Julia Operators and Complementation in Kreĭn Spaces

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Abstract. Operator methods are used to derive and in some ways extend the theory of complementation in Kreĭn spaces, especially in the direction of uniqueness questions. A close connection with the theory of Julia operators is established. An application is given to the characterization of the range of a Kreĭn space contraction operator.

Julia operators and complementation are familiar Hilbert space notions in invariant subspace theory and the study of contraction operators. Each has a Kreĭn space version that serves an additional purpose to circumvent difficulties in the indefinite setting, such as the failure of polar decompositions. The theory of complementation in Kreĭn spaces is due to de Branges [**5**] and is systematically applied in [**6**]. From a different direction, the authors [**14**] developed an abstract theory of Julia operators in Kreĭn spaces and used this to derive extension properties of contraction operators. As we shall see, the theories are essentially equivalent. In this paper, we use an elementary operator method to derive and in some ways extend the known complementation theory for Kreĭn spaces. We discuss uniqueness questions which arise in the indefinite setting. The result is a unified theory that has much the same completeness and flavor as in the Hilbert space case. Some of these results were announced in Appendix A in [**14**].

Julia operators and complementation have a long history, which we shall only give in part. Complementation is a geometric notion from invariant subspace theory as presented in de Branges and Rovnyak [**7,8**]. In the Hilbert space case, M. Rosenblum gave seminar lectures on an operator approach to complementation in the 1960's, and similar methods are adopted in the survey of operator ranges by Fillmore and Williams [**15**]. The operator approach to complementation is used by other authors such as Sarason [**19,20**] in applications in function theory. Schwartz [**21**] contains an elegant theory of operator ranges in both the Hilbert space and Kreĭn space settings and anticipates the ideas of complementation. The paper of Schwarz also makes a link with the very closely related point of view of reproducing kernels as in Alpay [**1**]. In Kreĭn spaces, it is not immediate that the spaces which arise in de Branges' complementation theory are operator ranges. This was first shown by Heinz Langer (private communication, 1988) using definitizable operators [**17**]. This point of view is pursued in joint work of Ćurgus and Langer [**10**]. The

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authors recently learned of work of T. Hara [**16**] that also features operator methods and extends the complementation theory of de Branges [**5**] in certain ways.

Julia operators and defect operators have long played an important role in the theory of contraction operators on Hilbert spaces (Sz.-Nagy and Foias [**23**]). The existence of Julia operators in the indefinite setting was first shown by Arsene, Constantinescu, and Gheondea [**2**]. In Kreĭn spaces, Julia operators appear explicitly and implicitly in a number of sources, including Constantinescu and Gheondea [**9**], Ćurgus, Dijksma, Langer, and de Snoo [**11**], as well as [**2**]. They are sometimes called elementary rotations or unitary colligations.

A **Kreĭn space** is a scalar product space $\mathcal{H}$ which is isomorphic with the direct sum of a Hilbert space and the anti-space of a Hilbert space. Any such isomorphism induces in a natural way a Hilbert space structure. The Hilbert space norms obtained in this way are equivalent and determine a strong topology on $\mathcal{H}$. All underlying spaces are assumed to be separable, although this is not an essential hypothesis for our main results (separability is convenient in keeping account of indices). The set of continuous linear operators on a Kreĭn space $\mathcal{H}$ to a Kreĭn space $\mathcal{K}$ is written $\mathbf{B}(\mathcal{H},\mathcal{K})$ or $\mathbf{B}(\mathcal{H})$ if $\mathcal{H}=\mathcal{K}$. The adjoint A^* of an operator $A\in\mathbf{B}(\mathcal{H},\mathcal{K})$ is the unique operator in $\mathbf{B}(\mathcal{K},\mathcal{H})$ such that $\langle Af,g\rangle_{\mathcal{K}}=\langle f,A^*g\rangle_{\mathcal{H}}$ for all f in $\mathcal{H}$ and g in $\mathcal{K}$. Isometric and selfadjoint operators are defined as for Hilbert spaces. By a **partial isometry** we mean an operator $A\in\mathbf{B}(\mathcal{H},\mathcal{K})$ whose kernel is a regular subspace (see below) such that the restriction of A to the orthogonal complement of the kernel is isometric. An operator $A\in\mathbf{B}(\mathcal{H},\mathcal{K})$ is contractive if

$$\langle Af,Af\rangle_{\mathcal{K}}\le\langle f,f\rangle_{\mathcal{H}},\qquad f\in\mathcal{H}.$$

A **projection operator** on a Kreĭn space $\mathcal{H}$ is a selfadjoint and idempotent operator P in $\mathbf{B}(\mathcal{H})$. A **regular subspace** of a Kreĭn space $\mathcal{H}$ is a closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is itself a Kreĭn space in the scalar product of $\mathcal{H}$. Regular subspaces are characterized as the ranges of projection operators. A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is **positive** if $\langle f,f\rangle_{\mathcal{H}}\ge 0$ for all f in $\mathcal{H}$, **maximal positive** if it is positive and not a proper subset of another positive subspace, and **maximal uniformly positive** if it is maximal positive and a Hilbert space in the scalar product of $\mathcal{H}$. Similar notions with "positive" replaced by "negative" are obtained by reversing the sense of the inequality; "definite" means "positive" or "negative" in this context. We shall assume familiarity with other basic notions such as fundamental decompositions, and positive and negative indices of a Kreĭn space. Notation and terminology generally follow that of [**14**]. For details and background material see also [**3,4**].

A Kreĭn space $\mathcal{P}$ is said to be **contained continuously**, **contractively**, or **isometrically** in a Kreĭn space $\mathcal{H}$ if $\mathcal{P}$ is a linear subspace of $\mathcal{H}$ and the inclusion mapping is continuous, contractive, or isometric, respectively. The prototype is a regular subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$. Viewed as a Kreĭn space in the scalar product of $\mathcal{H}$, $\mathcal{M}$ is contained continuously and isometrically in $\mathcal{H}$. The adjoint of the inclusion of $\mathcal{M}$ in $\mathcal{H}$ coincides with the projection P of $\mathcal{H}$ onto $\mathcal{M}$. The orthogonal complement $\mathcal{M}^{\perp}$ of $\mathcal{M}$ is also a regular subspace of $\mathcal{H}$, and $\mathcal{H}$ is the direct sum of $\mathcal{M}$ and $\mathcal{M}^{\perp}$. The adjoint of the inclusion of $\mathcal{M}^{\perp}$ in $\mathcal{H}$ coincides with the projection $1 - P$ of $\mathcal{H}$ onto $\mathcal{M}^{\perp}$. Complementation theory generalizes these properties of regular subspaces to other Kreĭn spaces $\mathcal{P}$ which are contained continuously in $\mathcal{H}$.

THEOREM 1. *Let $\mathcal{P}$ be a Kreĭn space which is contained continuously in a Kreĭn space $\mathcal{H}$, and let $A \in \mathbf{B}(\mathcal{P}, \mathcal{H})$ be the inclusion mapping. Then $P = AA^*$ is a selfadjoint operator on $\mathcal{H}$. The range of P is contained in $\mathcal{P}$ as a dense subspace, and*

$$\langle Pf, Pg\rangle_{\mathcal{P}} = \langle Pf, g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}.$$

The positive and negative indices of the Kreĭn space $\mathcal{P}$ coincide with the positive and negative hermitian indices of the selfadjoint operator P, respectively.

The **positive hermitian index** of a selfadjoint operator T on a Kreĭn space $\mathcal{H}$ is defined to be the supremum of all positive integers r such that there is a positive definite matrix of the form $[\langle Tf_j, f_k\rangle_{\mathcal{H}}]_{j,k=1}^{r}$ with vectors $f_1, \dots, f_r$ in $\mathcal{H}$, and zero if no such r exists. The **negative hermitian index** of T is defined in the same way with T replaced by $-T$.

Proof of Theorem 1. The operator P is selfadjoint by its form. Since A is the inclusion mapping and $A^* \in \mathbf{B}(\mathcal{H}, \mathcal{P})$, we have $Pf = A^*f \in \mathcal{P}$ for every f in $\mathcal{H}$. For any f and g in $\mathcal{H}$,

$$\langle Pf, Pg\rangle_{\mathcal{P}} = \langle A^*f, A^*g\rangle_{\mathcal{P}} = \langle AA^*f, g\rangle_{\mathcal{H}} = \langle Pf, g\rangle_{\mathcal{H}}.$$

The last assertion is a particular case of Theorem 1.2.1 of [**14**]. ■

In the situation of Theorem 1, we often call P the **selfadjoint operator which coincides with the adjoint of the inclusion** of $\mathcal{P}$ in $\mathcal{H}$. This operator generalizes the projection of a Kreĭn space onto a regular subspace. The view of P as a "generalized projection" is justified by the properties expressed in Theorem 3. A subtle uniqueness question arises. Although P determines certain properties of $\mathcal{P}$ and even a dense subspace of $\mathcal{P}$, the operator P does not completely determine the Kreĭn space $\mathcal{P}$. We shall address this question later and give an example of the phenomenon as well as sufficient conditions for uniqueness.

Every selfadjoint operator P on a Kreĭn space $\mathcal{H}$ arises from some continuous inclusion of Kreĭn spaces as in Theorem 1.

Theorem 2. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Then P has a factorization in the form $P = EE^*$, where $E \in \mathbf{B}(\mathcal{E}, \mathcal{H})$ for some Kreĭn space $\mathcal{E}$ and $\ker E = \{0\}$. For any such factorization, let $\mathcal{P}_E$ be the range of E in the scalar product which makes E an isomorphism from $\mathcal{E}$ onto $\mathcal{P}_E$. Then $\mathcal{P}_E$ is a Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{P}_E$ in $\mathcal{H}$ coincides with P.*

Proof. The existence of a factorization $P = EE^*$ with the required properties follows from Theorem 1.2.2 of [**14**]. The space $\mathcal{P}_E$ is a Kreĭn space because it is isomorphic with $\mathcal{E}$. If $f, f_1, f_2, \ldots \in \mathcal{E}$ and $Ef_n \to Ef$ in $\mathcal{P}_E$, then $f_n \to f$ in $\mathcal{E}$. Since $E \in \mathbf{B}(\mathcal{E}, \mathcal{H})$, $Ef_n \to Ef$ in $\mathcal{H}$, and so the inclusion of $\mathcal{P}_E$ in $\mathcal{H}$ is continuous. For any $f \in \mathcal{E}$ and $g \in \mathcal{H}$,

$$\langle Ef, g \rangle_{\mathcal{H}} = \langle f, E^* g \rangle_{\mathcal{E}} = \langle Ef, EE^* g \rangle_{\mathcal{P}_E}.$$

The adjoint of the inclusion of $\mathcal{P}_E$ in $\mathcal{H}$ thus coincides with P. ■

The next result summarizes some of the characteristic properties of complementation theory.

Theorem 3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be Kreĭn spaces which are contained continuously in a Kreĭn space $\mathcal{H}$, and let P and Q be the selfadjoint operators on $\mathcal{H}$ which coincide with the adjoints of the inclusions of $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{H}$. Assume that $P + Q = 1$.*

- *(i) The mapping $U : (f, g) \to f + g$ is a partial isometry from $\mathcal{P} \times \mathcal{Q}$ onto $\mathcal{H}$ with adjoint $U^* : h \to (Ph, Qh)$.*
- *(ii) The intersection $\mathcal{R}$ of $\mathcal{P}$ and $\mathcal{Q}$ is a Kreĭn space in the scalar product defined by*

$$\langle f, g \rangle_{\mathcal{R}} = \langle f, g \rangle_{\mathcal{P}} + \langle f, g \rangle_{\mathcal{Q}}, \qquad f, g \in \mathcal{R}.$$

  The space $\mathcal{R}$ is called the **overlapping space** *for $\mathcal{P}$ and $\mathcal{Q}$. It is contained continuously in $\mathcal{H}$, and the adjoint of the inclusion of $\mathcal{R}$ in $\mathcal{H}$ coincides with PQ.*
- *(iii) These conditions are equivalent:*
  - *(a) $\mathcal{P}$ is contained contractively in $\mathcal{H}$,*
  - *(b) $\mathcal{Q}$ is contained contractively in $\mathcal{H}$,*
  - *(c) $P^2 \le P$,*
  - *(d) $Q^2 \le Q$,*
  - *(e) U is a contraction, and*
  - *(f) the overlapping space $\mathcal{R}$ is a Hilbert space.*
- *(iv) These conditions are equivalent:*
  - *(a) $\mathcal{P}$ and $\mathcal{Q}$ are contained isometrically in $\mathcal{H}$ as regular subspaces with $\mathcal{Q} = \mathcal{P}^{\perp}$,*

(b) $P^2 = P$,
(c) $Q^2 = Q$,
(d) *U is an isometry, and*
(e) *the overlapping space $\mathcal{R}$ contains no nonzero element.*

Proof. (i) The continuity of U follows from the continuity of the inclusions of $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{H}$, and the formula for U^* is verified in a routine way. Since $P + Q = 1$, we obtain $UU^*U = U$, and therefore U is a partial isometry (see [**14**], Theorem 1.1.3). Its range is $\mathcal{H}$ because $U : (Ph, Qh) \to h$ for any h in $\mathcal{H}$.

(ii) The kernel $\mathcal{N}$ of U, that is, the set of pairs $(f, -f)$ with f in $\mathcal{R}$, is a regular subspace of $\mathcal{P} \times \mathcal{Q}$ by the definition of a partial isometry, and hence $\mathcal{N}$ is a Kreĭn space in the scalar product of $\mathcal{P} \times \mathcal{Q}$. Therefore $\mathcal{R}$ is a Kreĭn space which is isomorphic with $\mathcal{N}$ by means of the correspondence $f \to (f, -f)$. We omit the easy verification that the inclusion of $\mathcal{R}$ in $\mathcal{H}$ is continuous. If $u \in \mathcal{R}$ and $h \in \mathcal{H}$, then $PQh = QPh$ belongs to $\mathcal{R}$, and

$$\begin{aligned}\langle u, h\rangle_{\mathcal{H}} &= \langle u, Qh\rangle_{\mathcal{H}} + \langle u, Ph\rangle_{\mathcal{H}} \\ &= \langle u, PQh\rangle_{\mathcal{P}} + \langle u, QPh\rangle_{\mathcal{Q}} \\ &= \langle u, PQh\rangle_{\mathcal{R}}.\end{aligned}$$

Thus the adjoint of the inclusion of $\mathcal{R}$ in $\mathcal{H}$ coincides with PQ.

(iii) A formula for scalar products on dense sets in $\mathcal{P}$ and $\mathcal{Q}$ (the ranges of P and Q), is given in Theorem 1. Using this formula we easily verify the equivalence of (a)–(d). We assume (c) and prove (e). The assertion that U is a contraction is equivalent to showing that

$$\langle Pu + Qv, Pu + Qv\rangle_{\mathcal{H}} \leq \langle Pu, u\rangle_{\mathcal{H}} + \langle Qv, v\rangle_{\mathcal{H}}, \qquad u, v \in \mathcal{H}.$$

An algebraic calculation yields

$$\begin{aligned}&\langle Pu + Qv, Pu + Qv\rangle_{\mathcal{H}} - \langle Pu, u\rangle_{\mathcal{H}} - \langle Qv, v\rangle_{\mathcal{H}} \\ &= \left\langle P^2 u, u\right\rangle_{\mathcal{H}} + \langle (1-P)Pu, v\rangle_{\mathcal{H}} + \langle P(1-P)v, u\rangle_{\mathcal{H}} \\ &\qquad + \left\langle (1-P)^2 v, v\right\rangle_{\mathcal{H}} - \langle Pu, u\rangle_{\mathcal{H}} - \langle (1-P)v, v\rangle_{\mathcal{H}} \\ &= \left\langle (P^2 - P)(u - v), u - v\right\rangle_{\mathcal{H}}.\end{aligned}$$

This is nonpositive because $P^2 \leq P$ by (*c*). The remaining details are straightforward and omitted.

(iv) If $\mathcal{R}$ contains no nonzero element, then U is an isometry, and conversely. Thus (d) and (e) are equivalent. When these conditions hold, then $\mathcal{P} \times \mathcal{Q}$ is the orthogonal direct sum of $\mathcal{P} \times \{0\}$ and $\{0\} \times \mathcal{Q}$, and so $\mathcal{P}$ and $\mathcal{Q}$ are contained isometrically in $\mathcal{H}$ as regular subspaces with $\mathcal{Q} = \mathcal{P}^{\perp}$ by elementary properties of isometries. The converse assertion is clear, and therefore (d) and (e) are equivalent to (a). The equivalences with (b) and (c) are easily checked. ■

The main content of Theorem 3 can be expressed in more geometric language. Let Kreĭn spaces $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$ and operators P, Q be given as in Theorem 3. By a **minimal decomposition** of a vector h in $\mathcal{H}$ we mean a representation

$$h = f + g, \qquad f \in \mathcal{P}, g \in \mathcal{Q},$$

such that

$$\langle f, u \rangle_{\mathcal{P}} + \langle g, v \rangle_{\mathcal{Q}} = 0$$

whenever $0 = u + v$ with u in $\mathcal{P}$ and v in $\mathcal{Q}$. By Theorem 3, every vector in $\mathcal{H}$ has a unique minimal decomposition. If $h = f + g$ is a minimal decomposition of $h \in \mathcal{H}$ with $f \in \mathcal{P}$ and $g \in \mathcal{Q}$, the identity

$$\langle h, h' \rangle_{\mathcal{H}} = \langle f, f' \rangle_{\mathcal{P}} + \langle g, g' \rangle_{\mathcal{Q}}$$

holds for an arbitrary decomposition $h' = f' + g'$ with $f' \in \mathcal{P}$ and $g' \in \mathcal{Q}$. The minimal decomposition of any vector h in $\mathcal{H}$ is obtained with $f = Ph$ as the element of $\mathcal{P}$ and $g = Qh$ as the element of $\mathcal{Q}$. In the case of contractive inclusion, that is, when $P^2 \leq P$, then in addition we have

$$\langle h, h \rangle_{\mathcal{H}} \leq \langle f, f \rangle_{\mathcal{P}} + \langle g, g \rangle_{\mathcal{Q}}$$

for any decomposition $h = f + g$ with f in $\mathcal{P}$ and g in $\mathcal{Q}$, and equality holds only for the minimal decomposition. We then say that the spaces $\mathcal{P}$ and $\mathcal{Q}$ are complementary, or that $\mathcal{Q}$ is the complement of $\mathcal{P}$ and vice versa. This notion reduces to the standard concept of orthogonal complementation for regular subspaces in the case characterized in Part (iv) of Theorem 3.

By Theorem 1, a Kreĭn space $\mathcal{P}$ which is contained continuously in a Kreĭn space $\mathcal{H}$ uniquely determines a continuous selfadjoint operator P on $\mathcal{H}$ such that P coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The range $\mathcal{P}_0$ of P is a scalar product space with scalar product

$$\langle Pf, Pg \rangle_{\mathcal{P}_0} = \langle Pf, g \rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H},$$

and $\mathcal{P}$ contains $\mathcal{P}_0$ isometrically as a dense subspace. By Theorem 2, every selfadjoint operator $P \in \mathbf{B}(\mathcal{H})$ arises in this way from some Kreĭn space $\mathcal{P}$ which is contained continuously in $\mathcal{H}$. It may occur that P arises from distinct spaces $\mathcal{P}_1$ and $\mathcal{P}_2$, despite the fact that $\mathcal{P}_1$ and $\mathcal{P}_2$ each contains $\mathcal{P}_0$ isometrically as a dense subspace.

Example. *There exist (1) a continuous selfadjoint operator P on a Kreĭn space $\mathcal{H}$, and (2) Kreĭn spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ which are contained continuously in $\mathcal{H}$, such that the adjoints of the inclusions of $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{H}$ each coincide with P, yet $\mathcal{P}_1 \neq \mathcal{P}_2$.*

Choose $\mathcal{H}$ of the form $\mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ is an infinite dimensional Hilbert space $\mathcal{C}$ and $\mathcal{H}_-$ is the anti-space of $\mathcal{C}$. Let $K \in \mathbf{B}(\mathcal{C})$ be an operator such that $0 \leq K \leq 1$ in the partial ordering of Hilbert space selfadjoint operators on $\mathcal{C}$, $1 - K^2$ has zero kernel, but $1 - K^2$ is not invertible. If we represent the elements of $\mathcal{H}$ as column vectors of elements of $\mathcal{C}$, we may define a continuous selfadjoint operator P on $\mathcal{H}$ by setting

$$P = \begin{pmatrix} 1 - K^2 & 0 \\ 0 & 1 - K^2 \end{pmatrix}.$$

Then we have two factorizations,

$$P = AA^* = BB^*,$$

where

$$A = \begin{pmatrix} 1 & -K \\ -K & 1 \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} 1 & K \\ K & 1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & -K \\ K & -1 \end{pmatrix} \quad \text{and} \quad B^* = \begin{pmatrix} 1 & -K \\ K & -1 \end{pmatrix}.$$

The operators A and B are viewed as elements of $\mathbf{B}(\mathcal{H})$. Each has zero kernel. By Theorem 2, we may define Kreĭn spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ as the ranges of A and B, respectively, in the scalar products which make A and B isomorphisms from $\mathcal{H}$ onto $\mathcal{P}_1$ and $\mathcal{P}_2$. These spaces are contained continuously in $\mathcal{H}$, and in each case the adjoint of the inclusion coincides with P. However, $\mathcal{P}_1 \neq \mathcal{P}_2$. For if the spaces coincide, then the system of equations

$$\begin{aligned} u - Kv &= f - Kg \\ Ku - v &= -Kf + g \end{aligned}$$

has a solution u and v in $\mathcal{C}$ for any given f and g in $\mathcal{C}$. If $g = 0$, this says that the equation

$$(1 - K^2)u = (1 + K^2)f$$

has a solution u in $\mathcal{C}$ for every f in $\mathcal{C}$. This is not true, because $1 + K^2$ is invertible but $1 - K^2$ is not invertible by assumption. This example is adapted from a similar one in [**13**] which shows nonuniqueness of Julia operators.

The uniqueness question is treated in Schwarz [**21**], who gives a result similar to the following.

THEOREM 4. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Kreĭn spaces which are contained continuously in $\mathcal{H}$ such that the adjoints of the inclusions coincide with P. If $\mathcal{P}_1$ is contained continuously in $\mathcal{P}_2$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

Proof. Let A_1, A_2 be the inclusions of $\mathcal{P}_1, \mathcal{P}_2$ in $\mathcal{H}$, and let C be the inclusion of $\mathcal{P}_1$ in $\mathcal{P}_2$. Then $A_1 = A_2C$. Since

$$A_1A_1^* = A_2A_2^* = P$$

by hypothesis, where A_1 has zero kernel and A_2^* has dense range, CC^* is the identity operator on $\mathcal{P}_2$. Therefore C^* is an isometry. Since C has zero kernel, C is a Kreĭn space isomorphism. ■

We give a uniqueness criterion of a different nature.

THEOREM 5. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Let $\mathcal{P}_1$ be a Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{P}_1$ in $\mathcal{H}$ coincides with P. Assume that there is a maximal uniformly definite subspace $\mathcal{M}$ of $\mathcal{P}_1$ which is contained in the range of P. If $\mathcal{P}_2$ is a second Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{P}_2$ in $\mathcal{H}$ coincides with P, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

It can be shown that the sufficient condition for uniqueness given in Theorem 5 is necessary in the following sense. Assume that $\mathcal{P}_1$ is a Kreĭn space which is contained continuously in a Kreĭn space $\mathcal{H}$ and that the adjoint of the inclusion of $\mathcal{P}_1$ in $\mathcal{H}$ coincides with P. If there is no uniformly definite subspace of $\mathcal{P}_1$ which is contained in the range of P, then there is a Kreĭn space $\mathcal{P}_2$ which is not isometrically equal to $\mathcal{P}_1$ such that $\mathcal{P}_2$ is contained continuously in $\mathcal{H}$ and the adjoint of the inclusion coincides with P. See Dritschel [**13**].

Independently, T. Hara [**16**] has given a uniqueness condition of yet another kind, in terms of gaps in the spectra of Hilbert space operators associated with a fundamental decomposition of the underlying Kreĭn space. See also [**10**].

Proof. We assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy the conditions of the theorem and show that they coincide. For the purpose of the proof, we assume that $\mathcal{M}$ is maximal uniformly negative in $\mathcal{P}_1$, omitting the obvious modifications needed for the case in which $\mathcal{M}$ is maximal uniformly positive in $\mathcal{P}_1$.

Let $\mathcal{P}_0$ be the range of P in the scalar product

$$\langle Pf, Pg \rangle_{\mathcal{P}_0} = \langle Pf, g \rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}.$$

Then $\mathcal{P}_0$ is contained in both $\mathcal{P}_1$ and $\mathcal{P}_2$ and is dense in both spaces, and the three scalar products coincide on $\mathcal{P}_0$. Let W_0 be the identity mapping on $\mathcal{P}_0$, viewed as a densely defined isometry on $\mathcal{P}_1$ with dense range in $\mathcal{P}_2$. By hypothesis, dom W_0 includes the maximal uniformly negative subspace $\mathcal{M}$ of $\mathcal{P}_1$, and W_0 fixes $\mathcal{M}$. Notice also that, viewed as a closed subspace of $\mathcal{P}_1$, $\mathcal{M}$ is the continuous one-to-one image by P of some closed subspace $\mathcal{N}$ of $\mathcal{H}$.

We show that $\mathcal{M}$ is closed and uniformly negative in $\mathcal{P}_2$. The $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ scalar products coincide on $\mathcal{M}$ and make $\mathcal{M}$ into the anti-space of a Hilbert space. Temporarily write $\mathcal{K}$ for $\mathcal{M}$ viewed as the anti-space of a Hilbert space in this scalar product. Define U on $\mathcal{K}$ to $\mathcal{P}_2$ to be the identity mapping. Then $U = U_3 U_2 U_1$, where U_1 is the identity mapping from $\mathcal{K}$ into $\mathcal{P}_1$, U_2 is the inverse of $P|\mathcal{N}$ on $\mathcal{M}$ to $\mathcal{N}$, and U_3 is P on $\mathcal{N}$ into $\mathcal{P}_2$. Each factor is continuous, and so U is a continuous isometry on $\mathcal{K}$ into $\mathcal{P}_2$ with range $\mathcal{M}$. Therefore $\mathcal{M}$ is a closed and uniformly negative subspace of $\mathcal{P}_2$.

In fact, $\mathcal{M}$ is maximal uniformly negative in $\mathcal{P}_2$. For by what has been shown, $\mathcal{M}$ is regular in each of the spaces $\mathcal{P}_1$ and $\mathcal{P}_2$, it is closed and uniformly negative in both spaces, and it is maximal uniformly negative in $\mathcal{P}_1$. Since $\mathcal{P}_0$ is dense in both $\mathcal{P}_1$ and $\mathcal{P}_2$ and includes $\mathcal{M}$, the set of elements of $\mathcal{P}_0$ that are orthogonal to $\mathcal{M}$ in $\mathcal{P}_1$ is identical to the set of elements of $\mathcal{P}_0$ that are orthogonal to $\mathcal{M}$ in $\mathcal{P}_2$, and this set is dense in both $\mathcal{P}_1 \ominus \mathcal{M}$ and $\mathcal{P}_2 \ominus \mathcal{M}$. The orthogonal complement of $\mathcal{M}$ in $\mathcal{P}_0$ is therefore dense in both $\mathcal{P}_1 \ominus \mathcal{M}$ and $\mathcal{P}_2 \ominus \mathcal{M}$ with the former being a Hilbert space. Thus the latter is a Hilbert space, and so $\mathcal{M}$ is maximal uniformly negative in $\mathcal{P}_2$.

By a theorem of Shmul'yan [**22**] and Yan [**24**] (see also [**14**], Theorem 1.4.4), W_0 has a continuous extension to a unitary operator W on $\mathcal{P}_1$ to $\mathcal{P}_2$. Since W is the identity on $\mathcal{P}_0$, the spaces $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically. ■

When the uniqueness criterion of Theorem 5 is met, not only is the Kreĭn space $\mathcal{P}$ associated with the selfadjoint operator P unique, we also have a representation of $\mathcal{P}$ from an arbitrary factorization of the operator P in the form $P = EE^*$ with $\ker E = \{0\}$.

Corollary 6. *Let $\mathcal{P}$ be a Kreĭn space which is contained continuously in a Kreĭn space $\mathcal{H}$, and let $P \in \mathbf{B}(\mathcal{H})$ be the selfadjoint operator which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. Assume that there is a maximal uniformly definite subspace $\mathcal{M}$ in $\mathcal{P}$ which is contained in the range of P. Suppose that $P = EE^*$, where $E \in \mathbf{B}(\mathcal{E}, \mathcal{H})$ for some Kreĭn space $\mathcal{E}$ and E has zero kernel. Then E is a Kreĭn space isomorphism of $\mathcal{E}$ onto $\mathcal{P}$.*

Proof. Let $\mathcal{P}_E$ be the range of E viewed as a Kreĭn space in the scalar product that makes E a Kreĭn space isomorphism of $\mathcal{E}$ onto $\mathcal{P}_E$. By Theorem 2, $\mathcal{P}_E$

is a Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion coincides with P. By Theorem 6, $\mathcal{P}$ and $\mathcal{P}_E$ are equal isometrically. ■

The uniqueness condition in Theorem 5 is automatically met under certain circumstances. The following result strengthens one of de Branges [**5**].

THEOREM 7. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Kreĭn spaces which are contained continuously in a Kreĭn space $\mathcal{H}$ such that the adjoints of the inclusions each coincide with the selfadjoint operator P. If the inclusion of at least one of the spaces in $\mathcal{H}$ is contractive, then the spaces are equal isometrically.*

We use a technical result on the mapping properties of the adjoint of a contraction operator.

LEMMA 8. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and assume that $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction operator. There is a closed uniformly negative subspace $\mathcal{N}$ of $\mathcal{K}$ which is mapped in a one-to-one way by T^* onto a maximal uniformly negative subspace $\mathcal{M}$ of $\mathcal{H}$.*

Proof of Lemma 8. This result is due to Dritschel [**12**] and is given in Theorem 2.1.2 of [**14**]. ■

Proof of Theorem 7. Suppose that the inclusion of $\mathcal{P}_1$ in $\mathcal{H}$ is contractive, and let A_1 be the inclusion mapping. By Lemma 8, there is a closed uniformly negative subspace $\mathcal{N}$ of $\mathcal{H}$ which is mapped by A_1^* in a one-to-one way onto a maximal uniformly negative subspace $\mathcal{M}$ of $\mathcal{P}_1$. Then $\mathcal{M}$ is a subspace of the range of P which is maximal uniformly negative in $\mathcal{P}_1$, and the conclusion follows from Theorem 5. ■

THEOREM 9. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Kreĭn spaces which are contained continuously in a Kreĭn space $\mathcal{H}$ such that the adjoints of the inclusions each coincide with the selfadjoint operator P. If either the positive or negative hermitian index of the selfadjoint operator P is finite, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

LEMMA 10. *Let $\mathcal{D}$ be a dense subspace of a Kreĭn space $\mathcal{K}$, and let r be a positive integer. Assume that there exist r and no more than r linearly independent vectors in $\mathcal{D}$ whose Gram matrix is nonpositive. Then $\mathcal{K}$ is a Pontryagin space with negative index r, and there exists a maximal uniformly negative subspace in $\mathcal{K}$ which is contained in $\mathcal{D}$.*

Proof of Lemma 10. The result is probably in the literature, but we do not know a reference. It is an easy variant of Lemma B5 of [**14**]. ■

Proof of Theorem 9. It is sufficient to give the proof when the negative hermitian index of P is finite. Let $\mathcal{P}_0$ be the range of P in the scalar product

$$\langle Pf, Pg \rangle_{\mathcal{P}_0} = \langle Pf, g \rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}.$$

Then $\mathcal{P}_0$ is contained isometrically in $\mathcal{P}_1$ as a dense subspace. Since the negative hermitian index of P is finite by assumption, the maximum number of linearly independent vectors in $\mathcal{P}_0$ whose Gram matrix is nonpositive is finite. By Lemma 10, $\mathcal{P}_1$ is a Pontryagin space, and there is a maximal uniformly negative subspace of $\mathcal{P}_1$ which is contained in $\mathcal{P}_0$. The result therefore follows from Theorem 5. ∎

Assume that $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces. By a **defect operator** for T we mean any operator $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$, where $\tilde{\mathcal{D}}$ is a Kreĭn space, such that $\tilde{D}$ has zero kernel and $1 - T^*T = \tilde{D}\tilde{D}^*$. By a **Julia operator** for T we mean any unitary operator of the form

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are Kreĭn spaces and $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ are operators with zero kernels. In this case $\tilde{D}$ is a defect operator for T, D is a defect operator for T^*, and U^* is a Julia operator for T^*.

A close connection between Julia operators and complementation theory is established in the case of contractive inclusion. The result is a characterization of complementary spaces that does not presume knowledge of associated operators as in Theorem 3. The characterization is taken as the definition of complementation in de Branges [**5**].

THEOREM 11. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator such that $P^2 \leq P$. Let $\mathcal{P}$ be the unique Kreĭn space which is contained contractively and continuously in $\mathcal{H}$ such that the adjoint of the inclusion coincides with P. Let $\mathcal{Q}$ be the complementary space, that is, the unique Kreĭn space which is contained contractively and continuously in $\mathcal{H}$ such that the adjoint of the inclusion coincides with $Q = 1 - P$. Then a vector f in $\mathcal{H}$ belongs to $\mathcal{Q}$ if and only if*

$$\sup_{g \in \mathcal{P}} \left[\langle f + g, f + g \rangle_{\mathcal{H}} - \langle g, g \rangle_{\mathcal{P}} \right] < \infty,$$

and in this case the value of the supremum is $\langle f, f \rangle_{\mathcal{Q}}$.

Proof. Write $P = DD^*$, $Q = TT^*$ where $D \in \mathbf{B}(\mathcal{D}, \mathcal{H})$, $T \in \mathbf{B}(\mathcal{A}, \mathcal{H})$ for some Kreĭn spaces $\mathcal{D}, \mathcal{A}$ and the kernels of D, T are zero (see [**14**], Theorem 1.2.2).

Then D is a defect operator for T^*, and by Theorem 1.2.4 of [**14**] we may choose a Julia operator for T of the form

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{A} \oplus \mathcal{D}, \mathcal{H} \oplus \tilde{\mathcal{D}}).$$

By Theorems 2 and 7, D maps $\mathcal{D}$ isometrically onto $\mathcal{P}$, and T maps $\mathcal{A}$ isometrically onto $\mathcal{Q}$. Since U unitary,

$$\begin{aligned} T^*T + \tilde{D}\tilde{D}^* &= 1, & TT^* + DD^* &= 1, \\ D^*T + L^*\tilde{D}^* &= 0, & T\tilde{D} + DL^* &= 0, \\ D^*D + L^*L &= 1, & \tilde{D}^*\tilde{D} + LL^* &= 1. \end{aligned}$$

Our hypotheses imply that T is a contraction, and hence $\tilde{\mathcal{D}}$ is a Hilbert space.

The rest of the proof is similar to the Hilbert space case [**7,15**]. For any f in $\mathcal{H}$, set

$$\varphi(f) = \sup_{g \in \mathcal{P}} \left[\langle f+g, f+g \rangle_{\mathcal{H}} - \langle g, g \rangle_{\mathcal{P}} \right].$$

If $f \in \mathcal{Q}$ and $g \in \mathcal{P}$, we have $f = Th$ and $g = Du$, where $h \in \mathcal{A}$ and $u \in \mathcal{D}$, and

$$\begin{aligned} &\langle f+g, f+g \rangle_{\mathcal{H}} - \langle g, g \rangle_{\mathcal{P}} \\ &\qquad = \langle Th, Th \rangle_{\mathcal{H}} + 2\mathrm{Re}\, \langle Th, Du \rangle_{\mathcal{H}} + \langle Du, Du \rangle_{\mathcal{H}} - \langle u, u \rangle_{\mathcal{D}} \\ &\qquad = \langle T^*Th, h \rangle_{\mathcal{A}} + 2\mathrm{Re}\, \langle D^*Th, u \rangle_{\mathcal{D}} - \langle (1 - D^*D)u, u \rangle_{\mathcal{D}} \\ &\qquad = \langle h, h \rangle_{\mathcal{A}} - \left\langle \tilde{D}^*h, \tilde{D}^*h \right\rangle_{\tilde{\mathcal{D}}} - 2\mathrm{Re} \left\langle \tilde{D}^*h, Lu \right\rangle_{\tilde{\mathcal{D}}} - \langle Lu, Lu \rangle_{\tilde{\mathcal{D}}} \\ &\qquad = \langle f, f \rangle_{\mathcal{Q}} - \left\langle \tilde{D}^*h + Lu, \tilde{D}^*h + Lu \right\rangle_{\tilde{\mathcal{D}}} \\ &\qquad \le \langle f, f \rangle_{\mathcal{Q}}, \end{aligned}$$

since $\tilde{\mathcal{D}}$ is a Hilbert space. Therefore $\varphi(f) \le \langle f, f \rangle_{\mathcal{Q}} < \infty$.

In the other direction, suppose that $f \in \mathcal{H}$ and $\varphi(f) = C < \infty$. Then by the definition of $\varphi(f)$, for any $u \in \mathcal{D}$,

$$\langle f, f \rangle_{\mathcal{H}} + 2\mathrm{Re}\, \langle f, Du \rangle_{\mathcal{H}} + \langle Du, Du \rangle_{\mathcal{H}} \le C + \langle u, u \rangle_{\mathcal{D}},$$

and therefore

$$C - \langle f, f \rangle_{\mathcal{H}} - 2\mathrm{Re}\, \langle f, Du \rangle_{\mathcal{H}} + \langle Lu, Lu \rangle_{\tilde{\mathcal{D}}} \ge 0.$$

Set $B = C - \langle f, f \rangle_{\mathcal{H}}$. Notice that $B \ge 0$ because we can choose $g = 0$ in the supremum that defines $\varphi(f)$. A routine argument with quadratic forms yields

$$B + 2|\langle f, Du \rangle_{\mathcal{H}}| t + \langle Lu, Lu \rangle_{\tilde{\mathcal{D}}} t^2 \ge 0$$

for all real t, and therefore $|\langle f, Du\rangle_{\mathcal{H}}|^2 \le B\langle Lu, Lu\rangle_{\tilde{\mathcal{D}}}$. By the Riesz representation theorem there is a $v \in \tilde{\mathcal{D}}$ in the closure of the range of L such that

$$\langle f, Du\rangle_{\mathcal{H}} = \langle v, Lu\rangle_{\tilde{\mathcal{D}}}, \qquad u \in \mathcal{D}.$$

This implies that $D^*f = L^*v$, and

$$\begin{aligned} f &= (1 - TT^*)f + TT^*f = DD^*f + TT^*f \\ &= DL^*v + TT^*f = -T\tilde{D}v + TT^*f = Th, \end{aligned}$$

where $h = -\tilde{D}v + T^*f \in \mathcal{A}$. In particular, $f \in \mathcal{Q}$. By the first part of the proof, for any $g \in \mathcal{P}$ written as $g = Tu$ with $u \in \mathcal{D}$,

$$\langle f + g, f + g\rangle_{\mathcal{H}} - \langle g, g\rangle_{\mathcal{P}} = \langle f, f\rangle_{\mathcal{Q}} - \|\tilde{D}^*h + Lu\|^2_{\tilde{\mathcal{D}}}.$$

Now $\tilde{D}^*h = -\tilde{D}^*\tilde{D}v + \tilde{D}^*T^*f = (LL^* - 1)v - LD^*f$ is in the closure of the range of L, and so $\|\tilde{D}^*h + Lu\|^2_{\tilde{\mathcal{D}}}$ can be made arbitrarily small by appropriate choice of $u \in \mathcal{D}$. Therefore $\varphi(f) \ge \langle f, f\rangle_{\mathcal{Q}}$. The result follows since the reverse inequality was previously proved for any element of $\mathcal{Q}$. ■

As a consequence we obtain a characterization of the range of a Kreĭn space contraction operator.

Corollary 12. *Let T be a contraction operator on a Kreĭn space $\mathcal{H}$ to a Kreĭn space $\mathcal{K}$, and let $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ be a defect operator for T^*. Then an element g of $\mathcal{K}$ belongs to the range of T if and only if*

$$\sup_{u\in\mathcal{D}} \left[\langle g + Du, g + Du\rangle_{\mathcal{K}} - \langle u, u\rangle_{\mathcal{D}}\right] < \infty.$$

In this case, $g = Tf$ for an element f of $\mathcal{H}$ such that the value of the supremum is $\langle f, f\rangle_{\mathcal{H}}$.

Proof. Without loss of generality, we may assume that T has zero kernel. Let $\mathcal{P}$, $\mathcal{Q}$ be the ranges of D, T in the scalar products which make D, T isomorphisms of $\mathcal{D}$, $\mathcal{H}$ onto $\mathcal{P}$, $\mathcal{Q}$, respectively. The adjoints of the inclusion mappings coincide with $P = DD^* = 1 - TT^*$ and $Q = TT^*$. Since T is a contraction, $Q^2 \le Q$ and hence $P^2 \le P$. The conclusion follows from Theorem 11. ■

Theorem 11 uses Julia operators to say something about complementation. In turn, complementation theory may be used to construct Julia operators. Assume that T is a one-to-one operator in $\mathbf{B}(\mathcal{A}, \mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are Kreĭn spaces (with minor modifications we could weaken the hypothesis to allow the kernel of T to be a regular subspace of $\mathcal{H}$). Let $\mathcal{M}(T)$ be the range of T in the scalar product which makes T an isometry of $\mathcal{A}$ onto $\mathcal{M}(T)$. Then $\mathcal{M}(T)$ is contained continuously in $\mathcal{H}$, and the adjoint of the inclusion of $\mathcal{M}(T)$ in $\mathcal{H}$ coincides with TT^*. Let $\mathcal{H}(T)$ be any Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{H}(T)$ in $\mathcal{H}$ coincides with $1 - TT^*$. Let $\mathcal{R}$ be the overlapping space for $\mathcal{M}(T)$ and $\mathcal{H}(T)$ as defined in Theorem 3. Define $\mathcal{L}(T)$ to be the Kreĭn space of elements f of $\mathcal{A}$ such that Tf belongs to $\mathcal{R}$, in the scalar product which makes T an isomorphism of $\mathcal{L}(T)$ onto $\mathcal{R}$.

THEOREM 13. *Let $T \in \mathbf{B}(\mathcal{A}, \mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are Kreĭn spaces, and assume that T is one-to-one. If $\mathcal{M}(T)$, $\mathcal{H}(T)$, $\mathcal{L}(T)$ are as above, then a Julia operator for T is given by*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{A} \oplus \mathcal{D}, \mathcal{H} \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D} = \mathcal{H}(T)$ and D is the inclusion operator, $\tilde{\mathcal{D}} = \mathcal{L}(T)$ and $\tilde{D}$ is the inclusion operator, and $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ is the operator such that $L^* f = -Tf$ for all f in $\tilde{\mathcal{D}}$.

Proof. It is clear that D and $\tilde{D}$ have zero kernels. We establish the identities (i) $1 - TT^* = DD^*$, (ii) $1 - T^*T = \tilde{D}\tilde{D}^*$, and (iii) $DL^* = -T\tilde{D}$. The identity (i) is a restatement of the fact that the adjoint of the inclusion of $\mathcal{H}(T)$ in $\mathcal{H}$ coincides with $1 - TT^*$. To prove (ii), consider any $g \in \tilde{\mathcal{D}}$ and $f \in \mathcal{H}$. We have

$$\begin{aligned} \langle g, (1 - T^*T)f \rangle_{\tilde{\mathcal{D}}} &= \langle Tg, T(1 - T^*T)f \rangle_{\mathcal{R}} \\ &= \langle Tg, (1 - TT^*)Tf \rangle_{\mathcal{P}} + \langle Tg, T(1 - T^*T)f \rangle_{\mathcal{Q}} \\ &= \langle Tg, Tf \rangle_{\mathcal{K}} + \langle g, (1 - T^*T)f \rangle_{\mathcal{H}} \\ &= \langle g, f \rangle_{\mathcal{H}}, \end{aligned}$$

and (ii) follows. The identity (iii) is a consequence of the definition of L.

The identities (iv) $\tilde{D}L = -T^*D$, (v) $L^*L = 1 - D^*D$, (vi) $LL^* = 1 - \tilde{D}^*\tilde{D}$ are consequences of (i)-(iii) and the fact that D and $\tilde{D}$ have zero kernels (see [**14**], proof of Theorem B3):

$$\begin{aligned} &\tilde{D}LD^* = -\tilde{D}\tilde{D}^*T^* = -(1 - T^*T)T^* = -T^*(1 - TT^*) = -T^*DD^*, \\ &DL^*L = -T\tilde{D}L = TT^*D = (1 - DD^*)D = D(1 - D^*D), \\ &\tilde{D}LL^* = -T^*DL^* = T^*T\tilde{D} = (1 - \tilde{D}\tilde{D}^*)\tilde{D} = \tilde{D}(1 - \tilde{D}^*\tilde{D}). \end{aligned}$$

The identities (i)-(iv) imply that U is a Julia operator for T. ■

We show how the original setting of complementation theory in [**7**,**8**] is recaptured in the present scheme. In Theorem 13, choose both $\mathcal{H}$ and $\mathcal{K}$ to be the Hilbert space $\mathcal{C}(z)$ of square summable power series with complex coefficients, and let T be multiplication by a formal power series $B(z)$ which represents a function which is bounded by 1 in the unit disk. Then $\mathcal{M}(T) = \mathcal{M}(B)$ and $\mathcal{H}(T) = \mathcal{H}(B)$ are the Hilbert spaces with reproducing kernels

$$\frac{B(z)\overline{B(w)}}{1 - z\overline{w}} \quad \text{and} \quad \frac{1 - B(z)\overline{B(w)}}{1 - z\overline{w}},$$

respectively [**7**]. The space $\mathcal{L}(T) = \mathcal{L}_B$ appears in [**8**] and is called there the overlapping space, which is slightly different terminology than in both [**5**] and above. We show that $\mathcal{L}(T)$ has reproducing kernel

$$\begin{aligned} L(w, z) &= \int_\Gamma \frac{\Delta(\zeta)}{(1 - \zeta\bar{w})(1 - \bar{\zeta}z)}\, d\sigma(\zeta) \\ &= \frac{1}{2}\, \frac{\varphi(z) + \bar{\varphi}(w)}{1 - z\bar{w}}, \end{aligned}$$

where

$$\Delta(\zeta) = 1 - |B(\zeta)|^2$$

and

$$\varphi(z) = \int_\Gamma \frac{\zeta + z}{\zeta - z}\, \Delta(\zeta)\, d\sigma(\zeta).$$

Here ζ is a generic point on the unit circle Γ in the complex plane, and σ is normalized Lebesgue measure on the circle. We use the same notation for formal power series, functions which they represent in the unit disk, and boundary functions since the meaning is clear from context. To prove the assertions, observe that for α in the unit disk, $1 - T^*T$ maps $1/(1 - z\bar{\alpha})$ to $L(\alpha, z)$, and so

$$\begin{aligned} L(\alpha, \beta) &= \left\langle (1 - T^*T)\frac{1}{1 - z\bar{\alpha}}, \frac{1}{1 - z\bar{\beta}} \right\rangle_{\mathcal{C}(z)} \\ &= \int_\Gamma \frac{1 - |B(\zeta)|^2}{(1 - \zeta\bar{\alpha})(1 - \bar{\zeta}\beta)}\, d\sigma(\zeta). \end{aligned}$$

The idenfication of the reproducing kernel for $\mathcal{L}(T)$ allows us to give a concrete representation for the space. Let $L^2(\Delta)$ be the Lebesgue space of measurable complex valued functions $p(\zeta)$ on the unit circle such that

$$||p||_\Delta^2 = \int_\Gamma |p(\zeta)|^2\, d\sigma(\zeta) < \infty.$$

Let $L^2_+(\Delta)$ be the closed span in $L^2(\Delta)$ of all functions of the form $1/(1 - z\bar{w})$ with $|w| < 1$. Then an isometry from $L^2_+(\Delta)$ onto $\mathcal{L}(T)$ is defined by taking $p(\zeta)$ into $f(z)$, where

$$f(z) = \int_\Gamma \frac{p(\zeta)\Delta(\zeta)}{1 - \bar{\zeta}z}\, d\sigma(\zeta).$$

The proof of the last assertion consists in showing that the range of the mapping, viewed as a Hilbert space in the scalar product which makes the mapping an isometry, has the same reproducing kernel as $\mathcal{L}(T)$.

Similar constructions are possible in a Kreĭn space environment [**6**]. Other concrete examples are obtained from contractive substitution operators in Grunsky spaces induced by Riemann mappings [**18**] and will be taken up in future work.

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