ON THE COEFFICIENTS OF RIEMANN MAPPINGS OF THE UNIT DISK INTO ITSELF

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To Tsuyoshi Ando, with best wishes on the occasion of his 60-th birthday.

Two methods to construct inequalities are compared and investigated for their potential to characterize initial segments of coefficients. One is based on special function theory as in the proof of the Bieberbach conjecture and its power generalizations. While this method produces the best coefficient estimates to date, it is shown by an example that the method cannot, in its present form, characterize initial segments of coefficients.

1 Introduction

By a *Riemann mapping* we mean a function which is analytic and one-to-one on the open unit disk $|z| < 1$. Such a function is said to be *normalized* if it has value zero and positive derivative at the origin. We are concerned with normalized Riemann mappings of the unit disk into itself. The conformal contraction of the unit disk by such a mapping, say $B(z)$, implies analytical properties of the function. Of particular interest are inequalities involving the coefficients in the expansion $B(z) = B_1 z + B_2 z^2 + \cdots$. Such inequalities impose constraints on the numbers $B_1, \ldots, B_r$. It is an open problem to find necessary and sufficient conditions that given numbers arise as an initial segment of coefficients of a normalized Riemann mapping of the unit disk into itself.

We compare two existing methods which at first glance appear to be different but on closer inspection turn out to have a similar form. The first is of a geometric nature and draws on an analogy with the Carathéodory-Fejér interpolation problem [**3,4**]. It is based on a study of operators of the form $T : f(z) \mapsto f(B(z))$ acting on generalized power series $f(z) = \sum_{n=1}^{\infty} a_n z^{\nu+n}$. If $B(z)$ is a normalized Riemann mapping of the unit disk into itself, then substitution by $B(z)$ is contractive relative to certain quadratic forms. A proof using Loewner's differential equation is given in [**15**]. In this paper we give a new and more elementary argument in Theorem 2.2. Theorem 2.2 implies necessary conditions that given numbers $B_1, \ldots, B_r$ ($B_1 > 0$) occur as the initial segment of coefficients of a normalized Riemann mapping of the unit disk into itself. When $r = 2$, the conditions have a symmetry, and this is useful in proving that the necessary conditions are sufficient

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[**10,11**]. We extend the symmetry to an arbitrary positive integer r, but it is unknown if the necessary conditions are sufficient in general.

The second method is analytical and based on de Branges' proof of Milin's conjecture [**5**] which depends on Loewner's differential equation and special function theory. We use, in fact, stronger inequalities for powers or Riemann mappings [**6,15**]. Nevertheless the resulting conditions on coefficients have a similar character to those derived from the first method [**2**]. We answer, in the negative, the question if necessary conditions derived in this way are capable of characterizing initial segments of coefficients. This is done by exhibiting numbers B_1, B_2, B_3 ($B_1 > 0$) which satisfy all conditions obtainable by the method in its present form, yet which are not the first three coefficients of a normalized Riemann mapping of the unit disk into itself because they do not satisfy the conditions obtained by the first method.

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2 Contractive substitution property

For any real ν and sequence $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ of real numbers, the *Grunsky space* $\mathcal{G}_\sigma^\nu$ is the space of all generalized power series $f(z) = \sum_{n=1}^{\infty} a_n z^{\nu+n}$ with complex coefficients such that $\sum_{n=1}^{\infty} |(\nu+n)\sigma_n||a_n|^2 < \infty$. A scalar product is defined in the space by

$$\langle f(z), g(z)\rangle_{\mathcal{G}_\sigma^\nu} = \sum_{n=1}^{\infty} (\nu+n)\sigma_n a_n \overline{c_n}$$

if $f(z) = \sum_{n=1}^{\infty} a_n z^{\nu+n}$ and $g(z) = \sum_{n=1}^{\infty} c_n z^{\nu+n}$. We identify two elements if their coefficients coincide for all indices n such that $(\nu+n)\sigma_n \neq 0$. Then $\mathcal{G}_\sigma^\nu$ has the structure of a Kreĭn space [**1**]. Let $\mathcal{D}_\nu$ be $\mathcal{G}_\sigma^\nu$ with $\sigma_n = 1$ for all n, so $\mathcal{D}_0$ is the *Dirichlet space*. For any real number ν and positive integer r, let $\mathcal{D}_\nu^r$ be $\mathcal{G}_\sigma^\nu$ with $\sigma_n = 1$ or 0 according as $1 \leq n \leq r$ or $n > r$.

THEOREM 2.1 *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces having finite and equal negative indices, and let T be a continuous operator on $\mathcal{H}$ to $\mathcal{K}$. If T is a contraction, so is T^*.*

Our notation for Kreĭn spaces follows [**8**]. In Theorem 2.1, the *adjoint* T^* of T is defined so that $\langle Tf, g\rangle_{\mathcal{K}} = \langle f, T^* g\rangle_{\mathcal{H}}$ for all f in $\mathcal{H}$ and g in $\mathcal{K}$. We call T a *contraction* if $\langle Tf, Tf\rangle_{\mathcal{K}} \leq \langle f, f\rangle_{\mathcal{H}}$ for all f in $\mathcal{H}$. Theorem 2.1 is a well-known result proved in [**8, Theorem 1.3.7**] (for the essential case $\mathcal{H} = \mathcal{K}$).

We apply Theorem 2.1 to operators defined by substituting a formal power series $B(z) = B_1 z + B_2 z^2 + \cdots$ ($B_1 > 0$) in a generalized power series $f(z) = \sum_{n=1}^{\infty} a_n z^{\nu+n}$. The result is a series $f(B(z)) = \sum_{n=1}^{\infty} a_n B(z)^{\nu+n} = \sum_{n=1}^{\infty} b_n z^{\nu+n}$ of the same form. It is computed from the expansion

$$B(z)^\nu = B_0(\nu) z^\nu + B_1(\nu) z^{\nu+1} + \cdots, \tag{1}$$

where [**15**]

$$(2)\qquad \begin{cases} B_0(\nu) &= B_1^\nu, \\ B_1(\nu) &= \nu B_1^{\nu-1} B_2, \\ B_2(\nu) &= \nu B_1^{\nu-1}[B_3 + \frac{1}{2}(\nu-1)B_1^{-1}B_2^2], \\ & \cdots \end{cases}$$

In our applications, we shall not distinguish between a normalized Riemann mapping $B(z)$ and the formal power series determined from its sequence of Taylor coefficients.

THEOREM 2.2 *Let $B(z)$ be a normalized Riemann mapping of the unit disk into itself, and let $\mathcal{G}_\sigma^\nu$ be any Grunsky space with ν real and $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Then the operator*

$$(3)\qquad T : f(z) \;\mapsto\; f(B(z))$$

is an everywhere defined and continuous contraction of $\mathcal{G}_\sigma^\nu$ into itself.

Previous versions appear in [**2,15**]. The novel feature in our proof is the connection with Theorem 2.1. Interesting functional analysis aspects of such results and more complex methods have recently been discussed by V. I. Vasyunin and N. K. Nikol'skiĭ[**13**].

PROOF. Continuity is automatic by the closed graph theorem once it is known that the operator is everywhere defined. We first prove the result for $\mathcal{D}_\nu$ (following [**2**]). Let C_ρ be the positively oriented circle $|z| = \rho,\ 0 < \rho < 1$. By Schwarz's lemma, $B(C_\rho)$ lies inside C_ρ. Choose a radial slit S from a point of maximum modulus on $B(C_\rho)$ to C_ρ and a branch of z^ν on the region G bounded by $\Gamma = S + C_\rho - S - B(C_\rho)$. Then any element $f(z) = z^\nu \sum_{n=1}^\infty a_n z^n$ of $\mathcal{D}_\nu$ represents a function which is analytic on G and has a continuous extension to $\overline{G}$, and the derivative of the function has the same properties. Writing $f(B(z)) = z^\nu \sum_{n=1}^\infty b_n z^n$, we obtain by Green's Theorem [**12, p. 241**],

$$\begin{aligned} 0 &\leq \frac{1}{\pi} \iint_G |f'(z)|^2 \, dx\, dy \\ &= \frac{1}{2\pi i} \int_\Gamma f'(z)\overline{f(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{C_\rho} f'(z)\overline{f(z)}\, dz - \frac{1}{2\pi i} \int_{B(C_\rho)} f'(z)\overline{f(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{C_\rho} f'(z)\overline{f(z)}\, dz - \frac{1}{2\pi i} \int_{C_\rho} f'(B(w))\overline{f(B(w))}B'(w)\, dw \\ &= \sum_{n=1}^\infty (\nu+n)\rho^{2\nu+2n}(|a_n|^2 - |b_n|^2). \end{aligned}$$

Passage to the limit with $\rho \uparrow 1$ yields

$$(4)\qquad \langle f(B(z)), f(B(z))\rangle_{\mathcal{G}_\sigma^\nu} \;\leq\; \langle f(z), f(z)\rangle_{\mathcal{G}_\sigma^\nu}$$

when $\mathcal{G}_\sigma^\nu$ is $\mathcal{D}_\nu$.

Next let $\mathcal{G}_\sigma^\nu$ be $\mathcal{D}_\nu^r$ where r is a positive integer. Let T be the operator (3) viewed as acting on $\mathcal{D}_\nu$ and T_r the same operator but viewed as acting on $\mathcal{D}_\nu^r$. In a natural way, we may think of $\mathcal{D}_\nu^r$ as a subspace of $\mathcal{D}_\nu$. Let P_r be the projection of $\mathcal{D}_\nu$ on $\mathcal{D}_\nu^r$. Then $T_r = P_r T|\mathcal{D}_\nu^r$. The orthogonal complement of $\mathcal{D}_\nu^r$ in $\mathcal{D}_\nu$ is invariant under T, and so $\mathcal{D}_\nu^r$ is invariant under T^*. Therefore $T_r^* = T^*|\mathcal{D}_\nu^r$. Since T is a contraction by the first part of the proof, T^* is a contraction by Theorem 2.1. Hence its restriction T_r^* is a contraction, and so T_r is a contraction by Theorem 2.1. Thus the result holds for $\mathcal{D}_\nu^r$.

In the general case, it is enough to prove (4) when $\sigma_n = 0$ for large n. This follows from the identity

$$\sum_{n=1}^{\infty}(\nu+n)\sigma_n|c_n|^2 = \sum_{n=1}^{\infty}(\sigma_n-\sigma_{n+1})\sum_{j=1}^{n}(\nu+j)|c_j|^2 \tag{5}$$

and the special case of (4) for the spaces $\mathcal{D}_\nu^r$ proved above.

□

It is known that Theorem 2.2 applied only to the Dirichlet space is a weak result in the sense that it is not characteristic of the class of normalized Riemann mappings of the unit disk into itself. An example is $B(z) = \epsilon z(1+\rho z)$, where $\epsilon > 0, \rho > 1$, and $\epsilon^2(1+\rho)^2 \leq \frac{1}{3}$. This function is bounded by 1 in the unit disk but not a Riemann mapping. The operator (3) is a contraction on $\mathcal{D}_0$. For if $f(z) = \sum_{n=1}^{\infty} a_n z^{\nu+n}$ is in $\mathcal{D}_0$, then

$$f(B(z)) = \sum_{n=1}^{\infty}\sum_{j=1}^{n}\epsilon^j\rho^{n-j}\binom{j}{n-j}a_j z^n.$$

Writing $c_n = \sqrt{n}a_n$, we bring the inequality to be proved to the form

$$\sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\frac{1}{2}}\epsilon^j\rho^{n-j}\binom{j}{n-j}c_j\right|^2 \leq \sum_{n=1}^{\infty}|c_n|^2.$$

In fact, the Hilbert-Schmidt norm of the coefficient matrix is at most 1:

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left|\left(\frac{n}{j}\right)^{\frac{1}{2}}\epsilon^j\rho^{n-j}\binom{j}{n-j}\right|^2 = \sum_{j=1}^{\infty}\sum_{n=j}^{2j}\frac{n}{j}\epsilon^{2j}\rho^{2n-2j}\binom{j}{n-j}^2$$

$$\leq 2\sum_{j=1}^{\infty}\sum_{n=j}^{2j}\epsilon^{2j}\rho^{2n-2j}\binom{j}{n-j}^2 = 2\sum_{j=1}^{\infty}\epsilon^{2j}\sum_{n=0}^{j}\rho^{2n}\binom{j}{n}^2 \leq 2\sum_{j=1}^{\infty}\epsilon^{2j}(1+\rho)^{2j} \leq 1.$$

Therefore (3) is a contraction on $\mathcal{D}_0$. But $f(z) = z^{-1}$ is in $\mathcal{D}_{-2}$ and $f(B(z))$ is not in $\mathcal{D}_{-2}$, so the conclusion of Theorem 2.2 breaks down for $\mathcal{D}_{-2}$. In contrast, de Branges **[2]** has shown that the conclusion of Theorem 2.2 holds for every space $\mathcal{D}_\nu$ with ν a nonpositive integer only if $B(z)$ represents a normalized Riemann mapping of the unit disk into itself.

The contractive substitution property has an equivalent form as an estimate of $[B(z)^\nu - B'(0)^\nu z^\nu]/\nu$ for any real ν. If $\nu = 0$ this expression reduces to $\log B(z)/[zB'(0)]$, which appears in a related estimate equivalent to Milin's conjecture **[5]**.

THEOREM 2.3 *Let* $B(z) = B_1 z + B_2 z^2 + \cdots$ $(B_1 > 0)$ *be a formal power series, and consider sequences* $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ *and* $\tau = \{\sigma_2, \sigma_3, \ldots\}$ *where* $\sigma_1, \sigma_2, \ldots$ *are real numbers. The following conditions are equivalent.*
(i) For any real ν, $f(z)$ *in* $\mathcal{G}_\sigma^\nu$ *implies* $f(B(z))$ *in* $\mathcal{G}_\sigma^\nu$ *and*

$$\langle f(B(z)), f(B(z))\rangle_{\mathcal{G}_\sigma^\nu} \le \langle f(z), f(z)\rangle_{\mathcal{G}_\sigma^\nu}. \tag{6}$$

(ii) For any real ν, $g(z)$ *in* $\mathcal{G}_\tau^\nu$ *implies* $\nu^{-1}[B(z)^\nu - B'(0)^\nu z^\nu] + g(B(z))$ *in* $\mathcal{G}_\tau^\nu$ *and*

$$\left\langle \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)), \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)) \right\rangle_{\mathcal{G}_\tau^\nu} - \langle g(z), g(z)\rangle_{\mathcal{G}_\tau^\nu} \le \frac{1 - B'(0)^{2\nu}}{\nu}\sigma_1. \tag{7}$$

PROOF. Set $\mu = \nu + 1$. If $g(z)$ is in $\mathcal{G}_\tau^\mu$ and c_1 is a constant, then

$$f(z) \;=\; c_1 z^{\nu+1} + g(z) \tag{8}$$

belongs to $\mathcal{G}_\sigma^\nu$. Every element of $\mathcal{G}_\sigma^\nu$ has this form, and $\langle f(z), f(z)\rangle_{\mathcal{G}_\sigma^\nu}$ may be computed as $\mu\sigma_1|c_1|^2 + \langle g(z), g(z)\rangle_{\mathcal{G}_\tau^\mu}$. Assume (i). If $g(z)$ is in $\mathcal{G}_\tau^\mu$ and $\nu \ne -1$, we may choose $c_1 = 1/(\nu+1)$ in (8). Then by (i),

$$f(B(z)) = \frac{1}{\nu+1}B(z)^{\nu+1} + g(B(z)) = \frac{1}{\mu}B'(0)^\mu z^\mu + \frac{B(z)^\mu - B'(0)^\mu z^\mu}{\mu} + g(B(z))$$

belongs to $\mathcal{G}_\sigma^\nu$ and (6) holds. Therefore $\mu^{-1}[B(z)^\mu - B'(0)^\mu z^\mu] + g(B(z))$ belongs to $\mathcal{G}_\tau^\mu$ and

$$\frac{1}{\mu}B'(0)^{2\mu}\sigma_1 + \left\langle \frac{B(z)^\mu - B'(0)^\mu z^\mu}{\mu} + g(B(z)), \frac{B(z)^\mu - B'(0)^\mu z^\mu}{\mu} + g(B(z)) \right\rangle_{\mathcal{G}_\tau^\mu} \le \frac{1}{\mu}\sigma_1 + \langle g(z), g(z)\rangle_{\mathcal{G}_\tau^\mu} \quad .$$

This is equivalent to (7) with ν replaced by μ. When $\nu = -1$, that is $\mu = 0$, the same inequality holds as a limiting case. Therefore (ii) holds.

Assume (ii). Apply (ii) with ν replaced by μ and $g(z)$ by $(\nu+1)^{-1}c_1^{-1}g(z)$. Multiplying the inequality derived from (7) in this way by $(\nu+1)^2|c_1|^2$, we get

$$\begin{aligned}&\langle [B(z)^{\nu+1} - B'(0)^{\nu+1}z^{\nu+1}]c_1 + g(B(z)), [B(z)^{\nu+1} - B'(0)^{\nu+1}z^{\nu+1}]c_1 + g(B(z))\rangle_{\mathcal{G}_\tau^\mu} \\ &\quad - \langle g(z), g(z)\rangle_{\mathcal{G}_\tau^\mu} \le (\nu+1)\sigma_1|c_1|^2 - (\nu+1)\sigma_1 B'(0)^{2\nu+2}|c_1|^2.\end{aligned}$$

Thus if $f(z)$ is the element (8) of $\mathcal{G}_\sigma^\nu$, then

$$\begin{aligned} f(B(z)) &= c_1 B(z)^{\nu+1} + g(B(z)) \\ &= c_1 B'(0)^{\nu+1}z^{\nu+1} + [B(z)^{\nu+1} - B'(0)^{\nu+1}z^{\nu+1}]c_1 + g(B(z)) \end{aligned}$$

belongs to $\mathcal{G}_\sigma^\nu$ and (6) holds. The case $(\nu+1)c_1 = 0$ is handled by an approximation argument. Thus (ii) implies (i).

□

The preceding results may be used to simplify some arguments in [**15**]. Let $B(z)$ be a normalized Riemann mapping of the unit disk into itself. Let ν be any real number, and let $\mathcal{G}^\nu_\sigma$ be a Grunsky space such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Define $\mathcal{G}(\nu, \sigma, B)$ to be the space of all elements $f(z)$ of $\mathcal{G}^\nu_\sigma$ such that

$$\sup \left\{ \langle f(z) + g(B(z)), f(z) + g(B(z)) \rangle_{\mathcal{G}^\nu_\sigma} - \langle g(z), g(z) \rangle_{\mathcal{G}^\nu_\sigma} \right\} < \infty,$$

where the supremum is over all $g(z)$ in $\mathcal{G}^\nu_\sigma$. Then $\mathcal{G}(\nu, \sigma, B)$ is a Hilbert space in a norm such that the supremum is $||f(z)||^2_{\mathcal{G}(\nu,\sigma,B)}$. We can simplify the proof that $\mathcal{G}(\nu, \sigma, B)$ is a Hilbert space given in [**15, Cor. 3.8**]. By the discussion preceding [**15, Theorem 3.4**] or general properties of complementation [**7,9**], all that is needed is to show that $1 - TT^*$ is a nonnegative operator, where $T : f(z) \to f(B(z))$ in $\mathcal{G}^\nu_\sigma$. But T is contractive by Theorem 2.2, hence T^* is contractive by Theorem 2.1, and this gives the result.

Another simplification is to [**15, Theorem 3.4**]:

COROLLARY 2.4 *Let $B(z)$ be a normalized Riemann mapping of the unit disk into itself. For any real number ν, let $\mathcal{G}^\nu_\sigma$ be a Grunsky space such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Then $[B(z)^\nu - B'(0)^\nu z^\nu]/\nu$ belongs to $\mathcal{G}(\nu, \sigma, B)$ and*

$$\left\| \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} \right\|^2_{\mathcal{G}(\nu,\sigma,B)} \leq \frac{1 - B'(0)^{2\nu}}{\nu} \sigma_1. \tag{9}$$

PROOF. The hypotheses imply that conditions (i) and (ii) of Theorem 2.3 hold relative to the sequences $\{\sigma_1, \sigma_1, \sigma_2, \ldots\}$ and $\{\sigma_1, \sigma_2, \ldots\}$. The norm inequality (9) then follows on taking the supremum of the left hand side of (7) over all $g(z)$ in the space. □

Every formal power series $B(z) = B_1 z + B_2 z^2 + \cdots$ $(B_1 > 0)$ has a *compositional inverse*, which is defined as the unique formal power series $A(z)$ of the same form such that $A(B(z)) = B(A(z)) = z$. The coefficients in the expansions

$$\left\{ \begin{array}{rcl} A(z)^\nu & = & \sum_{n=0}^\infty A_n(\nu) z^{\nu+n}, \\ B(z)^\nu & = & \sum_{n=0}^\infty B_n(\nu) z^{\nu+n} \end{array} \right. \tag{10}$$

are connected by the Lagrange-Bürmann formula [**15, Theorem 3.5**]:

$$A_n(\nu) \quad = \quad \frac{\nu}{\nu + n} B_n(-\nu - n), \tag{11}$$

$n = 0, 1, 2, \ldots$. Write $B^*(z) = \overline{B}_1 z + \overline{B}_2 z^2 + \cdots$.

THEOREM 2.5 *Let $A(z)$ and $B(z)$ be as above. Let $\mathcal{G}^\nu_\sigma$ be a Grunsky space such that $\sigma_n = 0$ for $n > r$ and $\sigma_1 \cdots \sigma_r \neq 0$, and consider $\mathcal{G}^\mu_\tau$ where $\mu = -\nu - r - 1$ and $\tau = \{-\sigma_r^{-1}, -\sigma_{r-1}^{-1}, \ldots, -\sigma_1^{-1}, 0, 0, \ldots\}$. Then the mapping*

$$W : \sum_{n=1}^r a_n z^{\nu+n} \mapsto \sum_{n=1}^r \sigma_n a_n z^{-\nu-n}$$

is a Kreĭn space isomorphism from $\mathcal{G}^\nu_\sigma$ onto $\mathcal{G}^\mu_\tau$. If operators S and T are defined by $T : f(z) \mapsto f(B(z))$ on $\mathcal{G}^\nu_\sigma$ and $S : g(z) \mapsto g(A^*(z))$ on $\mathcal{G}^\mu_\tau$, then $T^* = W^{-1}SW$.

PROOF. The mapping W is an isomorphism from $\mathcal{G}_\sigma^\nu$ onto $\mathcal{G}_\tau^\mu$ since for any numbers $a_1, \dots, a_r$,

$$\begin{aligned}
&\left\langle \sum_{n=1}^r \sigma_n a_n z^{-\nu-n}, \sum_{n=1}^r \sigma_n a_n z^{-\nu-n} \right\rangle_{\mathcal{G}_\tau^\mu} \\
&\quad = \left\langle \sum_{n=1}^r \sigma_{r+1-n} a_{r+1-n} z^{-\nu-r-1+n}, \sum_{n=1}^r \sigma_{r+1-n} a_{r+1-n} z^{-\nu-r-1+n} \right\rangle_{\mathcal{G}_\tau^\mu} \\
&\quad = \sum_{n=1}^r (-\nu-r-1+n)(-\sigma_{r+1-n}^{-1})|\sigma_{r+1-n}a_{r+1-n}|^2 \\
&\quad = \sum_{n=1}^r (\nu+n)\sigma_n |a_n|^2 \\
&\quad = \left\langle \sum_{n=1}^r a_n z^{\nu+n}, \sum_{n=1}^r a_n z^{\nu+n} \right\rangle_{\mathcal{G}_\sigma^\nu}.
\end{aligned}$$

We show that WT^* and SW have the same action on any $f(z) = \sum_{n=1}^r a_n z^{\nu+n}$ in $\mathcal{G}_\sigma^\nu$. By [**15, Theorem 3.6**], $T^* : f(z) \mapsto \sum_{n=1}^r b_n z^{\nu+n}$, where for $n = 1, \dots, r$,

$$(\nu+n)\sigma_n b_n = \sum_{n=1}^r (\nu+j)\sigma_j \overline{B}_{j-n}(\nu+n)a_j.$$

Since $WT^* : f(z) \mapsto \sum_{n=1}^r \sigma_n b_n z^{-\nu-n}$ and $SW : f(z) \mapsto \sum_{n=1}^r \sigma_n a_n A^*(z)^{-\nu-n}$, it is required to show that, as elements of $\mathcal{G}_\tau^\mu$,

$$\sum_{n=1}^r \sigma_n b_n z^{-\nu-n} = \sum_{n=1}^r \sigma_n a_n A^*(z)^{-\nu-n}, \tag{12}$$

that is, the coefficients of $z^{-\nu-p}$ coincide for $p = 1, \dots, r$. By (10) and (11),

$$\begin{aligned}
\sum_{n=1}^r \sigma_n a_n A^*(z)^{-\nu-n} &= \sum_{n=1}^r \sigma_n a_n \sum_{j=0}^\infty \overline{A}_j(-\nu-n) z^{-\nu-n+j} \\
&= \sum_{n=1}^r \sigma_n a_n \sum_{j=0}^\infty \frac{-\nu-n}{-\nu-n+j} \overline{B}_j(\nu+n-j) z^{-\nu-n+j}.
\end{aligned}$$

For any $p = 1, \dots, r$, the coefficient of $z^{-\nu-p}$ in the last term on the right side is

$$\sum_{n-j=p} \frac{-\nu-n}{-\nu-n+j} \overline{B}_j(\nu+n-j)\sigma_n a_n = \sum_{n=p}^r \frac{\nu+n}{\nu+p} \sigma_n \overline{B}_{n-p}(\nu+p)a_n = \sigma_p b_p.$$

This proves (12). The possibility of division by zero when ν is a negative integer must be considered. The coefficient of $z^{-\nu-p}$ on the right side of (12) is a polynomial in ν, so this case follows by continuity.

□

The proof of Theorem 2.2 uses the full strength of Theorem 2.1. By taking advantage of Theorem 2.5, we can rewrite the argument so as to use only the finite-dimensional case of Theorem 2.1 given in [**14**].

VARIANT ON THE PROOF OF THEOREM 2.2. Prove the result for $\mathcal{D}_\nu$ as before. The case of any space $\mathcal{D}_\nu^r$ such that $\nu > -r-1$ then follows by truncation. For if $f(z) = \sum_{n=1}^r a_n z^{\nu+n}$ is in $\mathcal{D}_\nu^r$ and $f(B(z)) = \sum_{n=1}^\infty b_n z^{\nu+n}$, then by the result for $\mathcal{D}_\nu$,

$$\sum_{n=1}^{\infty} (\nu+n)|b_n|^2 \le \sum_{n=1}^{r} (\nu+n)|a_n|^2.$$

The terms for $n > r$ on the left are positive because $\nu + r + 1 > 0$, and so these terms can be dropped, yielding the result for $\mathcal{D}_\nu^r$ when $\nu > -r-1$.

In particular, the result holds for $\mathcal{D}_\nu^r$ when $\nu \ge -\frac{1}{2}(r+1)$. Now apply Theorem 2.5 with $\mathcal{G}_\sigma^\nu$ equal to $\mathcal{D}_\nu^r$. The operator T in Theorem 2.5 is a contraction, so T^* is a contraction by the finite-dimensional case of Theorem 2.1 . Therefore S is a contraction on $\mathcal{G}_\tau^\mu$, and S^{-1} is a contraction on the anti-space of $\mathcal{G}_\tau^\mu$, which is $\mathcal{D}_\mu^r$, $\mu = -\nu - r - 1 \le -\frac{1}{2}(r+1)$. But S^{-1} is substitution by $B^*(z)$. Equivalently, substitution by $B(z)$ is contractive in $\mathcal{D}_\nu^r$ when $\nu \le -\frac{1}{2}(r+1)$ and hence for all real ν. Once this is known, we may complete the argument as before using the identity (5).

□

3 Necessary conditions on segments of coefficients

Any property of the class of normalized Riemann mappings $B(z) = B_1 z + B_2 z^2 + \cdots$ of the unit disk into itself which depends only on an initial segment $B_1, \ldots, B_r$ of coefficients may be taken as a necessary condition that given numbers arise in this way. Theorem 2.2 may be used to generate one set of conditions, and similar but more complicated conditions occur in the theory of the Bieberbach conjecture. In this section we formulate these conditions and observe that either approach may be used to characterize the first two coefficients. It is unknown if the first approach can be used to characterize initial segments in general. For the second, this question is answered negatively in §3. In view of Theorem 2.3, the conditions derived from Theorem 2.2 may be stated in two equivalent forms.

DEFINITION 3.1 *We say that given numbers $B_1, \ldots, B_r$ $(B_1 > 0)$ satisfy the* **contractive substitution condition** *if the polynomial $B(z) = B_1 z + \cdots + B_r z^r$ has one and hence both of these properties:*
(i) The inequality

$$\langle f(B(z)), f(B(z))\rangle_{\mathcal{G}_\sigma^\nu} \le \langle f(z), f(z)\rangle_{\mathcal{G}_\sigma^\nu} \tag{13}$$

holds for every $f(z)$ in $\mathcal{G}_\sigma^\nu$ whenever ν is real, $\sigma_1 \ge \cdots \ge \sigma_r$, and $\sigma_n = 0$ for $n > r$.

(ii) The inequality

$$(14)\qquad \left\langle \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)), \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)) \right\rangle_{\mathcal{G}_\sigma^\nu} - \langle g(z), g(z) \rangle_{\mathcal{G}_\sigma^\nu}$$

$$\leq \frac{1 - B'(0)^{2\nu}}{\nu}\sigma_1$$

holds for every $g(z)$ in $\mathcal{G}_\sigma^\nu$ whenever ν is real, $\sigma_1 \geq \cdots \geq \sigma_{r-1}$, and $\sigma_n = 0$ for $n > r-1$.

A simpler form of the conditions may be given.

THEOREM 3.2 *For given numbers $B_1, \ldots, B_r$ ($B_1 > 0$) to satisfy the contractive substitution condition, it is sufficient that substitution by $B(z) = B_1 z + \cdots + B_r z^r$ is contractive in $\mathcal{D}_\nu^r$ for all $\nu \geq -\frac{1}{2}(r+1)$.*

PROOF. Let $T : f(z) \mapsto f(B(z))$ be contractive in $\mathcal{D}_\nu^r$ whenever $\nu \geq -\frac{1}{2}(r+1)$. Then Theorem 2.5 implies that T is contractive in $\mathcal{D}_\nu^r$ for all real ν (see the argument in the variant on the proof of Theorem 2.2 at the end of §2). We show next that the operator $T_k : f(z) \mapsto f(B(z))$ is contractive in $\mathcal{D}_\nu^k$ for any real ν and $k = 1, \ldots, r$. If we view $\mathcal{D}_\nu^k$ as a subspace of $\mathcal{D}_\nu^r$ with projection P_k, then $T_k = P_k T|_{\mathcal{D}_\nu^k}$ and $T_k^* = T^*|_{\mathcal{D}_\nu^k}$. By Theorem 2.1, T^* is a contraction. Therefore T_k^* and T_k are contractions. Finally, using the summation by parts identity (5), we verify condition (i) in Definition 3.1.

□

Inequalities similar to (13) and (14) appear in the proof of Milin's conjecture **[5]** and its generalizations **[6,15]**. Let ν be any real number, and let $\mathcal{G}_{\sigma(t)}^\nu$, $t \geq 1$, be a family of Grunsky spaces such that $\sigma_1(t), \sigma_2(t), \ldots$ are absolutely continuous functions of $t \geq 1$ and $\sigma_n(t)$ vanishes identically for all sufficiently large n. We say that the family $\mathcal{G}_{\sigma(t)}^\nu$, $t \geq 1$, is *admissible* if for all n,

$$(15)\qquad (2\nu + n)\sigma_n(t) + t\sigma_n'(t) = (2\nu + n)\sigma_{n+1}(t) - \frac{2\nu + n}{n+1} t\sigma_{n+1}'(t)$$

a.e. for $t \geq 1$ and the function

$$(16)\qquad \tau_n(t) = (\nu + n)\left[\sigma_n(t) - \sigma_{n+1}(t) + \frac{1}{n+1} t\sigma_n'(t)\right]$$

is nonnegative a.e. for $t \geq 1$. For an example, define $\sigma(t) = \{\sigma_1(t), \sigma_2(t), \ldots\}$ by

$$(17)\qquad \sigma_n(t) = \sum_{j=0}^{\infty} (-1)^j \frac{(2\nu + n)_j (2\nu + 2n + j + 1)_j}{j!(n+1)_j} \Delta_{n+j} t^{-2\nu - n - j},$$

$n = 1, 2, \ldots$, where

$$(18)\qquad \Delta_k = \frac{4^{-k}\Gamma(k+1)(2\nu + 2\lambda + r + 2)_{k-1}}{\Gamma(\nu + k + 1)\Gamma(\nu + \lambda + k + 1)\Gamma(2\nu + k + 1)\Gamma(r + 1 - k)},$$

$k = 1, 2, \ldots$. Here r is a fixed positive integer and $\lambda \geq 0$. We view $1/\Gamma(z)$ as an entire function vanishing at the nonpositive integers, and so $\Delta_k = 0, k > r$, and $\sigma_n(t) \equiv 0$ for $n > r$. The family is admissible if either $\nu > -\frac{3}{2}$ or 2ν is a negative integer **[6,15]**. As we shall see, there are examples of admissible families for arbitrary real ν.

For any admissible family $\mathcal{G}^{\nu}_{\sigma(t)}$, $t \geq 1$, and any normalized Riemann mapping $B(z)$ of the unit disk into itself, the inequality **[6,15]**

$$\left\langle \frac{B(z)^{\nu} - B'(0)^{\nu} z^{\nu}}{\nu} + g(B(z)), \frac{B(z)^{\nu} - B'(0)^{\nu} z^{\nu}}{\nu} + g(B(z)) \right\rangle_{\mathcal{G}^{\nu}_{\sigma(a)}} - \langle g(z), g(z) \rangle_{\mathcal{G}^{\nu}_{\sigma(b)}} \tag{19}$$

$$\leq 4b^{-2\nu} \sum_{n=1}^{\infty} (\nu + n) \left[\frac{(2\nu+1)_{n-1}}{n!} \right]^2 \left[a^{2\nu} \sigma_n(a) - b^{2\nu} \sigma_n(b) \right]$$

holds whenever

$$a \geq 1, \qquad b \geq 1, \qquad a/b = B'(0), \qquad g(z) = \sum_{n=1}^{\infty} c_n z^{\nu+n}. \tag{20}$$

Milin's conjecture is proved by choosing $\nu = 0$ with the example of an admissible family given above. With the same example but other values of ν, the method yields coefficient estimates for powers of Riemann mappings.

The inequality (19) may be viewed as a more delicate version of (14), and it can be used in a similar way to impose conditions on coefficients.

DEFINITION 3.3 *We say that given numbers $B_1, \ldots, B_r$ ($B_1 > 0$) satisfy the* **admissible families condition** *if the polynomial $B(z) = B_1 z + \cdots + B_r z^r$ satisfies (19) for every admissible family $\mathcal{G}^{\nu}_{\sigma(t)}$, $t \geq 1$, such that $\sigma_n(t) \equiv 0$ for $n > r - 1$.*

In the definition, it is understood that the quantities a, b, and $g(z)$ appearing in (19) are arbitrary except for the restrictions (20). The contractive substitution condition and admissible families condition give the same result when $r = 2$.

THEOREM 3.4 *If B_1, B_2 ($B_1 > 0$) are given numbers, the following conditions are equivalent :*
(i) B_1, B_2 satisfy the contractive substitution condition,
(ii) B_1, B_2 satisfy admissible families condition,
(iii) $B_1 \leq 1$ and $|B_2| \leq 2B_1(1 - B_1)$,
(iv) B_1, B_2 are the first two coefficients of a normalized Riemann mapping of the unit disk into itself.

PROOF. The equivalence of (i),(iii), and (iv) is proved in **[11, Theorem 2]**. Also (iii) implies (ii), because (iii) implies (iv) and (19) holds for any normalized Riemann mapping $B(z)$ of the unit disk into itself and any admissible family $\mathcal{G}^{\nu}_{\sigma(t)}$, $t \geq 1$. It remains to show that (ii) implies (iii).

Assume (ii). If $g(z) = \sum_{n=1}^{\infty} c_n z^{\nu+n}$, then

$$\frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)) = \left[\frac{1}{\nu} B_1(\nu) + B_0(\nu+1)c_1\right] z^{\nu+1} + \cdots.$$

For any real ν, an admissible family $\mathcal{G}^\nu_{\sigma(t)}$, $t \geq 1$, is obtained with $\sigma_1(t) = (\nu+1)t^{-2\nu-1}$ and $\sigma_n(t)$ identically zero otherwise. Then if $a \geq 1, b \geq 1$, and $a/b = B_1$, the inequality (19) asserts that

$$(\nu+1)\sigma_1(a) \left|\frac{1}{\nu} B_1(\nu) + B_0(\nu+1)c_1\right|^2 - (\nu+1)\sigma_1(b)|c_1|^2$$
$$\leq 4b^{-2\nu}(\nu+1)[a^{2\nu}\sigma_1(a) - b^{2\nu}\sigma_1(b)].$$

Using (2), we simplify the conditions on B_1, B_2 to the inequality

$$\left|B_1^{\frac{1}{2}} c_1 + B_1^{-\frac{3}{2}} B_2\right|^2 - |c_1|^2 \leq 4(B_1^{-1} - 1). \tag{21}$$

Now if p and q are complex numbers, the supremum sup $[|pc+q|^2 - |c|^2]$ over all complex numbers c is finite if and only if either $|p| < 1$ or $|p| = 1$ and $q = 0$. When $|p| < 1$ the value of the supremum is $|q|^2/(1-|p|^2)$. Applied to (21), this yields (iii).

□

4 Failure of the admissible families condition

We exhibit numbers B_1, B_2, B_3 ($B_1 > 0$) which satisfy the admissible families condition but are not the first three coefficients of a normalized Riemann mapping of the unit disk into itself. The example takes into account all admissible families $\mathcal{G}^\nu_{\sigma(t)}$, $t \geq 1$, with ν real and $\sigma(t)$ of the form $\{\sigma_1(t), \sigma_2(t), 0, 0, \ldots\}$. For any real ν, such a family is obtained with

$$\begin{cases} \sigma_1(t) &= (\nu+1)t^{-2\nu-1}, \\ \sigma_2(t) &= 0, \end{cases} \tag{22}$$

or

$$\begin{cases} \sigma_1(t) &= (\nu+2)^2(2\nu+2)t^{-2\nu-1} - (\nu+2)^2(2\nu+1)t^{-2\nu-2}, \\ \sigma_2(t) &= (\nu+2)t^{-2\nu-2}. \end{cases} \tag{23}$$

Apart from constant multiples, these sequences agree with those determined by (17) and (18) with $r = 1, 2$.

THEOREM 4.1 *In order that B_1, B_2, B_3 ($B_1 > 0$) satisfy the admissible families condition it is sufficient that $B(z) = B_1 z + B_2 z^2 + B_3 z^3$ satisfy* (19) *for the special families* (22) *and* (23) *in the case $a = 1$ and $b = 1/B'(0)$.*

PROOF. Assume that $B(z)$ satisfies (19) for the special families (22) and (23) with $a = 1$ and $b = 1/B'(0)$. It must be shown that $B(z)$ satisfies (19) in the general case. We exhibit all admissible families $\mathcal{G}^{\nu}_{\sigma(t)}$, $t \geq 1$, with ν real and $\sigma(t)$ of the form $\{\sigma_1(t), \sigma_2(t), 0, 0, \ldots\}$. For any real ν, an example is obtained with

$$\left\{ \begin{array}{rcl} \sigma_1(t) & = & (\nu+2)^2(2\nu+2)Dt^{-2\nu-1} - (\nu+2)^2(2\nu+1)Ct^{-2\nu-2}, \\ \sigma_2(t) & = & (\nu+2)Ct^{-2\nu-2}, \end{array} \right. \tag{24}$$

where $D \geq C \geq 0$. For $\nu = -1$, another example is

$$\sigma_1(t) = A + Bt \quad \text{and} \quad \sigma_2(t) = A, \tag{25}$$

where $A \geq 0$. For $\nu = -2$, another example is

$$\sigma_1(t) = Bt^3 \quad \text{and} \quad \sigma_2(t) = At^2, \tag{26}$$

where $B \leq 0$. This list exhausts all possibilities, because the definition of an admissible family requires that

$$\begin{array}{rcl} (2\nu+1)\sigma_1(t) + t\sigma_1'(t) & = & (2\nu+1)\sigma_2(t) - \dfrac{2\nu+1}{2}t\sigma_2'(t), \\ (2\nu+2)\sigma_2(t) + t\sigma_2'(t) & = & 0 \end{array}$$

and

$$\begin{array}{rcl} \tau_1(t) & = & (\nu+1)[\sigma_1(t) - \sigma_2(t) + \frac{1}{2}t\sigma_2'(t)] \geq 0, \\ \tau_2(t) & = & (\nu+2)\sigma_2(t) \geq 0 \end{array}$$

for $t \geq 1$. The differential equations have the general solution

$$\begin{array}{rcl} \sigma_1(t) & = & Bt^{-2\nu-1} - (\nu+2)(2\nu+1)At^{-2\nu-2}, \\ \sigma_2(t) & = & At^{-2\nu-2}, \end{array}$$

where A, B are real constants. If $\nu \neq -1, -2$, we may write

$$\begin{array}{rcl} A & = & (\nu+2)C, \\ B & = & (\nu+2)^2(2\nu+2)D, \end{array}$$

and the solutions has the form (24). The requirement that

$$\begin{array}{rcl} \tau_1(t) & = & 2(\nu+1)^2(\nu+2)^2t^{-2\nu-2}(Dt - C) \geq 0, \\ \tau_2(t) & = & (\nu+2)^2Ct^{-2\nu-2} \geq 0 \end{array}$$

for $t \geq 1$ implies $D \geq C \geq 0$. In a similar way, the solution reduces to (25) and (26) when $\nu = -1$ and -2.

If $g(z) = \sum_{n=1}^{\infty} c_n z^{\nu+n}$, then

$$\begin{array}{rcl} \dfrac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)) & = & \left[\dfrac{1}{\nu}B_1(\nu) + B_0(\nu+1)c_1\right] z^{\nu+1} \\ & & + \left[\dfrac{1}{\nu}B_2(\nu) + B_1(\nu+1)c_1 + B_0(\nu+2)c_2\right] z^{\nu+2} + \cdots. \end{array}$$

Thus (19) asserts

$$(27)\qquad \begin{aligned}&(\nu+1)\sigma_1(a)\left|\frac{1}{\nu}B_1(\nu)+B_0(\nu+1)c_1\right|^2\\ &+(\nu+2)\sigma_2(a)\left|\frac{1}{\nu}B_2(\nu)+B_1(\nu+1)c_1+B_0(\nu+2)c_2\right|^2\\ &-(\nu+1)\sigma_1(aB_1^{-1})|c_1|^2-(\nu+2)\sigma_2(aB_1^{-1})|c_2|^2\\ \leq\ & 4(\nu+1)[B_1^{2\nu}\sigma_1(a)-\sigma_1(aB_1^{-1})]+4(\nu+2)\left(\frac{2\nu+1}{2}\right)^2[B_1^{2\nu}\sigma_2(a)-\sigma_2(aB_1^{-1})].\end{aligned}$$

By hypothesis, (27) holds for the families (22) and (23) with $a=1$. We show that (27) holds for the families (22) and (23) when $a\geq 1$. The case of (22) is clear since we need only multiply the inequality for $a=1$ by a constant. In the case of (23), (27) becomes

$$\begin{aligned}&4(\nu+1)[B_1^{2\nu}(2\nu+2)a-B_1^{2\nu}(2\nu+1)-(2\nu+2)aB_1^{2\nu+1}+(2\nu+1)B_1^{2\nu+2}]\\ &-(\nu+1)[(2\nu+2)a-(2\nu+1)]\left|\frac{1}{\nu}B_1(\nu)+B_0(\nu+1)c_1\right|^2\\ &+(\nu+1)[(2\nu+2)aB_1^{2\nu+1}-(2\nu+1)B_1^{2\nu+2}]|c_1|^2+(2\nu+1)^2[B_1^{2\nu}-B_1^{2\nu+2}]\\ &+\left|\frac{1}{\nu}B_2(\nu)+B_1(\nu+1)c_1+B_0(\nu+2)c_2\right|^2+B_1^{2\nu+2}|c_2|^2\geq 0\end{aligned}$$

Since this holds for $a=1$, it is sufficient to show that the coefficient of a on the left side is nonnegative. The coefficient of a is nonnegative by (27) taken with the family (22), and so (27) holds for the families (22) and (23) and all $a\geq 1$.

It remains to verify (27) for the families (24)-(26). The functions (24) are a linear combination with nonnegative coefficients of those in (22) and (23). Therefore (27) holds for the family (24). Choosing $\nu=-1$ in the inequality, we obtained an inequality equivalent to (27) with the family (25). Choosing instead $D=-B$ and $C=0$, we obtain as a limiting case for $\nu=-2$ an inequality equivalent to (27) with the family (26).

□

THEOREM 4.2 *Given numbers* B_1, B_2, B_3 $(B_1>0)$ *which are not* $1,0,0$ *satisfy the admissible families condition if and only if* $B_1<1$, $|B_2|\leq 2B_1(1-B_1)$, *and*

$$(28)\qquad \left|B_3+\frac{1}{2}\left(\frac{\nu-1}{B_1}+\frac{1}{1-B_1}\right)B_2^2\right|^2\leq\frac{4B_1^2(1-B_1)^2\frac{1}{2}(B_1^{-1}+1)-|B_2|^2}{2B_1(1-B_1)}$$
$$\times\left\{8(\nu+1)^2B_1^2(1-B_1)-(2\nu+3)(2\nu+1)B_1^2(1-B_1^2)-\frac{2\nu+2-(2\nu+1)B_1}{2(1-B_1)}|B_2|^2\right\}$$

for all real ν.

PROOF. We use the supremum **[7,9]**

$$(29)\qquad \sup_{g\in\mathcal{H}}[\langle f+Tg,f+Tg\rangle_{\mathcal{K}}-\langle g,g\rangle_{\mathcal{H}}],$$

in the case where $\mathcal{H}$ and $\mathcal{K}$ are 2-dimensional Kreĭn spaces, T is a contraction on $\mathcal{H}$ to $\mathcal{K}$, and f is a vector in $\mathcal{K}$. The supremum may be evaluated in terms of any factorization

$$1 - TT^* = DD^*, \tag{30}$$

where D is an operator on a Kreĭn space $\mathcal{D}$ to $\mathcal{K}$ such that ker $D = \{0\}$. The supremum (29) is finite if and only if f is in the range of D, and if $f = Dh$ its value is

$$\langle h, h\rangle_{\mathcal{D}}. \tag{31}$$

We show that the conditions are necessary. Assume that B_1, B_2, B_3 satisfy the admissible families condition, and set $B(z) = B_1 z + B_2 z^2 + B_3 z^3$. By Theorem 3.4, $B_1 \leq 1$ and $|B_2| \leq 2B_1(1 - B_1)$. If $B_1 = 1$, then $B_2 = 0$ and $B_3 = 0$ (apply (27) with the family (23)). Since the numbers are not $1, 0, 0$, $B_1 < 1$. Notice that it is enough to prove (28) except for isolated values of ν, since then by continuity the inequality holds for all real ν.

Consider the supremum formed from the left side of (19),

$$\sup\left\{\left\langle \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)), \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z))\right\rangle_{\mathcal{G}^\nu_{\sigma(a)}} - \langle g(z), g(z)\rangle_{\mathcal{G}^\nu_{\sigma(b)}}\right\},$$

where for the admissible family we use (23). The supremum is over all generalized power series $g(z) = \sum_{n=1}^\infty c_n z^{\nu+n}$. It is not hard to see that $T : g(z) \mapsto g(B(z))$ is a contraction on $\mathcal{G}^\nu_{\sigma(b)}$ to $\mathcal{G}^\nu_{\sigma(a)}$. For if we apply (19) with $g(z)$ replaced by $\epsilon^{-1}g(z)$, multiply by ϵ^2, then let $\epsilon \to 0$, we get

$$\langle g(B(z)), g(B(z))\rangle_{\mathcal{G}^\nu_{\sigma(a)}} \leq \langle g(z), g(z)\rangle_{\mathcal{G}^\nu_{\sigma(b)}}$$

for any $g(z)$ in $\mathcal{G}^\nu_{\sigma(b)}$. The supremum is thus of the form (29). We calculate a factorization (30). Let C^2 be 2-dimensional Euclidean space with elements written as column vectors $c = [c_1, c_2]^t$. Set

$$D(t) = \begin{bmatrix} (\nu+1)\sigma_1(t) & 0 \\ 0 & (\nu+2)\sigma_2(t) \end{bmatrix}$$

for $t \geq 1$. Let $\mathcal{H}$ be C^2 in the scalar product $\langle c, c\rangle_{\mathcal{H}} = \langle D(b)c, c\rangle_{C^2}$, and let $\mathcal{K}$ be C^2 in the scalar product $\langle c, c\rangle_{\mathcal{K}} = \langle D(a)c, c\rangle_{C^2}$. Then $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces except for the isolated values of ν where $D(a)$ or $D(b)$ is singular. In an obvious way we may identify $\mathcal{G}^\nu_{\sigma(b)}$ with $\mathcal{H}$ and $\mathcal{G}^\nu_{\sigma(a)}$ with $\mathcal{K}$ by using the coefficients of generalized power series as entries of column vectors. The operator T acts as multiplication by the matrix

$$M = \begin{bmatrix} B_0(\nu+1) & 0 \\ B_1(\nu+1) & B_0(\nu+2) \end{bmatrix}$$

on $\mathcal{H}$ to $\mathcal{K}$. Making the usual identifications of operators and matrices, we have

$$T = M, \qquad T^* = D(b)^{-1}\overline{M}D(a),$$

where the bar of a matrix denotes its conjugate transpose. By matrix calculus there exists a matrix Q and a diagonal matrix J with diagonal entries either 1 or -1 such that

$$1 - TT^* = 1 - MD(b)^{-1}\overline{M}D(a) = D(a)^{-1}QJ\overline{Q}, \tag{32}$$

Define $\mathcal{D}$ to be C^2 in the scalar product $\langle c, c\rangle_{\mathcal{D}} = \langle Jc, c\rangle_{C^2}$. Viewing $D = D(a)^{-1}QJ$ as an operator on $\mathcal{D}$ to $\mathcal{K}$, one checks that $D^* = \overline{Q}$, and so (30) holds. From (29)-(31) we get

$$\text{(33)} \qquad \sup\left\{\left\langle \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)), \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z))\right\rangle_{\mathcal{G}^\nu_{\sigma(a)}} - \langle g(z), g(z)\rangle_{\mathcal{G}^\nu_{\sigma(b)}}\right\}$$

$$\begin{aligned} &= \left\langle D^{-1}\begin{bmatrix} B_1(\nu)/\nu \\ B_2(\nu)/\nu \end{bmatrix}, D^{-1}\begin{bmatrix} B_1(\nu)/\nu \\ B_2(\nu)/\nu \end{bmatrix}\right\rangle_{\mathcal{D}} \\ &= \left\langle JQ^{-1}D(a)\begin{bmatrix} B_1(\nu)/\nu \\ B_2(\nu)/\nu \end{bmatrix}, Q^{-1}D(a)\begin{bmatrix} B_1(\nu)/\nu \\ B_2(\nu)/\nu \end{bmatrix}\right\rangle_{C^2}. \end{aligned}$$

To make the calculation, we must exhibit matrices J and Q satisfying (32). Set

$$D(a) - D(a)MD(b)^{-1}\overline{M}D(a) = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix}.$$

Then

$$\begin{aligned} \alpha &= (\nu+1)\sigma_1(a)\left[1 - |B_0(\nu+1)|^2\frac{\sigma_1(a)}{\sigma_1(b)}\right] \\ &= 2(\nu+1)^2(\nu+2)^2a^{-2\nu-1}\frac{(2\nu+2)a - (2\nu+1)}{(2\nu+2)a - (2\nu+1)B_1}(1 - B_1), \\ \beta &= -B_0(\nu+1)\overline{B}_1(\nu+1)\cdot(\nu+2)\frac{\sigma_1(a)\sigma_2(a)}{\sigma_1(b)} \\ &= -(\nu+1)(\nu+2)^2\overline{B}_2\, a^{-2\nu-2}\frac{(2\nu+2)a - (2\nu+1)}{(2\nu+2)a - (2\nu+1)B_1}, \\ \gamma &= (\nu+2)\sigma_2(a)\left[1 - |B_1(\nu+1)|^2\frac{\nu+2}{\nu+1}\frac{\sigma_2(a)}{\sigma_1(b)} - |B_0(\nu+2)|^2\frac{\sigma_2(a)}{\sigma_2(b)}\right] \\ &= (\nu+2)^2a^{-2\nu-2}\left[1 - |B_2|^2B_1^{-1}\frac{\nu+1}{(2\nu+2)a - (2\nu+1)B_1} - B_1^2\right]. \end{aligned}$$

The determinant $\delta = \alpha\gamma - |\beta|^2$ is given by

$$\delta = (\nu+1)^2(\nu+2)^4a^{-4\nu-4}B_1^{-1}\frac{(2\nu+2)a - (2\nu+1)}{(2\nu+2)a - (2\nu+1)B_1}\left[2B_1^2(1-B_1)^2(B_1^{-1}+1)a - |B_2|^2\right].$$

When $\delta > 0$, (32) holds with

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \alpha^{-\frac{1}{2}}\begin{bmatrix} \alpha & 0 \\ \overline{\beta} & \delta^{\frac{1}{2}} \end{bmatrix}.$$

When $\delta < 0$, we obtain (32) with

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = |\alpha|^{-\frac{1}{2}} \begin{bmatrix} \alpha & 0 \\ \overline{\beta} & -|\delta|^{\frac{1}{2}} \end{bmatrix}.$$

Formula (33) yields the same result in both cases, namely

$$\begin{aligned} &\sup \left[\left\langle \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)), \frac{B(z)^\nu - B'(0)^\nu z^\nu}{\nu} + g(B(z)) \right\rangle_{\mathcal{G}^\nu_{\sigma(a)}} - \langle g(z), g(z) \rangle_{\mathcal{G}^\nu_{\sigma(b)}} \right] \\ &= \frac{1}{\alpha\delta} \left[\delta(\nu+1)^2 \sigma_1(a)^2 \left| \frac{B_1(\nu)}{\nu} \right|^2 + \left| -\overline{\beta}(\nu+1)\sigma_1(a) \frac{B_1(\nu)}{\nu} + \alpha(\nu+2)\sigma_2(a) \frac{B_2(\nu)}{\nu} \right|^2 \right]. \end{aligned}$$

Therefore (19) gives

$$\begin{aligned} &\frac{1}{\alpha\delta} \left[\delta(\nu+1)^2 \sigma_1(a)^2 \left| \frac{B_1(\nu)}{\nu} \right|^2 + \left| -\overline{\beta}(\nu+1)\sigma_1(a) \frac{B_1(\nu)}{\nu} + \alpha(\nu+2)\sigma_2(a) \frac{B_2(\nu)}{\nu} \right|^2 \right] \\ &\leq 4b^{-2\nu}(\nu+1) \left[a^{2\nu}\sigma_1(a) - b^{2\nu}\sigma_1(b) \right] + 4b^{-2\nu}(\nu+2) \frac{(2\nu+1)^2}{4} \left[a^{2\nu}\sigma_2(a) - b^{2\nu}\sigma_2(b) \right]. \end{aligned}$$

When $a = 1$ and $b = B_1^{-1}$, this simplifies to (28). We have shown that the conditions in the theorem are necessary.

Conversely, assume that $B_1 < 1$, $|B_2| \leq 2B_1(1 - B_1)$, and that (28) holds for all real ν. By Theorem 3.4, $B(z) = B_1 z + B_2 z^2 + B_3 z^3$ satisfies (19) with the family (22). We can reverse the preceding steps to show that $B(z)$ satisfies (19) for the family (23) in the special case $a = 1$ and $b = 1/B'(0)$. Then by Theorem 4.1, the numbers B_1, B_2, B_3 satisfy the admissible families condition.

□

We can now give an example of three numbers which satisfy the admissible families condition but which are not the first three coefficients of a normalized Riemann mapping of the unit disk into itself. They are

$$B_1 = \frac{1}{3}, \quad B_2 = \frac{1}{4} e^{i\pi/4}, \quad B_3 = \frac{1}{3} i. \tag{34}$$

Clearly $B_1 < 1$ and $|B_2| < 2B_1(1 - B_1)$. The inequality (28) has the form $uv - w \geq 0$, where

$$\begin{aligned} u &= \frac{4B_1^2(1-B_1)^2 \frac{1}{2}(B_1^{-1}+1) - |B_2|^2}{2B_1(1-B_1)}, \\ v &= 8(\nu+1)^2 B_1^2(1-B_1) - (2\nu+3)(2\nu+1)B_1^2(1-B_1^2) - \frac{2\nu+2-(2\nu+1)B_1}{2(1-B_1)} |B_2|^2, \\ w &= \left| B_3 + \frac{1}{2}\left(\frac{\nu-1}{B_1} + \frac{1}{1-B_1}\right) B_2^2 \right|^2. \end{aligned}$$

We find

$$9 \cdot 81(64)^2(uv - w) = p\nu^2 + q\nu + r,$$
$$p = 415100, \quad q = 582664, \quad r = 242436.$$

Since $q^2 - 4pr = -63043397504$ is negative, the inequality (28) holds for all real ν. Therefore B_1, B_2, B_3 satisfy the admissible families condition by Theorem 4.2.

We show that the numbers (34) do not satisfy the contractive substitution condition, and therefore they are not the first three coefficients of a normalized Riemann mapping of the unit disk into itself. Set $B(z) = B_1 z + B_2 z^2 + B_3 z^3$. The inequality

$$\langle g(B(z)), g(B(z)) \rangle_{\mathcal{D}_\nu^3} \leq \langle g(z), g(z) \rangle_{\mathcal{D}_\nu^3}$$

holds for all $g(z)$ in $\mathcal{D}_\nu^3$ if and only if $D - \overline{M}DM \geq 0$, where

$$D = \begin{bmatrix} \nu+1 & 0 & 0 \\ 0 & \nu+2 & 0 \\ 0 & 0 & \nu+3 \end{bmatrix},$$

$$M = \begin{bmatrix} B_0(\nu+1) & 0 & 0 \\ B_1(\nu+1) & B_0(\nu+2) & 0 \\ B_2(\nu+1) & B_1(\nu+2) & B_0(\nu+3) \end{bmatrix}.$$

The inequality fails when $\nu = -\frac{3}{2}$, because then $\det(D - \overline{M}DM) = -59363/1769472$.

The preceding calculations were carried out using MACSYMA and confirmed with a hand calculator. The example was found from numerical experiments. The experiments showed that, in fact, the contractive substitution condition and the admissible families condition each do a good job in constraining the first three coefficients. This was tested against large random sets of coefficients generated by composing Koebe mappings. The first numerical examples of the failure of the admissible families condition were dependent on the computer programs to verify, but with some adjustment of parameters we were able to obtain the simple numbers (34) for which the verification can be made by any method to do rational arithmetic.

Conclusions should not be drawn hastily. The admissible families method produces sharp coefficient estimates, and it could yet be used in the problem of characterizing initial segments of coefficients of normalized Riemann mappings of the unit disk into itself. Some modification of the method would, however, be required for this purpose.

Added in Proof. We thank the referee for pointing out that in a sequal to [13] titled "Operator-valued measures and coefficients of univalent functions", *Algebra i Analiz* 3 (6) (1991), 1-75; English translation, *St. Petersburg Math. J.* 3 (6) (1992), Vasyunin and Nikol'skiĭ have shown that a stronger condition than the "admissible families" condition is not sufficient for characterizing the initial segments of bounded univalent functions, and that this can be seen in other ways. Also, making use of an additional free parameter, which they called the *isometric trajectory*, they obtained more inequalities, which include inequality (28) in Theorem 4.2 as a special case. Unfortunately, the authors have not seen the key portions of this work and have only been able to refer to [13].

In a joint paper with Gene Christner, the authors have explicitly identified the inequalities corresponding to the contractive substitution condition. In a separate joint paper with Daniel Dreibelbis, making use of these inequalities, the authors have shown that the contractive substitution condition and the admissible families condition together are not sufficient to characterize the initial segments of the coefficients of bounded univalent functions on the unit disk. The first of these papers has been submitted to *Operator Theory: Advances and Applications*, and the second to the *Proceedings of the International Conference and Special Year on Complex Analysis (Nankai Institute of Mathematics, Tianjin, China, 1992).*

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