theory in the indefinite setting as well.

In Lecture 5 we present another approach to the Schur class, based on reproducing kernel Pontryagin spaces and colligations, which gives new information and leads to further generalizations.

Lecture 5: Reproducing Kernel Pontryagin Spaces and Colligations

Key ideas:

- For each nonnegative integer κ, a generalized Schur class $\mathbf{S}_\kappa(\mathfrak{A},\mathfrak{B})$ is defined in such a way that $\mathbf{S}_0(\mathfrak{A},\mathfrak{B}) = \mathbf{S}(\mathfrak{A},\mathfrak{B})$ is the Schur class defined in Lecture 4. The approach uses kernels such as $K_S(w,z) = [1 - S(z)S(w)^*]/(1 - z\bar{w})$.
- Such kernels and associated Hilbert and Pontryagin reproducing kernel spaces are studied by means of representations $S(z) = \Theta_V(z)$ as characteristic functions of colligations V.

The purpose of this lecture is to arrive at a definition of the generalized Schur class $\mathbf{S}_\kappa(\mathfrak{A},\mathfrak{B})$, where κ is any nonnegative integer. In the course of doing this, we shall simultaneously show a means to derive the main properties of such classes.

Vector-valued functions will take their values in Pontryagin spaces which are denoted $\mathfrak{A}$ and $\mathfrak{B}$ and called **coefficient spaces**. For simplicity, we assume that

$$\mathrm{ind}_- \mathfrak{A} = \mathrm{ind}_- \mathfrak{B}.$$

A coefficient space $\mathfrak{C}$ is also used and can be thought of as either $\mathfrak{A}$ or $\mathfrak{B}$, or their direct sum. For any coefficient space $\mathfrak{C}$, if $b \in \mathfrak{C}$, define b^* to be the linear functional on $\mathfrak{C}$ such that

$$b^*a = \langle a, b\rangle_{\mathfrak{C}}, \qquad a \in \mathfrak{C}.$$

If $\mathfrak{C}$ is a finite-dimensional space of column vectors, we can interpret b^* as the conjugate transpose of b and b^*a as a matrix product. The theory is mainly written for a more general situation in which coefficient spaces are Kreĭn spaces, and it is only in a few places where there is any difference. However, it will simplify the exposition to restrict the discussion throughout to Pontryagin spaces $\mathfrak{A}$ and $\mathfrak{B}$ having the same negative index.

Let $K(w,z)$ be a function with values in $\mathfrak{L}(\mathfrak{C})$ defined on $\Omega \times \Omega$ for some region Ω in the complex plane (a region is an open connected set). Assume that

$$K(w,z)^* = K(z,w), \qquad w, z \in \Omega,$$

and that $K(w,z)$ is analytic in z for each fixed w and analytic in $\bar{w}$ for each fixed z. We shall call such a function a **hermitian analytic kernel**, or for the purpose of this lecture just a **kernel**. We say that $K(w,z)$ has κ **negative squares on** Ω if for any finite set $w_1, \dots, w_n \in \Omega$ and vectors $c_1, \dots, c_n \in \mathfrak{C}$, the matrix

$$\left(c_i^* K(w_j, w_i) c_j\right)_{i,j=1}^n$$

has at most κ negative eigenvalues, and at least one such matrix has exactly κ negative eigenvalues. Eigenvalues are counted according to multiplicity. Write

$$\mathrm{sq}_- K = \kappa$$

if the condition holds and $\mathrm{sq}_- K = \infty$ if no such κ exists. A kernel $K(w,z)$ is said to be **positive** if $\mathrm{sq}_- K = 0$, that is,

$$\sum_{i,j=1}^{n} c_i^* K(w_j, w_i) c_j \geq 0$$

for any finite set $w_1, \ldots, w_n \in \Omega$ and vectors $c_1, \ldots, c_n \in \mathfrak{C}$.

Let $\mathfrak{H}$ be a Pontryagin space of $\mathfrak{C}$-valued holomorphic functions on a region Ω in the complex plane. Assume that for any point $w \in \Omega$,

$$(af + bg)(w) = af(w) + bg(w)$$

for all complex numbers a, b and all $f, g \in \mathfrak{H}$. A kernel $K(w,z)$ on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{C})$ is said to be a **reproducing kernel** for $\mathfrak{H}$ if for each $w \in \Omega$ and $c \in \mathfrak{C}$, $K(w,z)c$ belongs to $\mathfrak{H}$ as a function of z, and

$$\langle h(\cdot), K(w,\cdot)c \rangle_{\mathfrak{H}} = c^* h(w)$$

for every $h(\cdot)$ in $\mathfrak{H}$.

Theorem 5.1 (1) *A Pontryagin space $\mathfrak{H}$ of holomorphic $\mathfrak{C}$-valued functions on a region Ω has a reproducing kernel $K(w,z)$ if and only if for each $w \in \Omega$, the evaluation mapping $h \to h(w)$ acts continuously from $\mathfrak{H}$ into $\mathfrak{C}$. A reproducing kernel is unique when it exists.*

(2) *Let $K(w,z)$ be a kernel on $\Omega \times \Omega$ with values in $\mathfrak{L}(\mathfrak{C})$ having a finite number of negative squares. Then there is a unique Pontryagin space $\mathfrak{H}$ of holomorphic $\mathfrak{C}$-valued functions on Ω with reproducing kernel $K(w,z)$, and $\mathrm{ind}_- \mathfrak{H} = \mathrm{sq}_- K$.*

Most of the details are straightforward and follow the Hilbert space case. We prove one part where the indefiniteness of the inner product comes into play. See Sorjonen [1975] for a full treatment.

Proof, existence part of (2). We are given a kernel $K(w,z)$ having a finite number of negative squares, say κ, and we wish to construct a Pontryagin space $\mathfrak{H}$ for which $K(w,z)$ is a reproducing kernel and $\mathrm{sq}_- K = \kappa$. We take this as known in the case $\kappa = 0$ and reduce the general case to this.

Let $\mathfrak{H}_0$ be the linear span of functions $K(w,\cdot)c$ with w in Ω and $c \in \mathfrak{C}$. Define an inner product on $\mathfrak{H}_0$ by requiring that

$$\left\langle \sum_{j=1}^{n} K(w_j,\cdot)c_j, \sum_{i=1}^{n} K(w_i,\cdot)c_i \right\rangle_{\mathfrak{H}_0} = \sum_{i,j=1}^{n} c_i^* K(w_j, w_i) c_j$$

whenever $w_1, \ldots, w_n \in \Omega$ and $c_1, \ldots, c_n \in \mathfrak{C}$. As in the positive case, it is not hard to see that the inner product is well defined.

Since $K(w,z)$ has κ negative squares, we may choose $\alpha_1, \ldots, \alpha_p \in \Omega$ and $\varphi_1, \ldots, \varphi_p \in \mathfrak{C}$ such that the matrix $\left(\varphi_i^* K(\alpha_j, \alpha_i)\varphi_j\right)_{i,j=1}^{p}$ has κ negative eigenvalues $\lambda_1, \ldots, \lambda_\kappa$. Let

$$\begin{pmatrix} \gamma_{11} \\ \vdots \\ \gamma_{1p} \end{pmatrix}, \ldots, \begin{pmatrix} \gamma_{\kappa 1} \\ \vdots \\ \gamma_{\kappa p} \end{pmatrix}$$

be orthonormal eigenvectors for these eigenvalues. Then

$$\begin{aligned} u_1(\cdot) &= \gamma_{11} K(\alpha_1, \cdot)\varphi_1 + \cdots + \gamma_{1p} K(\alpha_p, \cdot)\varphi_p, \\ &\cdots \\ u_\kappa(\cdot) &= \gamma_{\kappa 1} K(\alpha_1, \cdot)\varphi_1 + \cdots + \gamma_{\kappa p} K(\alpha_p, \cdot)\varphi_p, \end{aligned}$$

belong to $\mathfrak{H}_0$ and satisfy

$$\langle u_j(\cdot), u_i(\cdot)\rangle_{\mathfrak{H}_0} = \lambda_j\, \delta_{ij}, \qquad i, j = 1, \dots, \kappa.$$

Hence these functions span a κ-dimensional subspace $\mathfrak{H}_-$ of $\mathfrak{H}_0$. Taken in the inner product of $\mathfrak{H}_0$, $\mathfrak{H}_-$ is the antispace of a Hilbert space with reproducing kernel

$$K_-(w, z) = \sum_{j=1}^{\kappa} \frac{1}{\lambda_j}\, u_j(z)\, u_j(w)^*, \qquad w, z \in \Omega.$$

In a similar way, we can show that because $\mathrm{sq}_- K = \kappa$, no subspace of $\mathfrak{H}_0$ of dimension greater than κ can be the antispace of a Hilbert space in the inner product of $\mathfrak{H}_0$.

Now define

$$K_+(w, z) = K(w, z) - K_-(w, z),$$

and let $\mathfrak{H}_{0+}$ be the span of functions $K_+(w, \cdot)c$ with w in Ω and c in $\mathfrak{C}$. This subspace is orthogonal to $\mathfrak{H}_-$ in $\mathfrak{H}_0$. If

$$f(\cdot) = \sum_{j=1}^{n} K_+(w_j, \cdot)a_j, \qquad g(\cdot) = \sum_{i=1}^{n} K_+(w_i, \cdot)b_i$$

where $w_1, \dots, w_n$ belong to Ω and $a_1, \dots, a_n, b_1, \dots, b_n$ are in $\mathfrak{C}$, then

$$\begin{aligned} \langle f(\cdot), g(\cdot)\rangle_{\mathfrak{H}_0} &= \Bigg\langle \sum_{j=1}^{n} K(w_j, \cdot)a_j - \sum_{j=1}^{n} K_-(w_j, \cdot)a_j, \\ &\qquad\qquad \sum_{i=1}^{n} K(w_i, \cdot)b_i - \sum_{i=1}^{n} K_-(w_i, \cdot)b_i \Bigg\rangle_{\mathfrak{H}_0} \\ &= \sum_{i,j=1}^{n} b_i^* K(w_j, w_i)a_j - \sum_{i,j=1}^{n} {b_i}^* K_-(w_j, w_i)a_j \\ &\qquad - \sum_{i,j=1}^{n} {b_i}^* K_-(w_j, w_i)a_j + \sum_{i,j=1}^{n} {b_i}^* K_-(w_j, w_i)a_j \\ &= \sum_{i,j=1}^{n} b_i^* K_+(w_j, w_i)a_j. \end{aligned}$$

If $K_+(w, z)$ is not positive, we can find $\beta_1, \dots, \beta_q$ in Ω and $\psi_1, \dots, \psi_q$ in $\mathfrak{C}$ such that the element $u_{\kappa+1}(\cdot) = \sum_{i=1}^{q} K_+(\beta_i, \cdot)\psi_i$ of $\mathfrak{H}_{0+}$ satisfies

$$\langle u_{\kappa+1}(\cdot), u_{\kappa+1}(\cdot)\rangle_{\mathfrak{H}_0} = \sum_{i,j=1}^{q} \psi_i^* K_+(\beta_i, \beta_j)\psi_j < 0.$$

Then the span of $\mathfrak{H}_-$ and $u_{\kappa+1}(\cdot)$ is a $(\kappa+1)$-dimensional subspace of $\mathfrak{H}_0$ which is the antispace of a Hilbert space. As noted above, this contradicts our hypothesis that $\mathrm{sq}_- K = \kappa$. Therefore $K_+(w,z)$ is positive.

Since we assume that the assertion is true in the case $\kappa = 0$, $\mathfrak{H}_{0+}$ is contained as a dense set in a Hilbert space $\mathfrak{H}_+$ of holomorphic functions on Ω with reproducing kernel $K_+(w,z)$. Define a Pontryagin space $\mathfrak{H}$ such that $\mathfrak{H}_\pm$ are isometrically contained in $\mathfrak{H}$ and $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. Since $\mathfrak{H}_\pm$ have reproducing kernels $K_\pm(w,z)$, $\mathfrak{H}$ has reproducing kernel

$$K(w,z) = K_+(w,z) + K_-(w,z).$$

This completes the existence part of the proof. $\square$

We are concerned with kernels determined by a holomorphic function $S(z)$ which is defined on a region $\Omega(S)$ in the complex plane whose values are operators in $\mathfrak{L}(\mathfrak{A},\mathfrak{B})$. Recall our standing assumption that $\mathfrak{A}$ and $\mathfrak{B}$ are Pontryagin spaces having the same negative index. We assume also that $\Omega(S)$ contains the origin and is a subset of the unit disk. Define

$$\tilde{S}(z) = S(\bar{z})^*, \qquad z \in \Omega(\tilde{S}),$$

where $\Omega(\tilde{S})$ is the set of conjugates of numbers in $\Omega(S)$. Then in a neighborhood of the origin,

$$\tilde{S}(z) = S_0^* + S_1^* z + S_2^* z^2 + \cdots$$

if $S(z) = S_0 + S_1 z + S_2 z^2 + \cdots$. Set

$$\begin{aligned} K_S(w,z) &= \frac{1 - S(z)S(w)^*}{1 - z\bar{w}}, \\ K_{\tilde{S}}(w,z) &= \frac{1 - \tilde{S}(z)\tilde{S}(w)^*}{1 - z\bar{w}}, \end{aligned}$$

and

$$D_S(w,z) = \begin{pmatrix} K_S(w,z) & \dfrac{S(z) - S(\bar{w})}{z - \bar{w}} \\ \dfrac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} & K_{\tilde{S}}(w,z) \end{pmatrix},$$

$$D_{\tilde{S}}(w,z) = \begin{pmatrix} K_{\tilde{S}}(w,z) & \dfrac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} \\ \dfrac{S(z) - S(\bar{w})}{z - \bar{w}} & K_S(w,z) \end{pmatrix}.$$

For example, the domain of $K_S(w,z)$ is the set of all pairs (w,z) with $w,z \in \Omega(S)$, and the values of $K_S(w,z)$ are operators in $\mathfrak{L}(\mathfrak{B})$. The domain of $D_S(w,z)$ is the set of all pairs (w,z) with $w,z \in \Omega(S) \cap \Omega(\tilde{S})$, and the values of $D_S(w,z)$ are operators in $\mathfrak{L}(\mathfrak{B} \oplus \mathfrak{A})$. Observe that

$$\mathrm{sq}_- D_{\tilde{S}} = \mathrm{sq}_- D_S \le \infty$$

and

$$\begin{aligned} \mathrm{sq}_- K_S &\le \mathrm{sq}_- D_S, \\ \mathrm{sq}_- K_{\tilde{S}} &\le \mathrm{sq}_- D_{\tilde{S}}. \end{aligned}$$

Definition 5.2 *In the preceding notation, define Pontryagin spaces*
(1) $\mathfrak{H}(S)$ *with reproducing kernel* $K_S(w,z)$ *if* $\mathrm{sq}_- K_S < \infty$, *and*
(2) $\mathfrak{D}(S)$ *with reproducing kernel* $D_S(w,z)$ *if* $\mathrm{sq}_- D_S < \infty$.
Spaces $\mathfrak{H}(\tilde{S})$ *and* $\mathfrak{D}(\tilde{S})$ *with reproducing kernels* $K_{\tilde{S}}(w,z)$ *and* $D_{\tilde{S}}(w,z)$ *are defined analogously when* $\mathrm{sq}_- K_{\tilde{S}} < \infty$ *and* $\mathrm{sq}_- D_{\tilde{S}} < \infty$.

Our account of these spaces follows a manuscript by Alpay, Dijksma, and de Snoo [1994]. A new version will appear in Alpay, Dijksma, Rovnyak, and de Snoo [in preparation]. The approach is based on the notion of a colligation. In operator theory, the term "colligation" has come to mean a particular group of operators which arise both in the study of nonselfadjoint operators and in the theory of linear systems. A standard source is the survey article by M. S. Brodskiĭ [1978]. Ćurgus, Dijksma, Langer, and de Snoo [1988] and Dijksma, Langer, and de Snoo [1986] adapt the theory to the indefinite setting.

An **operator colligation** is a quadruple $(\mathfrak{H}, \mathfrak{F}, \mathfrak{G}, V)$ consisting of three Kreĭn spaces, the **state space** $\mathfrak{H}$, the **inner** or **incoming space** $\mathfrak{F}$, and the **outer** or **outgoing space** $\mathfrak{G}$, and a **connecting operator**

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{F}, \mathfrak{H} \oplus \mathfrak{G}).$$

We call T the **main operator** for the colligation. We say that the colligation is **isometric, coisometric, or unitary** according as the connecting operator V is isometric, coisometric, or unitary. When there is no possibility of ambiguity, we sometimes identify the colligation $(\mathfrak{H}, \mathfrak{F}, \mathfrak{G}, V)$ by its connecting operator V. The characteristic function of the colligation is

$$\Theta_V(z) = H + zG(1 - zT)^{-1}F$$

which is a holomorphic function with values in $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ defined in a neighborhood of the origin. The precise domain of the characteristic function is taken to be the component of the origin in the set of complex numbers z such that

$$(1 - zT)^{-1}$$

exists as a continuous everywhere defined operator. A colligation $(\mathfrak{H}, \mathfrak{F}, \mathfrak{G}, V)$ is said to be **closely inner connected**, **closely outer connected**, and **closely connected** according as

$$\begin{aligned} \mathfrak{H} &= \overline{\mathrm{span}}\,\{\mathrm{ran}\, T^m F : m \geq 0\}, \\ \mathfrak{H} &= \overline{\mathrm{span}}\,\{\mathrm{ran}\, T^{*n} G^* : n \geq 0\}, \\ \mathfrak{H} &= \overline{\mathrm{span}}\,\{\mathrm{ran}\, T^m F,\ \mathrm{ran}\, T^{*n} G^* : m, n \geq 0\}. \end{aligned}$$

The terms "inner" and "outer" refer to the inner and outer spaces $\mathfrak{F}$ and $\mathfrak{G}$ of the colligation. In systems theory, the terms "controllable" and "observable" are also used for "closely inner connected" and "closely outer connected." The characteristic function of a colligation is also called a "transfer function."

We use colligations to construct examples of kernels having a finite number of negative squares.

Theorem 5.3 *Let*

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{A}, \mathfrak{H} \oplus \mathfrak{B}).$$

be a colligation such that $\mathfrak{H}$ *is a Pontryagin space. Assume that*

$$S(z) = \Theta_V(z), \qquad z \in \Omega(S),$$

is the restriction of the characteristic function of V *to a subregion* $\Omega(S)$ *of the unit disk which contains the origin.*

(1) *If* V^* *is isometric, then*

$$K_S(w,z) = G(1 - zT)^{-1}(1 - \bar{w}T^*)^{-1}G^*, \qquad w,z \in \Omega(S),$$

and $\operatorname{sq}_- K_S \leq \operatorname{ind}_- \mathfrak{H}$*. Equality holds if the colligation is closely outer connected, and in this case there is an isomorphism* W *mapping* $\mathfrak{H}$ *onto* $\mathfrak{H}(S)$ *such that*

$$W : (1 - \bar{w}T^*)^{-1}G^*b \to K_S(w,\cdot)b, \qquad w \in \Omega(S),\ b \in \mathfrak{B}.$$

(2) *If* V *is isometric, then*

$$K_{\tilde{S}}(w,z) = F^*(1 - zT^*)^{-1}(1 - \bar{w}T)^{-1}F, \qquad w,z \in \Omega(\tilde{S}),$$

and $\operatorname{sq}_- K_{\tilde{S}} \leq \operatorname{ind}_- \mathfrak{H}$*. Equality holds if the colligation is closely inner connected, and then there is an isomorphism* W *mapping* $\mathfrak{H}$ *onto* $\mathfrak{H}(\tilde{S})$ *such that*

$$W : (1 - \bar{w}T)^{-1}Fa \to K_{\tilde{S}}(w,\cdot)a, \qquad w \in \Omega(\tilde{S}),\ a \in \mathfrak{A}.$$

(3) *If* V *is unitary, then for all* $w,z \in \Omega(S) \cap \Omega(\tilde{S})$,

$$D_S(w,z) = \begin{pmatrix} G\,(1 - zT)^{-1} \\ F^*(1 - zT^*)^{-1} \end{pmatrix} \begin{pmatrix} (1 - \bar{w}T^*)^{-1}G^* & (1 - \bar{w}T)^{-1}F \end{pmatrix},$$
$$D_{\tilde{S}}(w,z) = \begin{pmatrix} F^*(1 - zT^*)^{-1} \\ G\,(1 - zT)^{-1} \end{pmatrix} \begin{pmatrix} (1 - \bar{w}T)^{-1}F & (1 - \bar{w}T^*)^{-1}G^* \end{pmatrix},$$

and $\operatorname{sq}_- D_S = \operatorname{sq}_- D_{\tilde{S}} \leq \operatorname{ind}_- \mathfrak{H}$*. Equality holds if the colligation is closely connected, and then there are isomorphisms* W *and* $\tilde{W}$ *mapping* $\mathfrak{H}$ *onto* $\mathfrak{D}(S)$ *and* $\mathfrak{D}(\tilde{S})$ *such that*

$$W : \begin{pmatrix} (1 - \bar{w}T^*)^{-1}G^* & (1 - \bar{w}T)^{-1}F \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \to D_S(w,\cdot) \begin{pmatrix} b \\ a \end{pmatrix},$$
$$\tilde{W} : \begin{pmatrix} (1 - \bar{w}T)^{-1}F & (1 - \bar{w}T^*)^{-1}G^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \to D_{\tilde{S}}(w,\cdot) \begin{pmatrix} a \\ b \end{pmatrix},$$

for all $w,z \in \Omega(S) \cap \Omega(\tilde{S})$, $a \in \mathfrak{A}$*, and* $b \in \mathfrak{B}$.

Proof of identities. Since V^* is isometric,

$$\begin{aligned} TT^* + FF^* &= 1, \\ GG^* + HH^* &= 1, \\ GT^* + HF^* &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} (1 - z\bar{w})K_S(w,z) & \\ &= 1 - \Big[H + zG(1-zT)^{-1}F\Big]\Big[H^* + \bar{w}F^*(1-\bar{w}T^*)^{-1}G^*\Big] \\ &= 1 - HH^* - \bar{w}HF^*(1-\bar{w}T^*)^{-1}G^* - zG(1-zT)^{-1}FH^* \\ &\quad - z\bar{w}G(1-zT)^{-1}FF^*(1-\bar{w}T^*)^{-1}G^* \\ &= GG^* + \bar{w}GT^*(1-\bar{w}T^*)^{-1}G^* + zG(1-zT)^{-1}TG^* \\ &\quad - z\bar{w}G(1-zT)^{-1}(1-TT^*)(1-\bar{w}T^*)^{-1}G^* \\ &= G(1-zT)^{-1}\Big\{(1-zT)(1-\bar{w}T^*) + (1-zT)\bar{w}T^* \\ &\quad zT(1-\bar{w}T^*) - z\bar{w}(1-TT^*)\Big\}(1-\bar{w}T^*)^{-1}G^* \\ &= (1-z\bar{w})G(1-zT)^{-1}(1-\bar{w}T^*)^{-1}G^* \end{aligned}$$

for all $w, z \in \Omega(S)$. This proves the identity in (1).

The identity in (2) follows on applying (1) with V replaced by V^*.

The formulas for $D_S(w,z)$ and $D_{\tilde{S}}(w,z)$ in (3) follow from (1) and (2) together with the identities

$$\begin{aligned} \frac{S(z) - S(\bar{w})}{z - \bar{w}} &= G(1-zT)^{-1}(1-\bar{w}T)^{-1}F, \\ \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} &= F^*(1-zT^*)^{-1}(1-\bar{w}T^*)^{-1}G^*, \end{aligned}$$

which we derive in a similar way. □

Given $S(z)$, the question arises if there exists a colligation V such that

$$S(z) = \Theta_V(z).$$

This is the realization problem. The existence question asks if such a colligation can be found which is isometric, coisometric, or unitary. The corresponding uniqueness question is to determine if two such colligations are essentially the same.

Theorem 5.4 *Let $S(z)$ be a holomorphic function defined on a subregion $\Omega(S)$ of the unit disk containing the origin and taking values in $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$. If $K_S(w,z)$ has a finite number of negative squares, there exists a coisometric and closely outer connected colligation*

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(S) \oplus \mathfrak{A}, \mathfrak{H}(S) \oplus \mathfrak{B})$$

such that

$$\begin{aligned}(Th)(z) &= \frac{h(z)-h(0)}{z},\\ (Fa)(z) &= \frac{S(z)-S(0)}{z}a,\\ Gh &= h(0),\\ Ha &= S(0)a,\end{aligned}$$

for all h in $\mathfrak{H}(S)$ and a in $\mathfrak{A}$. The characteristic function of the colligation coincides with $S(z)$ on the set where both are defined:

$$S(z) = \Theta_V(z), \qquad z \in \Omega(S) \cap \Omega(\Theta_V). \tag{5.1}$$

For all h in $\mathfrak{H}(S)$, the relation

$$((1-wT)^{-1}h)(z) = \frac{zh(z)-wh(w)}{z-w}$$

holds for all $z \in \Omega(S)$ and $w \in \Omega(S) \cap \Omega(\Theta_V)$. Moreover

$$(1-\bar{w}T^*)^{-1}G^*b = K_S(w,\cdot)b$$

for all w in $\Omega(S) \cap \Omega(\Theta_V)$ and $b \in \mathfrak{B}$.

The realization (5.1) is of the type which is described in Theorem 5.3(1). In a certain natural sense which we shall not make precise, there is an essentially unique coisometric closely outer connected colligation V whose characteristic function has the form (5.1). The colligation constructed in Theorem 5.4 is called the **canonical coisometric colligation** associated with $S(z)$.

Proof. Let $\mathbf{R}$ be the linear relation in $\big(\mathfrak{H}(S) \oplus \mathfrak{B}\big) \times \big(\mathfrak{H}(S) \oplus \mathfrak{A}\big)$ spanned by the pairs

$$\left(\begin{pmatrix} K_S(\alpha,\cdot)u_1 \\ \\ u_2 \end{pmatrix}, \begin{pmatrix} \dfrac{K_S(\alpha,\cdot)-K_S(0,\cdot)}{\bar{\alpha}}\,u_1 + K_S(0,\cdot)u_2 \\ \dfrac{S(\alpha)^*-S(0)^*}{\bar{\alpha}}\,u_1 + S(0)^*u_2 \end{pmatrix} \right)$$

with $0 \neq \alpha \in \Omega(S)$ and $u_1, u_2 \in \mathfrak{B}$. Let

$$\left(\begin{pmatrix} K_S(\beta,\cdot)v_1 \\ \\ v_2 \end{pmatrix}, \begin{pmatrix} \dfrac{K_S(\beta,\cdot)-K_S(0,\cdot)}{\bar{\beta}}\,v_1 + K_S(0,\cdot)v_2 \\ \dfrac{S(\beta)^*-S(0)^*}{\bar{\beta}}\,v_1 + S(0)^*v_2 \end{pmatrix} \right)$$

be a second pair with $0 \neq \beta \in \Omega(S)$ and $v_1, v_2 \in \mathfrak{B}$. Straightforward algebra yields the identity

$$\left\langle \begin{pmatrix} \dfrac{K_S(\alpha,\cdot) - K_S(0,\cdot)}{\bar\alpha} u_1 + K_S(0,\cdot) u_2 \\ \dfrac{S(\alpha)^* - S(0)^*}{\bar\alpha} u_1 + S(0)^* u_2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \dfrac{K_S(\beta,\cdot) - K_S(0,\cdot)}{\bar\beta} v_1 + K_S(0,\cdot) v_2 \\ \dfrac{S(\beta)^* - S(0)^*}{\bar\beta} v_1 + S(0)^* v_2 \end{pmatrix} \right\rangle_{\mathfrak{H}(S)\oplus\mathfrak{A}}$$

$$= \frac{1}{\bar\alpha\,\beta} \Big[v_1^* K_S(\alpha,\beta) u_1 - v_1^* K_S(0,\beta) u_1 - v_1^* K_S(\alpha,0) u_1 + v_1^* K_S(0,0) u_1 \Big]$$

$$+ \frac{1}{\beta} \Big[v_1^* K_S(0,\beta) u_2 - v_1^* K_S(0,0) u_2 \Big] + \frac{1}{\bar\alpha} \Big[v_2^* K_S(\alpha,0) u_1 - v_2^* K_S(0,0) u_1 \Big]$$

$$+ v_2^* K_S(0,0) u_2 + v_1^* \frac{S(\beta) - S(0)}{\beta} \frac{S(\alpha)^* - S(0)^*}{\bar\alpha} u_1$$

$$+ v_1^* \frac{S(\beta) - S(0)}{\beta} S(0)^* u_2 + v_2^* S(0) \frac{S(\alpha)^* - S(0)^*}{\bar\alpha} u_1 + v_2^* S(0) S(0)^* u_2$$

$$= v_1^* K_S(\alpha,\beta) u_1 + v_2^* u_2$$

$$= \left\langle \begin{pmatrix} K_S(\alpha,\cdot) u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} K_S(\beta,\cdot) v_1 \\ v_2 \end{pmatrix} \right\rangle_{\mathfrak{H}(S)\oplus\mathfrak{B}}.$$

It follows that $\mathbf{R}$ is an isometric relation in

$$\big(\mathfrak{H}(S) \oplus \mathfrak{B}\big) \times \big(\mathfrak{H}(S) \oplus \mathfrak{A}\big).$$

The domain of $\mathbf{R}$ is easily seen to be dense in $\mathfrak{H}(S) \oplus \mathfrak{B}$. Since we assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Pontryagin spaces having the same negative index, by the version of Shmul′yan's theorem given in Theorem 2.10 (alternative form), the closure of $\mathbf{R}$ is the graph of an isometry

$$V^* = \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(S) \oplus \mathfrak{B}, \mathfrak{H}(S) \oplus \mathfrak{A}).$$

This completes the definition of the colligation.

Suppose that

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} h \\ a \end{pmatrix} \to \begin{pmatrix} k \\ b \end{pmatrix},$$

where $h, k \in \mathfrak{H}(S)$, $a \in \mathfrak{A}$, and $b \in \mathfrak{B}$. Then for any $0 \neq \alpha \in \Omega(\Theta)$ and $u_1, u_2 \in \mathfrak{B}$,

$$\begin{aligned} u_1^* k(\alpha) + u_2^* b &= \left\langle \begin{pmatrix} k(\cdot) \\ b \end{pmatrix}, \begin{pmatrix} K_S(\alpha,\cdot) u_1 \\ u_2 \end{pmatrix} \right\rangle_{\mathfrak{H}(S)\oplus\mathfrak{B}} \\ &= \left\langle \begin{pmatrix} h(\cdot) \\ \\ a \end{pmatrix}, \begin{pmatrix} \dfrac{K_S(\alpha,\cdot) - K_S(0,\cdot)}{\bar\alpha} u_1 + K_S(0,\cdot) u_2 \\ \dfrac{S(\alpha)^* - S(0)^*}{\bar\alpha} u_1 + S(0)^* u_2 \end{pmatrix} \right\rangle_{\mathfrak{H}(S)\oplus\mathfrak{A}} \\ &= u_1^* \frac{h(\alpha) - h(0)}{\alpha} + u_2^* h(0) + u_1^* \frac{S(\alpha) - S(0)}{\alpha} a + u_2^* S(0) a. \end{aligned}$$

By the arbitrariness of u_1, u_2, the formulas for T, F, G, H in the statement of the theorem follow.

To see that V is closely outer connected, we must show that the only element h in $\mathfrak{H}(S)$ which is orthogonal to the range of $T^{*n}G^*$ for all $n \geq 0$ is the zero element. The conditions imply that

$$T^n h \perp K_S(0,\cdot) b, \qquad n \geq 0,\ b \in \mathfrak{B}.$$

Since T acts as the difference quotient operator at the origin, this implies that h and all of its derivatives are zero at the origin. Therefore $h \equiv 0$ in a neighborhood of the origin, and since $\Omega(S)$ is a region and hence connected, h is the zero element of $\mathfrak{H}(S)$. Thus V is closely outer connected.

If $0 \neq w \in \Omega(S) \cap \Omega(\Theta_V)$ and h in $\mathfrak{H}(S)$ and $(1 - wT)^{-1} h = g$, then

$$h(z) = g(z) - w \frac{g(z) - g(0)}{z}$$

on $\Omega(S)$. Setting $z = w$, we find that $g(0) = h(w)$ and hence

$$g(z) = \frac{z h(z) - w h(w)}{z - w}$$

on $\Omega(S)$. The same conclusion holds trivially for $w = 0$. Thus for any $h \in \mathfrak{H}(S)$, $b \in \mathfrak{B}$, and $w \in \Omega(S) \cap \Omega(\Theta_V)$,

$$\begin{aligned} \left\langle h(\cdot), (1 - \bar w T^*)^{-1} G^* b \right\rangle_{\mathfrak{H}(S)} &= \left\langle G(1 - wT)^{-1} h, b \right\rangle_{\mathfrak{B}} \\ &= b^* \left\{ \frac{z h(z) - w h(w)}{z - w} \right\}_{z=0} \\ &= b^* h(w). \end{aligned}$$

This proves the two formulas at the end of the theorem.

It remains to prove (5.1). For any $w \in \Omega(\Theta) \cap \Omega(\Theta_V)$ and $a \in \mathfrak{A}$,

$$\begin{aligned} \Theta_V(w)a &= \big(H + wG(1 - wT)^{-1}F\big)a \\ &= S(0)a + w \left\{ \frac{z\, \dfrac{S(z) - S(0)}{z}\, a - w\, \dfrac{S(w) - S(0)}{w}\, a}{z - w} \right\}_{z=0} \\ &= S(0)a + \frac{S(w) - S(0)}{w}\, a \\ &= S(w)a, \end{aligned}$$

as asserted. □

The **canonical isometric colligation** associated with $S(z)$ is obtained as an immediate consequence and does not need a separate construction. It is described in the next result.

Theorem 5.5 *Let $S(z)$ be a holomorphic function defined on a subregion $\Omega(S)$ of the unit disk containing the origin and taking values in $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$. If $K_{\tilde{S}}(w, z)$ has a finite number of negative squares, there exists an isometric and closely inner connected colligation*

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(\tilde{S}) \oplus \mathfrak{A}, \mathfrak{H}(\tilde{S}) \oplus \mathfrak{B})$$

such that

$$\begin{aligned} (T^* h)(z) &= \frac{h(z) - h(0)}{z}, \\ (G^* b)(z) &= \frac{\tilde{S}(z) - \tilde{S}(0)}{z} b, \\ F^* h &= h(0), \\ H^* b &= \tilde{S}(0) b, \end{aligned}$$

for all h in $\mathfrak{H}(\tilde{S})$ and b in $\mathfrak{B}$. The characteristic function of the colligation coincides with $S(z)$ on the set where both are defined:

$$S(z) = \Theta_V(z), \qquad z \in \Omega(S) \cap \Omega(\Theta_V). \tag{5.2}$$

For all h in $\mathfrak{H}(\tilde{S})$, the relation

$$((1 - wT^*)^{-1} h)(z) = \frac{z h(z) - w h(w)}{z - w}$$

holds for all $z \in \Omega(\tilde{S})$ and $w \in \Omega(\tilde{S}) \cap \overline{\Omega(\Theta_V)}$. Moreover

$$(1 - \bar{w} T)^{-1} F a = K_{\tilde{S}}(w, \cdot) a$$

for all w in $\Omega(\tilde{S}) \cap \overline{\Omega(\Theta_V)}$ and $a \in \mathfrak{A}$.

Proof. Let

$$U = \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(\tilde{S}) \oplus \mathfrak{B}, \mathfrak{H}(\tilde{S}) \oplus \mathfrak{A})$$

be the canonical coisometric colligation associated with $\tilde{S}(z)$, as constructed in Theorem 5.4. Define

$$V = U^* = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{H}(\tilde{S}) \oplus \mathfrak{A}, \mathfrak{H}(\tilde{S}) \oplus \mathfrak{B}).$$

Then V is an isometric and closely inner connected colligation. By Theorem 5.4 and the definition of a characteristic function of a colligation,

$$\Theta_U(z) = H^* + zF^*(1 - zT^*)^{-1}G^*,$$

and so

$$S(z) = \tilde{\Theta}_U(z) = H + zG(1 - zT)^{-1}F = \Theta_V(z)$$

whenever the functions are defined. The remaining assertions of the theorem follow from Theorem 5.4. $\square$

Theorem 5.6 *Under the assumptions of Theorem 5.5, there is a Hilbert space $\mathfrak{E}$ and a closely connected unitary colligation of the form*

$$U = \begin{pmatrix} \begin{pmatrix} A & 0 \\ B & T \end{pmatrix} & \begin{pmatrix} 0 \\ F \end{pmatrix} \\ \begin{pmatrix} C & G \end{pmatrix} & \begin{pmatrix} H \end{pmatrix} \end{pmatrix} \in \mathfrak{L}((\mathfrak{E} \oplus \mathfrak{H}(\tilde{S})) \oplus \mathfrak{A}, (\mathfrak{E} \oplus \mathfrak{H}(\tilde{S})) \oplus \mathfrak{B})$$

such that

$$S(z) = \Theta_U(z), \qquad z \in \Omega(S) \cap \Omega(\Theta_U).$$

We omit the proof of Theorem 5.6. A similar result is proved for Hilbert spaces by Dijksma, Langer, and de Snoo [1986], Lemma 6.2.

We are now able to derive a special case of a result of Alpay, Dijksma, van der Ploeg, and de Snoo [1992].

Theorem 5.7 *Let $S(z)$ be a holomorphic function defined on a subregion $\Omega(S)$ of the unit disk containing the origin and taking values in $\mathfrak{L}(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Pontryagin spaces having the same negative index. If either $K_S(w,z)$ or $K_{\tilde{S}}(w,z)$ has a finite number of negative squares, then all four of the kernels*

$$K_S(w,z),\ K_{\tilde{S}}(w,z),\ D_S(w,z),\ D_{\tilde{S}}(w,z)$$

have a finite number of squares, and

$$\mathrm{sq}_- K_S = \mathrm{sq}_- K_{\tilde{S}} = \mathrm{sq}_- D_S = \mathrm{sq}_- D_{\tilde{S}}. \tag{5.3}$$

In this case, the Pontryagin spaces $\mathfrak{H}(S), \mathfrak{H}(\tilde{S}), \mathfrak{D}(S), \mathfrak{D}(\tilde{S})$ have the same negative index. In the statement of Theorem 5.7, for emphasis we have repeated our standing hypothesis that $\mathfrak{A}$ and $\mathfrak{B}$ are Pontryagin spaces having the same negative index. The reason is that there is a version of Theorem 5.7 for Kreĭn spaces, and the statement is not exactly the same.

Proof. Suppose that $K_{\tilde{S}}(w,z)$ has a finite number of negative squares. Construct V as in Theorem 5.5 and U as in Theorem 5.6. By Theorem 5.3(3),

$$\mathrm{sq}_-\, D_S = \mathrm{sq}_-\, D_{\tilde{S}} = \mathrm{ind}_-\,(\mathfrak{E} \oplus \mathfrak{H}(\tilde{S})).$$

Since $\mathfrak{E}$ is a Hilbert space,

$$\mathrm{ind}_-\,(\mathfrak{E} \oplus \mathfrak{H}(\tilde{S})) = \mathrm{ind}_-\,\mathfrak{H}(\tilde{S}) = \mathrm{sq}_-\, K_{\tilde{S}}.$$

Thus

$$\mathrm{sq}_-\, D_S = \mathrm{sq}_-\, D_{\tilde{S}} = \mathrm{sq}_-\, K_{\tilde{S}}.$$

As noted before Definition 5.2, $\mathrm{sq}_-\, K_S \leq \mathrm{sq}_-\, D_S$, and applying what we have shown with the roles of $S(z)$ and $\tilde{S}(z)$ interchanged, we obtain the equality of all four indices. □

Finally we arrive at:

Definition 5.8 *For any nonnegative integer κ, we define the* ***generalized Schur class*** $\mathbf{S}_\kappa(\mathfrak{A},\mathfrak{B})$ *to be the set of all holomorphic functions $S(z)$ defined on a subregion $\Omega(S)$ of the unit disk containing the origin and taking values in $\mathfrak{L}(\mathfrak{A},\mathfrak{B})$ such that the indices in* (5.3) *have the common value κ.*

The idea for such a definition is due to Kreĭn and Langer [1973], [1977], [1981]. The next result relates the generalized Schur class to the Schur class introduced in Lecture 4.

Theorem 5.9 *The generalized Schur class $\mathbf{S}_0(\mathfrak{A},\mathfrak{B})$ coincides with the Schur class $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ defined in Lecture 4.*

Sketch of proof. Suppose that $S(z)$ is in $\mathbf{S}_0(\mathfrak{A},\mathfrak{B})$, so $K_S(w,z)$ is positive. Write

$$K_S(w,z) = \frac{1 - S(z)S(w)^*}{1 - \bar{w}z} = \sum_{m,n=0}^{\infty} C_{mn}\, z^m \bar{w}^n.$$

An algebraic calculation shows that for any nonnegative integer k,

$$[C_{mn}]_{m,n=0}^{k} = 1 - T_k T_k^*,$$

where

$$T_k = \begin{pmatrix} S_0 & & & \\ S_1 & S_0 & 0 & \\ & \ddots & \ddots & \\ S_k & & S_1 & S_0 \end{pmatrix}.$$

The positivity of the kernel shows that the Toeplitz matrices have contractive adjoints. Since we assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Pontryagin spaces having the same negative index, the Toeplitz matrices are bicontractions, so $S(z)$ belongs to $\mathbf{S}(\mathfrak{A},\mathfrak{B})$.

The other direction follows on reversing the argument to show that if $S(z)$ belongs to $\mathbf{S}(\mathfrak{A},\mathfrak{B})$, then $K_S(w,z)$ is positive. Hence by Theorem 5.7, $S(z)$ belongs to $\mathbf{S}_0(\mathfrak{A},\mathfrak{B})$. □

The Schur class $\mathbf{S}(\mathfrak{A},\mathfrak{B})$ is defined for any Kreĭn spaces $\mathfrak{A}$ and $\mathfrak{B}$, and so it is natural to extend Definition 5.8 to this generality. We define $\mathbf{S}_\kappa(\mathfrak{A},\mathfrak{B})$ for any Kreĭn spaces $\mathfrak{A}$ and $\mathfrak{B}$ to be the set of all holomorphic functions $S(z)$ defined on a subregion $\Omega(S)$ of the unit disk containing the origin and taking values in $\mathfrak{L}(\mathfrak{A},\mathfrak{B})$ such that the indices of the four kernels in (5.3) are equal and have the common value κ.

The proof of Theorem 5.9 is not quite adequate to obtain the conclusion when $\mathfrak{A}$ and $\mathfrak{B}$ are Kreĭn spaces. What is missing is a proof that the kernels $D_S(w,z)$ and $D_{\tilde{S}}(w,z)$ are positive if $S(z)$ belongs to $\mathbf{S}(\mathfrak{A},\mathfrak{B})$. This may be seen in various ways. One is to use the Kreĭn space generalization of Theorem 5.7. The results at hand are sufficient, however, if we write the linear fractional representation of $S(z)$ given in Theorem 4.7 in the form

$$S(z) = \alpha + z\beta S'(z)\bigl(1 - z\delta S'(z)\bigr)^{-1}\gamma,$$

where $\alpha = S_0$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{L}(\mathfrak{A}\oplus\mathfrak{B}', \mathfrak{B}\oplus\mathfrak{A}')$$

is a Julia operator, and $S'(z)$ is in the Schur class $\mathbf{S}(\mathfrak{A}',\mathfrak{B}')$. Here $\mathfrak{A}'$ and $\mathfrak{B}'$ are Hilbert spaces because α is a bicontraction. A short calculation shows that

$$\begin{aligned} D_S(w,z) &= \begin{pmatrix} \dfrac{1 - S(z)S(w)^*}{1 - z\bar{w}} & \dfrac{S(z) - S(\bar{w})}{z - \bar{w}} \\ \dfrac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} & \dfrac{1 - \tilde{S}(z)\tilde{S}(w)^*}{1 - z\bar{w}} \end{pmatrix} \\ &= \begin{pmatrix} \beta\bigl(1 - zS'(z)\delta\bigr)^{-1} & 0 \\ 0 & \gamma^*\bigl(1 - z\tilde{S}'(z)\delta^*\bigr)^{-1} \end{pmatrix} \\ &\qquad\times \begin{pmatrix} \dfrac{1 - zS'(z)\bar{w}S'(w)^*}{1 - z\bar{w}} & \dfrac{zS'(z) - \bar{w}S'(\bar{w})}{z - \bar{w}} \\ \dfrac{z\tilde{S}'(z) - \bar{w}\tilde{S}'(\bar{w})}{z - \bar{w}} & \dfrac{1 - z\tilde{S}'(z)\bar{w}\tilde{S}'(w)^*}{1 - z\bar{w}} \end{pmatrix} \\ &\qquad\times \begin{pmatrix} \bigl(1 - \bar{w}\delta^* S'(w)^*\bigr)^{-1}\beta^* & 0 \\ 0 & \bigl(1 - \bar{w}\delta S'(\bar{w})\bigr)^{-1}\gamma \end{pmatrix}. \end{aligned}$$

Now we can appeal to the Hilbert space case of Theorem 5.7 to conclude that the kernels $D_S(w,z)$ and $D_{\tilde{S}}(w,z)$ are positive. Therefore Theorem 5.9 holds when $\mathfrak{A}$ and $\mathfrak{B}$ are Kreĭn spaces.

Our purpose in Lectures 4 and 5 has been to motivate generalizations of the Schur class in the indefinite setting. Many issues arise which we have not discussed. Some of these will be taken up in Alpay, Dijksma, Rovnyak, and de Snoo [in preparation]. In other areas, we do not have the answers at this time, and it seems likely that generalizations of the Schur class will remain an interesting subject for future work.

Spaces $\mathfrak{H}(S)$ and $\mathfrak{D}(S)$ are studied in many places for the indefinite case as well as the scalar and Hilbert space cases. A geometric viewpoint based on a concept of complementation is used in de Branges and Rovnyak [1966a], [1966b], de Branges [1988b], [1991], [1994], and Yang [1994]. The theory of complementation in Kreĭn spaces is due to de Branges [1988a] and is further developed by Dritschel and Rovnyak [1991]. The approach of Ball and Cohen [1991] is operator theoretic and features colligations. Operator methods are also used in Andô [1990] and Marcantognini [1990]. Sarason [1994] treats the scalar case and shows applications to function theory on the unit disk. The interaction with canonical models is explored in Ball and Kriete [1987] and Nikol′skiĭ and Vasyunin [1986], [1989]. Similar notions arise in the model theory of noncontractions on Hilbert spaces, as, for example, in McEnnis [1981]. Dym [1989] approaches the subject from the point of view of reproducing kernel spaces and shows many applications. For connections with engineering and linear systems, see also Bart, Gohberg, and Kaashoek [1979] and Helton [1987]. Applications in interpolation theory are given in Ball, Gohberg, and Rodman [1990].

Lecture 6: Invariant Subspaces

Key ideas:

- There is a natural homomorphism between commutants of selfadjoint operators which is useful in the study of hyperinvariant subspaces and maximal dual pairs of invariant subspaces for positive and definitizable operators.
- Certain compact perturbations of definitizable operators are shown to have nontrivial hyperinvariant subspaces. This implies the existence of nontrivial hyperinvariant subspaces for Hilbert space operators of the form $A(B+K)$ with A selfadjoint, B positive, and K selfadjoint and compact and in a Schatten class.

There is a long tradition of invariant subspace theorems in operator theory on spaces with indefinite inner product (see, for example, Azizov and Iokhvidov [1989], Bognár [1974], and Iokhvidov, Kreĭn, and Langer [1982]). We shall not attempt a survey of this large field, but instead highlight techniques found in Dritschel [1993b] and Dritschel [to appear]. The lecture is divided into three parts:

§A. The commutant of a selfadjoint operator.

§B. Dual pairs of invariant subspaces for definitizable operators.

§C. Compact perturbations of definitizable operators.

Proofs of results in §A are mostly complete and self-contained. Some results in §B and §C have complete proofs, but others draw on outside material and are only sketched.

A. The commutant of a selfadjoint operator

A generalization of the notion of a Julia operator is used to construct special homomorphisms of the commutant of a selfadjoint operator (Theorem 6.4). Some examples of the use of such homomorphisms in the study of invariant subspaces will appear in Theorems 6.5 and 6.6.

Our starting point is again the Bognár-Krámli factorization, but an additional property is needed. We do not know if the property holds for every factorization, but it holds for those produced by the method of proof of Theorem 1.1.

Theorem 6.1 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator on a Kreĭn space $\mathfrak{H}$, and construct a factorization*

$$A = DD^*, \qquad D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H}),\ \ker D = \{0\}, \tag{6.1}$$

by the method of proof of Theorem 1.1. If $X \in \mathfrak{L}(\mathfrak{H})$ and XA is selfadjoint, there is a unique selfadjoint $Y \in \mathfrak{L}(\mathfrak{D})$ such that $XD = DY$.

A factorization (6.1) of a selfadjoint operator $A \in \mathfrak{L}(\mathfrak{H})$ which is constructed by the method of proof of Theorem 1.1 will be called a **polar factorization**.

Proof. Let $X \in \mathfrak{L}(\mathfrak{H})$ and assume that XA is selfadjoint. Assume for the moment that we can factor

$$XA = EE^*, \qquad E \in \mathfrak{L}(\mathfrak{E}, \mathfrak{H}), \text{ ker } E = \{0\}, \tag{6.2}$$

such that $E = DZ$ for some $Z \in \mathfrak{L}(\mathfrak{E}, \mathfrak{D})$. This gives

$$XDD^* = XA = EE^* = DZZ^*D^*.$$

Setting $Y = ZZ^*$, we get $XD = DY$ since D^* has dense range. Clearly Y is selfadjoint. It is unique because D has zero kernel.

It remains to show that (6.2) can be so chosen. In the notation of the proof of Theorem 1.1, ran D = ran $R^{1/2}$, where $AJ_{\mathfrak{H}} = UR$ is the polar decomposition of the Hilbert space selfadjoint operator $AJ_{\mathfrak{H}}$. Construct $XA = EE^*$ in the same way with the same fundamental symmetry. Then ran E = ran $S^{1/2}$, where $XAJ_{\mathfrak{H}} = VS$ is the polar decomposition of the Hilbert space selfadjoint operator $XAJ_{\mathfrak{H}}$. Since

$$VS = XAJ_{\mathfrak{H}} = XUR,$$

we obtain

$$S^2 = SV^{\times}VS = RU^{\times}X^{\times}XUR \leq \|X\|^2 R^2,$$

and therefore $S \leq \|X\| R$ by Loewner's theorem (Rosenblum and Rovnyak [1985], p. 41). Factor $S^{1/2} = R^{1/2}C$ as in Douglas' lemma. The required operator Z is the restriction of C to the closure of the range of R. $\square$

Theorem 6.2 *Let $A \in \mathfrak{L}(\mathfrak{H})$ and $B \in \mathfrak{L}(\mathfrak{K})$ be selfadjoint operators on Kreĭn spaces $\mathfrak{H}$ and $\mathfrak{K}$ having polar factorizations*

$$A = \tilde{D}\tilde{D}^* \qquad \text{and} \qquad B = DD^*$$

with $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$ and $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$. For any $S, T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ such that

$$TA = BS,$$

there is a unique $L \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{D})$ satisfying

$$T\tilde{D} = DL \qquad \text{and} \qquad S^*D = \tilde{D}L^*.$$

If $S = T$, then also

$$L\hat{A} = \hat{B}L,$$

where $\hat{A} = \tilde{D}^*\tilde{D}$ and $\hat{B} = D^*D$.

Proof. Set

$$X = \begin{pmatrix} 0 & S^* \\ T & 0 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Our hypotheses imply that C and XC are selfadjoint operators on $\mathfrak{H} \oplus \mathfrak{K}$. We apply Theorem 6.1 to the polar factorization

$$C = \begin{pmatrix} \tilde{D} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \tilde{D}^* & 0 \\ 0 & D^* \end{pmatrix}.$$

Matrix multiplication shows that the operator Y in Theorem 6.1 has the form

$$Y = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix},$$

where $L \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{D})$. The equations $T\tilde{D} = DL$ and $S^*D = \tilde{D}L^*$ now follow. The last statement is a consequence of the first. □

Theorem 6.2 generalizes the notion of a Julia operator. This is most easily seen when $S = T$. In this case, put

$$U = U_{A,B}(T) = \begin{pmatrix} T & D \\ \tilde{D}^* & -L^* \end{pmatrix}.$$

The relations $T\tilde{D} = DL$ and $T^*D = \tilde{D}L^*$ then merely express the fact that the off diagonal terms of U^*U and UU^* are zero. We recover the usual Julia operator for T with $A = 1 - T^*T$ and $B = 1 - TT^*$.

For any operator $A \in \mathfrak{L}(\mathfrak{H})$, we write $\{A\}'$ for the **commutant** of A, that is, the set of operators $T \in \mathfrak{L}(\mathfrak{H})$ such that $AT = TA$. The **double commutant** of A is the set $\{A\}''$ of operators $S \in \mathfrak{L}(\mathfrak{H})$ such that $ST = TS$ for all $T \in \{A\}'$.

Corollary 6.3 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator on a Kreĭn space $\mathfrak{H}$, and let $A = DD^*$ and $\hat{A} = D^*D$ for some polar factorization* (6.1). *Then there is a mapping*

$$\pi : \{A\}' \to \{\hat{A}\}' \tag{6.3}$$

such that for each $T \in \{A\}'$, $\pi(T) = \hat{T}$ is the unique operator in $\mathfrak{L}(\mathfrak{D})$ such that

$$TD = D\hat{T} \qquad \text{and} \qquad T^*D = D\hat{T}^*.$$

Proof. Apply Theorem 6.2 with $A = B = DD^*$ and $S = T$. Continuity follows from the closed graph theorem. □

Theorem 6.4 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator on a Kreĭn space $\mathfrak{H}$, and let $A = DD^*$ and $\hat{A} = D^*D$ for some polar factorization* (6.1). *Then the mapping* (6.3) *is a $*$-homomorphism: for all $X, Y \in \{A\}'$ and all scalars a and b,*

(1) $\pi(aX + bY) = a\pi(X) + b\pi(Y)$,

(2) $\pi(XY) = \pi(X)\pi(Y)$,

(3) $\pi(X^*) = \pi(X)^*$.

Moreover, $\pi(1_{\mathfrak{H}}) = 1_{\mathfrak{D}}$, and $\pi(A) = \hat{A}$. The mapping takes isometric, partially isometric, unitary, selfadjoint, and projection operators to operators of the same type. For $T \in \{A\}'$, $\sigma(\pi(T)) \subseteq \sigma(T)$ and $\sigma_p(\pi(T)) \subseteq \sigma_p(T)$. If further $\varphi(T)$ is defined by the Riesz functional calculus for some holomorphic function φ, then $\varphi(\pi(T))$ is defined and $\pi(\varphi(T)) = \varphi(\pi(T))$.

We write $\sigma(T)$ and $\rho(T)$ for the spectrum and resolvent set for an operator T. The point spectrum is denoted $\sigma_p(T)$.

Proof. Most of the statements are immediate. We only prove the last two assertions.

Fix $T \in \{A\}'$. If $\lambda \in \rho(T)$, then $(T-\lambda)^{-1} \in \{A\}'$. Hence there is an $X \in \{\hat{A}\}'$ such that $(T-\lambda)^{-1}D = DX$ and $(T^* - \bar{\lambda})^{-1}D = DX^*$. If $\hat{T} = \pi(T)$, then

$$D = (T-\lambda)(T-\lambda)^{-1}D = D(\hat{T}-\lambda)X,$$

and since D has zero kernel, X is a right inverse for $\hat{T}-\lambda$. It is similarly a left inverse. Thus $\lambda \in \rho(\pi(T))$ and $\pi((T-\lambda)^{-1}) = X = (\pi(T)-\lambda)^{-1}$. In particular, $\sigma(\pi(T)) \subseteq \sigma(T)$. On the other hand, if there exists $f \neq 0$ such that $\hat{T}f = \lambda f$, then $Df \neq 0$ and $TDf = D\hat{T}f = \lambda Df$. Therefore $\sigma_p(\pi(T)) \subseteq \sigma_p(T)$.

The last assertion is verified above when $\varphi(z) = 1/(z-\lambda)$ for some $\lambda \in \rho(T)$. Suppose that φ is holomorphic on an open set containing $\sigma(T)$, so that

$$\varphi(T) = \frac{1}{2\pi i}\int_\Gamma \varphi(z)(z-T)^{-1}\,dz$$

is defined. Here Γ is a suitable system of curves as in Conway [1990], p. 201. Since $\sigma(\pi(T)) \subseteq \sigma(T)$, $\varphi(\pi(T))$ is also defined and given by

$$\varphi(\pi(T)) = \frac{1}{2\pi i}\int_\Gamma \varphi(z)(z-\pi(T))^{-1}\,dz.$$

We then get $\pi(\varphi(T)) = \varphi(\pi(T))$ by approximating the integrals by Riemann sums and using the result for the special case of resolvents. □

It can be shown that in cases of interest, the mapping π in Theorem 6.4 is essentially unique. What essential uniqueness means in this context is that if π' is a second such mapping defined from a polar factorization

$$A = D'D'^*, \qquad D' \in \mathfrak{L}(\mathfrak{D}', \mathfrak{H}),\ \ker D' = \{0\}, \tag{6.4}$$

then

$$\pi'(T) = W^*\pi(T)W, \qquad T \in \{A\}', \tag{6.5}$$

for some Kreĭn space isomorphism W from $\mathfrak{D}'$ onto $\mathfrak{D}$. A sufficient condition for the essential uniqueness of π is that A is a positive operator, that is, $A \geq 0$. In this case, the space $\mathfrak{D}$ in any polar factorization (6.1) is a Hilbert space, and standard operator methods can be used to show that (6.1) and (6.4) are related by

$$D' = DW$$

for some isomorphism W from $\mathfrak{D}'$ onto $\mathfrak{D}$. Now suppose that $T \in \{A\}'$, and put

$$\hat{T} = \pi(T) \qquad \text{and} \qquad \hat{T}' = \pi'(T).$$

From the relations $TD = D\hat{T}$ and $TD' = D'\hat{T}'$, we then get $TDW = DW\hat{T}'$ and so $W\hat{T}'W^* = \hat{T}$, yielding (6.5). More generally, there will be essential uniqueness if $\mathfrak{D}$ is a Pontryagin space for some polar factorization (6.1), or, more generally still, if ran D^* contains a maximal uniformly definite subspace (see Dritschel [1993a]). We make no use of these results and omit their proofs.

Let $\mathfrak{M}$ be a closed subspace of a Kreĭn space $\mathfrak{H}$. The subspace $\mathfrak{M}$ is said to be **proper** if it is not the zero subspace or the whole space. We say that $\mathfrak{M}$ is **invariant** for an operator $A \in \mathfrak{L}(\mathfrak{H})$ if $A\mathfrak{M} \subseteq \mathfrak{M}$ and **hyperinvariant** if $\mathfrak{M}$ is invariant for each operator in the commutant of A. We call $\mathfrak{M}$ **reducing** for A if it is invariant for A and A^*, or, equivalently, the subspace and its orthogonal companion are invariant for A. A hyperinvariant subspace for a selfadjoint operator

$A \in \mathfrak{L}(\mathfrak{H})$ is a reducing subspace for every $T \in \{A\}'$, because the commutant of a selfadjoint operator A contains T^* whenever it contains T.

The following result illustrates the use of Theorem 6.4 in the study of invariant subspaces.

Theorem 6.5 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint operator on a Kreĭn space $\mathfrak{H}$, not a constant multiple of the identity. Let $A = DD^*$ and $\hat{A} = D^*D$ for some polar factorization* (6.1).

(1) *The operator A has a proper closed hyperinvariant subspace if $p(\hat{A}) = 0$ for some nonzero polynomial p.*
(2) *The operator A has a proper closed hyperinvariant subspace if $\varphi(\hat{A})$ has such a subspace for some holomorphic function φ which is defined on an open set containing the spectrum of A.*
(3) *The operator A has a pair of orthogonal proper closed hyperinvariant subspaces if $\varphi(\hat{A})$ has such a pair for some holomorphic function φ which is defined on an open set containing the spectrum of A.*

Proof. (1) For any nonnegative integer n, $D\hat{A}^n D^* = D(D^*D)^n D^* = A^{n+1}$, and so $Ap(A) = Dp(\hat{A})D^* = 0$. Since $q(A) = 0$ for a nonconstant polynomial q, A has a proper closed hyperinvariant subspace by an elementary argument.

(2) We adopt the notation of Corollary 6.3 and Theorem 6.4. Let $\mathfrak{M}$ be a proper closed hyperinvariant subspace for $\varphi(\hat{A})$, where φ is a holomorphic function which is defined on an open set containing the spectrum of A. Set $\mathfrak{N} = \overline{D\mathfrak{M}}$. We show that $\mathfrak{N}$ is hyperinvariant for A. Suppose $T \in \{A\}'$ and $\pi(T) = \hat{T}$. Then T commutes with $\varphi(A)$, and therefore $\hat{T} = \pi(T)$ commutes with $\varphi(\hat{A}) = \pi(\varphi(A))$ by Theorem 6.4. Since $\mathfrak{M}$ is hyperinvariant for $\varphi(\hat{A})$, $\hat{T}\mathfrak{M} \subseteq \mathfrak{M}$. Thus for $f \in \mathfrak{M}$,

$$TDf = D\hat{T}f \in D\mathfrak{M} \subseteq \mathfrak{N}.$$

By the continuity of T, $T\mathfrak{N} \subseteq \mathfrak{N}$. This shows that $\mathfrak{N}$ is hyperinvariant for A.

Observe that $\mathfrak{N}$ is nonzero since $\mathfrak{M}$ is nonzero and $\ker D = \{0\}$. If $\mathfrak{N} = \mathfrak{H}$, then $D\mathfrak{M}$ is dense in $\mathfrak{H}$ and hence $D^*D\mathfrak{M}$ is dense in $\mathfrak{D}$. But $D^*D\mathfrak{M} = \hat{A}\mathfrak{M} \subseteq \mathfrak{M}$ since $\mathfrak{M}$ is hyperinvariant for $\varphi(\hat{A})$ and $\hat{A}$ commutes with $\varphi(\hat{A})$. This contradicts the assumption that $\mathfrak{M}$ is proper. Therefore $\mathfrak{N}$ is a proper subspace of $\mathfrak{H}$.

(3) Now suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are orthogonal proper closed hyperinvariant subspaces for $\varphi(\hat{A})$ for φ as above. By the first part of the proof, $\mathfrak{N}_1 = \overline{D\mathfrak{M}_1}$ and $\mathfrak{N}_2 = \overline{D\mathfrak{M}_2}$ are proper closed hyperinvariant subspaces for A. If $f \in \mathfrak{M}_1$ and $g \in \mathfrak{M}_2$, then

$$\langle Df, Dg\rangle_{\mathfrak{H}} = \langle D^*Df, g\rangle_{\mathfrak{D}} = 0,$$

since $\mathfrak{M}$ is invariant for $\hat{A} = D^*D$. It follows that $\mathfrak{N}_1 \perp \mathfrak{N}_2$. □

Theorem 6.5 can be used to give a quick proof that a positive operator has nontrivial hyperinvariant subspaces. More is true in this case.

Theorem 6.6 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a positive operator, not a constant multiple of the identity. Then A has a proper closed hyperinvariant subspace. Moreover, there exist closed hyperinvariant subspaces $\mathfrak{N}_+$ and $\mathfrak{N}_-$ which are positive and negative, respectively, such that*

(1) $\mathfrak{N}_- \perp \mathfrak{N}_+$,
(2) $\mathfrak{N}_+ + \mathfrak{N}_- \supseteq \operatorname{ran} A$, *and*
(3) $\mathfrak{N}_\pm^\perp \cap \operatorname{ran} A \subseteq \mathfrak{N}_\mp$.

Proof. Let $A = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$, be a polar factorization, and let $\hat{A} = D^*D$. Since A is positive, $\mathfrak{D}$ is a Hilbert space. The projections in the range of the spectral measure $E(\cdot)$ for $\hat{A}$ commute with every operator T that commutes with $\hat{A}$ (Halmos [1951], p. 68). If $\hat{A}$ is not a constant multiple of the identity, we obtain proper closed hyperinvariant subspaces for A by part (2) of Theorem 6.5. If $\hat{A}$ is a constant multiple of the identity, the same conclusion follows from part (1) of Theorem 6.5.

For any real Borel set Δ, the closure of the range of $DE(\Delta)$ is a hyperinvariant subspace for A by the proof of Theorem 6.5. Let $\mathfrak{N}_+$ and $\mathfrak{N}_-$ be these spaces for $\Delta_+ = [0, \infty)$ and $\Delta_- = (-\infty, 0]$. If $f \in \mathfrak{D}$, then

$$\langle DE(\Delta_+)f, DE(\Delta_+)f \rangle_{\mathfrak{H}} = \left\langle \hat{A}E(\Delta_+)f, E(\Delta_+)f \right\rangle_{\mathfrak{D}} \geq 0.$$

Thus $\mathfrak{N}_+$ is a positive subspace. In a similar way, we see that $\mathfrak{N}_-$ is a negative subspace. If $f, g \in \mathfrak{D}$, then

$$\langle DE(\Delta_+)f, DE(\Delta_-)g \rangle_{\mathfrak{H}} = \left\langle \hat{A}E(\Delta_+)f, E(\Delta_-)g \right\rangle_{\mathfrak{D}} = 0,$$

so (1) holds. Since the span of the ranges of $E(\Delta_+)$ and $E(\Delta_-)$ is all of $\mathfrak{D}$,

$$\mathfrak{N}_+ + \mathfrak{N}_- \supseteq \operatorname{ran} D \supseteq \operatorname{ran} A,$$

which yields (2). Suppose $f \in \mathfrak{N}_+^\perp \cap \operatorname{ran} A$. Then $f = Ag = DD^*g$ and $f \perp DE(\Delta_+)u$ for all $u \in \mathfrak{D}$. Thus for all $u \in \mathfrak{D}$,

$$0 = \langle f, DE(\Delta_+)u \rangle_{\mathfrak{H}} = \langle D^*g, D^*DE(\Delta_+)u \rangle_{\mathfrak{D}} = \left\langle D^*g, \hat{A}E(\Delta_+)u \right\rangle_{\mathfrak{D}}.$$

It follows that $D^*g \in \operatorname{ran} E(\Delta_-)$, and therefore $f = DD^*g \in \mathfrak{N}_-$. We have shown that $\mathfrak{N}_+^\perp \cap \operatorname{ran} A \subseteq \mathfrak{N}_-$. In a similar way, $\mathfrak{N}_-^\perp \cap \operatorname{ran} A \subseteq \mathfrak{N}_+$, and (3) follows. □

Theorem 6.7 *Let $\mathfrak{H}$ be a Kreĭn space, and let $A \in \mathfrak{L}(\mathfrak{H})$ be a positive operator. Assume that* $\ker A = \{0\}$ *and $\sigma(A) \subseteq [0, \infty)$. Then $\mathfrak{H}$ is a Hilbert space.*

Proof. Let $A = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$, be a polar factorization. Then $\mathfrak{D}$ is a Hilbert space, and $\hat{A} = D^*D \in \mathfrak{L}(\mathfrak{D})$ is a positive operator because $\hat{A}$ is selfadjoint and $\sigma(\hat{A}) \subseteq \sigma(A) \subseteq [0, \infty)$ by Theorem 6.4. For any $f \in \mathfrak{D}$,

$$\langle Df, Df \rangle_{\mathfrak{H}} = \langle \hat{A}f, f \rangle_{\mathfrak{D}} \geq 0.$$

Since A has zero kernel, it has dense range. Therefore D has dense range, and so $\mathfrak{H}$ is a Hilbert space. □

Norm estimates are implicit in Theorems 6.1, 6.2, and Corollary 6.3. Recall that any two operator norms on $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ are equivalent. Operator norms are determined by fundamental decompositions, and sometimes these can be chosen to produce special effects. With slightly greater care, the proof of Theorem 6.1 produces the inequality

$$\|Y\| \leq \|X\|$$

for suitable operator norms on $\mathfrak{L}(\mathfrak{H})$ and $\mathfrak{L}(\mathfrak{D})$. In the proof of Theorem 6.2 it is possible to show that there exist operator norms on $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and $\mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{D})$ such that

$$\|L\| \leq \max\,(\|S\|, \|T\|).$$

In Corollary 6.3, the polar factorization of A similarly determines operator norms on $\mathfrak{L}(\mathfrak{H})$ and $\mathfrak{L}(\mathfrak{D})$ such that

$$\|\pi(T)\| \le \|T\|$$

for every $T \in \{A\}'$.

A particular instance of Theorem 6.2, as noted in the remarks following the proof, yields another proof of the existence of a Julia operator (2.2) for any given operator $T \in \mathfrak{L}(\mathfrak{H}, \mathfrak{K})$. By this method, operator norms may be chosen such that in addition

$$\|L\| \le \|T\|,$$
$$\max\left(\|D\|, \|\tilde{D}\|\right) \le \left(1 + \|T\|^2\right)^{1/2}.$$

Such estimates were obtained by Arsene, Constantinescu, and Gheondea [1987] in their original construction of a Julia operator (see Dritschel and Rovnyak [1990], p. 296).

B. Dual pairs of invariant subspaces for definitizable operators

For applications it is often useful to know if an operator on a Kreĭn space has an invariant subspace which is maximal negative or maximal positive. This is a large field, going back to L. S. Pontryagin [1944]. An extensive account with bibliographic notes may be found in Azizov and Iokhvidov [1989]. In this section, we examine some of the ideas in Langer [1965], [1971], [1982] from the perspective of the constructions in §A. The main results of the section are due to Langer and involve, in particular, the spectral theory of definitizable operators. The methods of §A give an alternative approach to these ideas.

Define a **dual pair** of subspaces of a Kreĭn space $\mathfrak{H}$ as a pair consisting of a closed positive subspace $\mathfrak{M}_+$ and a closed negative subspace $\mathfrak{M}_-$ of $\mathfrak{H}$ such that $\mathfrak{M}_+ \perp \mathfrak{M}_-$. A **maximal dual pair** is a dual pair consisting of a maximal positive subspace and a maximal negative subspace. A theorem of R. S. Phillips [1961] assures the existence of a maximal dual pair extending any given dual pair. A particular case was previously verified in Theorem 1.6(5).

Phillips' Theorem. *For any dual pair $\mathfrak{N}_+$, $\mathfrak{N}_-$ of subspaces of a Kreĭn space $\mathfrak{H}$, there is a maximal dual pair $\mathfrak{M}_+$, $\mathfrak{M}_-$ such that $\mathfrak{N}_\pm \subseteq \mathfrak{M}_\pm$.*

Proof. (Arsene and Gheondea [1982]). Let $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ be a fundamental decomposition. By the graph representation of definite subspaces, we may write

$$\mathfrak{H}_+ = \mathfrak{S}_+ \oplus \mathfrak{R}_+, \qquad |\mathfrak{H}_-| = \mathfrak{S}_- \oplus \mathfrak{R}_- \tag{6.6}$$

and choose Hilbert space contractions

$$C : \mathfrak{S}_+ \to |\mathfrak{H}_-| = \mathfrak{S}_- \oplus \mathfrak{R}_-,$$
$$R^\times : \mathfrak{S}_- \to \mathfrak{H}_+ = \mathfrak{S}_+ \oplus \mathfrak{R}_+,$$

such that

$$\mathfrak{N}_+ = \left\{ \begin{pmatrix} f \\ Cf \end{pmatrix} : f \in \mathfrak{S}_+ \right\} \qquad \text{and} \qquad \mathfrak{N}_- = \left\{ \begin{pmatrix} R^\times g \\ g \end{pmatrix} : g \in \mathfrak{S}_- \right\}.$$

Write $R = \begin{pmatrix} T & F \end{pmatrix}$ and $C^\times = \begin{pmatrix} S^\times & G^\times \end{pmatrix}$ relative to the decompositions (6.6). The orthogonality of $\mathfrak{N}_+$ and $\mathfrak{N}_-$ means that for all $f \in \mathfrak{S}_+$ and $g \in \mathfrak{S}_-$,

$$0 = \left\langle \begin{pmatrix} f \\ Cf \end{pmatrix}, \begin{pmatrix} R^\times g \\ g \end{pmatrix} \right\rangle_{\mathfrak{H}} = \langle f, R^\times g \rangle_{\mathfrak{H}_+} - \langle Cf, g \rangle_{|\mathfrak{H}_-|} = \langle f, T^\times g \rangle_{\mathfrak{S}_+} - \langle Sf, g \rangle_{\mathfrak{S}_-},$$

and so $S = T$. By the Hilbert space cases of Theorems 3.2–3.4, we may find an operator $H \in \mathfrak{L}(\mathfrak{R}_+, \mathfrak{R}_-)$ such that

$$A = \begin{pmatrix} T & F \\ G & H \end{pmatrix} \in \mathfrak{L}(\mathfrak{S}_+ \oplus \mathfrak{R}_+, \mathfrak{S}_- \oplus \mathfrak{R}_-)$$

is a contraction. Without difficulty we check that the subspaces

$$\mathfrak{M}_+ = \left\{ \begin{pmatrix} f \\ Af \end{pmatrix} : f \in \mathfrak{H}_+ \right\} \qquad \text{and} \qquad \mathfrak{M}_- = \left\{ \begin{pmatrix} A^\times g \\ g \end{pmatrix} : g \in |\mathfrak{H}_-| \right\}$$

have the required properties. □

We are interested in criteria for the existence of maximal positive or maximal negative subspaces which are invariant for a selfadjoint operator on a Kreĭn space. Such subspaces automatically occur in dual pairs. For example, if $\mathfrak{M}_-$ is a maximal negative subspace of a Kreĭn space $\mathfrak{H}$ which is invariant for a selfadjoint operator $A \in \mathfrak{L}(\mathfrak{H})$, then $\mathfrak{M}_+ = \mathfrak{M}_-^\perp$ is a maximal positive subspace which is invariant for A. The roles of positive and negative subspaces can obviously be interchanged here. It is thus natural to speak of maximal dual pairs of invariant subspaces of a selfadjoint operator, and we state existence criteria in these terms (Theorems 6.8 and 6.11).

In one situation, Theorem 6.6 assures the existence of nontrivial dual pairs of invariant subspaces, which, however, are not necessarily maximal. Using Phillips' theorem, we then obtain maximal dual pairs of invariant subspaces. This method follows Langer [1982], p. 44, except that in place of spectral theory we employ Theorem 6.6.

Theorem 6.8 *Every positive operator A on a Kreĭn space $\mathfrak{H}$ has a maximal dual pair of invariant subspaces.*

Proof. The conclusion is trivial if A is a constant multiple of the identity, so assume that this is not the case. Then by Theorem 6.6, there exists a dual pair of invariant subspaces $\mathfrak{N}_+$, $\mathfrak{N}_-$ such that $\mathfrak{N}_\pm^\perp \cap \operatorname{ran} A \subseteq \mathfrak{N}_\mp$. Use Phillips' theorem to extend to a maximal dual pair $\mathfrak{M}_\pm \supseteq \mathfrak{N}_\pm$. Then

$$A\mathfrak{M}_+ \subseteq A(\mathfrak{N}_-^\perp) \subseteq \mathfrak{N}_-^\perp \cap \operatorname{ran} A \subseteq \mathfrak{N}_+ \subseteq \mathfrak{M}_+.$$

In a similar way, $A\mathfrak{M}_- \subseteq \mathfrak{M}_-$. □

The full class of Kreĭn space selfadjoint operators has no structure theory which is comparable to the Hilbert space case. It is not difficult to understand why this should be so. Let $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{K}$, where $\mathfrak{K}$ is a Hilbert space. We view $\mathfrak{H}$ as a Kreĭn space with fundamental symmetry

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Any selfadjoint operator on $\mathfrak{H}$ is easily seen to have the form

$$A = \begin{pmatrix} T & B \\ C & T^{\times} \end{pmatrix} \tag{6.7}$$

where $B, C \in \mathfrak{L}(\mathfrak{K})$ are Hilbert space selfadjoint operators, $T \in \mathfrak{L}(\mathfrak{K})$, and $T^{\times}$ is the Hilbert space adjoint of T. Considering such operators with $B = C = 0$, we observe that knowing the structure of Kreĭn space selfadjoint operators is equivalent to knowing the structure of an arbitrary bounded operator on a Hilbert space.

Special classes of selfadjoint operators do have a detailed structure theory, however. The positive operators provide an example. In fact, the structure of positive operators can be explicitly described in a manner similar to the Hilbert space case. The analysis lends itself to a broader class of operators. A selfadjoint operator A on a Kreĭn space $\mathfrak{H}$ is said to be **definitizable** if there is a nonconstant polynomial p having real coefficients such that $p(A) \geq 0$. We call any such polynomial p a **definitizing polynomial** for A. A definitizable operator thus commutes with a positive operator.

Figure 6 shows a typical picture for the spectrum of a definitizable operator. The nonreal spectrum is a subset of the zeros of the definitizing polynomial. To describe this picture, we need some standard notions from spectral theory. Assume $A \in \mathfrak{L}(\mathfrak{H})$ for some Kreĭn space $\mathfrak{H}$, and let λ_0 be an isolated point in $\sigma(A)$. We call

$$P_{\lambda_0} = \frac{1}{2\pi i} \int_{\gamma} (z - A)^{-1} \, dz$$

the **Riesz idempotent** for the point λ_0 (the term "projection" is reserved for a selfadjoint idempotent as in Lecture 1). Here γ is any sufficiently small circle around λ_0 with counterclockwise orientation. The Riesz idempotent P_{λ_0} satisfies $P_{\lambda_0}^2 = P_{\lambda_0}$ but is not necessarily selfadjoint. The range $\mathfrak{M}_{\lambda_0}$ of P_{λ_0} is a closed subspace of $\mathfrak{H}$ which is invariant under A. We say that λ_0 has finite **Riesz index** equal to the positive integer ν if $(\lambda_0 - A)^{\nu} \mathfrak{M}_{\lambda_0} = \{0\}$ and no smaller positive integer has this property. Such a point is necessarily an eigenvalue, but it could have infinite multiplicity. If no such integer ν exists, the Riesz index is considered to be infinite. For example, see Conway [1990], pp. 210–211.

Theorem 6.9 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a selfadjoint definitizable operator with definitizing polynomial p. The nonreal spectrum of A consists of a finite number of conjugate pairs of points $\lambda_j, \bar{\lambda}_j$, $j = 1, \dots, r$, which are zeros of p. Each such point is an eigenvalue of finite Riesz index not greater than the order of the point as a zero of p. If λ is one of the nonreal eigenvalues, let P_λ be the associated Riesz projection and $\mathfrak{M}_\lambda$ its range.*

(1) *For each nonreal eigenvalue λ, $P_\lambda^* = P_{\bar{\lambda}}$ and $P_\lambda P_{\bar{\lambda}} = P_{\bar{\lambda}} P_\lambda = 0$. The subspaces $\mathfrak{M}_\lambda, \mathfrak{M}_{\bar{\lambda}}$ are neutral, and their sum $\mathfrak{M}_\lambda + \mathfrak{M}_{\bar{\lambda}}$ is a Kreĭn subspace which reduces A and coincides with the range of the projection $P_\lambda + P_{\bar{\lambda}}$.*

(2) *The subspace $\mathfrak{M}_{\lambda_j} + \mathfrak{M}_{\bar{\lambda}_j}$ is orthogonal to $\mathfrak{M}_{\lambda_k} + \mathfrak{M}_{\bar{\lambda}_k}$ whenever $j \neq k$, and*

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \sum_{j=1}^{r} \left(\mathfrak{M}_{\lambda_j} + \mathfrak{M}_{\bar{\lambda}_j} \right),$$

where $\mathfrak{H}_0$ reduces A and $A_0 = A|_{\mathfrak{H}_0}$ is definitizable with a definitizing polynomial p_0 which has only real zeros.

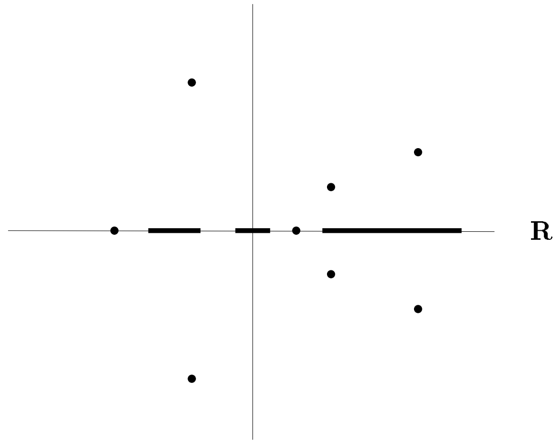

Figure 6

Proof. If $\sigma(A)$ contains a nonreal point, so does $\sigma_{ap}(A)$, the approximate point spectrum, since the boundary of $\sigma(A)$ is contained in $\sigma_{ap}(A)$ (Conway [1990], p. 210; $\sigma_{ap}(A)$ is defined as in the Hilbert space case for any choice of norm for the Kreĭn space). The first assertion will be proved by showing that a nonreal point λ in $\sigma_{ap}(A)$ is a zero of p.

Let $p(A) = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$, be a polar factorization. Since $p(A)$ is positive by hypothesis, $\mathfrak{D}$ is a Hilbert space. Define π as in Corollary 6.3, and set $\hat{A} = \pi(A)$. Then $D^*(A - \lambda) = (\hat{A} - \lambda)D^*$. Since $\hat{A}$ is selfadjoint, the spectrum of $\hat{A}$ is real, and so $\hat{A} - \lambda$ is invertible. Choose a norm for $\mathfrak{H}$ and a sequence $\{f_n\}_{n=1}^{\infty}$ of unit vectors such that $\|(A - \lambda)f_n\| \to 0$. Since $p(A) = D(\hat{A} - \lambda)^{-1}D^*(A - \lambda)$, we have $\|p(A)f_n\| \to 0$. Write $p(z) = c(z - \mu_1)\cdots(z - \mu_r)$, where $\mu_1, \dots, \mu_r$ are roots of p and c is a constant. If $p_1(z)$ is the polynomial $p(z)/(z - \mu_1)$, then

$$\|p(A)f_n\| = \|p_1(A)[(A - \lambda) + (\lambda - \mu_1)]f_n\| \to 0.$$

Since $\|p_1(A)(A - \lambda)f_n\| \to 0$, either $\lambda = \mu_1$ or $\|p_1(A)f_n\| \to 0$. In the latter case, we repeat the argument. Continuing in this way, we see that λ must be one of the roots $\mu_1, \dots, \mu_r$, since otherwise we obtain the contradiction that $\|f_n\| \to 0$. The first assertion has thus been proved.

Let λ be any nonreal point of $\sigma(A)$. Its Riesz idempotent

$$P_\lambda = \frac{1}{2\pi i}\int_{\gamma(\lambda)} (z - A)^{-1}\, dz$$

can be formed by integrating over any circle centered at λ and containing no other point of $\sigma(A)$ in its interior. Choosing this circle so that it does not meet the real axis, we obtain by Theorem 6.4,

$$\pi(P_\lambda) = \frac{1}{2\pi i}\int_{\gamma(\lambda)} (z - \hat{A})^{-1}\, dz = 0,$$

since $(z - \hat{A})^{-1}$ is holomorphic off the real axis. Therefore

$$p(A)P_\lambda = P_\lambda DD^* = D\pi(P_\lambda)D^* = 0.$$

According to the Riesz functional calculus, if λ is a root of p of order k, we can factor $p(A) = (A-\lambda)^k q(A)$, where q is a polynomial and the restriction of $q(A)$ to $\mathfrak{M}_\lambda$ is invertible. Thus $(A-\lambda)^k \mathfrak{M}_\lambda = \{0\}$, and so λ has Riesz index not greater than k. As a consequence, λ is an eigenvalue for A.

In the same situation, by the selfadjointness of A, $\bar{\lambda}$ also belongs to $\sigma(A)$. The relations $P_\lambda^* = P_{\bar{\lambda}}$ and $P_\lambda P_{\bar{\lambda}} = P_{\bar{\lambda}} P_\lambda = 0$ follow from the Riesz functional calculus. In particular, $P_\lambda^* P_\lambda = P_{\bar{\lambda}} P_\lambda = 0$, and hence $\mathfrak{M}_\lambda$ is a neutral subspace of $\mathfrak{H}$. The same assertion holds for $\mathfrak{M}_{\bar{\lambda}}$. The sum $P_\lambda + P_{\bar{\lambda}}$ is a selfadjoint idempotent and hence a projection. Its range is $\mathfrak{M}_\lambda + \mathfrak{M}_{\bar{\lambda}}$, which is therefore a Kreĭn subspace of $\mathfrak{H}$. The subspace reduces A because each of the summands is invariant under A and A is selfadjoint. This proves (1).

The assertions in (2) follow from straightforward arguments using the Riesz functional calculus. □

It is therefore sufficient to restrict attention to definitizable operators whose definitizing polynomials have only real zeros. Such operators have real spectrum as in the Hilbert space case.

Theorem 6.10 *Let $A \in \mathfrak{L}(\mathfrak{H})$ be a definitizable selfadjoint operator with definitizing polynomial p having only real zeros. Let $\mathfrak{Z}$ be the set of zeros of p, and let Ω be the Boolean algebra of all Borel subsets Δ of $\mathbf{R}$ such that $\partial\Delta \cap \mathfrak{Z} = \emptyset$. Then there exist a function $E : \Omega \to \mathfrak{L}(\mathfrak{H})$ and positive operators $N_\nu \in \mathfrak{L}(\mathfrak{H})$, $\nu \in \mathfrak{Z}$, such that $N_\nu N_\mu = 0$ for all $\mu, \nu \in \mathfrak{Z}$ (including $\mu = \nu$), and $E(\Delta)N_\nu = N_\nu E(\Delta) = 0$ whenever $\Delta \in \Omega$, $\nu \in \mathfrak{Z}$, and $\nu \notin \Delta$, and such that the following properties hold:*

1. *For $\Delta \in \Omega$, $E(\Delta)$ is a projection in the the double commutant $\{A\}''$ of A, $E(\emptyset) = 0$, and $E(\mathbf{R}) = 1$.*
2. *For all $\Delta, \Delta' \in \Omega$, $E(\Delta \cap \Delta') = E(\Delta)E(\Delta')$.*
3. *If $\{\Delta_n\}_{n=1}^\infty \subset \Omega$ are pairwise disjoint and $\Delta = \cup_{n=1}^\infty \Delta_n \in \Omega$, then $E(\Delta) = \sum_{n=1}^\infty E(\Delta_n)$ with convergence in the strong operator topology.*
4. *If $\Delta \in \Omega$ and $p > 0$ ($p < 0$) on Δ, then the range of $E(\Delta)$ is uniformly positive (uniformly negative).*
5. *The spectrum of the restriction of A to the range of $E(\Delta)$ is contained in $\bar{\Delta}$ for each $\Delta \in \Omega$.*
6. *If φ is a Borel measurable function which is bounded on $\sigma(A)$, then*

$$\int_{\sigma(A)\setminus\mathfrak{Z}} \varphi(\lambda)p(\lambda)\, E(d\lambda) = \lim_{\epsilon \downarrow 0} \int_{S_\epsilon} \varphi(\lambda)p(\lambda)\, E(d\lambda) \tag{6.8}$$

exists in the strong operator topology, where S_ϵ is the set of points in $\sigma(A)$ at a distance greater than ϵ from $\mathfrak{Z}$. An operator $\varphi(A)p(A)$ may be defined for all such φ and expressed as

$$\varphi(A)p(A) = \int_{\sigma(A)\setminus\mathfrak{Z}} \varphi(\lambda)p(\lambda)\, E(d\lambda) + \sum_{\nu\in\mathfrak{Z}} \varphi(\nu)N_\nu. \tag{6.9}$$

The definition of $\varphi(A)p(A)$ is made in such a way that if φ is holomorphic on an open set containing $\sigma(A)$ and $\varphi(A)$ is defined by the Riesz functional calculus, $\varphi(A)p(A)$ is the product of $\varphi(A)$ and $p(A)$.

The function $E(\cdot)$ is unique and called the **spectral function** for A. The functional calculus is adequate for our purposes, but it has an obvious limitation in that it does not define $\varphi(A)$ directly but only speaks of $\varphi(A)p(A)$. See Jonas [1981] for a more refined analysis which overcomes this difficulty for functions which are sufficiently smooth in a neighborhood of what are called "critical points" for A (the critical points are a subset of the zeros of the definitizing polynomial). Also, there may be choices of definitizing polynomial with fewer zeros. The article of Langer [1982] contains further discussions of this point.

Sketch of proof. By assumption, $p(A)$ is positive. Therefore it has a polar factorization $p(A) = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$, where $\mathfrak{D}$ is a Hilbert space. Let π be the associated mapping constructed in Corollary 6.3. The operator $\hat{A} = \pi(A)$ is Hilbert space selfadjoint and therefore has a spectral representation

$$\hat{A} = \int t\, F(dt)$$

for some spectral measure $F(\cdot)$ on the Borel subsets of the real line.

Every $\Delta \in \Omega$ can be written as a disjoint union $\Delta = I_{\nu_1} \cup \cdots \cup I_{\nu_r} \cup \Delta_0$, where each I_{ν_k} is a closed interval containing a single point of $\mathfrak{Z}$ in its interior and Δ_0 is a Borel set at a positive distance from $\mathfrak{Z}$. Possibly there are no intervals I_{ν_k}, in which case that part of the union is the empty set. We show how to define $E(\Delta')$ if Δ' is one of the sets in the union and then take

$$E(\Delta) = E(I_{\nu_1}) + \cdots + E(I_{\nu_k}) + E(\Delta_0)$$

in the general case. The representation Δ as a union of special sets is not unique, and a full account should include a verification that the definition is independent of the representation.

For sets of the form Δ_0, we may write $\Delta_0 = \Delta_0^+ \cup \Delta_0^-$, where $p(t) > \delta$ for all $t \in \Delta_0^+$ and $p(t) < -\delta$ for all $t \in \Delta_0^-$ for some positive number δ. Here essential use is made of the fact that Δ_0 lies at a positive distance from the zeros of p. Put $\mathfrak{F}(\Delta_0^\pm) = \text{ran } F(\Delta_0^\pm)$. For $f \in \mathfrak{F}(\Delta_0^+)$,

$$\langle Df, Df\rangle_{\mathfrak{H}} = \langle D^*Df, f\rangle_{\mathfrak{D}} = \langle p(\hat{A})f, f\rangle_{\mathfrak{D}} \geq \delta\langle f, f\rangle_{\mathfrak{D}} \geq \delta\|D\|^{-2}\|Df\|^2.$$

It follows that $\mathfrak{E}(\Delta_0^+) = D\mathfrak{F}(\Delta_0^+)$ is a closed uniformly positive subspace of $\mathfrak{H}$. In a similar way, $\mathfrak{E}(\Delta_0^-) = D\mathfrak{F}(\Delta_0^-)$ is closed and uniformly negative. We easily check that these subspaces are orthogonal and hyperinvariant for A (similar arguments appear in the proofs of Theorems 6.5 and 6.6). Therefore

$$\mathfrak{E}(\Delta_0) = \mathfrak{E}(\Delta_0^+) \oplus \mathfrak{E}(\Delta_0^-)$$

is a Kreĭn subspace of $\mathfrak{H}$ which is hyperinvariant for A. Define $E(\Delta_0)$ to be the projection with range $\mathfrak{E}(\Delta_0)$. Since A is selfadjoint and $\mathfrak{E}(\Delta_0)$ is hyperinvariant for A, $E(\Delta_0)$ is in the double commutant of A. By construction,

$$E(\Delta_0)Df = DF(\Delta_0)f, \qquad f \in \mathfrak{F}(\Delta_0^\pm).$$

It is not hard to see that $E(\Delta_0)Df = 0 = DF(\Delta_0)f$ for f orthogonal to $\mathfrak{F}(\Delta_0^\pm)$, and therefore

$$E(\Delta_0)D = DF(\Delta_0).$$

In other words, $\pi(E(\Delta_0)) = F(\Delta_0)$. The subspaces in the decomposition

$$\mathfrak{H} = \mathfrak{E}(\Delta_0) \oplus \mathfrak{E}(\Delta_0)^\perp$$

reduce A. The restrictions of A to the summands can be shown to have spectra contained in $\overline{\Delta_0}$ and $\overline{\mathbf{R} \setminus \Delta_0}$, respectively.

These constructions apply in particular when Δ_0 is the complement of $\cup_{\nu \in \mathfrak{Z}} I_\nu$, where I_ν, $\nu \in \mathfrak{Z}$, are disjoint closed intervals each of which contains exactly one zero of p in its interior (it is important here to exhaust the zeros of p so that Δ_0 lies at a positive distance from $\mathfrak{Z}$). Then the spectrum of $A|_{\mathfrak{E}(\Delta_0)^\perp}$ is contained in $\cup_{\nu \in \mathfrak{Z}} I_\nu$. The appropriate definition of $E(I_{\nu_k})$ turns out to be the projection obtained from the Riesz functional calculus using a symmetric contour that surrounds I_{ν_k} and no other interval.

Let φ be a bounded Borel measurable function on $\sigma(A)$. By Theorem 6.4, $\sigma(\hat{A}) \subseteq \sigma(A)$. Therefore $\varphi(\hat{A})$ is defined by the functional calculus for selfadjoint operators on a Hilbert space. In this situation, we define

$$\varphi(A)p(A) = D\varphi(\hat{A})D^*.$$

For any $\nu \in \mathfrak{Z}$, define

$$N_\nu = D\varphi_\nu(\hat{A})D^*,$$

where φ_ν is the characteristic function of the singleton $\{\nu\}$. For any two zeros μ and ν of p (possibly $\mu = \nu$),

$$N_\mu N_\nu = D\varphi_\mu(\hat{A})D^* D\varphi_\nu(\hat{A})D^* = D\varphi_\mu(\hat{A})p(\hat{A})\varphi_\nu(\hat{A})D^* = 0.$$

With these definitions, the proof is completed by checking the various assertions. For example, (6.9) holds when φ is the characteristic function of a point in $\mathfrak{Z}$ by the definition of the operators N_ν, $\nu \in \mathfrak{Z}$. Using $\varphi = \chi_{\Delta_0}$ for a Borel set Δ_0 at a positive distance from $\mathfrak{Z}$, we obtain

$$\chi_{\Delta_0}(A)p(A) = D\chi_{\Delta_0}(\hat{A})D^* = DF(\Delta_0)D^* = E(\Delta_0)DD^* = E(\Delta_0)p(A).$$

This verifies (6.9) when $\varphi = \chi_{\Delta_0}$.

For additional details, see Dritschel [1993b]. Also of interest is Bognár [1987], which foreshadows this work. □

Theorem 6.11 *Every definitizable selfadjoint operator $A \in \mathfrak{L}(\mathfrak{H})$ has a maximal dual pair of invariant subspaces.*

Sketch of proof. We first make some reductions. Let p be a definitizing polynomial, and let $\mathfrak{Z}$ be the set of zeros of p. Let $\Delta = \cup_{\nu \in \mathfrak{Z}} I_\nu$, where I_ν, $\nu \in \mathfrak{Z}$, are disjoint closed intervals each of which contains exactly one zero of p in its interior. Writing $\mathfrak{E}(\Delta) = \text{ran } E(\Delta)$ with $E(\cdot)$ as in Theorem 6.10, we obtain a decomposition

$$\mathfrak{H} = \mathfrak{E}(\Delta) \oplus \mathfrak{E}(\Delta)^\perp$$

into Kreĭn subspaces which reduce A. The conclusion is easily obtained for the restriction of A to $\mathfrak{E}(\Delta)^\perp$, and it is thus sufficient to consider the case in which the spectrum of A is contained in $\Delta = \cup_{\nu \in \mathfrak{Z}} I_\nu$.

By considering a decomposition corresponding to the separate intervals that make up Δ, we may assume further that the spectrum of A is contained in a single interval, say I_μ. There is only one zero of p in I_μ, but p may have other zeros not in I_μ. We show that p can be replaced by a monomial. Let λ be a zero of p not in I_μ, and suppose that its order is m. For definiteness, suppose that λ lies to the left of I_μ. Then the spectrum of $(A - \lambda)^m$ is contained in $(0, \infty)$. Write

$p(z) = q(z)(z-\lambda)^m$. Let $q(A) = \tilde{D}\tilde{D}^*$, $\tilde{D} \in \mathfrak{L}(\tilde{\mathfrak{D}}, \mathfrak{H})$, be a polar factorization. Define π as in Corollary 6.3. Since

$$p(A) = (A-\lambda)^m q(A) = \tilde{D}\pi((A-\lambda)^m)\tilde{D}^*$$

is a positive operator on $\mathfrak{H}$ and $\tilde{D}^*$ has dense range, $\pi((A-\lambda)^m)$ is a positive operator on $\tilde{\mathfrak{D}}$. By Theorem 6.4, the spectrum of $\pi((A-\lambda)^m)$ is contained in the spectrum of $(A-\lambda)^m$ and hence in $(0,\infty)$. By Theorem 6.7, $\tilde{\mathfrak{D}}$ is a Hilbert space, and thus $q(A)$ is positive. In the case that λ lies to the right of I_μ, then a similar argument can be applied to $-q(A)$. In this way, we reduce the theorem to the case in which the definitizing polynomial is a monomial. Without loss of generality, we may suppose that A^n is positive for some $n \geq 2$ (the case $n = 1$ is handled by Theorem 6.8).

For any selfadjoint operator $C \in \mathfrak{L}(\mathfrak{H})$ there is a neutral invariant subspace $\mathfrak{N}$ which is maximal in the sense that there is no other neutral invariant subspace strictly containing it. If J is a fundamental symmetry on $\mathfrak{H}$, then $\mathfrak{K} = \mathfrak{N} + J\mathfrak{N}$ is a Kreĭn subspace of $\mathfrak{H}$. The compression $\tilde{C}$ of C to $\tilde{\mathfrak{H}} = \mathfrak{K}^\perp$ is selfadjoint, and if $\mathfrak{M}$ is an invariant subspace for $\tilde{C}$, then $\mathfrak{M} + \mathfrak{N}$ is invariant for C. In particular, $\tilde{C}$ has no nonzero neutral invariant subspace. Also, $\mathfrak{M} + \mathfrak{N}$ is maximal positive or maximal negative in $\mathfrak{H}$ if $\mathfrak{M}$ is maximal positive or maximal negative in $\tilde{\mathfrak{H}}$. Furthermore, if $C^n \geq 0$, then $\tilde{C}^n \geq 0$. Hence the problem is reduced to the case where A has no nonzero neutral invariant subspace. Then both $\ker A$ and $(\ker A)^\perp$ are invariant under A, and so is their intersection, which is a neutral subspace. Thus

$$\ker A \cap \overline{\operatorname{ran} A} = \{0\}.$$

This can be used to show that $\ker A + \operatorname{ran} A$ is dense in $\mathfrak{H}$, and that $\ker A^k = \ker A$ and $\overline{\operatorname{ran} A^k} = \overline{\operatorname{ran} A}$, $k \geq 1$.

If n is even, then

$$\left\langle A^{n/2} f, A^{n/2} f \right\rangle \geq 0$$

for all $f \in \mathfrak{H}$, and so $\overline{\operatorname{ran} A}$ is a positive subspace. Also $\ker A$ is either strictly positive or strictly negative, since A has no nonzero invariant neutral subspace. If $\ker A$ is positive, then since $\ker A + \operatorname{ran} A$ is dense in $\mathfrak{H}$, $\mathfrak{H}$ is a Hilbert space. The result is trivially true in this case, since then the whole space $\mathfrak{H}$ is maximal positive and invariant for A. If $\ker A$ is negative, then it is maximal negative, since $(\ker A)^\perp = \overline{\operatorname{ran} A}$ is positive in $\mathfrak{H}$ (this follows from the graph representation). It then follows that $\overline{\operatorname{ran} A}$ is maximal positive. In either case, we have a maximal dual pair of invariant subspaces for A.

If n is odd, then for all $f \in \mathfrak{H}$, $g \in \ker A$,

$$\left\langle A(A^{(n-1)/2} f + g), A^{(n-1)/2} f + g \right\rangle \geq 0,$$

and since the vectors $A^{(n-1)/2} f + g$ span a dense subspace of $\mathfrak{H}$, it follows that $A \geq 0$. The result is then true by Theorem 6.8. $\square$

C. Compact perturbations of definitizable operators

Some of the results for definitizable operators discussed in §B are true for other classes including certain compact perturbations of definitizable operators. This area has also been intensively investigated. For example, see Jonas [1982], [1988], [1993], Jonas and Langer [1979], [1983], Langer and Najman [1983], and Najman [1980]. We sketch the following result from Dritschel [to appear].

Compact operators and subclasses such as the Schatten class $\mathfrak{C}_p$ on a Kreĭn space are defined as in the Hilbert space case. The definitions can be made relative to any Hilbert space norm for the Kreĭn space, and since any two such norms are equivalent the definitions are independent of choice of norm.

Theorem 6.12 *Let $\mathfrak{H}$ be a Kreĭn space, A a definitizable selfadjoint operator in $\mathfrak{L}(\mathfrak{H})$, K a compact operator in $\mathfrak{C}_p$, $1 \leq p < \infty$. Then if $T = A + K$ is not a constant multiple of the identity, there is a proper closed subspace of $\mathfrak{H}$ which is hyperinvariant for T.*

To prove this we use an idea in Radjavi and Rosenthal [1973], Chapter 6, which is based in part on work of J. T. Schwartz. The following theorems will be needed. They appear in Radjavi and Rosenthal [1973], pp. 7, 97, 158, for Hilbert spaces, and they apply without change when the underlying spaces are Kreĭn spaces.

Weyl's Theorem. *If $A \in \mathfrak{L}(\mathfrak{H})$ and K is compact, then*

$$\sigma(A+K) \subseteq \sigma(A) \cup \sigma_p(A+K).$$

Lomonosov's Theorem. *Every operator commuting with a nonzero compact operator and which is not a constant multiple of the identity has a proper closed hyperinvariant subspace.*

Theorem on Exposed Arcs. *Let $T \in \mathfrak{L}(\mathfrak{H})$ be such that $\sigma(T)$ contains an exposed arc Γ and let k be a positive integer. Suppose that for each point $z_0 \in \Gamma$ and each closed line segment Λ which meets $\sigma(T)$ only at $\{z_0\}$ and which is not tangent to Γ, there exists a constant $\tilde{C}$ such that*

$$\left\|(T-z)^{-1}\right\| \leq \exp\left(\tilde{C}\,|z-z_0|^{-k}\right)$$

for all $z \in \Lambda$ not equal to z_0. Then T has a proper closed hyperinvariant subspace.

A few explanations are needed to make the last theorem clear. Some choice of operator norm is understood in the resolvent estimate. A **smooth Jordan arc** is (the range of) a one-to-one function $z(t) = x(t) + iy(t)$ mapping the interval $(0,1)$ to the complex numbers which is twice differentiable on $(0,1)$. A bounded operator T is said to have an **exposed arc** Γ in its spectrum if there is an open disk Δ such that $\Delta \cap \sigma(T) = \Gamma$ is a smooth Jordan arc.

Sketch of proof of Theorem 6.12. It may be assumed that $\sigma_p(T) = \emptyset$, since if T has an eigenvalue the conclusion is elementary.

As a first case, suppose that $\sigma(T)$ contains more than one point. We may take $\sigma(T)$ to be connected, because otherwise a nontrivial Riesz projection exists, and the range of such a projection is a proper closed hyperinvariant subspace for T. By Weyl's theorem $\sigma(T) \subseteq \sigma(A)$, and so, by Theorem 6.9, $\sigma(T)$ must be a closed and bounded interval with nonempty interior. Thus the spectrum of T has an exposed arc. The resolvent estimate required to apply the Theorem on Exposed Arcs holds. In fact, in the situation required by that theorem,

$$\left\|(T-z)^{-1}\right\| \le \exp\left(\tilde{C}\,|z-z_0|^{-p(k_A+1)-1}\right),$$

where k_A is the degree of a definitizing polynomial for A. The derivation of this estimate follows a method in Radjavi and Rosenthal [1973] (proof of Lemma 1.7) and is justified by estimates in Dunford and Schwartz which are cited in the same place. The only difference is that we use an estimate of the form

$$\left\|(A-z)^{-1}\right\| \le M|z-z_0|^{-k_A-1},$$

which follows from the spectral theory for definitizable selfadjoint operators (see Langer [1982]). Thus the Theorem on Exposed Arcs applies, and the result follows in the case that $\sigma(T)$ contains more than one point.

Before taking up the case in which the spectrum of T is a single point, we make a preliminary observation that if $C \in \mathfrak{L}(\mathfrak{K})$ is a positive operator on a Kreĭn space $\mathfrak{K}$ and $\sigma(C) = \{\lambda\}$, then either $\lambda \neq 0$ and $C = \lambda$ or $\lambda = 0$ and $C^2 = 0$. For C has a polar factorization $C = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$. If $\hat{C} = D^*D$, then by Theorem 6.4, $\sigma(\hat{C}) \subseteq \sigma(C) = \{\lambda\}$. Thus $\hat{C}$ is a Hilbert space selfadjoint operator with spectrum $\{\lambda\}$, and so $\hat{C} = \lambda$. Hence

$$CD = \lambda D.$$

If $\lambda \neq 0$, then C is invertible, the range of D is all of $\mathfrak{K}$, and $C = \lambda$. If $\lambda = 0$, then $C^2 = D\hat{C}D^* = 0$, which proves the claim.

We also use some properties of the Calkin algebra in the Kreĭn space setting. The Calkin algebra is defined as $\mathfrak{L}(\mathfrak{H})/\mathfrak{C}(\mathfrak{H})$, where $\mathfrak{C}(\mathfrak{H})$ is the set of compact operators on $\mathfrak{H}$. The quotient mapping $\varphi : \mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})/\mathfrak{C}(\mathfrak{H})$ takes any fundamental symmetry on $\mathfrak{H}$ into a selfadjoint and unitary element of the algebra. If we represent the algebra as a space of operators on a Hilbert space $\mathfrak{K}$, we induce in this way a fundmental symmetry and hence a Kreĭn space structure for $\mathfrak{K}$. In this representation, the image of a Kreĭn space selfadjoint or positive operator on $\mathfrak{H}$ is again Kreĭn space selfadjoint or positive.

Now suppose that the spectrum of T reduces to a point, say $\sigma(T) = \{\lambda\}$. It can be assumed that $\mathfrak{H}$ is infinite dimensional, since otherwise the result is clear. For any polynomial p, $p(T) = p(A) + S$ with S compact, and hence $\varphi(p(A)) = \varphi(p(T))$. The spectrum of $p(T)$ is $\{p(\lambda)\}$, and since the Calkin algebra is nontrivial under our assumptions, the spectrum of $\varphi(p(T))$ is nonempty. Since $\sigma(\varphi(p(T))) \subseteq \sigma(p(T))$, it follows that $\sigma(\varphi(p(T))) = \{p(\lambda)\}$. Thus

$$\sigma(\varphi(p(A))) = \sigma(\varphi(p(T))) = \{p(\lambda)\}.$$

The choice $p(z) = z$ shows that $\sigma(\varphi(A)) = \{\lambda\}$. Since $\varphi(A)$ is selfadjoint, λ is real. Next take p to be a definitizing polynomial for A. Then $\varphi(p(A))$ is positive and its spectrum is $\{p(\lambda)\}$. By the preliminary observation about positive Kreĭn space operators with such spectra, either $p(A) - p(\lambda)$ or $p(A)^2$ is compact, since in both cases the image under φ is zero. But then T is polynomially compact, and the result follows from Lomonosov's theorem. □

Theorem 6.12 provides a natural framework for studying invariant subspace problems for Hilbert space operators as well. The next theorem generalizes a result of Radjavi and Rosenthal [1974].

Theorem 6.13 *Suppose that A and B are bounded operators on a Hilbert space $\mathfrak{K}$ with A selfadjoint and $B = C + K$, where $C \geq 0$ and K is in $\mathfrak{C}_p$, $1 \leq p < \infty$. Assume that AB is not a constant multiple of the identity, and that either B is selfadjoint or both A and B have dense range and zero kernel. Then AB and BA have proper closed hyperinvariant subspaces.*

Proof. For the case in which B is selfadjoint, if AB fails to have dense range or zero kernel, it is not hard to produce proper closed hyperinvariant subspaces for AB and $BA = (AB)^*$. Hence we may assume that AB has dense range and zero kernel. Then by the selfadjointness of A and B, both A and B have dense range and zero kernel. So we assume that this is the case.

Use the Bognár-Krámli factorization (Theorem 1.1) to write $A = DD^*$, where $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{K})$ for some Kreĭn space $\mathfrak{D}$ and $\ker D = \{0\}$. Then

$$(AB)D = D(D^*BD) \qquad D^*B(AB) = (D^*BD)D^*B,$$

and

$$(BA)BD = BD(D^*BD) \qquad D^*(BA) = (D^*BD)D^*.$$

Our hypotheses imply that the operators D, D^*, BD, D^*B have dense range and zero kernel. Therefore the relations show that AB and BA are both quasisimilar to D^*BD. Since $DD^*BD = ABD$ and AB is not a constant multiple of the identity, D^*BD is not a constant multiple of the identity. Now

$$D^*BD = D^*CD + D^*KD,$$

where D^*CD is a positive selfadjoint operator on the Kreĭn space $\mathfrak{D}$ and D^*KD in the same Schatten p-class as K. Hence by Theorem 6.12, D^*BD has a proper closed hyperinvariant subspace. By a theorem of Sz.-Nagy and Foiaş (see Radjavi Rosenthal [1973], p. 108), this implies that AB and BA also have proper closed hyperinvariant subspaces. $\square$

Theorem 6.14 *Suppose that A and B are bounded operators on a Hilbert space $\mathfrak{K}$ with A selfadjoint and B the sum of a positive operator and a compact operator in $\mathfrak{C}_p$, $1 \leq p < \infty$. Assume that A and B are not both zero. Set*

$$T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Then T has a proper closed hyperinvariant subspace.

Proof. The nontrivial case is when T has zero kernel and dense range. Then A and B both have dense range and zero kernel. We may assume that AB is not a constant multiple of the identity. For if $AB = c$, then $ABA = cA$ and so $BA = c$; thus $T^2 - c = 0$, and the result is elementary in this case.

By the identities

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ AB & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & A \\ AB & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ AB & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ AB & 0 \end{pmatrix},$$

T is quasisimilar to

$$R = \begin{pmatrix} 0 & 1 \\ AB & 0 \end{pmatrix}.$$

We show that R has a proper closed hyperinvariant subspace. By Theorem 6.13, AB has a proper closed hyperinvariant subspace $\mathfrak{N}$. We show that $\mathfrak{N} \oplus \mathfrak{N}$ is hyperinvariant for R. In fact, suppose that

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

commutes with R. Then

$$X_1 = X_4, \qquad X_4 AB = ABX_1, \qquad X_3 = ABX_2, \qquad X_2 AB = X_3.$$

So X_1, X_2, X_3, and X_4 all commute with AB and consequently leave $\mathfrak{N}$ invariant. Thus X leaves $\mathfrak{N} \oplus \mathfrak{N}$ invariant, and the result is proved. □

The operator matrix which appears in Theorem 6.14 may also be viewed as a compact perturbation of a Kreĭn space selfadjoint operator (see (6.7)). Such operator matrices are considered by Jonas [1993] in the study of the Klein-Gordon equation.

In Theorem 6.13, if $A = J$ is selfadjoint and unitary and B is selfadjoint, then AB is Kreĭn space selfadjoint relative to the Kreĭn space structure induced by J. In view of the very general nature of Kreĭn space selfadjoint operators (again see (6.7)), this points out an apparent need for some strong hypothesis in order to say anything meaningful about structural properties of products of Hilbert space selfadjoint operators.

Addendum: Nondefinitizable operators

There exist selfadjoint operators which are not definitizable and, in fact, which are not in the commutant of any nonzero positive operator. Examples can be constructed which further have a rich invariant subspace structure. Such examples underscore the difference between Hilbert space and Kreĭn space selfadjoint operators.

Let T be a normal operator on a Hilbert space $\mathfrak{K}$. Let $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{K}$ viewed as a Kreĭn space with fundamental symmetry

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The operator defined on $\mathfrak{H}$ by

$$A = \begin{pmatrix} T & 0 \\ 0 & T^{\times} \end{pmatrix}$$

is a special case of (6.7) and is therefore selfadjoint. Observe that the spectrum of A is the union of the spectrum of T and the spectrum of $T^{\times}$, which consists of conjugates of all points in the spectrum of T. Assume that the spectrum of T does not meet the real axis. Then the spectrum of A does not meet the real axis either. Now suppose that $B \in \mathfrak{L}(\mathfrak{H})$ is a positive operator and $A \in \{B\}'$. Choose a polar factorization $B = DD^*$, $D \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H})$. Let $\hat{A} = \pi(A)$, where π is the mapping defined in Corollary 6.3. Then $\mathfrak{D}$ is a Hilbert space since B is positive, and therefore $\hat{A} \in \mathfrak{L}(\mathfrak{D})$ is Hilbert space selfadjoint by Theorem 6.4. By the same result, $\sigma(\hat{A}) \subseteq \sigma(A) \cap \mathbf{R}$. This means that $\hat{A}$ has empty spectrum. It follows that $\mathfrak{D} = \{0\}$, and this implies that $B = 0$. In other words, A does not commute with any nonzero positive operator.

More generally, the same conclusion follows if $\sigma(T) \cap \mathbf{R}$ is a finite set and these points are not eigenvalues for T. For in the preceding notation, $\sigma(A) \cap \mathbf{R} = \sigma(T) \cap \mathbf{R}$, and no point in this set is an eigenvalue for A. Since $\sigma(\hat{A}) \subseteq \sigma(A) \cap \mathbf{R}$, the spectrum of $\hat{A}$ consists of eigenvalues, and by Theorem 6.4, $\sigma_p(\hat{A}) \subseteq \sigma_p(A) \cap \mathbf{R}$. Therefore the spectrum of $\hat{A}$ is empty, and the conclusion follows as before.

Bibliography

Alpay, D., Dijksma, A., van der Ploeg, J. and de Snoo, H. S. V. [1992] *Holomorphic operators between Krein spaces and the number of squares of associated kernels*, Oper. Theory: Adv. Appl., vol. 59, Birkhäuser, Basel-Boston, pp. 11–29.

Alpay, D., Dijksma, A., Rovnyak, J. and de Snoo, H. S. V. [in preparation] *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*.

Alpay, D., Dijksma, A. and de Snoo, H. S. V. [1994] *Reproducing kernel Pontryagin spaces, operator colligations and operator models for quasicontractions*, preprint.

Alpay, D. and Dym, H. [1986] *On applications of reproducing kernel spaces to the Schur algorithm and rational J unitary factorization*, Oper. Theory: Adv. Appl., vol. 18, Birkhäuser, Basel-Boston, pp. 89–159; MR 89g:46051.

_____ [1992] *On reproducing kernel spaces, the Schur algorithm, and interpolation in a general class of domains*, Oper. Theory: Adv. Appl., vol. 59, Birkhäuser, Basel-Boston, pp. 30–37; MR 94j:46034.

Andô, T. [1979] *Linear Operators on Krein Spaces*, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo.

_____ [1990] *de Branges Spaces and Analytic Operator Functions*, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo.

Arov, D. Z. and Grossman, I. V. [1983] *Scattering matrices in the theory of extensions of isometric operators*, Dokl. Akad. Nauk. SSSR **270**, 17–20; English transl. in Soviet Math. Dokl. **27** (1983), 518–522; MR 85c:47008.

_____ [1992] *Scattering matrices in the theory of unitary extensions of isometric operators*, Math. Nachr. **157**, 105–123; MR 94i:47019.

Arsene, Gr., Constantinescu, T. and Gheondea, A. [1987] *Lifting of operators and prescribed numbers of negative squares*, Michigan Math. J. **34**, 201–216; MR 88j:47008.

Arsene, Gr., and Gheondea, A. [1982] *Completing matrix contractions*, J. Operator Theory **7**, 179–189; MR 83i:47010.

Azizov, T. Ya. and Iokhvidov, I. S. [1989] *Linear Operators in Spaces with an Indefinite Metric*, "Nauka", Moscow, 1986; MR 88g:47070; Engl. transl., Wiley, New York; MR 90j:47042.

Bakonyi, M. and Constantinescu, T. [1992] *Schur's Algorithm and Several Applications*, Pitman Research Notes, vol. 261, Longman Scientific and Technical, Harlow (co-publisher, Wiley, New York); MR 94b:47022.

Ball, J. A. and Cohen, N. [1991] *de Branges-Rovnyak operator models and systems theory: a survey*, Oper. Theory: Adv. Appl., vol. 50, Birkhäuser, Basel-Boston, pp. 93–136; MR 93a:47011.

Ball, J. A., Gohberg, I. and Rodman, L. [1990] *Interpolation of Rational Matrix Functions*, Oper. Theory: Adv. Appl., vol. 45, Birkhäuser, Basel-Boston; MR 92m:47027.

Ball, J. A. and Helton, J.W. [1983] *A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most interpolation theory*, J. Operator Theory **9**, 107–142; MR 84m:47046.

Ball, J. A. and Kriete, T. L. III [1987] *Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators*, Integral Equations and Operator Theory **10**, 17–61; MR 88a:47013.

Bart, H., Gohberg, I. and Kaashoek, M. A. [1979] *Minimal factorization of matrix and operator functions*, Oper. Theory: Adv. Appl., vol. 1, Birkhäuser, Basel-Boston; MR 81a:47001.

Bognár, J. [1974] *Indefinite Inner Product Spaces*, Ergeb. Math. Grenzgeb., Bd. 78, Springer-Verlag, Berlin-New York; MR 57#7125.

——— [1987] *An approach to the spectral decomposition of J-positizable operators*, J. Operator Theory **17**, 309–326 MR 88g:47071.

de Branges, L. [1988a] *Complementation in Krein spaces*, Trans. Amer. Math. Soc. **305**, 277–291; MR 89c:46034.

——— [1988b] *Krein spaces of analytic functions*, J. Funct. Anal. **81**, 219–259; MR 90c:47060.

——— [1991] *A construction of Krein spaces of analytic functions*, J. Funct. Anal. **98**, 1–41; MR 92k:46031.

——— [1994] *Factorization in Krein spaces*, J. Funct. Anal. **124**, 228–262.

de Branges, L. and Rovnyak, J. [1966a] *Square Summable Power Series*, Holt, Rinehart and Winston, New York; MR 35#5909.

——— [1966b] *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), pp. 295–392; MR 39#6109.

Brodskiĭ, M. S. [1978] *Unitary operator colligations and their characteristic functions*, Uspekhi Mat. Nauk **33**, no. 4, 141–168; English transl. Russian Math. Surveys **33** (1978), no. 4, 159–191; MR 80e:47010.

Christner, G. [1993] *Applications of the Extension Properties of Operators on Krein Spaces*, Dissertation, University of Virginia.

Christner, G., Li, Kin Y. and Rovnyak, J. [1994] *Julia Operators and coefficient problems*, Oper. Theory Adv. Appl., vol. 73, Birkhäuser, Basel-Boston, pp. 138–181.

Christner, G. and Rovnyak, J. [to appear] *Julia operators and the Schur algorithm*, Proceedings of International Conference on Operator Theory and Harmonic Analysis (Caracas, 1994).

Constantinescu, T. and Gheondea, A. [1989] *Minimal signature in lifting of operators. I*, J. Operator Theory **22**, 345–367; MR 91b:47013.

_____[1992] *Minimal signature in lifting of operators. II*, J. Funct. Anal. **103**, 317–351; MR 93c:47041.

_____[1993] *Elementary rotations of linear operators in Krein spaces*, J. Operator Theory **29**, 167–203.

Conway, J. B. [1990] *A Course in Functional Analysis*, 2nd ed., Springer-Verlag, Berlin-New York; MR 91e:46001.

Ćurgus, B., Dijksma, A., Langer, H. and de Snoo, H. S. V. [1988] *Characteristic functions of unitary colligations and of bounded operators in Krein spaces*, Oper. Theory: Adv. Appl., vol. 41, Birkhäuser, Basel-Boston, pp. 125–152; MR 91c:47020.

Davis, Ch. [1970] *J-unitary dilation of a general operator*, Acta Sci. Math. (Szeged) **31**, 75–86; MR 41#9032.

_____[1976] *An extremal problem for extensions of a sesquilinear form*, Linear Algebra and Appl. **13**, 91–102; MR 52#15068.

Dijksma, A., Dritschel, M. A., Marcantognini, S. and de Snoo, H. S. V. [1993] *The commutant lifting theorem for contractions on Krein spaces*, Oper. Theory: Adv. Appl., vol. 61, Birkhäuser, Basel-Boston, pp. 65–83.

Dijksma, A., Langer, H. and de Snoo, H. S. V. [1986] *Characteristic functions of unitary operator colligations in Π_κ-spaces*, Oper. Theory: Adv. Appl., vol. 19, Birkhäuser, Basel-Boston, pp. 125–194; MR 89a:47053.

Dijksma, A., Marcantognini, S. A. M. and de Snoo, H. S. V. [1994] *A Schur type analysis of the minimal unitary Hilbert space extensions of a Kreĭn space isometry whose defect subspaces are Hilbert spaces*, Z. Anal. Anwendungen **13**, 233–260.

Dritschel, M. A. [1993a] *The essential uniqueness property for linear operators in Kreĭn spaces*, J. Funct. Anal. **118**, 198–248; MR 94j:47051.

_____[1993b] *A method for constructing invariant subspaces for some operators on Krein spaces*, Oper. Theory: Adv. Appl., vol. 61, Birkhäuser, Basel-Boston, pp. 85–113.

_____[1993c] *Commutant lifting when the intertwining operator is not necessarily a contraction*, unpublished manuscript.

_____[to appear] *Compact perturbations of operators on Kreĭn spaces*, Proceedings of International Conference on Operator Theory and Harmonic Analysis (Caracas, 1994).

Dritschel, M. A. and Rovnyak, J. [1990] *Extension theorems for contraction operators on Kreĭn spaces*, Oper. Theory: Adv. Appl., vol. 47, Birkhäuser, Basel-Boston, pp. 221–305; MR 92m:47068.

——— [1991] *Julia operators and complementation in Kreĭn spaces*, Indiana Univ. Math. J. **40**, 885–901; MR 93b:46042.

Dubovoj, V. K., Fritzsche, B. and Kirstein, B. [1992] *Matricial Version of the Classical Schur Problem*, Teubner-Texte zur Mathematik, vol. 129, B. G. Teubner Verlagsgesellschaft, Stuttgart; MR 93e:47021.

Dym, H. [1989] *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces, and Interpolation*, CBMS Regional Conference Series in Mathematics, vol. 71, Amer. Math. Soc., Providence, RI; MR 90g:47003.

Foiaş, C. and Frazho, A. [1990] *The Commutant Lifting Approach to Interpolation Problems*, Oper. Theory: Adv. Appl., vol. 44, Birkhäuser, Basel-Boston; MR 92k:47033.

Gohberg, I., Lancaster, P. and Rodman, L. [1983] *Matrices and Indefinite Scalar Products*, Oper. Theory: Adv. Appl., vol. 8, Birkhäuser, Basel-Boston; MR 87j:15001.

Halmos, P. R. [1951] *Introduction to Hilbert Spaces*, Chelsea, New York;MR 13,563a.

Hassibi, B., Sayed, A. H. and Kailath, T. [1994] *Kalman filtering in Krein spaces-Parts I: Theory; II: Applications*, preprint.

Helton, J. W., with the assistance of J. A. Ball, C. R. Johnson, and J. N. Palmer [1987] *Operator Theory, Analytic Functions, Matrices, and Electrical Engineering*, CBMS Regional Conference Series in Mathematics, vol. 68, Amer. Math. Soc., Providence, RI; MR 89f:47001.

Herglotz, G., Schur, I. et al. [1991] *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis* (B. Fritzsche and B. Kirstein, eds.), B. G. Teubner, Stuttgart; MR 93d:01095.

Iokhvidov, I. S., Kreĭn, M. G. and Langer, H. [1982] *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag, Berlin; MR 85g:47050.

Ivanchenko, T. [1979] *The Schur problem for J-contractive matrix functions*, Technical Report, 31/7/79, VINITI #2868–79.

Jonas, P. [1981] *On the functional calculus and the spectral function for definitizable operators in Kreĭn space*, Beiträge Anal. **16**, 121–135; MR 83i:47047.

——— [1982] *Compact perturbations of definitizable operators. II*, J. Operator Theory **8**, 3–18; MR 84d:47046.

——— [1988] *On a class of selfadjoint operators in Krein space and their compact perturbations*, Integral Equations and Operator Theory **11**, 351–384; MR 89f:47052.

——— [1993] *On the spectral theory of operators associated with perturbed Klein-Gordon and wave type equations*, J. Operator Theory **29**, 207–224.

Jonas, P. and Langer, H. [1979] *Compact perturbations of definitizable operators*, J. Operator Theory **2**, 63–77; MR 81c:47039.

——— [1983] *Some questions in the perturbation theory of J-nonnegative operators in Kreĭn spaces*, Math. Nachr. **114**, 205–226; MR 85j:47045.

Kreĭn , M. G. and Langer, H. [1973] *Über die Q-Funktion eines π-hermiteschen Operators im Raume Π_κ*, Acta Sci. Math. (Szeged) **34**, 191–230; MR 47#7504.

——[1977] *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhangen. I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. **77**, 187–236; MR 57#1173; *II. Verallgemeinerte Resolventen, u-Resolventen und ganze Operatoren*, J. Funct. Anal. **30**, 390–447; MR 80h:47045; III. *Indefinite analogues of the Hamburger and Stieltjes moment problems. Part* I, Beiträge Anal. **14**, 25–40; MR 83b:47047a; III. *Indefinite analogues of the Hamburger and Stieltjes moment problems. Part* II, Beiträge Anal. **15**, 27–45; MR 83b: 47047b; IV. *Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions*, J. Operator Theory **13**, 299–417; MR 87h:47084.

——[1981] *Some propositions on analytic matrix functions related to the theory of operators in the space Π_κ*, Acta Sci. Math. (Szeged) **43**, 181–205; MR 82i:47053.

Langer, H. [1965] *Spektraltheorie Linearer Operatoren in J-Räumen und Einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 I + \lambda B + C$*, Habilitationsschrift, Fakultät für Mathematik und Naturwissenschaften der Technischen Universität Dresden, Dresden.

——[1971] *Invariante Teilräume definisierbarer J-selbstadjungierter Operatoren*, Ann. Acad. Sci. Fenn. Ser. A I Math., no. 475, 23 pages; MR 44#4553.

——[1982] *Spectral functions of definitizable operators in Krein spaces*, Functional Analysis (Dubrovnik, 1981) Lecture Notes in Math., vol. 948, Springer-Verlag, Berlin-New York, pp. 1–46; MR 84g:47034.

Langer, H. and Najman, B. [1983] *Perturbation theory for definitizable operators in Krein spaces*, J. Operator Theory **9**, 297–317; MR 85e:47053.

McEnnis, B. [1981] *Models for operators with bounded characteristic function*, Acta Sci. Math. (Szeged) **43**, 71–90; MR 83b:47012.

——[1990] *Shifts on Krein spaces*, Operator Theory: Operator Algebras and Applications, Part 2 (Durham, NH, 1988), Amer. Math. Soc., Providence, RI, pp. 201–211; MR 91j:47038.

Marcantognini, S. A. M. [1990] *Unitary colligations of operators in Krein spaces*, Integral Equations and Operator Theory **13**, 701–727; MR 92f:47010.

——[1992] *The commutant lifting theorem in the Krein space setting: a proof based on the coupling method*, Indiana Univ. Math. J. **41**, 1303–1314; MR 93k:47009.

Najman, B. [1980] *Perturbation theory for selfadjoint operators in Pontrjagin spaces. I*, Glas. Mat. Ser. III **15(35)**, 351–371; MR 82e:47048.

Nikol′skiĭ , N. K. and Vasyunin, V. I. [1986] *Notes on two function models*, The Bieberbach Conjecture (West Lafayette, Ind., 1985), Amer. Math. Soc., Providence, RI, pp. 113–141; MR 91c:47017.

——[1989] *A unified approach to function models, and the transcription problem*, Oper. Theory: Adv. Appl., vol. 41, Birkhäuser, Basel-Boston, pp. 405–434; MR 91c:47017.

Parrott, S. [1978] *On a quotient norm and the Sz.-Nagy-Foiaş lifting theorem*, J. Funct. Anal. **30**, 311–328; MR 81h:47006.

Phillips, R. S. [1961] *The extension of dual subspaces invariant under an algebra*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem (Pergamon, Oxford), pp. 366–398; MR 24#A3512.

Pontryagin, L. S. [1944] *Hermitian operators in spaces with indefinite metric*, Izv. Akad. Nauk SSSR Ser. Mat. **8**, 243–280; MR 6,273.

Radjavi, H. and Rosenthal, P. [1973] *Invariant Subspaces*, Ergeb. Math. Grenzgeb., Bd. 77, Springer-Verlag, New York-Heidelberg; MR 51#3924.

——— [1974] *Invariant subspaces for products of Hermitian operators*, Proc. Amer. Math. Soc. **43**, 483–484; MR 58#23663.

Rosenblum, M. and Rovnyak, J. [1985] *Hardy Classes and Operator Theory*, Oxford Mathematical Monographs, Oxford University Press, New York; MR 87e:47001.

Rovnyak, J. [1994] Review of *Schur's Algorithm and Several Applications* by M. Bakonyi and T. Constantinescu, Bull. Amer. Math. Soc. (N.S.) **30**, 270–276.

Rudin, W. [1974] *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York; MR 49#8783.

Sarason, D. [1994] *Sub-Hardy Hilbert Spaces in the Unit Disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York.

Schur, I. [1917] *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind I*, J. Reine Angew. Math. **147**, 205–232; II, ibid. **148** (1918), Gesammelte Abhandlungen, vol. II, nos. 29, 30.

Shmul′yan, Yu. L. [1967] *Division in the class of J-expansive operators*, Mat. Sb. (N.S.) **74 (116)**, 516–525; Math. USSR-Sbornik **3**, 471–479; MR 37#784.

Sorjonen, P. [1975] *Pontrjaginräume mit einem reproduzierenden kern*, Ann. Acad. Sci. Fenn. Ser. A I Math., no. 594, 30 pages; MR 53#8875.

Sz.-Nagy, B. and Foiaş, C. [1970] *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York; MR 43#947.

Yang, A. [1994] *A construction of unitary linear systems*, Integral Equations and Operator Theory **19**, 477–499.

Index