Extension Theorems for Contraction Operators on Kreĭn Spaces

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To the memory of Mark Grigor'evič Kreĭn

Abstract notions of Julia and defect operators are used as a foundation for a theory of matrix extension and commutant lifting problems for contraction operators on Kreĭn spaces. The account includes a self-contained treatment of key propositions from the theory of Potapov, Ginsburg, Kreĭn, and Shmul'yan on the behavior of a contraction operator on negative subspaces. This theory is extended by an analysis of the behavior of the adjoint of a contraction operator on negative subspaces. Together, these results provide the technical input for the main extension theorems.

Contents

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Introduction

Kreĭn spaces are indefinite generalizations of Hilbert spaces which are important in both abstract operator theory and its applications. We are concerned with everywhere defined and continuous linear operators on a Kreĭn space $\mathcal{H}$ to a Kreĭn space $\mathcal{K}$. The set of all such operators is denoted $\mathbf{B}(\mathcal{H}, \mathcal{K})$. An operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction if

$$\langle Tf, Tf \rangle_{\mathcal{K}} \leq \langle f, f \rangle_{\mathcal{H}}$$

for all vectors f in $\mathcal{H}$. If both T and T^* are contractions, then T is said to be a bicontraction.

Our purpose is to show the possibility of proving extension theorems for contraction operators on Kreĭn spaces which are strikingly similar to those of the Hilbert space case. By an extension of an operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ we mean, for example, a row extension

$$\begin{pmatrix} T & E \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K}),$$

column extension

$$\begin{pmatrix} T \\ F \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \mathcal{F}),$$

or a two-by-two matrix extension

$$\begin{pmatrix} T & E \\ F & G \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \mathcal{F}),$$

where $\mathcal{E}$ and $\mathcal{F}$ are Kreĭn spaces. Many theorems have the hypothesis that $\mathcal{F}$ is a Hilbert space. In applications, this hypothesis is often assured whenever T is a contraction, and so the hypothesis is not as restrictive as might appear. Applications include commutant lifting theorems, in which a commutator relation is extended to minimal isometric and minimal unitary dilations.

We assume familiarity with operator theory on Hilbert spaces, but we do not presume that the reader is necessarily at ease in the indefinite environment of Kreĭn spaces. Our aim has been to give a self-contained treatment for such readers, and experts may view parts of the paper as expository. Background material is summarized in §1.1. While it is essential to master concepts of regular subspaces, projections, negative subspaces, isometries, and direct sums, if a few things are taken for granted or gleaned from standard monographs, not much more is needed as preparation. Kreĭn space extensions of the Hilbert space notions of defect and Julia operators are introduced in §1.2. Julia operators are a kind of unitary two-by-two matrix extension of a given operator and are needed to formulate certain of the extension theorems in Chapters 2 and 3.

The backbone of the study of contraction operators on Kreĭn spaces is a theory, due to Ginsburg, Kreĭn, and Shmul'yan, which analyzes how a contraction operator acts with respect to negative subspaces. One should also mention Potapov, whose treatment of the finite dimensional case helped to motivate developments. The discussion in §1.3 includes the Potapov-Ginsburg transform and its interpretation as a scattering operator, characterizations of bicontractions, and mappings of operator spheres.

The main new results are those of Chapters 2 and 3. These characterize contractive and bicontractive extensions of given contractive and bicontractive operators. Theorems on row, column, and two-by-two matrix extensions, as well as commutant lifting theorems, are proved for Kreĭn spaces in forms essentially identical to the Hilbert space case. Similar results were previously obtained by Alpay, de Branges, Constantinescu and Ghéondea, and Dritschel, using other hypotheses and somewhat different methods. Our approach is based on an analysis of the adjoint of a contraction relative to negative subspaces in §2.1.

The principal methods and main results of Chapters 2 and 3 appear in the first author's doctoral dissertation (University of Virginia, May, 1989). The present account is an expanded version of the dissertation which includes simplifications and additional results.

Chapter 1: Operator Theory on Kreĭn spaces

1.1 Definitions and Preliminaries

Kreĭn spaces are generalizations of Hilbert spaces and are the fundamental underlying objects in our study. Our philosophy on notation is to reserve the simplest notation for Kreĭn spaces and operators which act on them. This is generally standard Hilbert space notation.

A. Kreĭn Spaces and Continuous Operators

A Kreĭn space $\mathcal{H}$ is a scalar product space which is isomorphic to the direct sum of a Hilbert space and the anti-space of a Hilbert space. By a *scalar product space* we mean a complex vector space $\mathcal{H}$ together with a scalar product $\langle \cdot, \cdot \rangle$ which obeys the same axioms of linearity and symmetry as for Hilbert spaces and is nondegenerate in the sense that the only vector f in $\mathcal{H}$ such that $\langle f, g \rangle = 0$ for all g in $\mathcal{H}$ is $f = 0$. The anti-space of a scalar product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is $(\mathcal{H}, -\langle \cdot, \cdot \rangle)$. Notions of isomorphism, subspace, orthogonality, direct sum, and linear operator are defined as in linear algebra. Orthogonality is indicated by $\perp$, direct sum by $\dot{+}$, and orthogonal direct sum by $\oplus$. Occasionally, a subscript $\mathcal{H}$ is used to show dependence on the underlying space.

A *fundamental decomposition* of a Kreĭn space $\mathcal{H}$ is a direct sum representation $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of $\mathcal{H}$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are subspaces of $\mathcal{H}$ such that $\mathcal{H}_+$ is a Hilbert space and $\mathcal{H}_-$ the anti-space of a Hilbert space in the scalar product of $\mathcal{H}$. In general, fundamental decompositions are not unique. The choice of a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ induces a Hilbert space inner product, norm, and strong topology on $\mathcal{H}$. Namely, the Hilbert space inner product of $f_+ + f_-$ and $g_+ + g_-$ ($f_+, g_+ \in \mathcal{H}_+$ and $f_-, g_- \in \mathcal{H}_-$) is $\langle f_+, f_+ \rangle - \langle f_-, f_- \rangle$. The strong topology of this Hilbert space is independent of the choice of fundamental decomposition and is also called the *Mackey topology* of $\mathcal{H}$. It is used to define convergence and continuity in the usual way. The norm of the Hilbert space depends on the choice of fundamental decomposition, but two such norms are equivalent.

If $\mathcal{H}$ is a Kreĭn space, the dimensions of $\mathcal{H}_+$ and $\mathcal{H}_-$ in any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ are independent of the choice of fundamental decomposition. These dimensions are called the *positive* and *negative indices* and of $\mathcal{H}$. A *Pontryagin space* is a Kreĭn space with finite negative index. Two Kreĭn spaces are isomorphic if and only if they have the same positive and negative indices.

Given a Kreĭn space $\mathcal{H}$ with fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we define operators J, P_+, P_- on $\mathcal{H}$ by

$$Jf = f_1 - f_2, \qquad P_+ f = f_1, \qquad P_- f = f_2,$$

whenever

$$f = f_1 + f_2, \qquad f_1 \in \mathcal{H}_+,\ f_2 \in \mathcal{H}_-.$$

We call J the *signature operator* or *fundamental symmetry*, and P_+ and P_- the *associated projections* for the given fundamental decomposition. The signature operator J serves to identify the fundamental decomposition. We write $\mathcal{H}_J$ for $\mathcal{H}$ viewed as a Hilbert space relative to the given fundamental decomposition. Thus,

$$\langle f, g\rangle_{\mathcal{H}_J} = \langle Jf, g\rangle_{\mathcal{H}}$$

for any vectors f and g in $\mathcal{H}$. The *absolute value* of $\mathcal{H}_-$, $|\mathcal{H}_-|$, is a Hilbert space defined as the anti-space of $\mathcal{H}_-$. In this notation, $\mathcal{H}_J = \mathcal{H}_+ \oplus |\mathcal{H}_-|$.

If $\mathcal{H}$ is a Kreĭn space and J_1 and J_2 are signature operators for two fundamental decompositions of $\mathcal{H}$, then $J_2 = U^{-1}J_1U$ where U is an isomorphism of $\mathcal{H}$ onto itself.

If $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces, $\mathbf{B}(\mathcal{H})$ and $\mathbf{B}(\mathcal{H}, \mathcal{K})$ denote the sets of everywhere defined continuous operators on $\mathcal{H}$ to itself and on $\mathcal{H}$ to $\mathcal{K}$, respectively. Every $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ has a unique *adjoint* $A^* \in \mathbf{B}(\mathcal{K}, \mathcal{H})$ satisfying

$$\langle Af, g\rangle_{\mathcal{K}} = \langle f, A^*g\rangle_{\mathcal{H}}, \qquad f \in \mathcal{H}, g \in \mathcal{K}.$$

The identity operator is written 1. By viewing $\mathcal{H}$ and $\mathcal{K}$ as Hilbert spaces relative to some fundamental decompositions, we may induce norm, weak operator, and strong operator topologies on $\mathbf{B}(\mathcal{H}, \mathcal{K})$ which are independent of the choice of fundamental decompositions. Any two operator norms obtained in this way are equivalent.

Kreĭn space adjoints and Hilbert space adjoints must be distinguished. Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$. The Kreĭn space adjoint of an operator $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is an operator $A^* \in \mathbf{B}(\mathcal{K}, \mathcal{H})$. The Hilbert space adjoint of $A \in \mathbf{B}(\mathcal{H}_{J_{\mathcal{H}}}, \mathcal{K}_{J_{\mathcal{K}}})$ is an operator $A^\times \in \mathbf{B}(\mathcal{K}_{J_{\mathcal{K}}}, \mathcal{H}_{J_{\mathcal{H}}})$ related to A^* by

$$A^* = J_{\mathcal{H}} A^\times J_{\mathcal{K}}.$$

In general, we reserve $*$ for Kreĭn space adjoints and $\times$ for Hilbert space adjoints.

Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces. As in the Hilbert space case, we say that

(i) $A \in \mathbf{B}(\mathcal{H})$ is *selfadjoint* if $A^* = A$,
(ii) $A \in \mathbf{B}(\mathcal{H})$ is a *projection* if A is selfadjoint and $A^2 = A$,
(iii) $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is *isometric* if $A^*A = 1$, and
(iv) $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is *unitary* if both A and A^* are isometric.

A partial ordering of selfadjoint operators is defined in the usual way: if $A, B \in \mathbf{B}(\mathcal{H})$ are selfadjoint, $A \geq 0$ means that $\langle Af, f\rangle_{\mathcal{H}} \geq 0$ for all f in $\mathcal{H}$, and $A \geq B$ means that $A - B \geq 0$.

Note that the associated projections $P_\pm$ for a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ are projections in the sense of the preceding definition.

B. Subspaces and Projections

A *subspace* of a Kreĭn space $\mathcal{H}$ is a nonempty linear set $\mathcal{M}$ in $\mathcal{H}$ (not necessarily closed). The Kreĭn space orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ coincides with the Hilbert space orthogonal complement of $J\mathcal{M}$ in $\mathcal{H}_J$ for any signature operator J for $\mathcal{H}$. For any subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$,

$$\mathcal{M}^{\perp\perp} = \overline{\mathcal{M}} \qquad \text{and} \qquad (\mathcal{M}+\mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp,$$

where $\overline{\mathcal{M}}$ is the closure of $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are closed, then also

$$(\mathcal{M}\cap\mathcal{N})^\perp = \overline{(\mathcal{M}^\perp + \mathcal{N}^\perp)}.$$

A subspace $\mathcal{M}$ of $\mathcal{H}$ is dense in $\mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$.

In contrast with the Hilbert space case, the relation $\mathcal{M} + \mathcal{M}^\perp = \mathcal{H}$ may fail for a closed subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$. Moreover, a closed subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ need not itself be a Kreĭn space in the scalar product of $\mathcal{H}$. These pathologies are excluded in an important class of subspaces. By a *regular subspace* of a Kreĭn space $\mathcal{H}$ we mean a closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is a Kreĭn space in the scalar product of $\mathcal{H}$. An analogue of the projection theorem for Hilbert spaces holds for regular subspaces of a Kreĭn space.

Theorem 1.1.1. *If $\mathcal{M}$ is a closed subspace of a Kreĭn space $\mathcal{H}$, the following assertions are equivalent:*

(i) *$\mathcal{M}$ is regular;*
(ii) *$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$;*
(iii) *$\mathcal{M}$ is the range of a projection operator P.*

In this case, if $\mathcal{M}$ is viewed as a Kreĭn space in the scalar product of $\mathcal{H}$, the strong topology of $\mathcal{M}$ coincides with the restriction of the strong topology of $\mathcal{H}$ to $\mathcal{M}$, and the inclusion of $\mathcal{M}$ in $\mathcal{H}$ is continuous.

We occasionally write $\mathrm{Pr}^{\mathcal{H}}_{\mathcal{M}}$ or simply $\mathrm{Pr}_{\mathcal{M}}$ for the projection operator on a Kreĭn space $\mathcal{H}$ whose range is the regular subspace $\mathcal{M}$. If P is a projection on $\mathcal{H}$ with range $\mathcal{M}$, then $1-P$ is a projection with range $\mathcal{M}^\perp$. Therefore $\mathcal{M}^\perp$ is regular whenever $\mathcal{M}$ is regular. The class of regular subspaces of a Kreĭn space $\mathcal{H}$ is not in general closed under intersection and union.

A subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ is

(i) *negative* if $\langle f, f\rangle_{\mathcal{H}} \le 0$ for all f in $\mathcal{M}$,
(ii) *maximal negative* if $\mathcal{M}$ is negative and not a proper subset of another negative subspace,

(iii) *uniformly negative* if for some (and hence any) fundamental symmetry J on $\mathcal{H}$, there is a $\delta_J > 0$ such that

$$\langle f, f\rangle_{\mathcal{H}} \le -\delta_J \|f\|^2_{\mathcal{H}_J}$$

for all f in $\mathcal{M}$, and

(iv) *maximal uniformly negative* if $\mathcal{M}$ is uniformly negative and not a proper subset of another uniformly negative subspace.

An equivalent form of (iv) is that $\mathcal{M}$ is

(iv′) *maximal uniformly negative* if $\mathcal{M}$ is maximal negative and uniformly negative.

By reversing the sense of inequalities of scalar products, we obtain parallel definitions for a subspace to be *positive*, *maximal positive*, *uniformly positive*, *and maximal uniformly positive*. A subspace which is either positive or negative is said to be *definite*. Maximal positive and maximal negative subspaces are closed.

Properties of definite subspaces are derived from a graph representation. Let $\mathcal{M}$ be a negative subspace of a Kreĭn space $\mathcal{H}$. Fix a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of $\mathcal{H}$. If $h = g + f$, $g \in \mathcal{H}_+$, $f \in \mathcal{H}_-$, we also write

$$h = \begin{pmatrix} g \\ f \end{pmatrix}.$$

In this representation, no nonzero element of $\mathcal{M}$ has the form $\binom{g}{0}$, and so $\mathcal{M}$ is the graph

$$\mathcal{G}(K) = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} : f \in \operatorname{dom} K \right\}$$

of a Hilbert space contraction operator K, with domain $\operatorname{dom} K \subset |\mathcal{H}_-|$ and range $\operatorname{ran} K \subset \mathcal{H}_+$, which is called the *angle operator* for $\mathcal{M}$. Every Hilbert space contraction operator K with $\operatorname{dom} K \subset |\mathcal{H}_-|$ and $\operatorname{ran} K \subset \mathcal{H}_+$ is the angle operator of some negative subspace $\mathcal{M}$ of $\mathcal{H}$. The following properties are more or less immediate:

(i) $\mathcal{M}$ is closed if and only if $\operatorname{dom} K$ is closed in $|\mathcal{H}_-|$;

(ii) $\mathcal{M}$ is maximal negative if and only if $\operatorname{dom} K = |\mathcal{H}_-|$;

(iii) $\mathcal{M}$ is uniformly negative if and only if $\|K\| < 1$;

(iv) $\mathcal{M}$ is maximal uniformly negative if and only if both $\operatorname{dom} K = |\mathcal{H}_-|$ and $\|K\| < 1$.

Positive subspaces have a similar graph representation, and parallel results hold.

THEOREM 1.1.2. *Let $\mathcal{H}$ be a Kreĭn space.*

(i) A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is maximal negative if and only if $\mathcal{M}^{\perp}$ is maximal positive in $\mathcal{H}$.

(ii) A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is maximal uniformly negative if and only if $\mathcal{M}^{\perp}$ is maximal uniformly positive in $\mathcal{H}$.

(iii) Every negative subspace of $\mathcal{H}$ is contained in a maximal negative subspace of $\mathcal{H}$.

(iv) Every uniformly negative subspace of $\mathcal{H}$ is contained in a maximal uniformly negative subspace of $\mathcal{H}$.

Moreover, each of these statements remains true if the words "positive" and "negative" are interchanged.

The condition for a negative subspace $\mathcal{M}$ of a Kreĭn space $\mathcal{H}$ to be maximal negative is frequently used in this form: For some and hence any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of $\mathcal{H}$, $\mathrm{Pr}_{\mathcal{H}_-}\mathcal{M} = \mathcal{H}_-$.

A closed positive subspace of a Kreĭn space $\mathcal{H}$ is a Hilbert space if and only if it is uniformly positive. An example is $\mathcal{H}_+$ in any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of $\mathcal{H}$. This subspace is also maximal uniformly positive. Conversely, if $\mathcal{H}_+$ is a maximal uniformly positive subspace of a Kreĭn space $\mathcal{H}$ and $\mathcal{H}_- = \mathcal{H}_+^{\perp}$, then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a fundamental decomposition of $\mathcal{H}$.

C. ISOMETRIES AND PARTIAL ISOMETRIES

The definition of a partial isometry in the Kreĭn space setting is similar to that for Hilbert spaces. Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces. We call A a *partial isometry* if there exist regular subspaces $\mathcal{M}$ of $\mathcal{H}$ and $\mathcal{N}$ of $\mathcal{K}$ such that A maps $\mathcal{M}$ isometrically onto $\mathcal{N}$ and $\ker A = \mathcal{M}^{\perp}$. We call $\mathcal{M}$ the *initial space* and $\mathcal{N}$ the *final space* of A in this situation. The following properties are immediate.

(i) For any f and g in $\mathcal{H}$, the identity

$$\langle Af, Ag\rangle_{\mathcal{K}} = \langle f, g\rangle_{\mathcal{H}}$$

holds if either f or g is in $\mathcal{M}$.

(ii) The adjoint A^* of the operator A is a partial isometry with initial space $\mathcal{N}$ and final space $\mathcal{M}$.

(iii) If P is the projection of $\mathcal{H}$ on $\mathcal{M}$ and Q is the projection of $\mathcal{K}$ on $\mathcal{N}$, then $A^*A = P$, $\ker A = \ker P$, and $AA^* = Q$, $\ker A^* = \ker Q$.

It is not evident from the definition of a partial isometry that an isometry is a partial isometry. This is true and a consequence of the following nonspatial characterization of partial isometries.

THEOREM 1.1.3. *Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces. Then A is a partial isometry if and only if $AA^*A = A$.*

Proof. Suppose that A is a partial isometry with initial space $\mathcal{M}$ and final space $\mathcal{N}$. If f is in $\mathcal{M}^\perp = \ker A$, then

$$AA^*Af = 0 = Af.$$

If f is in $\mathcal{M}$, then $A^*Af = f$ by what was noted above, and hence again $AA^*Af = Af$. By linearity, $AA^*A = A$.

Conversely, assume that $AA^*A = A$, and set $P = A^*A$. Then P is selfadjoint and idempotent:

$$P^2 - P = A^*(AA^*A - A) = 0.$$

Hence P is a projection operator on $\mathcal{H}$. In a similar way, $A^*AA^* = A^*$ and $Q = AA^*$ is a projection operator on $\mathcal{K}$. By Theorem 1.1.1, $\mathcal{M} = P\mathcal{H}$ and $\mathcal{N} = Q\mathcal{K}$ are regular subspaces of $\mathcal{H}$ and $\mathcal{K}$. Now

$$A\mathcal{M} = AA^*A\mathcal{M} \subset AA^*\mathcal{K} = Q\mathcal{K} = \mathcal{N}$$

and

$$A^*\mathcal{N} = A^*AA^*\mathcal{N} \subset A^*A\mathcal{H} = P\mathcal{H} = \mathcal{M}.$$

Hence $A\mathcal{M} = \mathcal{N}$ and $A|\mathcal{M}$ maps $\mathcal{M}$ isometrically onto $\mathcal{N}$. It remains to show that $\ker A = \mathcal{M}^\perp$. Let $f \in \mathcal{H}$. If $Af = 0$, then $Pf = A^*Af = 0$, and so $f \in \mathcal{M}^\perp$. Conversely, if $f \in \mathcal{M}^\perp$, then $A^*Af = Pf = 0$ and $Af = AA^*Af = 0$. Thus A is a partial isometry. ■

COROLLARY 1.1.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces. An isometry $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a partial isometry with initial space $\mathcal{H}$. In particular, the range of an isometry $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a regular subspace of $\mathcal{K}$.*

Proof. If $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is an isometry, then $A^*A = 1$ and so $AA^*A = A$. Therefore A is a partial isometry by Theorem 1.1.3. The initial space of A is $(A^*A)\mathcal{H} = \mathcal{H}$. ■

THEOREM 1.1.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be an isometry.*

(i) If $\mathcal{M}$ is a closed uniformly positive or closed uniformly negative subspace of $\mathcal{H}$, $A\mathcal{M}$ is of the same type in $\mathcal{K}$.

(ii) If $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a fundamental decomposition of $\mathcal{H}$, then $A\mathcal{H} = A\mathcal{H}_+ \oplus A\mathcal{H}_-$ is a fundamental decomposition of $A\mathcal{H}$.

Theorem 1.1.6. *Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, where $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces. The following assertions are equivalent:*

(i) *A is a partial isometry;*
(ii) *A^*A is a projection operator and* $\ker A^*A = \ker A$*;*
(iii) *AA^* is a projection operator and* $\ker AA^* = \ker A^*$*.*

Proofs of these results are not difficult and omitted. An example clarifies the condition on kernels in parts (ii) and (iii) of Theorem 1.1.6. Let $\mathcal{H}$ and $\mathcal{K}$ each be the space of pairs $\binom{a}{b}$ of complex numbers with

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = |a|^2 - |b|^2.$$

If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $A^* = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, and $A^*A = 0$ is trivially a projection. But A is not a partial isometry because $\ker A^*A$ and $\ker A$ do not coincide.

In contrast with the Hilbert space case, densely defined isometries, that is, linear mappings which preserve scalar products, do not necessarily have continuous extensions to everywhere defined isometries. There even exist everywhere defined isometries on Kreĭn spaces which are not continuous (Bognár [**12**], p. 125). A simple condition for continuity, given in the next result, is sometimes useful.

Theorem 1.1.7. *Let C be a densely defined isometry from a Hilbert space $\mathcal{H}$ to a Kreĭn space $\mathcal{K}$. Then C has an extension to an operator $\hat{C} \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ if and only if* $\operatorname{ran} C$ *is uniformly positive.*

Proof. Necessity follows from Theorem 1.1.5. Conversely, if $\operatorname{ran} C$ is uniformly positive, then $\overline{\operatorname{ran} C}$ is a Hilbert space in the scalar product of $\mathcal{K}$. We can extend C by continuity to a Hilbert space isometry $\hat{C}$ from $\mathcal{H}$ to $\overline{\operatorname{ran} C}$ viewed as a Hilbert space in the scalar product of $\mathcal{K}$. Since the inclusion of $\overline{\operatorname{ran} C}$ in $\mathcal{K}$ is continuous by Theorem 1.1.1, $\hat{C}$ is continuous as an operator on $\mathcal{H}$ to $\mathcal{K}$. ■

D. Orthogonal Direct Sums

Orthogonal direct sums of Kreĭn spaces are convenient for matrix representations of operators. If a Kreĭn space $\mathcal{H}$ is represented as an orthogonal direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then for any f in $\mathcal{H}_1$ and g in $\mathcal{H}_2$, the expressions $h = \binom{f}{g}$ and $h = f + g$ are used interchangeably. If a second Kreĭn space $\mathcal{K}$ is represented as $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, then in the usual way any operator $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ has a representation in matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

This representation extends to orthogonal direct sums with any finite number of summands, and the usual rules of matrix calculus are valid. In fact, we can extend the matrix calculus to orthogonal direct sums with countably many summands. We shall only indicate the appropriate definitions of external and internal orthogonal direct sums of sequences, leaving the rest to the reader.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, ... be given Kreĭn spaces with fundamental decompositions $\mathcal{H}_1^+ \oplus \mathcal{H}_1^-$, $\mathcal{H}_2^+ \oplus \mathcal{H}_2^-$, ... Define

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

to be the space of sequences $f = \{f_1, f_2, \ldots\}$ such that $f_n \in \mathcal{H}_n$ for all n and

$$\sum_{n=1}^{\infty} \|f_n\|^2 < \infty,$$

where norms are computed relative to the given fundamental decompositions. If $f = \{f_1, f_2, \ldots\}$ and $g = \{g_1, g_2, \ldots\}$ are in $\mathcal{H}$, set

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \langle f_n, g_n\rangle_{\mathcal{H}_n}.$$

Then $\mathcal{H}$ is a Kreĭn space. We call $\mathcal{H}$ an *external orthogonal direct sum* or simply *direct sum* of $\mathcal{H}_1$, $\mathcal{H}_2$, ... For each n, $\mathcal{H}_n$ has a natural embedding in $\mathcal{H}$ as a regular subspace. If $P_n^{\pm}$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_n^{\pm}$ for every n, then

$$\sup_n \left\| \sum_{k=1}^{n} P_k^+ \right\| < \infty \qquad \text{and} \qquad \sup_n \left\| \sum_{k=1}^{n} P_k^- \right\| < \infty,$$

where norms are computed relative to any fundamental decomposition of $\mathcal{H}$. It is sufficient to prove this for a definite choice of fundamental decomposition of $\mathcal{H}$. Choosing $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^{\pm} = \mathcal{H}_1^{\pm} \oplus \mathcal{H}_2^{\pm} \oplus \cdots$, we have, in fact,

$$\left\| \sum_{k=1}^{n} P_k^{\pm} \right\| \leq 1$$

for all n. It should be noted that the definition of $\mathcal{H}$ depends on the choice of fundamental decompositions for the summands. However, any two spaces obtained in this way with different choices of fundamental decompositions for the summands are naturally isomorphic.

There are several possibilities for the definition of an internal orthogonal direct sum. We use a strong hypothesis which results in a notion which is isomorphic with the previous definition of an external orthogonal direct sum.

THEOREM 1.1.8. *Let $\mathcal{H}$ be a Kreĭn space, and let $\mathcal{M}_1$, $\mathcal{M}_2$, ... be given pairwise orthogonal regular subspaces of $\mathcal{H}$ with fundamental decompositions $\mathcal{M}_1^+ \oplus \mathcal{M}_1^-$, $\mathcal{M}_2^+ \oplus \mathcal{M}_2^-$, ... Assume that the projections $P_1^{\pm}$, $P_2^{\pm}$, ... on the subspaces $\mathcal{M}_1^{\pm}$, $\mathcal{M}_2^{\pm}$, ... satisfy*

$$\sup_n \left\| \sum_{k=1}^{n} P_k^+ \right\| < \infty \qquad \text{and} \qquad \sup_n \left\| \sum_{k=1}^{n} P_k^- \right\| < \infty,$$

where norms are computed relative to any fundamental decomposition of $\mathcal{H}$. Then:

(i) The closed span $\mathcal{M}$ of $\mathcal{M}_1$, $\mathcal{M}_2$, ... is a regular subspace of $\mathcal{H}$. The closed span $\mathcal{M}^{\pm}$ of $\mathcal{M}_1^{\pm}$, $\mathcal{M}_2^{\pm}$, ... is uniformly positive with the plus signs and uniformly negative with the minus signs, and $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$.

(ii) Every $f \in \mathcal{M}$ has a norm convergent representation $f = f_1 + f_2 + \cdots$, where $f_n \in \mathcal{M}_n$ for all n and $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$ with norms computed relative to any fundamental decomposition of $\mathcal{H}$. The representation is unique. Conversely, every square summable sequence of elements of $\mathcal{M}_1$, $\mathcal{M}_2$, ... determines an element of $\mathcal{M}$ in this way.

(iii) If P, $P^{\pm}$, $P_1^{\pm}$, $P_2^{\pm}$, ... are the projections of $\mathcal{H}$ on the subspaces $\mathcal{M}$, $\mathcal{M}^{\pm}$, $\mathcal{M}_1^{\pm}$, $\mathcal{M}_2^{\pm}$, ..., then $P = \sum_{n=1}^{\infty} P_n$ and $P^{\pm} = \sum_{n=1}^{\infty} P_n^{\pm}$ with convergence in the strong operator topology.

In this situation, we write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$ and call $\mathcal{M}$ the *internal orthogonal direct sum* or *direct sum* of $\mathcal{M}_1$, $\mathcal{M}_2$, ... An internal orthogonal direct sum is defined whenever there exist projections satisfying the hypotheses of the theorem, but the direct sum itself is independent of the choice of projections. The role of the hypothesis on projections is to insure that the difference between internal and external orthogonal direct sums is essentially a notational one. Every internal orthogonal direct sum is naturally isomorphic to an external orthogonal direct sum, and conversely. In practice, the distinction between internal and external orthogonal direct sums is typically ignored.

LEMMA 1.1.9. *Let $\mathcal{N}_1$, $\mathcal{N}_2$, ... be regular subspaces of a Kreĭn space $\mathcal{H}$ such that $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \cdots$. Assume that the projections Q_1, Q_2, ... of $\mathcal{H}$ onto the subspaces $\mathcal{N}_1$, $\mathcal{N}_2$, ... satisfy $\sup_n \|Q_n\| < \infty$, where norms are computed relative to any fundamental decomposition of $\mathcal{H}$. Then*

$$\mathcal{N} = \bigcap_1^\infty \mathcal{N}_n \qquad \text{and} \qquad \mathcal{M} = \bigvee_1^\infty \mathcal{N}_n^\perp$$

are regular subspaces of $\mathcal{H}$ with $\mathcal{N} = \mathcal{M}^\perp$. If Q is the projection of $\mathcal{H}$ onto $\mathcal{N}$, then $Q = \lim_{n\to\infty} Q_n$ with convergence in the strong operator topology.

Proof of Lemma 1.1.9. For any $f \in \mathcal{H}$, $Q_1 f$, $Q_2 f$, ... is a bounded sequence, and hence $g = \lim_{k\to\infty} Q_{n_k} f$ exists weakly for some subsequence. Elementary Hilbert space considerations show that $g \in \mathcal{N}$. Write $h = f - g$, so that

$$h = \lim_{k\to\infty} (1 - Q_{n_k}) f$$

weakly. Since

$$(1 - Q_{n_k}) f \in \mathcal{N}_{n_k}^\perp \subset \mathcal{N}^\perp$$

for every k, we have $h \in \mathcal{N}^\perp$. Thus $f = g + h$ with $g \in \mathcal{N}$ and $h \in \mathcal{N}^\perp$. Therefore $\mathcal{N} + \mathcal{N}^\perp = \mathcal{H}$. This implies that $\mathcal{N} \cap \mathcal{N}^\perp = \{0\}$, because any vector in $\mathcal{N} \cap \mathcal{N}^\perp$ is orthogonal to $\mathcal{N} + \mathcal{N}^\perp = \mathcal{H}$. By Theorem 1.1.1, $\mathcal{N}$ is a regular subspace of $\mathcal{H}$.

In the preceding construction, suppose that $f \in \mathcal{N}^\perp$. Then $g = 0$ because $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. Therefore

$$f = h = \lim_{k\to\infty} (1 - Q_{n_k}) f$$

belongs to $\mathcal{M} = \vee_1^\infty \mathcal{N}_n^\perp$, and so $\mathcal{N}^\perp \subset \mathcal{M}$. The reverse inclusion is obvious, and so $\mathcal{N}^\perp = \mathcal{M}$. In particular, $\mathcal{M}$ is a regular subspace of $\mathcal{H}$. We clearly have

$$\lim_{n\to\infty} Q_n f = Qf$$

if f is in the span of $\mathcal{N}$, $\mathcal{N}_1^\perp$, $\mathcal{N}_2^\perp$, ... Since this span is dense in $\mathcal{H}$ and the projections Q_1, Q_2, ... are bounded in norm, the relation holds for all $f \in \mathcal{H}$ by a routine approximation. ■

Proof of Theorem 1.1.8. For each $n \geq 1$, let $\mathcal{N}_n$ be the orthogonal complement of $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ in $\mathcal{H}$. The projection Q_n of $\mathcal{H}$ onto $\mathcal{N}_n$ is $Q_n = 1 - P_1 - \cdots - P_n$. By Lemma 1.1.9,

$$\mathcal{N} = \bigcap_1^\infty \mathcal{N}_n \qquad \text{and} \qquad \mathcal{M} = \bigvee_1^\infty \mathcal{N}_n^\perp$$

are regular subspaces of $\mathcal{H}$ with $\mathcal{N} = \mathcal{M}^{\perp}$. If Q is the projection of $\mathcal{H}$ onto $\mathcal{N}$, then $Q = \lim_{n\to\infty} Q_n$ in the strong operator topology. By the definition of the subspaces $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{M}$ is the closed span of $\mathcal{M}_1, \mathcal{M}_2, \ldots$. Moreover, if P is the projection of $\mathcal{H}$ onto $\mathcal{M}$, then

$$P = 1 - Q = \lim_{n\to\infty}(1 - Q_n) = \lim_{n\to\infty}\sum_{k=1}^{n} P_k$$

with convergence in the strong operator topology.

We repeat the construction with $\mathcal{M}_1, \mathcal{M}_2, \ldots$ replaced by $\mathcal{M}_1^{\pm}, \mathcal{M}_2^{\pm}, \ldots$. Thus the closed span of $\mathcal{M}_1^{\pm}, \mathcal{M}_2^{\pm}, \ldots$ is a regular subspace $\mathcal{M}^{\pm}$ of $\mathcal{H}$. Since $\mathcal{M}^{\pm}$ is positive/negative and regular, $\mathcal{M}^{\pm}$ is uniformly positive/uniformly negative. If $P^{\pm}$ is the projection of $\mathcal{H}$ onto $\mathcal{M}^{\pm}$, then

$$P^{\pm} = \lim_{n\to\infty}\sum_{k=1}^{n} P_k^{\pm}$$

with convergence in the strong operator topology. Clearly $P = P^+ + P^-$ and $P^+P^- = P^-P^+ = 0$. Thus $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$. The assertions (i) and (iii) are now proved.

The assertion (ii) is independent of the choice of fundamental decomposition of $\mathcal{H}$. It is convenient to choose a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of $\mathcal{H}$ such that $\mathcal{H}_\pm \supset \mathcal{M}^{\pm}$. Then the subspaces $\mathcal{M}_1^+, \mathcal{M}_2^+, \ldots$ and $\mathcal{M}_1^-, \mathcal{M}_2^-, \ldots$ are pairwise orthogonal in both the Kreĭn space sense and the Hilbert space sense. In this situation, (ii) is clear. ■

1.2 Defect Operators and Julia Operators

As preparation for the study of operators on Kreĭn spaces, we consider two related problems. One is to represent a given selfadjoint operator H in the form $H = AA^*$, where A is an operator with zero kernel. The other is to embed any given operator T in a unitary matrix $U = \begin{pmatrix} T & * \\ * & * \end{pmatrix}$. These constructions are needed to circumvent certain Hilbert space notions which do not have adequate counterparts in the indefinite case.

Let $\mathcal{H}$ be a Kreĭn space, and let $H \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. By $h_+(H)$ we mean the supremum of all $r = 1, 2, \ldots$ such that there exists a nonnegative and invertible matrix of the form

$$\left[\langle Hf_j, f_k\rangle_{\mathcal{H}}\right]_{j,k=1}^{r}, \qquad f_1, \ldots, f_r \in \mathcal{H}.$$

Set $h_+(H) = 0$ if no such r exists, and $h_-(H) = h_+(-H)$. We call $h_\pm(H)$ the *positive* and *negative hermitian indices* of H. Notice that $H \geq 0$ if and only if $h_-(H) = 0$, and $H \leq 0$ if and only if $h_+(H) = 0$.

THEOREM 1.2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be Kreĭn spaces, $\mathcal{A}$ separable, and let $A \in \mathbf{B}(\mathcal{A}, \mathcal{H})$. If A has zero kernel, then the positive and negative indices of $\mathcal{A}$ coincide with $h_+(AA^*)$ and $h_-(AA^*)$, respectively.*

Proof. Let $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ be a fundamental decomposition. It is sufficient to show that $h_+(AA^*) = \dim \mathcal{A}_+$, since we then obtain $h_-(AA^*) = \dim \mathcal{A}_-$ by suitably reversing signs.

The dimension of $\mathcal{A}_+$ is the supremum of all $r = 1, 2, \ldots$ such that there exists a nonnegative and invertible matrix of the form

$$\left[\langle g_j, g_k \rangle_{\mathcal{A}}\right]_{j,k=1}^{r}, \qquad g_1, \ldots, g_r \in \mathcal{A},$$

and zero if no such r exists. If $f_1, \ldots, f_r \in \mathcal{H}$,

$$\left[\langle AA^* f_j, f_k \rangle_{\mathcal{H}}\right]_{j,k=1}^{r} = \left[\langle A^* f_j, A^* f_k \rangle_{\mathcal{A}}\right]_{j,k=1}^{r},$$

and so $h_+(AA^*) \leq \dim \mathcal{A}_+$.

To see that equality holds, consider vectors $g_1, \ldots, g_r$ in $\mathcal{A}$ such that

$$\left[\langle g_j, g_k \rangle_{\mathcal{A}}\right]_{j,k=1}^{r}$$

is nonnegative and invertible. Since $\ker A = \{0\}$, the range of A^* is dense in $\mathcal{A}$, and there exist vectors $f_{1n}, \ldots, f_{rn}$ in $\mathcal{H}$ such that

$$\lim_{n \to \infty} A^* f_{jn} = g_j, \qquad j = 1, \ldots, r.$$

Hence for all sufficiently large n, the matrix

$$\left[\langle A^* f_{jn}, A^* f_{kn} \rangle_{\mathcal{A}}\right]_{j,k=1}^{r} = \left[\langle AA^* f_{jn}, f_{kn} \rangle_{\mathcal{H}}\right]_{j,k=1}^{r}$$

is nonnegative and invertible, and so $h_+(AA^*) \geq \dim \mathcal{A}_+$. Thus equality holds. ∎

THEOREM 1.2.2. *Let $\mathcal{H}$ be a Kreĭn space, and let $H \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Then there is a Kreĭn space $\mathcal{A}$ and an operator $A \in \mathbf{B}(\mathcal{A}, \mathcal{H})$ with zero kernel such that $H = AA^*$.*

Proof. Let $J_{\mathcal{H}}$ be the signature operator for some fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Then $HJ_{\mathcal{H}}$ is a selfadjoint operator on the Hilbert space $\mathcal{H}_+ \oplus |\mathcal{H}_-|$. Let its spectral decomposition be $HJ_{\mathcal{H}} = \int \lambda dE(\lambda)$, and set $R = \int |\lambda|^{1/2} dE(\lambda)$, $\mathcal{M}_+ = E((0, \infty))$, and $\mathcal{M}_- = E((-\infty, 0))$.

Let $\mathcal{A}$ be a Kreĭn space with fundamental decomposition $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ and signature operator $J_{\mathcal{A}}$ such that $\dim \mathcal{A}_\pm = \dim \mathcal{M}_\pm$. Choose an isometry

W on $\mathcal{A}_+ \oplus |\mathcal{A}_-|$ to $\mathcal{H}_+ \oplus |\mathcal{H}_-|$ such that $W\mathcal{A}_\pm = \mathcal{M}_\pm$. Define $A \in \mathbf{B}(\mathcal{A}, \mathcal{H})$ by $A = RW$. Then $\ker A = \{0\}$, and $AA^* = RWW^*R^* = (RWJ_{\mathcal{A}}W^\times R)J_{\mathcal{H}} = (HJ_{\mathcal{H}})J_{\mathcal{H}} = H$. ■

DEFINITION 1.2.3. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces.*

(i) By a **defect operator** *for T we mean an operator $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$, where $\tilde{\mathcal{D}}$ a Kreĭn space, such that $\tilde{D}$ has zero kernel and $1 - T^*T = \tilde{D}\tilde{D}^*$.*

(ii) By a **Julia operator** *for T we mean a unitary operator U having the form*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are Kreĭn spaces, $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ have zero kernels, and $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$.

In (i), the positive and negative indices of $\tilde{\mathcal{D}}$ are determined by T and given by $h_\pm(1 - T^*T)$ when $\tilde{\mathcal{D}}$ is separable by Theorem 1.2.1. In (ii), $\tilde{D}$ is a defect operator for T and D is a defect operator for T^*. Hence in (ii), the positive and negative indices of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are determined by T and coincide when these spaces are separable with $h_\pm(1 - TT^*)$ and $h_\pm(1 - T^*T)$, respectively. It is also easy to see that if U is a Julia operator for T, then U^* is a Julia operator for T^*.

For an example, let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a partial isometry with initial space $\mathcal{M}$ and final space $\mathcal{N}$. Choose $\mathcal{D} = \mathcal{N}^\perp$ and $\tilde{\mathcal{D}} = \mathcal{M}^\perp$ in the scalar products of $\mathcal{K}$ and $\mathcal{H}$, respectively. Define $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ to be the inclusion mappings, and let $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ be the zero operator. Then $\left(\begin{smallmatrix} T & D \\ \tilde{D}^* & L \end{smallmatrix}\right) \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}})$ is a Julia operator for T.

Theorem 1.2.2 insures the existence of a defect operator for any given operator T. As we show next, it also implies the existence of a Julia operator for any given operator T.

THEOREM 1.2.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. If $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ is a defect operator for T, there exists a Julia operator of the form*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

Proof. We seek a Kreĭn space $\mathcal{D}$ and operators $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ such that D has zero kernel and $U = \left(\begin{smallmatrix} T & D \\ \tilde{D}^* & L \end{smallmatrix}\right)$ is unitary, that is,

$$\left.\begin{aligned} T^*T + \tilde{D}\tilde{D}^* &= 1, \\ T^*D + \tilde{D}L &= 0, \\ D^*D + L^*L &= 1, \end{aligned}\right\} \tag{1.2.1a, b, c}$$

and

$$\left.\begin{aligned} TT^* + DD^* &= 1, \\ \tilde{D}^* T^* + LD^* &= 0, \\ \tilde{D}^* \tilde{D} + LL^* &= 1. \end{aligned}\right\} \qquad (1.2.2a, b, c)$$

The relation (1.2.1a) holds by the assumption that $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ is a defect operator for T. It implies that

$$V = \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

is an isometry. Then $1 - VV^*$ is the projection onto $\ker V^*$. Factor $1 - VV^* = BB^*$, where $B \in \mathbf{B}(\mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}})$ for some Kreĭn space $\mathcal{D}$ and $\ker B = \{0\}$. Since BB^* is a projection and $\ker B^* = \ker BB^*$, B is a partial isometry by Theorem 1.1.6. In fact, it is an isometry with range $\ker V^*$. Thus $V^* B = 0$ and $B^* B = 1$.

If we write elements of $\mathcal{K} \oplus \tilde{\mathcal{D}}$ in column form, then

$$B = \begin{pmatrix} D \\ L \end{pmatrix},$$

where $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$. The relations (1.2.1b,c) follow from $V^* B = 0$ and $B^* B = 1$. The relations (1.2.2a,b,c) follow from $1 - VV^* = BB^*$. It remains to verify that D has zero kernel. If $Df = 0$ for some vector f in $\mathcal{D}$, then $Lf = 0$ by (1.2.1b) because $\tilde{D}$ has zero kernel by assumption. Hence $Bf = 0$ and $f = 0$. ■

1.3 Contraction and Bicontraction Operators

Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces. An operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is said to be a *contraction* if

$$\langle Tf, Tf \rangle_{\mathcal{K}} \leq \langle f, f \rangle_{\mathcal{H}}, \qquad f \in \mathcal{H},$$

and a *bicontraction* if both T and T^* are contractions. An example of a contraction which is not a bicontraction is the embedding of a Hilbert space into its direct sum with the anti-space of a nonzero Hilbert space. The structural properties of contractions depend on the way in which such operators map negative subspaces.

THEOREM 1.3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Choose fundamental decompositions for $\mathcal{H}$ and $\mathcal{K}$, and let norms be computed with respect to the associated Hilbert spaces. Set*

$$\delta = \left\{ \|T\| + \left[1 + \|T\|^2 \right]^{1/2} \right\}^{-1}.$$

(i) For any $f \in \mathcal{H}$ with $\langle f, f \rangle_{\mathcal{H}} \leq 0$, $\|Tf\| \geq \delta \|f\|$.
(ii) The kernel of T is a closed uniformly positive subspace of $\mathcal{H}$.

Proof. Let J be the chosen fundamental symmetry on $\mathcal{H}$. Since T is a contraction, $1 - T^*T \geq 0$ in the partial ordering of selfadjoint operators on $\mathcal{H}$. Therefore $C = J(1 - T^*T)$ is nonnegative as an operator on $\mathcal{H}_J$. For any $f \in \mathcal{H}$,

$$\|f\| - \|T\| \|Tf\| \leq \|Cf\| \leq \|C\|^{1/2} \langle Cf, f \rangle_{\mathcal{H}_J}^{1/2} \leq \left[1 + \|T\|^2 \right]^{1/2} \langle Cf, f \rangle_{\mathcal{H}_J}^{1/2}.$$

If $\langle f, f \rangle_{\mathcal{H}} \leq 0$, then also

$$\langle Cf, f \rangle_{\mathcal{H}_J} = \langle f, f \rangle_{\mathcal{H}} - \langle Tf, Tf \rangle_{\mathcal{K}} \leq -\langle Tf, Tf \rangle_{\mathcal{K}} \leq \|Tf\|^2.$$

Combining these inequalities, we obtain

$$\|f\| - \|T\| \|Tf\| \leq \left[1 + \|T\|^2 \right]^{1/2} \|Tf\|,$$

which proves (i). If $g \in \ker T$, then $Cg = J(1 - T^*T)g = Jg$ and

$$\|g\|^2 = \|Cg\|^2 \leq \|C\| \langle Cg, g \rangle_{\mathcal{H}_J} = \|C\| \langle g, g \rangle_{\mathcal{H}}.$$

Thus (ii) follows. ■

Corollary 1.3.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Then*

(i) T maps any closed negative subspace of $\mathcal{H}$ in a one-to-one way onto a closed negative subspace of $\mathcal{K}$, and

(ii) T maps any closed uniformly negative subspace of $\mathcal{H}$ in a one-to-one way onto a closed uniformly negative subspace of $\mathcal{K}$.

Proof. Part (i) follows from Theorem 1.3.1. To prove (ii), we must show that if $\mathcal{M}$ is a uniformly negative subspace of $\mathcal{H}$, then $T\mathcal{M}$ is a uniformly negative subspace of $\mathcal{K}$.

Assume that fundamental symmetries are chosen for $\mathcal{H}$ and $\mathcal{K}$ and norms are computed with respect to the associated Hilbert spaces. If $\mathcal{M}$ is uniformly negative in $\mathcal{H}$, there is an $\eta > 0$ such that

$$\langle f, f\rangle_{\mathcal{H}} \le -\eta \|f\|^2, \qquad f \in \mathcal{M}.$$

Hence for $f \in \mathcal{M}$,

$$\langle Tf, Tf\rangle_{\mathcal{K}} \le \langle f, f\rangle_{\mathcal{H}} \le -\eta\|f\|^2 \le -\eta\|T\|^{-2}\|Tf\|^2,$$

and $T\mathcal{M}$ is uniformly negative in $\mathcal{K}$. This proves (ii). ■

The main results on contraction operators are derived using a scattering formalism. The formalism uses two Kreĭn spaces $\mathcal{H}$ and $\mathcal{K}$ and fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. In addition, let $\mathcal{L}$ be the direct sum of the anti-space of $\mathcal{H}$ together with $\mathcal{K}$. Thus $\mathcal{L}$ is the space of pairs $\binom{f}{g}$ with $f \in \mathcal{H}$, $g \in \mathcal{K}$, and

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{L}} = -\langle f, f\rangle_{\mathcal{H}} + \langle g, g\rangle_{\mathcal{K}}.$$

We use the fundamental decomposition of $\mathcal{L}$ given by $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$,

$$\mathcal{L}_+ = \mathcal{K}_+ \oplus \mathcal{H}_- \qquad \text{and} \qquad \mathcal{L}_- = \mathcal{H}_+ \oplus \mathcal{K}_-.$$

The elements of $\mathcal{L}$ are also represented as pairs $\binom{v}{u}$ with v in $\mathcal{L}_+$ and u in $\mathcal{L}_-$.

If $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction, its graph

$$\mathcal{G}(T) = \left\{ \begin{pmatrix} f \\ Tf \end{pmatrix} : f \in \mathcal{H} \right\}$$

is a closed negative subspace of $\mathcal{L}$. It is negative because for any $f \in \mathcal{H}$,

$$\left\langle \begin{pmatrix} f \\ Tf \end{pmatrix}, \begin{pmatrix} f \\ Tf \end{pmatrix} \right\rangle_{\mathcal{L}} = -\langle f, f\rangle_{\mathcal{H}} + \langle Tf, Tf\rangle_{\mathcal{K}} \le 0.$$

It is closed because the domain of T is closed. By the *scattering operator* or *Potapov-Ginsburg transform* of T we mean the angle operator S for $\mathcal{G}(T)$ relative to the fundamental decomposition $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$. Thus

$$\mathcal{G}(T) = \left\{ \begin{pmatrix} Su \\ u \end{pmatrix} : u \in \operatorname{dom} S \right\} \subset \mathcal{L}_+ \oplus \mathcal{L}_-.$$

By construction, the scattering operator S is a Hilbert space contraction with closed domain $\operatorname{dom} S \subset |\mathcal{L}_-|$ and $\operatorname{ran} S \subset \mathcal{L}_+$.

Write T in matrix form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus \mathcal{H}_-, \mathcal{K}_+ \oplus \mathcal{K}_-). \tag{1.3.1}$$

Denote the signature operators and projections for the given fundamental decompositions of $\mathcal{H}$ and $\mathcal{K}$ by $J_\mathcal{H}$, $J_\mathcal{K}$ and

$$P_\pm : \mathcal{H} \to \mathcal{H}_\pm, \qquad Q_\pm : \mathcal{K} \to \mathcal{K}_\pm. \tag{1.3.2}$$

Define operators

$$Q_+T + P_- = \begin{pmatrix} T_{11} & T_{12} \\ 0 & 1 \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus \mathcal{H}_-, \mathcal{K}_+ \oplus \mathcal{H}_-), \tag{1.3.3}$$

and

$$P_+ + Q_-T = \begin{pmatrix} 1 & 0 \\ T_{21} & T_{22} \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus \mathcal{H}_-, \mathcal{H}_+ \oplus \mathcal{K}_-). \tag{1.3.4}$$

The preceding notation is assumed in Theorems 1.3.3–1.3.5.

THEOREM 1.3.3. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction with scattering operator S. The domain of S is equal to* $\operatorname{ran}(P_+ + Q_-T)$, *and*

$$S = (Q_+T + P_-)(P_+ + Q_-T)^{-1} | \operatorname{dom} S.$$

Proof. We show that $P_+ + Q_-T$ is one-to-one and has closed range. First note that T_{22} is contraction on $\mathcal{H}_-$ to $\mathcal{K}_-$. For if $f \in \mathcal{H}_-$, then

$$\langle T_{22}f, T_{22}f \rangle_{\mathcal{K}_-} = \langle Q_-Tf, Q_-Tf \rangle_\mathcal{K} \le \langle Tf, Tf \rangle_\mathcal{K} \le \langle f, f \rangle_\mathcal{H} = \langle f, f \rangle_{\mathcal{H}_-}.$$

By Theorem 1.3.1, T_{22} is one-to-one and has closed range. By (1.3.4), $P_+ + Q_-T$ is one-to-one and has closed range.

For any $f \in \mathcal{H}$,

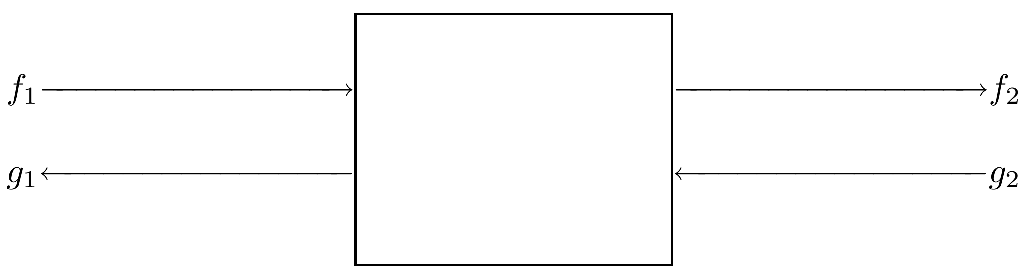
$$\Pr_{\mathcal{L}_-}\begin{pmatrix} f \\ Tf \end{pmatrix} = \begin{pmatrix} P_+ f \\ Q_- T(P_+ + P_-)f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} P_+ f \\ P_- f \end{pmatrix} = (P_+ + Q_- T)f$$

and

$$\Pr_{\mathcal{L}_+}\begin{pmatrix} f \\ Tf \end{pmatrix} = \begin{pmatrix} Q_+ T(P_+ + P_-)f \\ P_- f \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} P_+ f \\ P_- f \end{pmatrix} = (Q_+ T + P_-)f.$$

The computation of S is immediate from these identities and the definition of the angle operator. ■

The scattering operator S models a system in which a left input f_1 is partially transmitted in f_2, partially reflected in g_1, and partially absorbed. A right input g_2 is partially transmitted in g_1, partially reflected in f_2, and partially absorbed.



Here f_1, f_2, g_1, g_2 belong to $\mathcal{H}_+$, $\mathcal{K}_+$, $|\mathcal{H}_-|$, $|\mathcal{K}_-|$, respectively. The matrix entries in

$$S = \begin{pmatrix} T_\ell & R_r \\ R_\ell & T_r \end{pmatrix},$$

represent left and right transmission and reflection coefficients. The scattering operator

$$S : \begin{pmatrix} f_1 \\ g_2 \end{pmatrix} \to \begin{pmatrix} f_2 \\ g_1 \end{pmatrix}$$

associates net output to net input. The original operator

$$T : \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \to \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$$

is a *cascade scattering operator* for the system. *Passivity* of the system means that energy is not increased, or, what is the same thing, S is contractive in the Hilbert space sense:

$$\|f_2\|^2 + \|g_1\|^2 \le \|f_1\|^2 + \|g_2\|^2.$$

This is equivalent to the inequality

$$\|f_2\|^2 - \|g_2\|^2 \le \|f_1\|^2 - \|g_1\|^2,$$

which holds because T is a contraction. When $\operatorname{dom} S \neq |\mathcal{L}_-|$, only certain inputs and outputs occur in the system.

The scattering formalism is particularly simple when $\operatorname{dom} S = |\mathcal{L}_-|$. In this case, from Theorem 1.3.3 and the identities (1.3.3) and (1.3.4), we see that T_{22} is invertible and

$$S = \begin{pmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & T_{12}T_{22}^{-1} \\ -T_{22}^{-1}T_{21} & T_{22}^{-1} \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus |\mathcal{K}_-|, \mathcal{K}_+ \oplus |\mathcal{H}_-|). \tag{1.3.5}$$

Identities involving the scattering operator can therefore be reduced to straightforward matrix calculations.

THEOREM 1.3.4. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction with scattering operator S. The following assertions are equivalent:*

(i) *T is a bicontraction;*
(ii) *the entry T_{22} in (1.3.1) is invertible;*
(iii) $\operatorname{dom} S = |\mathcal{L}_-|$.

If T is a bicontraction, then

$$T = (Q_+S + Q_-)(P_+ + P_-S)^{-1}, \tag{1.3.6}$$

and the scattering operator for T^* is $S^\times$. Moreover, in this case,

$$T^* = (P_+S^\times + P_-)(Q_+ + Q_-S^\times)^{-1} \tag{1.3.7}$$

and

$$1 - T^*T = J_{\mathcal{H}}(P_+ + Q_-T)^\times(1 - S^\times S)(P_+ + Q_-T), \tag{1.3.8}$$

$$1 - TT^* = J_{\mathcal{K}}(Q_+ + P_-T^*)^\times(1 - SS^\times)(Q_+ + P_-T^*). \tag{1.3.9}$$

Recall that $*$ indicates Kreĭn space adjoint and $\times$ Hilbert space adjoint. Notation in (1.3.6)–(1.3.9) is similar to that used in (1.3.3) and (1.3.4). Explicitly, if

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

then

$$Q_+S + Q_- = \begin{pmatrix} S_{11} & S_{12} \\ 0 & 1 \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus \mathcal{K}_-, \mathcal{K}_+ \oplus \mathcal{K}_-),$$
$$P_+ + P_-S = \begin{pmatrix} 1 & 0 \\ S_{21} & S_{22} \end{pmatrix} \in \mathbf{B}(\mathcal{H}_+ \oplus \mathcal{K}_-, \mathcal{H}_+ \oplus \mathcal{H}_-),$$

and,

$$P_+S^\times + P_- = \begin{pmatrix} S_{11}^\times & S_{21}^\times \\ 0 & 1 \end{pmatrix} \in \mathbf{B}(\mathcal{K}_+ \oplus \mathcal{H}_-, \mathcal{H}_+ \oplus \mathcal{H}_-),$$
$$Q_+ + Q_-S^\times = \begin{pmatrix} 1 & 0 \\ S_{12}^\times & S_{22}^\times \end{pmatrix} \in \mathbf{B}(\mathcal{K}_+ \oplus \mathcal{H}_-, \mathcal{K}_+ \oplus \mathcal{K}_-).$$

Proof of Theorem 1.3.4. (i) $\Rightarrow$ (ii) Assume that T is a bicontraction. As in the proof of Theorem 1.3.3, $T_{22} = Q_-TP_-|\mathcal{H}_-$ is one-to-one and has closed range as an operator on $\mathcal{H}_-$ to $\mathcal{K}_-$. Suppose that f is in $\mathcal{K}_-$ and $f \perp Q_-TP_-\mathcal{H}_-$. Then $f \perp T\mathcal{H}_-$ and $T^*f \perp \mathcal{H}_-$. So $T^*f \in \mathcal{H}_+$, and since T is bicontractive,

$$\langle T^*f, T^*f\rangle_{\mathcal{H}} \geq 0 \geq \langle f, f\rangle_{\mathcal{K}} \geq \langle T^*f, T^*f\rangle_{\mathcal{H}}.$$

Therefore equality holds throughout, and since $f \in \mathcal{K}_-$ we obtain $f = 0$. Hence $\operatorname{ran} T_{22} = \mathcal{K}_-$, and T_{22} is invertible.

(ii) $\Rightarrow$ (iii) If T_{22} is invertible, so is the operator (1.3.4). Then by Theorem 1.3.3, $\operatorname{dom} S = \operatorname{ran}(P_+ + Q_-T) = |\mathcal{L}_-|$.

(iii) $\Rightarrow$ (i). If $\operatorname{dom} S = |\mathcal{L}_-|$, the scattering operator S is given by (1.3.5). Multiplication of operator matrices shows that

$$S^\times = (P_+T^* + Q_-)(Q_+ + P_-T^*)^{-1}, \tag{1.3.10}$$

where

$$P_+T^* + Q_- = \begin{pmatrix} T_{11}^\times & -T_{21}^\times \\ 0 & 1 \end{pmatrix} \in \mathbf{B}(\mathcal{K}_+ \oplus |\mathcal{K}_-|, \mathcal{H}_+ \oplus |\mathcal{K}_-|),$$
$$Q_+ + P_-T^* = \begin{pmatrix} 1 & 0 \\ -T_{12}^\times & T_{22}^\times \end{pmatrix} \in \mathbf{B}(\mathcal{K}_+ \oplus |\mathcal{K}_-|, \mathcal{K}_+ \oplus |\mathcal{H}_-|).$$

Since $S^\times$ is a contraction, for all $g \in \mathcal{K}$,

$$\|(P_+T^* + Q_-)g\|^2_{\mathcal{H}_+ \oplus |\mathcal{K}_-|} \leq \|(Q_+ + P_-T^*)g\|^2_{\mathcal{K}_+ \oplus |\mathcal{H}_-|}.$$

Therefore

$$\|P_+T^*g\|^2 + \|Q_+g\|^2 \leq \|Q_+g\|^2 + \|P_-T^*g\|^2$$

and

$$\|P_+T^*g\|^2 - \|P_-T^*g\|^2 \leq \|Q_+g\|^2 - \|Q_-g\|^2,$$

where the norms are computed in $\mathcal{H}_+$, $|\mathcal{H}_-|$, $\mathcal{K}_+$, $|\mathcal{K}_-|$. It follows that T^* is contractive, and hence T is bicontractive.

Now assume that (i) - (iii) hold. Then (1.3.10) says that the scattering operator for T^* is $S^\times$, and (1.3.6) and (1.3.7) follow by multiplying matrices.

The identities (1.3.8) and (1.3.9) are equivalent because the scattering operator for T^* is $S^\times$. To prove (1.3.8), calculate using the identity for S in Theorem 1.3.3:

$$\begin{aligned}(P_+ + Q_-T)^\times & (1 - S^\times S)(P_+ + Q_-T) \\ &= (P_+ + Q_-T)^\times (P_+ + Q_-T) - (Q_+T + P_-)^\times (Q_+T + P_-) \\ &= \begin{pmatrix} 1 & T_{21}^\times \\ 0 & T_{22}^\times \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_{21} & T_{22} \end{pmatrix} - \begin{pmatrix} T_{11}^\times & 0 \\ T_{12}^\times & 1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

By multiplication of matrices, this is the same as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} T_{11}^\times & T_{21}^\times \\ T_{12}^\times & T_{22}^\times \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ -T_{21} & -T_{22} \end{pmatrix} = J_{\mathcal{H}}(1 - T^*T),$$

which yields the result. ■

The next result characterizes the Hilbert space contractions S which occur as scattering operators of bicontractions.

THEOREM 1.3.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces with fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. Let $S \in \mathbf{B}(\mathcal{H}_+ \oplus |\mathcal{K}_-|, \mathcal{K}_+ \oplus |\mathcal{H}_-|)$ be a contraction with matrix*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Then S is the scattering operator for a bicontraction $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ if and only if S_{22} is invertible.

Proof. Necessity follows from Theorem 1.3.4 and (1.3.5). Conversely, suppose that S_{22} is invertible. Define $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ by

$$T = \begin{pmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{pmatrix}.$$

Multiplication of matrices verifies the identity

$$S = (Q_+T + P_-)(P_+ + Q_-T)^{-1}.$$

Since S is a contraction, for any $f \in \mathcal{H}$,

$$\|(Q_+T + P_-)f\|^2_{\mathcal{K}_+ \oplus |\mathcal{H}_-|} \leq \|(P_+ + Q_-T)f\|^2_{\mathcal{H}_+ \oplus |\mathcal{K}_-|},$$

equivalently,

$$\|Q_+Tf\|^2 + \|P_-f\|^2 \leq \|P_+f\|^2 + \|Q_-Tf\|^2,$$
$$\|Q_+Tf\|^2 - \|Q_-Tf\|^2 \leq \|P_+f\|^2 - \|P_-f\|^2,$$

which means that $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction. Since $T_{22} = S_{22}^{-1}$ is invertible, T is a bicontraction by Theorem 1.3.4. By construction, S is the scattering operator for T. ■

Theorem 1.3.1 describes how a contraction maps negative subspaces. Bicontractions are characterized by how they map maximal negative subspaces.

THEOREM 1.3.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces. If $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction, the following assertions are equivalent:*

(i) T is a bicontraction;
(ii) αT^* is a contraction for some positive number α;
(iii) T maps some maximal negative subspace of $\mathcal{H}$ onto a maximal negative subspace of $\mathcal{K}$;
(iv) T maps every maximal negative subspace of $\mathcal{H}$ onto a maximal negative subspace of $\mathcal{K}$.

In this case, T maps any maximal uniformly negative subspace of $\mathcal{H}$ in a one-to-one way onto a maximal uniformly negative subspace of $\mathcal{K}$.

Proof. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ be fundamental decompositions of the spaces $\mathcal{H}$ and $\mathcal{K}$.

(i) $\Rightarrow$ (ii) If (i) holds, (ii) holds with $\alpha = 1$.

(ii) $\Rightarrow$ (iv) Assume (ii), and let $\mathcal{M}$ be a maximal negative subspace of $\mathcal{H}$. By Corollary 1.3.2, $T\mathcal{M}$ is a closed negative subspace of $\mathcal{K}$. By the graph representation of negative subspaces, $T\mathcal{M}$ is maximal negative if and only if there is no nonzero vector in $\mathcal{K}_-$ which is orthogonal to $T\mathcal{M}$. If f is in $\mathcal{K}_-$ and orthogonal to $T\mathcal{M}$, then for g in $\mathcal{M}$,

$$\langle \alpha T^* f, g \rangle_{\mathcal{H}} = \langle f, \alpha T g \rangle_{\mathcal{K}} = 0.$$

It follows that $\alpha T^* f$ is in $\mathcal{M}^{\perp}$, which is maximal positive since $\mathcal{M}$ is maximal negative. Therefore,

$$\langle \alpha T^* f, \alpha T^* f \rangle_{\mathcal{H}} \geq 0 \geq \langle f, f \rangle_{\mathcal{K}}.$$

Since αT^* is contractive by assumption and f is in $\mathcal{K}_-$, $\langle f, f \rangle_{\mathcal{K}} = 0$ and $f = 0$. Hence $T\mathcal{M}$ is maximal negative.

(iv) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (i) Assume that there is a maximal negative subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T\mathcal{M}$ is maximal negative in $\mathcal{K}$. We show that the entry T_{22} in (1.3.1) is invertible, and hence T is a bicontraction by Theorem 1.3.4.

Represent $\mathcal{M}$ as the graph $\mathcal{G}(K)$ of a contraction operator K on $|\mathcal{H}_-|$ to $\mathcal{H}_+$. Then sK is a contraction and its graph $\mathcal{G}(sK)$ is a maximal negative subspace of $\mathcal{H}$ for $0 \le s \le 1$. Since T is contractive, so is Q_-T where Q_- is the projection of $\mathcal{K}$ onto $\mathcal{K}_-$. Choose $\delta > 0$ for Q_-T as in Theorem 1.3.1. For all f in $\mathcal{H}_-$,

$$\|(T_{21}sK + T_{22})f\|^2_{|\mathcal{K}_-|} = \|Q_-T(sKf + f)\|^2_{|\mathcal{K}|} \ge \delta\|sKf + f\|^2_{|\mathcal{H}|} \ge \delta\|f\|^2_{|\mathcal{H}_-|}$$

uniformly for $0 \le s \le 1$. The range of $T_{21}K + T_{22}$ is all of $\mathcal{K}_-$ because T maps $\mathcal{M} = \mathcal{G}(K)$ onto a maximal negative subspace of $\mathcal{K}$ by assumption. Therefore $T_{21}K + T_{22}$ is an invertible operator on $\mathcal{K}_-$. By a Neumann series argument, $T_{21}sK + T_{22}$ is invertible for $0 \le s \le 1$. In particular, T_{22} is invertible.

We have proved the equivalence of the statements (i)–(iv). The last assertion of the theorem follows from Corollary 1.3.2. ■

Theorem 1.3.7. *If $\mathcal{H}$ is a Pontryagin space, every contraction operator T from $\mathcal{H}$ into itself is bicontractive.*

Proof. In any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{H}_-$ has finite dimension. If P_- is the projection onto $\mathcal{H}_-$, then P_-T is a contraction because T is a contraction. By Corollary 1.3.2, $T_{22} = P_-TP_-|\mathcal{H}_-$ is a one-to-one mapping of $\mathcal{H}_-$ into itself. Hence T_{22} is invertible, and T is bicontractive by Theorem 1.3.6. ■

Examples 1.3.8. (i) Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces. An isometry $V \in \mathbf{B}(\mathcal{H},\mathcal{K})$ is a bicontraction if and only if $\ker V^*$ is a uniformly positive subspace of $\mathcal{K}$. Necessity of the condition follows from Theorem 1.3.1. For the other direction, note that V is a partial isometry with initial space $\mathcal{H}$ and final space $V\mathcal{H}$ by Corollary 1.1.4. Since $\mathcal{K} = V\mathcal{H} \oplus \ker V^*$ with $\ker V^*$ uniformly positive, sufficiency of the condition follows from elementary properties of partial isometries.

(ii) Theorem 1.3.7 fails for Kreĭn spaces. Consider the Kreĭn space $\mathcal{H}$ of square summable sequences $a = (a_0, a_1, a_2, \ldots)$ with

$$\langle a, a\rangle_{\mathcal{H}} = |a_0|^2 - |a_1|^2 - |a_2|^2 - \cdots.$$

The operator $T \in \mathbf{B}(\mathcal{H})$ defined by

$$T : (a_0, a_1, a_2, \ldots) \to (0, a_0, a_1, \ldots)$$

is contractive, but its adjoint

$$T^* : (a_0, a_1, a_2, \ldots) \to (-a_1, a_2, a_3, \ldots)$$

is not contractive. The range of T is a maximal negative subspace of $\mathcal{H}$ which is mapped by T onto a proper subspace of itself and hence onto a negative subspace which is not maximal.

(iii) In connection with Theorem 1.3.6, we note a counter-example to a related statement in Andô ([**4**], p. 31, Corollary 3.3.2). In our language, the statement is that an operator $T \in \mathbf{B}(\mathcal{H})$ is a scalar multiple of a bicontraction if it maps maximal negative subspaces onto maximal negative subspaces. A counterexample is $T = V^* \in \mathbf{B}(\mathcal{H})$, where $\mathcal{H}$ is the anti-space of a Hilbert space and V is an isometry whose range is not all of $\mathcal{H}$. The only maximal negative subspace of $\mathcal{H}$ is $\mathcal{H}$, and this mapped by T onto itself. No constant multiple of T is a contraction, since the kernel of T includes a vector f with $\langle f, f\rangle_{\mathcal{H}} < 0$.

Bicontractions are also characterized in terms of mappings of operator spheres.

THEOREM 1.3.9. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction with matrix (1.3.1) relative to some fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. Then T is a bicontraction if and only if whenever $X \in \mathbf{B}(|\mathcal{H}_-|, \mathcal{H}_+)$ is a contraction, then $T_{21}X + T_{22}$ is an invertible operator on $|\mathcal{H}_-|$ to $|\mathcal{K}_-|$ and*

$$Y = (T_{11}X + T_{12})(T_{21}X + T_{22})^{-1} \tag{1.3.11}$$

is a contraction in $\mathbf{B}(|\mathcal{K}_-|, \mathcal{K}_+)$.

Proof. If T is a bicontraction and $X \in \mathbf{B}(|\mathcal{H}_-|, \mathcal{H}_+)$ is a contraction, then

$$\mathcal{G}(X) = \left\{ \begin{pmatrix} Xf \\ f \end{pmatrix} : f \in \mathcal{H}_- \right\}$$

is a maximal negative subspace of $\mathcal{H}$ and

$$T\mathcal{G}(X) = \left\{ \begin{pmatrix} T_{11}Xf + T_{12}f \\ T_{21}Xf + T_{22}f \end{pmatrix} : f \in \mathcal{H}_- \right\}$$

is a maximal negative subspace of $\mathcal{K}$ by Theorem 1.3.6. Let Q_- be the projection of $\mathcal{K}$ onto $\mathcal{K}_-$. Then Q_-T is contractive, and $T_{21}X + T_{22}$ is one-to-one by Corollary 1.3.2. Since $Q_-T\mathcal{G}(X) = \mathcal{K}_-$, $T_{21}X + T_{22}$ is invertible. Therefore (1.3.11) defines an operator $Y \in \mathbf{B}(|\mathcal{K}_-|, \mathcal{K}_+)$. Since $T\mathcal{G}(X)$ is a negative subspace of $\mathcal{K}$,

$$\|(T_{11}X + T_{12})f\|^2_{\mathcal{K}_+} \le \|(T_{21}X + T_{22})f\|^2_{|\mathcal{K}_-|}, \qquad f \in \mathcal{H}_-,$$

and Y is a contraction.

In the other direction, if the condition holds for $X = 0$, then T_{22} is invertible and T is a bicontraction by Theorem 1.3.4. ■

In the next result, the domain and range spaces $\mathcal{H}$ and $\mathcal{K}$ coincide.

THEOREM 1.3.10. *Let $\mathcal{H}$ be a Kreĭn space with fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and let $T \in \mathbf{B}(\mathcal{H})$ be a bicontraction. Define a mapping Φ of the set $\mathcal{C}$ of all contractions $X \in \mathbf{B}(|\mathcal{H}_-|, \mathcal{H}_+)$ into itself by*

$$\Phi(X) = (T_{11}X + T_{12})(T_{21}X + T_{22})^{-1}, \qquad X \in \mathcal{C}.$$

Then for $X \in \mathcal{C}$, we have $\Phi(X) = X$ if and only if the graph

$$\mathcal{M} = \left\{ \begin{pmatrix} Xf \\ f \end{pmatrix} : f \in \mathcal{H}_- \right\}$$

of X is invariant under T.

Proof. By its form, $\mathcal{M}$ is a maximal negative subspace of $\mathcal{H}$. If $\mathcal{M}$ is invariant under T, then $T\mathcal{M} = \mathcal{M}$ by Theorem 1.3.6. Since

$$T\mathcal{M} = \left\{ \begin{pmatrix} T_{11}Xf + T_{12}f \\ T_{21}Xf + T_{22}f \end{pmatrix} : f \in \mathcal{H}_- \right\},$$

we have $T_{11}X + T_{12} = X(T_{21}X + T_{22})$, that is, $\Phi(X) = X$. The other direction is obtained by reversing these steps. ■

Existence theorems for definite invariant subspaces follow with the aid of Theorem 1.3.10. A full treatment is beyond the scope of our discussion. But having come this far, we show the connection by giving one of the early results of the subject which is due to Kreĭn (Theorem 1.3.11). See the notes at the end of the chapter for literature references. Note that the set $\mathcal{C}$ in Theorem 1.3.10 is convex and compact in the weak operator topology of $\mathbf{B}(|\mathcal{H}_-|, \mathcal{H}_+)$. We also recall the SCHAUDER-TYCHONOFF THEOREM: *Every continuous mapping of a convex and compact subset of a locally convex linear topological vector space has a fixed point.* The Schauder-Tychonoff theorem is proved, for example, in Dunford and Schwarz ([**31**], p. 456). It is applicable in the situation of Theorem 1.3.10 whenever Φ is continuous. This is automatic in the case of a Pontryagin space.

THEOREM 1.3.11. *Let $\mathcal{H}$ be a Pontryagin space. If $T \in \mathbf{B}(\mathcal{H})$ is a contraction (and hence a bicontraction), there exists a maximal negative subspace $\mathcal{M}$ of $\mathcal{H}$ which is mapped by T onto itself.*

Proof. It is sufficient to show that the mapping Φ in Theorem 1.3.10 is continuous.

Let $\{X_\alpha\}_{\alpha\in\mathcal{D}}$ be a generalized sequence in $\mathcal{C}$ which converges in the weak operator topology to $X \in \mathcal{C}$. Then

$$\lim_\alpha (T_{11}X_\alpha + T_{12}) = T_{11}X + T_{12}$$

in the weak operator topology of $\mathbf{B}(|\mathcal{H}_-|, \mathcal{K}_+)$.

Let P_- be the projection of $\mathcal{H}$ onto $\mathcal{H}_-$. Choose $\delta > 0$ for the contraction $P_-T \in \mathbf{B}(\mathcal{H})$ as in Theorem 1.3.1. Then

$$\|T_{21}X_\alpha f + T_{22}f\| \geq \delta\|f\|, \qquad \alpha \in \mathcal{D}, f \in \mathcal{H}_-.$$

Therefore $\left\|(T_{21}X_\alpha + T_{22})^{-1}\right\| \leq 1/\delta$ for all $\alpha \in \mathcal{D}$. Now

$$\lim_\alpha (T_{21}X_\alpha + T_{22}) = T_{21}X + T_{22}$$

in the weak operator topology of $\mathbf{B}(|\mathcal{H}_-|)$ and hence also in the norm topology because $\mathcal{H}_-$ is finite dimensional. By Theorem 1.3.9, $T_{21}X + T_{22}$ is invertible, and

$$\lim_\alpha (T_{21}X_\alpha + T_{22})^{-1} = (T_{21}X + T_{22})^{-1}$$

in the operator norm of $\mathbf{B}(|\mathcal{H}_-|)$. Thus

$$\lim_\alpha \Phi(X_\alpha) = \Phi(X)$$

in the weak operator topology of $\mathbf{B}(|\mathcal{H}_-|, \mathcal{H}_+)$, and so Φ is continuous. ■

1.4 Additional Results on Contractions and Bicontractions

The results here catalog properties of defect operators and Julia operators of contractions. We also give a useful theorem on the existence of a bicontractive extension of a densely defined contraction.

THEOREM 1.4.1. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces, and let*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

be a Julia operator for T. The adjoint of the operator $(T \quad D) \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K})$ is an isometry. The following assertions are equivalent:

(i) T is a contraction;
(ii) $\tilde{\mathcal{D}}$ is a Hilbert space;
(iii) $(T \quad D)$ is a contraction;
(iv) $(T \quad D)$ is a bicontraction;
(v) the kernel of $(T \quad D)$ is uniformly positive in $\mathcal{H} \oplus D$.

Proof. In the proof, we write $R = (T \quad D)$ and make repeated use of the relations (1.2.1) and (1.2.2). The operator R^* is an isometry because $RR^* = TT^* + DD^* = 1$ by the definition of a Julia operator.

(i) ⇒ (ii) If T is a contraction, then $\tilde{D}\tilde{D}^* = 1 - T^*T \geq 0$. Since $\tilde{D}$ has zero kernel, the range of $\tilde{D}^*$ is dense in $\tilde{\mathcal{D}}$. The inequality $\tilde{D}\tilde{D}^* \geq 0$ then implies that $\langle g, g\rangle_{\tilde{\mathcal{D}}} \geq 0$ for all $g \in \mathcal{D}$, and therefore $\tilde{\mathcal{D}}$ is a Hilbert space.

(ii) ⇒ (iii) If $\tilde{\mathcal{D}}$ is a Hilbert space, then

$$1 - R^*R = \begin{pmatrix} 1 - T^*T & -T^*D \\ -D^*T & 1 - D^*D \end{pmatrix} = \begin{pmatrix} \tilde{D}\tilde{D}^* & \tilde{D}L \\ L^*\tilde{D}^* & L^*L \end{pmatrix} = \begin{pmatrix} \tilde{D} \\ L^* \end{pmatrix} \begin{pmatrix} \tilde{D}^* & L \end{pmatrix} \geq 0,$$

because $(\tilde{D}^* \quad L) \in \mathbf{B}(\mathcal{H} \oplus D, \tilde{\mathcal{D}})$. Thus R is a contraction.

(iii) ⇒ (iv) Since R^* is an isometry, if R is a contraction it is automatically a bicontraction.

(iv) ⇒ (v) This follows from Theorem 1.3.1.

(v) ⇒ (i) Since R^* is isometric, this follows from Example 1.3.8 (i). ■

Corollary 1.4.2. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces, and let $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ be a defect operator for T^*. Then the operator $(T \quad D)$ has an isometric adjoint, and the following assertions are equivalent:*

(i) T is a contraction;
(ii) $(T \quad D)$ is a contraction;
(iii) $(T \quad D)$ is a bicontraction;
(iv) the kernel of $(T \quad D)$ is uniformly positive in $\mathcal{H} \oplus D$.

If these equivalent conditions are satisfied, then for any defect operator $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ of T, $\tilde{\mathcal{D}}$ is a Hilbert space.

Proof. By Theorem 1.2.4, this is a consequence of Theorem 1.4.1. ■

Corollary 1.4.3. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces, and let*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

be a Julia operator for T. If T is a bicontraction, then $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are Hilbert spaces, and both

$$\begin{pmatrix} T & D \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K}) \qquad \text{and} \qquad \begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

are bicontractions with isometric adjoints.

Proof. Since T^* has Julia operator

$$U^* = \begin{pmatrix} T^* & \tilde{D} \\ D^* & L^* \end{pmatrix} \in \mathbf{B}(\mathcal{K} \oplus \tilde{\mathcal{D}}, \mathcal{H} \oplus D),$$

the result follows by applying Theorem 3.1.1 to both T and T^*. ■

In Hilbert spaces, densely defined contractions extend automatically by continuity to everywhere defined contractions. This is not true in Kreĭn spaces, but with an extra condition the existence of a continuous bicontractive extension is assured. The result can be used, for example, to construct continuous isometric operators from densely defined isometries.

THEOREM 1.4.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let T_0 be a densely defined linear mapping from $\mathcal{H}$ to $\mathcal{K}$. Assume that*

$$\langle T_0 f, T_0 f\rangle_{\mathcal{K}} \leq \langle f, f\rangle_{\mathcal{H}}, \qquad f \in \operatorname{dom} T_0.$$

Assume also that $\operatorname{dom} T_0$ *contains a maximal uniformly negative subspace $\mathcal{M}$ of $\mathcal{H}$ and that $T_0\mathcal{M}$ is a maximal uniformly negative subspace of $\mathcal{K}$. Then T_0 has an extension by continuity to a bicontractive operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$.*

LEMMA 1.4.5. *Let $\mathcal{H}$ be a Kreĭn space with fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Set $\alpha = \sqrt{2} - 1$. If $f \in \mathcal{H}_-$, $\|f\| \geq 1$, and $g \in \mathcal{H}$, $\|g\| < \alpha$, then*

$$\langle f + g, f + g\rangle < 0.$$

The norms in the lemma are computed with respect to the given fundamental decomposition of $\mathcal{H}$.

Proof of Lemma 1.4.5. Fix $f \in \mathcal{H}_-$, $\|f\| \geq 1$, and $g \in \mathcal{H}$, $\|g\| < \alpha$. Then

$$\begin{aligned}\langle f + g, f + g\rangle &= -\|f\|^2 + 2\mathrm{Re}\langle f, g\rangle + \langle g, g\rangle \\ &< -\|f\|^2 + 2\alpha\|f\| + \alpha^2.\end{aligned}$$

The function $\phi(x) = -x^2 + 2\alpha x + \alpha^2$ attains its maximum at $x = \alpha$, and therefore $\phi(x) \leq \phi(1)$ for $x \geq 1$. Thus $\langle f + g, f + g\rangle < -1 + 2\alpha + \alpha^2 = 0$. ■

Proof of Theorem 1.4.4. Choose fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ such that

$$\mathcal{H}_+ = \mathcal{M}^\perp, \qquad \mathcal{H}_- = \mathcal{M},$$

and

$$\mathcal{K}_+ = (T_0\mathcal{M})^\perp, \qquad \mathcal{K}_- = T_0\mathcal{M}.$$

The conclusion is immediate if $\mathcal{H}$ is a Hilbert space, and we exclude this case in what follows. Let S be the restriction of T_0 to $\mathcal{H}_-$, viewed as a linear mapping of $\mathcal{H}_-$ onto $\mathcal{K}_-$. For $f \in \mathcal{H}_-$,

$$\|Sf\|^2 = -\langle T_0 f, T_0 f\rangle_{\mathcal{K}} \geq -\langle f, f\rangle_{\mathcal{H}} = \|f\|^2.$$

Therefore S has a continuous inverse, and so $S \in \mathbf{B}(\mathcal{H}_-, \mathcal{K}_-)$ by the open mapping theorem.

Set $\alpha = \sqrt{2} - 1$. We show that for $g \in \operatorname{dom} T_0$, $\|g\| < \alpha$,

$$\left\|\Pr_{\mathcal{K}_-} T_0 g\right\| < \|S\|.$$

Argue by contradiction, assuming that the inequality is not true. Choose $h \in \mathcal{H}_-$ such that $Sh = \Pr_{\mathcal{K}_-} T_0 g$. Then $\|S\| \leq \left\|\Pr_{\mathcal{K}_-} T_0 g\right\| = \|Sh\| \leq \|S\|\|h\|$, and so $\|h\| \geq 1$. By Lemma 1.4.5,

$$0 > \langle h - g, h - g\rangle_{\mathcal{H}} \geq \langle T_0(h-g), T_0(h-g)\rangle_{\mathcal{K}},$$

a contradiction, because $T_0(h-g)$ has zero projection in $\mathcal{K}_-$ and hence is in $\mathcal{K}_+$.

We show that for $g \in \operatorname{dom} T_0, \|g\| < \alpha$,

$$\left\|\Pr_{\mathcal{K}_+} T_0 g\right\| < 2\|S\|.$$

Since $\mathcal{H}$ is not a Hilbert space, we can choose $h \in \mathcal{H}_-$ with $\|h\| = 1$. By Lemma 1.4.5,

$$\begin{aligned} 0 &> \langle h - g, h - g\rangle_{\mathcal{H}} \geq \langle T_0(h-g), T_0(h-g)\rangle_{\mathcal{K}} \\ &= \left\|\Pr_{\mathcal{K}_+} T_0(h-g)\right\|^2 - \left\|\Pr_{\mathcal{K}_-} T_0(h-g)\right\|^2 \\ &= \left\|\Pr_{\mathcal{K}_+} T_0 g\right\|^2 - \left\|\Pr_{\mathcal{K}_-} T_0(h-g)\right\|^2. \end{aligned}$$

Therefore

$$\left\|\Pr_{\mathcal{K}_+} T_0 g\right\|^2 < \left\|\Pr_{\mathcal{K}_-} T_0(h-g)\right\|^2 \leq \left[\left\|\Pr_{\mathcal{K}_-} T_0 h\right\| + \left\|\Pr_{\mathcal{K}_-} T_0 g\right\|\right]^2 < 4\|S\|^2,$$

which proves the assertion.

By what we have shown, for any $g \in \operatorname{dom} T_0$, $\|g\| < \alpha$,

$$\|T_0 g\|^2 = \left\|\Pr_{\mathcal{K}_+} T_0 g\right\|^2 + \left\|\Pr_{\mathcal{K}_-} T_0 g\right\|^2 < 5\|S\|^2.$$

It follows that T_0 has an extension to an operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ with

$$\|T\| < \frac{\sqrt{5}}{\sqrt{2} - 1}\|S\|.$$

Clearly T is a contraction. Since T maps a maximal negative subspace of $\mathcal{H}$ onto a maximal negative subspace of $\mathcal{K}$, T is a bicontraction. ■

Notes on Chapter 1

Accounts of Kreĭn spaces and operators on them are given in Andô [**4**], Azizov and Iokhvidov [**10**], Bognár [**12**], and Kreĭn [**43**]. The book by Iokhvidov, Kreĭn, and Langer [**40**] is another useful source: it emphasizes Pontryagin spaces, but the results and methods often apply to general Kreĭn spaces. Kreĭn [**43**] is an authoritative source for the early history of the subject and applications that motivated its development. Azizov and Iokhvidov [**9**] survey literature in the years 1953–1978. For the finite dimensional case, see Gohberg, Lancaster, and Rodman [**38**] and Potapov [**55**].

Additional information on isometries and direct sums may be found in Bognár [**12**], Ghéondea [**36**], and McEnnis [**52,53**]. In particular, Theorem 1.1.8 on internal orthogonal direct sums is not as strong as possible. Ghéondea [**36**] gives necessary and sufficient conditions for the closed span of pairwise orthogonal regular subspaces to be regular. This condition is weaker than the uniform boundedness condition we give on the associated projections, which he shows to be equivalent to both convergence in the strong and the weak operator topologies of the sums given in 1.1.8(iii). McEnnis [**53**] gives necessary and sufficient conditions for the square summability of the representation of elements of an internal direct sum given in 1.1.8(ii).

Factorizations of selfadjoint operators in the form $H = AA^*$, discussed in §1.2, appear in many places, including Bognár [**12**]. The existence of a Julia operator in the Kreĭn space setting was first shown by Arsene, Constantinescu, and Ghéondea [**8**] by a different method (see Appendix B). This is an important result and plays a key role in what follows. The abstract definition of a Julia operator and existence proof in Theorem 1.2.4 seem to be new; uniqueness is discussed in Chapter 2 and Appendix B. For the Hilbert space case as well as references to the original papers of Julia and Halmos, see Sz.-Nagy and Foiaş [**66**].

The main results of §1.3 are due to Ginsburg [**37**] and Kreĭn and Shmul'yan [**45,47**]. In the finite dimensional case, some ideas go back to Potapov [**55**]. Additional results are given in Kreĭn and Shmul'yan [**46**].

Theorem 1.3.11 is the simplest of its kind. The method is due to Kreĭn [**42**], but the problem was known previously. The history of the problem is given in the survey of Azizov and Iokhvidov [**9**]. See their monograph [**10**] for a recent account. Additional information may be found in Iokhvidov, Kreĭn, and Langer [**40**]. Other sources are Kreĭn [**43**], Andô [**4**], and Bognár [**12**]. Best possible conditions for the existence of definite invariant subspaces are not known.

Literature citations for the Potapov-Ginsburg transform are given in Azizov and Iokhvidov [**9**]. Our treatment is adapted from Iokhvidov, Kreĭn, and Langer [**40**]. The transform is used in other situations, such as Arov [**5**], Dym [**32**], and Kreĭn and Langer [**44**]. The scattering interpretation appears in Alpay [**3**] and Dym [**32**].

Theorem 1.4.4 is due to Shmul'yan [**64**]. A similar result is proved by Yan [**67**].

Chapter 2: Matrix Extensions of Contraction Operators

2.1 The Adjoint of a Contraction

The theory of Ginsburg, Kreĭn, and Shmul'yan discussed in Chapter 1 is based on an analysis of how a contraction operator maps negative subspaces. In the case of bicontractions, the adjoint of a given contraction is again a contraction and subject to the same analysis. Contraction operators which are not bicontractions are also of interest. The main point of this section is that one can decompose the domain space of the adjoint of any Kreĭn space contraction in such a way as to discard a uniformly negative subspace, and then the restriction of the adjoint to what remains is a bicontraction. In this way we are able to apply the strong results on bicontractions to arbitrary contractions. In particular, we obtain a new characterization of bicontractions in terms of adjoint operators (Theorem 2.1.5).

To begin, we assume that bicontractive restrictions of the adjoint of a contraction operator exist, and we determine some consequences of this fact. Chiefly, there exist maximal uniformly negative subspaces of the range space with special properties. Then we shall reverse the process.

Theorem 2.1.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction operator. Let $\hat{\mathcal{K}}$ be a regular subspace of $\mathcal{K}$ such that $\hat{\mathcal{K}}^\perp$ is uniformly negative and $T^*|\hat{\mathcal{K}}$ is a bicontraction on $\hat{\mathcal{K}}$ to $\mathcal{H}$. Let $\mathcal{L}_-$ be a uniformly negative subspace of $\hat{\mathcal{K}}$ which is maximal in $\hat{\mathcal{K}}$, and let $\mathcal{K}_- = \mathcal{L}_- \oplus \hat{\mathcal{K}}^\perp$. Define*

$$\mathcal{L}_+ = \{v : v \in \mathcal{K}_- \text{ and } T^*v \perp T^*\mathcal{L}_-\}.$$

Then $\mathcal{K}_-$ is a maximal uniformly negative subspace of $\mathcal{K}$, and $\mathcal{K}_- = \mathcal{L}_- \dot{+} \mathcal{L}_+$.

Recall that $\dot{+}$ is the symbol for a direct sum which is not necessarily orthogonal.

Proof. Since $\hat{\mathcal{K}}^\perp$ is uniformly negative, $\mathcal{K}_- = \mathcal{L}_- \oplus \hat{\mathcal{K}}^\perp$ is closed and uniformly negative. Since $\mathcal{L}_-$ is a maximal uniformly negative subspace in $\hat{\mathcal{K}}$, the subspace defined by $\mathcal{K}_+ = \hat{\mathcal{K}} \ominus \mathcal{L}_-$ is closed and uniformly positive. Now

$$\mathcal{K} = \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp = (\mathcal{K}_+ \oplus \mathcal{L}_-) \oplus \hat{\mathcal{K}}^\perp = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

and so $\mathcal{K}_-$ is maximal uniformly negative in $\mathcal{K}$.

Since $T^*|\hat{\mathcal{K}}$ is a bicontraction and $\mathcal{L}_-$ is maximal uniformly negative in $\hat{\mathcal{K}}$, $T^*\mathcal{L}_-$ is a maximal uniformly negative subspace of $\mathcal{H}$ by Theorem 1.3.6. Choose a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by setting $\mathcal{H}_- = T^*\mathcal{L}_-$ and $\mathcal{H}_+ = (T^*\mathcal{L}_-)^\perp$.

Clearly $\mathcal{K}_- \supset \mathcal{L}_- + \mathcal{L}_+$. Let $k \in \mathcal{K}_-$. Write $T^*k = h_- + h_+$, $h_\pm \in \mathcal{H}_\pm$. Then $h_- = T^*u$ for some $u \in \mathcal{L}_-$, and so $h_+ = T^*v$, where $v = k - u \in \mathcal{K}_-$ and $T^*v \perp T^*\mathcal{L}_-$. Thus $k = u + v$ where $u \in \mathcal{L}_-$ and $v \in \mathcal{L}_+$. Therefore $\mathcal{K}_- = \mathcal{L}_- + \mathcal{L}_+$.

If $f \in \mathcal{L}_- \cap \mathcal{L}_+$, then $T^*f \in T^*\mathcal{L}_- \cap T^*\mathcal{L}_+ \subset \mathcal{H}_- \cap \mathcal{H}_+$ and $T^*f = 0$. Since $T^*|\mathcal{L}_-$ is one-to-one by Corollary 1.3.2, $f = 0$. Therefore the sum is direct. ■

Notice that in the situation of Theorem 2.1.1, $T^*|\mathcal{L}_-$ is one-to-one, its range is a maximal uniformly negative subspace of $\mathcal{H}$, and $T^*\mathcal{L}_+ \perp T^*\mathcal{L}_-$. We show that for any contraction $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces, there exist subspaces of $\mathcal{K}$ having these properties.

THEOREM 2.1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Let $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ be a defect operator for T^*, and assume that $\mathcal{K}_-$, $\mathcal{D}_-$ are maximal uniformly negative subspaces of $\mathcal{K}$, $\mathcal{D}$. Then there exist closed subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\mathcal{K}_-$ such that*

$$\mathcal{K}_- = \mathcal{L}_- \dot{+} \mathcal{L}_+,$$

and T^* maps $\mathcal{L}_-$ in a one-to-one way onto a maximal uniformly negative subspace of $\mathcal{H}$, $T^*\mathcal{L}_+ \perp T^*\mathcal{L}_-$, and $D^*\mathcal{L}_- \perp \mathcal{D}_-$.

LEMMA 2.1.3. *Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a bicontraction, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces. Let $\hat{\mathcal{H}}$ be a regular subspace of $\mathcal{H}$ such that $\hat{\mathcal{H}}^\perp$ is uniformly positive. Then $\hat{T} = T|\hat{\mathcal{H}}$ is a bicontraction on $\hat{\mathcal{H}}$ to $\mathcal{K}$.*

Proof of Lemma 2.1.3. Since T is a contraction and $\hat{T}$ is a restriction of T, $\hat{T}$ is a contraction. For any $g \in \mathcal{K}$,

$$\begin{aligned}
\left\langle \hat{T}^*g, \hat{T}^*g \right\rangle_{\mathcal{H}} &= \left\langle \mathrm{Pr}_{\hat{\mathcal{H}}} T^*g, \mathrm{Pr}_{\hat{\mathcal{H}}} T^*g \right\rangle_{\mathcal{H}} \\
&\leq \left\langle \mathrm{Pr}_{\hat{\mathcal{H}}} T^*g, \mathrm{Pr}_{\hat{\mathcal{H}}} T^*g \right\rangle_{\mathcal{H}} + \left\langle \mathrm{Pr}_{\mathcal{H}\ominus\hat{\mathcal{H}}} T^*g, \mathrm{Pr}_{\mathcal{H}\ominus\hat{\mathcal{H}}} T^*g \right\rangle_{\mathcal{H}} \\
&= \langle T^*g, T^*g \rangle_{\mathcal{H}} \\
&\leq \langle g, g \rangle_{\mathcal{K}}
\end{aligned}$$

because $\mathcal{H} \ominus \hat{\mathcal{H}} = \hat{\mathcal{H}}^\perp$ is uniformly positive. ■

Proof of Theorem 2.1.2. Define $D_- = D|\mathcal{D}_- \in \mathbf{B}(\mathcal{D}_-, \mathcal{K})$. The operator

$$R = (T \quad D) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K})$$

is a bicontraction by Corollary 1.4.2, and

$$R_- = \begin{pmatrix} T & D_- \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}_-, \mathcal{K})$$

is a bicontraction by Lemma 2.1.3. Hence $R_-^*\mathcal{K}_-$ is a maximal uniformly negative subspace of $\mathcal{H} \oplus \mathcal{D}_-$ by Theorem 1.3.6.

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fundamental decomposition, and write the graph representation of $R_-^*\mathcal{K}_-$ as

$$R_-^*\mathcal{K}_- = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} : f \in \mathcal{H}_- \oplus \mathcal{D}_- \right\},$$

where $K \in \mathbf{B}(|\mathcal{H}_- \oplus \mathcal{D}_-|, \mathcal{H}_+)$ and $\|K\| < 1$. Then

$$\tilde{\mathcal{H}}_- = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} : f \in \mathcal{H}_- \right\}$$

is a maximal uniformly negative subspace of $\mathcal{H}$. Set $\tilde{\mathcal{H}}_+ = \mathcal{H} \ominus \tilde{\mathcal{H}}_-$. Since $R_-^*|\mathcal{K}_-$ is one-to-one and $\tilde{\mathcal{H}}_- \subset R_-^*\mathcal{K}_-$, there is a subspace $\mathcal{L}_-$ of $\mathcal{K}_-$ which is mapped by R_-^* in a one-to-one way onto $\tilde{\mathcal{H}}_-$.

We show that $\mathcal{L}_-$ is closed. Let $f_1, f_2, \ldots \in \mathcal{L}_-$ and let $f_n \to f$ for some $f \in \mathcal{K}_-$. Then $R_-^* f_n \to R_-^* f$, and $R_-^* f \in \tilde{\mathcal{H}}_-$ because $R_-^* f_n \in \tilde{\mathcal{H}}_-$ for all n. By the definition of $\mathcal{L}_-$, we can write $R_-^* f = R_-^* g$ where $g \in \mathcal{L}_-$. Since $R_-^*|\mathcal{K}_-$ is one-to-one, $f = g \in \mathcal{L}_-$, and $\mathcal{L}_-$ is closed.

Next note that $T^*|\mathcal{L}_- = R_-^*|\mathcal{L}_-$, because

$$T^* f = \mathrm{Pr}_{\mathcal{H}}\, R_-^* f = R_-^* f$$

for any $f \in \mathcal{L}_-$ by the definition of $\mathcal{L}_-$. Since $R_-^*|\mathcal{L}_-$ is one-to-one and maps $\mathcal{L}_-$ onto $\tilde{\mathcal{H}}_-$, $T^*|\mathcal{L}_-$ is one-to-one and maps $\mathcal{L}_-$ onto $\tilde{\mathcal{H}}_-$. The preceding identity also shows that $D_-^* f = 0$ for any $f \in \mathcal{L}_-$, which implies that $D^*\mathcal{L}_- \perp \mathcal{D}_-$.

Define $\mathcal{L}_+ = \{v : v \in \mathcal{K}_- \text{ and } T^*v \perp T^*\mathcal{L}_-\}$. The subspace $T^*\mathcal{L}_- = \tilde{\mathcal{H}}_-$ is maximal uniformly negative, and exactly as in the proof of Theorem 2.1.1 we can show that $\mathcal{K}_- = \mathcal{L}_- \dot{+} \mathcal{L}_+$. By construction, $T^*\mathcal{L}_+ \perp T^*\mathcal{L}_-$. ■

The existence of bicontractive restrictions of the adjoint of any contraction operator follows.

THEOREM 2.1.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H},\mathcal{K})$ be a contraction. There exists a regular subspace $\hat{\mathcal{K}}$ of $\mathcal{K}$ such that $\hat{\mathcal{K}}^{\perp}$ is uniformly negative and $T^*|\hat{\mathcal{K}}$ is a bicontraction. If $\hat{\mathcal{K}}'$ is a second such subspace with $\hat{\mathcal{K}}' \supset \hat{\mathcal{K}}$, then $\hat{\mathcal{K}}' = \hat{\mathcal{K}}$.*

Proof. Choose a defect operator $D \in \mathbf{B}(\mathcal{D},\mathcal{K})$ for T^*. Let $\mathcal{D}_-$ be a maximal uniformly negative subspace of $\mathcal{D}$, and define $D_- = D|\mathcal{D}_-$. By Corollary 1.4.2, $(T \quad D)$ is a bicontraction. In particular, D is a contraction and $D_-\mathcal{D}_- = D\mathcal{D}_-$ is a closed uniformly negative subspace of $\mathcal{K}$. Choose a maximal uniformly negative subspace $\mathcal{K}_-$ of $\mathcal{K}$ which contains $D_-\mathcal{D}_-$.

Choose closed subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\mathcal{K}_-$ having the properties specified in Theorem 2.1.2. Define $\hat{\mathcal{K}} = \mathcal{L}_- \oplus \mathcal{K}_+$, where $\mathcal{K}_+ = \mathcal{K}_-^{\perp}$. We show that $\hat{\mathcal{K}}$ has the required properties. By construction, $\hat{\mathcal{K}}^{\perp}$ is uniformly negative.

For $f \in \hat{\mathcal{K}}$,

$$\begin{aligned} \langle f, f\rangle_{\hat{\mathcal{K}}} - \langle T^*f, T^*f\rangle_{\mathcal{H}} &= \langle (1-TT^*)f, f\rangle_{\mathcal{K}} \\ &= \langle D^*f, D^*f\rangle_{\mathcal{D}} \\ &\geq 0. \end{aligned}$$

The inequality is proved by contradiction in this way. If $\langle D^*f, D^*f\rangle_{\mathcal{K}} < 0$, then

$$D_-^*f = \mathrm{Pr}_{\mathcal{D}_-} D^*f \neq 0.$$

But $D_-^*\hat{\mathcal{K}} = D_-^*(\mathcal{L}_- \oplus \mathcal{K}_+) = \{0\}$ because $D^*\mathcal{L}_- \perp \mathcal{D}_-$ by the choice of $\mathcal{L}_-$ and $D_-\mathcal{D}_- \subset \mathcal{K}_-$ by the choice of $\mathcal{K}_-$. In view of this contradiction, we have shown that $T^*|\hat{\mathcal{K}}$ is a contraction from $\hat{\mathcal{K}}$ into $\mathcal{H}$.

The subspace $\mathcal{L}_-$ is maximal negative in $\hat{\mathcal{K}}$, and it is mapped by $T^*|\hat{\mathcal{K}}$ onto a maximal negative subspace of $\mathcal{H}$ by the choice of $\mathcal{L}_-$. Therefore $T^*|\hat{\mathcal{K}}$ is a bicontraction by Theorem 1.3.6. The last assertion is clear, since otherwise Theorem 1.3.6 would be violated. ■

The methods of this section yield a new characterization of bicontraction operators in the spirit of Theorem 1.3.6, but now with a condition on how adjoints map negative subspaces.

THEOREM 2.1.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H},\mathcal{K})$ be a contraction. Then T is a bicontraction if and only if T^* maps some maximal negative subspace of $\mathcal{K}$ in a one-to-one way into a negative subspace of $\mathcal{H}$.*

Proof. Necessity follows from Theorem 1.3.6.

Conversely, assume that T^* maps the maximal negative subspace $\mathcal{N}$ of $\mathcal{K}$ into a negative subspace of $\mathcal{H}$. Let $D \in \mathbf{B}(\mathcal{D},\mathcal{K})$ be a defect operator for T^*. Let

$\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ be fundamental decompositions. The fundamental decomposition of $\mathcal{K}$ can be chosen so that $D\mathcal{D}_- \subset \mathcal{K}_-$, because $D\mathcal{D}_-$ is uniformly negative by an argument in the proof of Theorem 2.1.4. Set $D_- = D|\mathcal{D}_-$. Since the range of D_- is contained in $\mathcal{K}_-$, $D_-^*|\mathcal{K}_+ = 0$.

As in the proof of Theorem 2.1.2, the operator

$$R_-^* = \begin{pmatrix} T & D_- \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}_-, \mathcal{K})$$

is a bicontraction. By Theorem 1.3.6, R_-^* maps $\mathcal{N}$ in a one-to-one way onto a maximal negative subspace of $\mathcal{H} \oplus \mathcal{D}_-$. Therefore

$$R_-^*\mathcal{N} = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} : f \in \mathcal{H}_- \oplus \mathcal{D}_- \right\}$$

for some contraction $K \in \mathbf{B}(|\mathcal{H}_- \oplus \mathcal{D}_-|, \mathcal{H}_+)$. Define

$$\mathcal{M} = \left\{ \begin{pmatrix} Kf \\ f \end{pmatrix} : f \in \mathcal{H}_- \right\}.$$

By its form, $\mathcal{M}$ is a maximal negative subspace of $\mathcal{H}$.

Since the range of $R_-^*|\mathcal{N}$ includes $\mathcal{M}$, there is a subspace $\mathcal{L}$ of $\mathcal{N}$ which is mapped by R_-^* onto $\mathcal{M}$. For $f \in \mathcal{L}$,

$$T^*f = \mathrm{Pr}_{\mathcal{H}}\, R_-^* f = R_-^* f,$$

because $R_-^* f \in \mathcal{M} \subset \mathcal{H}$. In other words, $T^*|\mathcal{L} = R_-^*|\mathcal{L}$. Now by assumption, T^* maps $\mathcal{N}$ in a one-to-one way into a negative subspace of $\mathcal{H}$. Since $\mathcal{L} \subset \mathcal{N}$ and $T^*\mathcal{L} = \mathcal{M}$ is maximal negative, we conclude that $\mathcal{L} = \mathcal{N}$. We therefore have $D_-^*|\mathcal{N} = 0$, because for $f \in \mathcal{N}$,

$$T^*f = R_-^*f = T^*f + D_-^*f$$

as elements of $\mathcal{H} \oplus \mathcal{D}_-$.

But $D_-^*|\mathcal{K}_+ = 0$, and so D_-^* annihilates $\mathcal{K}_+ + \mathcal{N}$. This is all of $\mathcal{K}$. In fact, $\mathcal{N}$ is maximal negative and has a representation as the set of vectors $Nu + u$, $u \in \mathcal{K}_-$, where $N \in \mathbf{B}(|\mathcal{K}_-|, \mathcal{K}_+)$ is a contraction. Therefore any u in $\mathcal{K}_-$ has a representation $u = -Nu + (Nu + u)$ with $-Nu \in \mathcal{K}_+$ and $Nu + u \in \mathcal{N}$. So $\mathcal{K}_- \subset \mathcal{K}_+ + \mathcal{N}$, and hence $\mathcal{K}_+ + \mathcal{N} = \mathcal{K}$. We have shown that $D_-^* = 0$.

Since $\mathrm{Pr}_{\mathcal{D}_-} D^* = D_-^* = 0$, the range of D^* is contained in $\mathcal{D}_+$. Therefore

$$\langle (1 - TT^*)f, f\rangle_{\mathcal{K}} = \langle D^*f, D^*f\rangle_{\mathcal{D}} \geq 0$$

for every f in $\mathcal{K}$, and T^* is a contraction. Since T is a contraction by assumption, T is a bicontraction. ■

2.2 Column Extensions

Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. A *column extension* of T is an operator of the form

$$C = \begin{pmatrix} T \\ \tilde{E}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}}),$$

where $\tilde{\mathcal{E}}$ is a Kreĭn space and $\tilde{E} \in \mathbf{B}(\tilde{\mathcal{E}}, \mathcal{H})$. In this section we assume that $\tilde{\mathcal{E}}$ is a Hilbert space. We characterize the contractive column extensions of T when T is a contraction. We also characterize the bicontractive column extensions of T when T is a bicontraction.

The motivating example is obtained from a defect operator $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ for T. If T is a contraction,$\tilde{\mathcal{D}}$ is a Hilbert space by Corollary 1.4.2. The operator

$$\begin{pmatrix} T \\ \tilde{D}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

is an isometry and hence a contractive column extension of T. If T is a bicontraction, then $\left(\begin{smallmatrix} T \\ \tilde{D}^* \end{smallmatrix}\right)$ is a bicontraction by Corollary 1.4.3.

THEOREM 2.2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a given operator with defect operator $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$. Let*

$$C = \begin{pmatrix} T \\ \tilde{E}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

be a column extension of T with $\tilde{\mathcal{E}}$ a Hilbert space.

(i) If T is a contraction, then C is a contraction if and only if $\tilde{E} = \tilde{D}\tilde{G}$ where $\tilde{G} \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a contraction.

(ii) If T is a bicontraction, then C is a bicontraction if and only if $\tilde{E} = \tilde{D}\tilde{G}$ where $\tilde{G} \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a contraction.

Proof. (i) Assume that T is a contraction. Then $\tilde{\mathcal{D}}$ is a Hilbert space by Corollary 1.4.2. If C is a contraction, then

$$0 \leq 1 - C^*C = 1 - T^*T - \tilde{E}\tilde{E}^* = \tilde{D}\tilde{D}^* - \tilde{E}\tilde{E}^*.$$

Hence

$$\|\tilde{E}^* f\|_{\tilde{\mathcal{E}}}^2 \leq \|\tilde{D}^* f\|_{\tilde{\mathcal{D}}}^2, \qquad f \in \mathcal{H}.$$

So $\tilde{E}^* = \tilde{G}^*\tilde{D}^*$ for a contraction $\tilde{G}^* \in \mathbf{B}(\tilde{\mathcal{D}}, \tilde{\mathcal{E}})$. Thus the condition is necessary. Sufficiency is verified by reversing these steps.

(ii) Assume that T is a bicontraction. Then T is a contraction, and the necessity of the condition follows from (i).

Conversely, let $\tilde{E} = \tilde{D}\tilde{G}$ where $\tilde{G} \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a contraction. By (i), C is a contraction. We prove that C is a bicontraction by showing that it maps any maximal negative subspace $\mathcal{N}$ of $\mathcal{H}$ onto a maximal negative subspace of $\mathcal{K} \oplus \tilde{\mathcal{E}}$. Since $\tilde{\mathcal{E}}$ is a Hilbert space, any maximal uniformly negative subspace $\mathcal{K}_-$ of $\mathcal{K}$ is a maximal uniformly negative subspace of $\mathcal{K} \oplus \tilde{\mathcal{E}}$. We have

$$C\mathcal{N} = \left\{ \begin{pmatrix} Tf \\ \tilde{E}^* f \end{pmatrix} : f \in \mathcal{N} \right\},$$

and so by Theorem 1.3.6,

$$\mathrm{Pr}_{\mathcal{K}_-} C\mathcal{N} = \mathrm{Pr}_{\mathcal{K}_-} T\mathcal{N} = \mathcal{K}_-$$

because T is a bicontraction. Hence $C\mathcal{N}$ is maximal negative in $\mathcal{K} \oplus \tilde{\mathcal{E}}$, and another application of Theorem 1.3.6 shows that C is a bicontraction. ■

Corollary 2.2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a bicontraction. Let*

$$C = \begin{pmatrix} T \\ \tilde{E}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

be a column extension of T with $\tilde{\mathcal{E}}$ a Hilbert space. If C is a contraction, then C is a bicontraction.

Proof. Choose a defect operator for T as in Theorem 2.2.1. If C is a contraction, then by part (i) of the theorem, $\tilde{E} = \tilde{D}\tilde{G}$ where $\tilde{G} \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a contraction. But then by part (ii) of the theorem, C is a bicontraction. ■

We note a norm estimate which will be useful in the commutant lifting theorem to be proved later.

Corollary 2.2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Let*

$$C = \begin{pmatrix} T \\ \tilde{E}^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

be a contractive column extension of T with $\tilde{\mathcal{E}}$ a Hilbert space. If norms are computed relative to fixed fundamental decompositions of $\mathcal{H}$ and $\mathcal{K}$ and the induced fundamental decomposition of $\mathcal{K} \oplus \tilde{\mathcal{E}}$, then

$$\|C\|^2 \leq 1 + 2\|T\|^2.$$

Proof. Choose a defect operator for T and factor $\tilde{E} = \tilde{D}\tilde{G}$ as in Theorem 2.2.1 (i). Since T is a contraction, $\tilde{\mathcal{D}}$ is a Hilbert space by Corollary 1.4.2. For any $f \in \mathcal{H}$,

$$\left\|\tilde{D}^* f\right\|^2 = \left\langle \tilde{D}^* f, \tilde{D}^* f \right\rangle_{\tilde{\mathcal{D}}} = \left\langle \tilde{D}\tilde{D}^* f, f \right\rangle_{\mathcal{H}} = \langle (1 - T^*T)f, f\rangle_{\mathcal{H}}$$
$$\leq \left[1 + \|T\|^2\right] \|f\|^2.$$

Since $\tilde{G}$ is a Hilbert space contraction, for any $f \in \mathcal{H}$,

$$\|Cf\|^2 = \|Tf\|^2 + \|\tilde{E}^* f\|^2 = \|Tf\|^2 + \|\tilde{G}^*\tilde{D}^* f\|^2 \leq \left[1 + 2\|T\|^2\right] \|f\|^2,$$

as required. ■

2.3 Row Extensions

Let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{H}$ and $\mathcal{K}$ Kreĭn spaces. By a *row extension* of T we mean an operator of the form

$$R = (\,T \quad E\,) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K}),$$

where $\mathcal{E}$ is a Kreĭn space and $E \in \mathbf{B}(\mathcal{E}, \mathcal{K})$. If T is a contraction, then there exist bicontractive row extensions of T. In fact, if $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ is a defect operator for T^*, the operator

$$R = (T \quad D) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K})$$

is a bicontractive row extension of T by Corollary 1.4.2. We characterize all such extensions. It is convenient to begin with the special case of an extension by a Hilbert space.

THEOREM 2.3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ a bicontraction, and $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ a defect operator for T^*. Let*

$$R = (\,T \quad E\,) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$$

be a row extension of T with $\mathcal{E}$ a Hilbert space. Then R is a bicontraction if and only if $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction.

Note that $\mathcal{D}$ is a Hilbert space by the hypothesis that T is a bicontraction.

Proof. If R is a bicontraction, then the operators

$$R^* = \begin{pmatrix} T^* \\ E^* \end{pmatrix} \in \mathbf{B}(\mathcal{K}, \mathcal{H} \oplus \mathcal{E})$$

and T^* are contractions. By Theorem 2.2.1 (i) applied to T^*, $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction.

Conversely, assume that $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction. By Theorem 2.2.1 (ii) applied to T^*, the operator R^* is a bicontraction, hence R is a bicontraction. ■

A technical result is needed. It describes how bicontractive row extensions act with respect to the decompositions obtained in Theorem 2.1.2. This information will be used to determine the general form of any bicontractive row extension of a contraction.

THEOREM 2.3.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Let $\mathcal{K}_-$ be a maximal uniformly negative subspace of $\mathcal{K}$, and choose closed subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\mathcal{K}_-$ such that*

$$\mathcal{K}_- = \mathcal{L}_- \dot{+} \mathcal{L}_+,$$

T^* maps $\mathcal{L}_-$ in a one-to-one way onto a maximal uniformly negative subspace of $\mathcal{H}$, and $T^*\mathcal{L}_+ \perp T^*\mathcal{L}_-$. Let

$$R = \begin{pmatrix} T & E \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$$

be any bicontractive row extension of T, where $\mathcal{E}$ a Kreĭn space and $E \in \mathbf{B}(\mathcal{E}, \mathcal{K})$. Then E^* maps $\mathcal{L}_+$ in a one-to-one way onto a maximal uniformly negative subspace of $\mathcal{E}$.

The existence of subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ with the required properties follows from Theorem 2.1.2.

Proof. Choose fundamental decompositions $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_- = T^*\mathcal{L}_-$ and $\mathcal{H}_+ = (T^*\mathcal{L}_-)^\perp$. Let $\mathcal{K}_+ = \mathcal{K}_-^\perp$.

Since R^* is a bicontraction, $R^*\mathcal{K}_-$ is a maximal uniformly negative subspace of $\mathcal{H} \oplus \mathcal{E}$. If $b \in \mathcal{L}_+$, then $T^*b \in \mathcal{H}_+$ and for some positive number ϵ which does not depend on b,

$$\begin{aligned} \langle E^*b, E^*b\rangle_{\mathcal{E}} &\le \langle T^*b, T^*b\rangle_{\mathcal{H}} + \langle E^*b, E^*b\rangle_{\mathcal{E}} = \langle R^*b, R^*b\rangle_{\mathcal{H}\oplus\mathcal{E}} \\ &\le -\epsilon\|R^*b\|^2 = -\epsilon\|T^*b\|^2 - \epsilon\|E^*b\|^2 \\ &\le -\epsilon\|E^*b\|^2, \end{aligned}$$

where norms are computed in a suitable way. It follows that E^* maps $\mathcal{L}_+$ in a one-to-one way onto a uniformly negative subspace of $\mathcal{E}$.

We show that the projection of $E^*\mathcal{L}_+$ into $\mathcal{E}_-$ is all of $\mathcal{E}_-$. Since $R^*\mathcal{K}_-$ is maximal uniformly negative in $\mathcal{H} \oplus \mathcal{E}$,

$$\mathrm{Pr}_{\mathcal{H}_- \oplus \mathcal{E}_-} R^*\mathcal{K}_- = \mathcal{H}_- \oplus \mathcal{E}_-.$$

Given $e \in \mathcal{E}_-$, we may therefore choose $b \in \mathcal{K}_-$ such that

$$\mathrm{Pr}_{\mathcal{H}_- \oplus \mathcal{E}_-} R^*b = e.$$

Applying the projection onto $\mathcal{E}_-$, we obtain

$$\mathrm{Pr}_{\mathcal{E}_-} E^*b = \mathrm{Pr}_{\mathcal{E}_-} R^*b = e.$$

Applying instead the projection onto $\mathcal{H}_-$, we see that

$$\mathrm{Pr}_{\mathcal{H}_-} T^*b = \mathrm{Pr}_{\mathcal{H}_-} R^*b = 0,$$

that is, $T^*b \in \mathcal{H}_+$. Therefore if $b = u + v$, $u \in \mathcal{L}_-$, $v \in \mathcal{L}_+$, we have

$$T^*u = T^*b - T^*v \in \mathcal{H}_- \cap \mathcal{H}_+ = \{0\}.$$

Since $T^*|\mathcal{L}_-$ is one-to-one, $u = 0$ and $b = v \in \mathcal{L}_+$. We have shown that the projection of $E^*\mathcal{L}_+$ into $\mathcal{E}_-$ is all of $\mathcal{E}_-$, and so $E^*\mathcal{L}_+$ is maximal uniformly negative by the graph representation of negative subspaces. ■

The main result of this section follows.

THEOREM 2.3.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ a contraction, and $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ a defect operator for T^*. Let $E \in \mathbf{B}(\mathcal{E}, \mathcal{K})$, where $\mathcal{E}$ is a Kreĭn space. Then the operator*

$$R = \begin{pmatrix} T & E \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$$

is a bicontraction if and only if $E = DG$, where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a bicontraction.

Proof of Sufficiency. Assume that $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a bicontraction. Since $1 - GG^* \geq 0$,

$$1 - RR^* = 1 - TT^* - DGG^*D^* = D(1 - GG^*)D^* \geq 0,$$

and R^* is a contraction. By Theorem 1.2.4, T has a Julia operator of the form

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

Thus

$$\begin{aligned} 1 - R^*R &= 1 - \begin{pmatrix} T^* \\ G^*D^* \end{pmatrix} \begin{pmatrix} T & DG \end{pmatrix} \\ &= \begin{pmatrix} 1 - T^*T & -T^*DG \\ -G^*D^*T & 1 - G^*D^*DG \end{pmatrix} \\ &= \begin{pmatrix} \tilde{D}\tilde{D}^* & \tilde{D}LG \\ G^*L^*\tilde{D}^* & G^*L^*LG \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 - G^*D^*DG - G^*L^*LG \end{pmatrix} \\ &= \begin{pmatrix} \tilde{D} \\ G^*L^* \end{pmatrix} \begin{pmatrix} \tilde{D}^* & LG \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 - G^*G \end{pmatrix} \\ &\geq 0, \end{aligned}$$

because $\tilde{\mathcal{D}}$ is a Hilbert space by Theorem 1.4.1 and $1 - G^*G \geq 0$. Thus R is a bicontraction.

Proof of Necessity. Assume now that R is a bicontraction. Choose fundamental decompositions

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \quad \mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-, \quad \mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-.$$

Define $E_- = E|\mathcal{E}_- \in \mathbf{B}(\mathcal{E}_-, \mathcal{K})$. The operator

$$R_- = \begin{pmatrix} T & E_- \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}_-, \mathcal{K})$$

is a bicontraction by Lemma 2.1.3. Choose closed subspaces $\mathcal{L}_-$ and $\mathcal{L}_+$ of $\mathcal{K}_-$ as in Theorem 2.1.2. In particular, $\mathcal{K}_- = \mathcal{L}_- \dot{+} \mathcal{L}_+$.

By Theorem 2.3.2, E^* maps $\mathcal{L}_+$ in a one-to-one way onto a maximal uniformly negative subspace of $\mathcal{E}$. The projection of $\mathcal{E}$ onto $\mathcal{E}_-$ maps any maximal negative subspace of $\mathcal{E}$ onto $\mathcal{E}_-$ by the graph representation. Therefore E_-^* maps $\mathcal{L}_+$ in a one-to-one way onto $\mathcal{E}_-$.

We show that

$$\mathcal{K} = \mathcal{L}_+ \dot{+} \ker E_-^*.$$

If $k \in \mathcal{K}$, then $E_-^* k \in \mathcal{E}_-$ and there exists $u \in \mathcal{L}_+$ such that $E_-^* k = E_-^* u$. Then $k = u + v$ where $u \in \mathcal{L}_+$ and $v = k - u \in \ker E_-^*$. If $u \in \mathcal{L}_+ \cap \ker E_-^*$, then $u = 0$ because $E_-^*|\mathcal{L}_+$ is one-to-one and u is an element of $\mathcal{L}_+$, which is annihilated by E_-^*. Thus $\mathcal{K}$ is the direct sum of $\mathcal{L}_+$ and $\ker E_-^*$.

We next define a bicontractive operator $G_- \in \mathbf{B}(\mathcal{E}_-, \mathcal{D})$ such that $E_- = DG_-$. To this end first define a mapping X on the dense set $\operatorname{ran} D^*$ in $\mathcal{D}$ to $\mathcal{E}_-$ by

$$XD^*f = E_-^*f, \qquad f \in \mathcal{K}.$$

The mapping is well defined. For suppose $f \in \mathcal{K}$ and $D^*f = 0$. Write $f = u + v$, $u \in \mathcal{L}_+$, $v \in \ker E_-^*$. Then $D^*u = -D^*v$. By Corollary 1.4.2, $(T \quad D)$ is a bicontractive row extension of T. Applying Theorem 2.3.2 to $(T \quad D)$, we obtain

$$\langle D^*u, D^*u\rangle_{\mathcal{D}} \le 0,$$

with equality only for $u = 0$. Since R_- is a bicontraction and $E_-^*v = 0$,

$$\begin{aligned} 0 &\le \big\langle (1 - R_-R_-^*)v, v\big\rangle_{\mathcal{K}} = \big\langle (1 - TT^* - E_-E_-^*)v, v\big\rangle_{\mathcal{K}} \\ &= \langle D^*v, D^*v\rangle_{\mathcal{D}} = \langle D^*u, D^*u\rangle_{\mathcal{D}} \le 0. \end{aligned}$$

Therefore equality holds throughout and u = 0. Thus $E_-^*f = E_-^*v = 0$, and X is well defined.

As far as it is defined, X satisfies

$$\langle Xg, Xg\rangle_{\mathcal{E}_-} \le \langle g, g\rangle_{\mathcal{D}}, \qquad g \in \operatorname{dom} X.$$

For if $g = D^*f$, $f \in \mathcal{K}$,

$$\begin{aligned} 0 &\le \big\langle (1 - R_-R_-^*)f, f\big\rangle_{\mathcal{K}} = \big\langle (1 - TT^* - E_-E_-^*)f, f\big\rangle_{\mathcal{K}} \\ &= \langle D^*f, D^*f\rangle_{\mathcal{D}} - \big\langle E_-^*f, E_-^*f\big\rangle_{\mathcal{E}_-} = \langle g, g\rangle_{\mathcal{D}} - \langle Xg, Xg\rangle_{\mathcal{E}_-}. \end{aligned}$$

Theorem 2.3.2 applied to $(T \quad D)$ implies that $D^*\mathcal{L}_+$ is a maximal uniformly negative subspace of $\mathcal{D}$. By the definition of X, $D^*\mathcal{L}_+ \subset \operatorname{dom} X$ and X maps $D^*\mathcal{L}_+$ onto the maximal uniformly negative subspace $E^*\mathcal{L}_+$ of $\mathcal{E}$. Therefore by Theorem 1.4.4, X has an extension by continuity to a bicontraction which we denote G_-^*. This completes the construction of a bicontractive operator $G_- \in \mathbf{B}(\mathcal{E}_-, \mathcal{D})$ such that $E_- = DG_-$.

Let $D_{R_-} \in \mathbf{B}(\mathcal{D}_{R_-}, \mathcal{K})$ and $D_{G_-} \in \mathbf{B}(\mathcal{D}_{G_-}, \mathcal{D})$ be defect operators for R_-^* and G_-^*. Then

$$\begin{aligned} D_{R_-}D_{R_-}^* &= 1 - R_-R_-^* = 1 - TT^* - E_-E_-^* \\ &= DD^* - DG_-G_-^*D^* = D(1 - G_-G_-^*)D^* \\ &= DD_{G_-}D_{G_-}^*D^*. \end{aligned}$$

Therefore there is a unitary operator $V \in \mathbf{B}(\mathcal{D}_{R_-}, \mathcal{D}_{G_-})$ such that

$$D_{R_-} = DD_{G_-}V.$$

Write $R = (T \quad E_- \quad E_+)$ where $E_+ = E|\mathcal{E}_+ \in \mathbf{B}(\mathcal{E}_+, \mathcal{K})$. The 1×3 row matrix acts on $\mathcal{H} \oplus \mathcal{E}_- \oplus \mathcal{E}_+$ to $\mathcal{K}$. Its restriction to $\mathcal{H} \oplus \mathcal{E}_-$ is the bicontraction R_-, and so by Theorem 2.3.1, $E_+ = D_{R_-} G_+$, where $G_+ \in \mathbf{B}(\mathcal{E}_+, \mathcal{D}_{R_-})$ is a contraction. Finally,

$$R = (\, T \quad DG_- \quad D_{R_-} G_+ \,) = (\, T \quad DG_- \quad DD_{G_-} V G_+ \,) = (\, T \quad DG \,),$$

where $G = \big(G_- \quad D_{G_-} V G_+\big) \in \mathbf{B}(\mathcal{E}_- \oplus \mathcal{E}_+, \mathcal{D}) = \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a bicontraction by Theorem 2.3.1. ■

A similar theorem on contractive row extensions follows as a consequence.

THEOREM 2.3.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ a contraction, and $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ a defect operator for T^*. Let $\mathcal{E}$ be a Kreĭn space, and let $E \in \mathbf{B}(\mathcal{E}, \mathcal{K})$. Then the operator*

$$R = (\, T \quad E \,) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$$

is a contraction if and only if $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction.

Proof of Sufficiency. If $E = DG$ where $G \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction, we show that $1 - R^* R \geq 0$ exactly as in the sufficiency part of the proof of Theorem 2.3.3.

Proof of Necessity. Assume that R is a contraction, and choose a defect operator $D_R \in \mathbf{B}(\mathcal{D}_R, \mathcal{K})$ for R^*. By Corollary 1.4.2, $\hat{R} = (R \quad D_R)$ is a bicontraction in $\mathbf{B}((\mathcal{H} \oplus \mathcal{E}) \oplus \mathcal{D}_R, \mathcal{K})$. In another view,

$$\hat{R} = (\, T \quad (\, E \quad D_R \,)\,) \in \mathbf{B}(\mathcal{H} \oplus (\mathcal{E} \oplus \mathcal{D}_R), \mathcal{K}),$$

is a bicontractive row extension of T, and we may apply the necessity part of Theorem 2.3.3. It follows that

$$(\, E \quad D_R \,) = D\Gamma,$$

where $\Gamma \in \mathbf{B}(\mathcal{E} \oplus \mathcal{D}_R, \mathcal{D})$ is a bicontraction. Thus

$$R = \hat{R}|(\mathcal{H} \oplus \mathcal{E}) = (\, T \quad DG \,)$$

where $G = \Gamma|\mathcal{E}$ is a contraction. ■

2.4 Two-by-Two Matrix Completions

Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. We label all contractive two-by-two matrices with T in the upper left entry. The extension space for the range is required to be a Hilbert space.

Theorem 2.4.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction with Julia operator*

$$\begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

Let $\mathcal{E}$ be a Kreĭn space, and let $\tilde{\mathcal{E}}$ be a Hilbert space. If

$$Q = \begin{pmatrix} T & Q_1 \\ Q_2 & Q_3 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

is a contraction, then

$$Q = \begin{pmatrix} T & DG_1 \\ G_2^* \tilde{D}^* & G_2^* L G_1 + \tilde{D}_{G_2} G_3 \tilde{D}_{G_1}^* \end{pmatrix},$$

where

(i) $G_1 \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a Kreĭn space contraction with defect operator $\tilde{D}_{G_1} \in \mathbf{B}(\tilde{\mathcal{D}}_{G_1}, \mathcal{E})$,
(ii) $G_2 \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a Hilbert space contraction with defect operator $\tilde{D}_{G_2} \in \mathbf{B}(\tilde{\mathcal{D}}_{G_2}, \tilde{\mathcal{E}})$, and
(iii) $G_3 \in \mathbf{B}(\tilde{\mathcal{D}}_{G_1}, \tilde{\mathcal{D}}_{G_2})$ is a Hilbert space contraction.

Conversely, every operator $Q \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$ of this form is a contraction.

Proof. Assume that Q is a contraction. Then so is its restriction

$$\begin{pmatrix} T \\ Q_2 \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}}).$$

Since T is a contraction and $\tilde{\mathcal{E}}$ is a Hilbert space, we can apply Theorem 2.2.1 (i) to write $Q_2 = G_2^* \tilde{D}^*$ where $G_2 \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a Hilbert space contraction. Let $\tilde{D}_{G_2} \in \mathbf{B}(\tilde{\mathcal{D}}_{G_2}, \tilde{\mathcal{E}})$ be any defect operator for G_2.

Since $\tilde{\mathcal{E}}$ is a Hilbert space, $(T \quad Q_1) = \Pr_{\mathcal{K}} Q \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$ is a contraction. By Theorem 2.3.4, $Q_1 = DG_1$ where $G_1 \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a Kreĭn space contraction. Let $\tilde{D}_{G_1} \in \mathbf{B}(\tilde{\mathcal{D}}_{G_1}, \mathcal{E})$ be any defect operator for G_1. Thus far we have

$$Q = \begin{pmatrix} T & DG_1 \\ G_2^* \tilde{D}^* & Q_3 \end{pmatrix}.$$

After a short calculation, the inequality $1 - Q^*Q \geq 0$ yields

$$0 \leq \begin{pmatrix} 1 - T^*T - \tilde{D}G_2G_2^*\tilde{D}^* & -T^*DG_1 - \tilde{D}G_2Q_3 \\ -G_1^*D^*T - Q_3^*G_2^*\tilde{D}^* & 1 - G_1^*D^*DG_1 - Q_3^*Q_3 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{D}\tilde{D}^* - \tilde{D}G_2G_2^*\tilde{D}^* & \tilde{D}LG_1 - \tilde{D}G_2Q_3 \\ G_1^*L^*\tilde{D}^* - Q_3^*G_2^*\tilde{D}^* & 1 - G_1^*G_1 + G_1^*(1 - D^*D)G_1 - Q_3^*Q_3 \end{pmatrix} .$$

$$= \begin{pmatrix} \tilde{D} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - G_2G_2^* & LG_1 - G_2Q_3 \\ G_1^*L^* - Q_3^*G_2^* & \tilde{D}_{G_1}\tilde{D}_{G_1}^* + G_1^*L^*LG_1 - Q_3^*Q_3 \end{pmatrix} \begin{pmatrix} \tilde{D}^* & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\tilde{D}$ is one-to-one, the range of $\tilde{D}^*$ is dense in $\tilde{\mathcal{D}}$. Therefore

$$\begin{pmatrix} 1 - G_2G_2^* & LG_1 - G_2Q_3 \\ G_1^*L^* - Q_3^*G_2^* & \tilde{D}_{G_1}\tilde{D}_{G_1}^* + G_1^*L^*LG_1 - Q_3^*Q_3 \end{pmatrix} \geq 0$$

as operators in $\mathbf{B}(\tilde{\mathcal{D}} \oplus \mathcal{E})$. A straightforward calculation brings this to the form $A^*A \geq B^*B$, where

$$A = \begin{pmatrix} 1 & LG_1 \\ 0 & \tilde{D}_{G_1}^* \end{pmatrix} \in \mathbf{B}(\tilde{\mathcal{D}} \oplus \mathcal{E}, \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}_{G_1})$$

and

$$B = \begin{pmatrix} G_2^* & Q_3 \end{pmatrix} \in \mathbf{B}(\tilde{\mathcal{D}} \oplus \mathcal{E}, \tilde{\mathcal{E}}).$$

Since $\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}_{G_1}$ and $\tilde{\mathcal{E}}$ are Hilbert spaces, there is a Hilbert space contraction

$$K = \begin{pmatrix} K_0 & K_1 \end{pmatrix} \in \mathbf{B}(\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}_{G_1}, \tilde{\mathcal{E}})$$

such that $B = KA$, that is,

$$\begin{pmatrix} G_2^* & Q_3 \end{pmatrix} = \begin{pmatrix} K_0 & K_1 \end{pmatrix} \begin{pmatrix} 1 & LG_1 \\ 0 & \tilde{D}_{G_1}^* \end{pmatrix}$$
$$= \begin{pmatrix} K_0 & K_0LG_1 + K_1\tilde{D}_{G_1}^* \end{pmatrix}.$$

Thus $K_0 = G_2^*$ and $Q_3 = K_0LG_1 + K_1\tilde{D}_{G_1}^*$. Applying Theorem 2.3.1 in the Hilbert space case, we obtain

$$\begin{pmatrix} K_0 & K_1 \end{pmatrix} = \begin{pmatrix} G_2^* & \tilde{D}_{G_2}G_3 \end{pmatrix}$$

where $G_3 \in \mathbf{B}(\tilde{\mathcal{D}}_{G_1}, \tilde{\mathcal{D}}_{G_2})$ is a Hilbert space contraction. Therefore

$$Q_3 = K_0LG_1 + K_1\tilde{D}_{G_1}^* = G_2^*LG_1 + \tilde{D}_{G_2}G_3\tilde{D}_{G_1}^*.$$

We have shown that Q has the required form if Q is a contraction. In the other direction, if Q has the stated form, we simply reverse the steps to show that $1 - Q^*Q \geq 0$. ■

A similar result characterizes bicontractive two-by-two matrices with T in the upper left entry when T is a contraction and the extension space for the range is a Hilbert space.

THEOREM 2.4.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction with Julia operator*

$$\begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

Let $\mathcal{E}$ be a Kreĭn space, and let $\tilde{\mathcal{E}}$ be a Hilbert space. If

$$Q = \begin{pmatrix} T & Q_1 \\ Q_2 & Q_3 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

is a bicontraction, then Q can be represented as in Theorem 2.4.1 with G_1 a bicontraction. Conversely, every $Q \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$ of the form described in Theorem 2.4.1 with G_1 a bicontraction is a bicontraction.

Proof. Assume first that Q has the form described in Theorem 2.4.1 with G_1 a bicontraction. By Theorem 2.3.3,

$$(T \quad Q_1) = (T \quad DG_1) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$$

is a bicontraction. Now view

$$Q = \begin{pmatrix} (T \quad Q_1) \\ (Q_2 \quad Q_3) \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

as a contractive column extension of the bicontraction $(T \quad Q_1)$ with extension space $\tilde{\mathcal{E}}$ a Hilbert space. By Corollary 2.2.2, Q is a bicontraction.

Conversely, let Q be a bicontraction. If $\mathcal{N}$ is a maximal negative subspace of $\mathcal{H} \oplus \mathcal{E}$, then $Q\mathcal{N}$ is a maximal negative subspace of $\mathcal{K} \oplus \tilde{\mathcal{E}}$. Since $\tilde{\mathcal{E}}$ is a Hilbert space, $\mathrm{Pr}_{\mathcal{K}}\, Q\mathcal{N} = (T \quad Q_1)\mathcal{N}$ is a maximal negative subspace of $\mathcal{K}$. As in the proof of Theorem 2.4.1, $(T \quad Q_1) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K})$ is a contraction, and hence it is a bicontraction by Theorem 1.3.6. By Theorem 2.3.3, $Q_1 = DG_1'$ where $G_1' \in \mathbf{B}(\mathcal{E}, \tilde{\mathcal{D}})$ is a bicontraction. Since D has zero kernel, $G_1 = G_1'$ is a bicontraction. ■

COROLLARY 2.4.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a bicontraction. Let $\mathcal{E}$ be a Kreĭn space, and let $\tilde{\mathcal{E}}$ be a Hilbert space. If*

$$Q = \begin{pmatrix} T & Q_1 \\ Q_2 & Q_3 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

is a contraction, then $\mathcal{E}$ is also a Hilbert space and Q is a bicontraction.

Proof. Choose a Julia operator

$$\begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

for T, and represent Q as in Theorem 2.4.1. Since T is a bicontraction,$\mathcal{D}$ is a Hilbert space by Corollary 1.4.3. Since $G_1 \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a contraction, $\mathcal{E}$ is a Hilbert space and G_1 is a bicontraction. So Q is a bicontraction by Theorem 2.4.2. ■

We recast the results of this section in a form that is convenient for application to the commutant lifting problem.

THEOREM 2.4.4. *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{E}$ be Kreĭn spaces, $\tilde{\mathcal{E}}$ a Hilbert space. Assume that*

$$(C_{11} \quad C_{12}) \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K}) \qquad \text{and} \qquad \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

are contraction operators. There exists an operator $C_{22} \in \mathbf{B}(\mathcal{E}, \tilde{\mathcal{E}})$ such that

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{K} \oplus \tilde{\mathcal{E}})$$

is a contraction. If $(C_{11} \quad C_{12})$ is a bicontraction, C_{22} may be chosen so that C is a bicontraction.

Proof. Since $(C_{11} \quad C_{12})$ is a contraction, so is its restriction C_{11}. Write $T = C_{11}$, and let

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}})$$

be a Julia operator for T. By Theorem 2.3.4, $C_{12} = DG_1$, where $G_1 \in \mathbf{B}(\mathcal{E}, \mathcal{D})$ is a Kreĭn space contraction. By Theorem 2.2.1 (i), $C_{21} = G_2^* \tilde{D}^*$, where $G_2 \in \mathbf{B}(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a Hilbert space contraction. We may now invoke Theorem 2.4.1 with any choice of G_3, such as, $G_3 = 0$ (which is a contraction since the underlying spaces are Hilbert spaces), to produce an operator C_{22} such that C is a contraction.

If $(C_{11} \quad C_{12})$ is a bicontraction, then in the preceding argument we can use Theorem 2.3.3 in place of Theorem 2.3.4 to choose G_1 to be a bicontraction. In place of Theorem 2.4.1, we may use Theorem 2.4.2 to produce C_{22} such that C is a bicontraction. ■

To conclude this section, we use Theorem 2.4.1 to show that Julia operators for contractions, or operators whose adjoints are contractions, are essentially unique. See Appendix B for a more general result.

THEOREM 2.4.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H},\mathcal{K})$ have two Julia operators*

$$\begin{pmatrix} T & D_j \\ \tilde{D}_j^* & L_j \end{pmatrix} \in \mathbf{B}(\mathcal{H}\oplus\mathcal{D}_j, \mathcal{K}\oplus\tilde{\mathcal{D}}_j), \qquad j = 1,2.$$

If either T or T^* is a contraction, then there exist unitary operators $V \in \mathbf{B}(\mathcal{D}_2,\mathcal{D}_1)$ and $\tilde{V} \in \mathbf{B}(\tilde{\mathcal{D}}_2,\tilde{\mathcal{D}}_1)$ such that

$$\begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}^* \end{pmatrix}\begin{pmatrix} T & D_1 \\ \tilde{D}_1^* & L_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}.$$

Proof. Since the adjoint of a Julia operator for T is a Julia operator for T^*, it is sufficient to give the proof in the case in which T is a contraction. In this case, Theorem 2.4.1 is applicable and yields contractions $V \in \mathbf{B}(\mathcal{D}_2,\mathcal{D}_1)$ and $\tilde{V} \in \mathbf{B}(\tilde{\mathcal{D}}_2,\tilde{\mathcal{D}}_1)$ such that

$$\begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}^* \end{pmatrix}\begin{pmatrix} T & D_1 \\ \tilde{D}_1^* & L_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}_{\tilde{V}} G \tilde{D}_V^* \end{pmatrix},$$

where $\tilde{D}_V \in \mathbf{B}(\tilde{\mathcal{D}}_V,\mathcal{D}_2)$ and $\tilde{D}_{\tilde{V}} \in \mathbf{B}(\tilde{\mathcal{D}}_{\tilde{V}},\tilde{\mathcal{D}}_2)$ are defect operators for V and $\tilde{V}$ and $G \in \mathbf{B}(\tilde{\mathcal{D}}_V,\tilde{\mathcal{D}}_{\tilde{V}})$ is a contraction. In particular, $D_2 = D_1 V$ and

$$D_1 D_1^* = 1 - TT^* = D_2 D_2^* = D_1 V V^* D_1^*.$$

Since D_1 has zero kernel, VV^* is the identity on $\mathcal{D}_1$. Since D_2 has zero kernel, V is a unitary operator. In a similar way, $\tilde{D}_2 = \tilde{D}_1\tilde{V}$ and

$$\tilde{D}_1 \tilde{D}_1^* = 1 - T^*T = \tilde{D}_2 \tilde{D}_2^* = \tilde{D}_1 \tilde{V}\tilde{V}^* \tilde{D}_1^*.$$

Since $\tilde{D}_1$ and $\tilde{D}_2$ have zero kernels, $\tilde{V}$ is a unitary operator. In particular, $\tilde{\mathcal{D}}_V$ and $\tilde{\mathcal{D}}_{\tilde{V}}$ each contain only the zero vector, and $\tilde{D}_{\tilde{V}} G \tilde{D}_V^* = 0$. ■

Notes on Chapter 2

The methods of Chapter 2 follow Dritschel [**30**]. Theorems 2.1.1 and 2.1.4 generalize a result of Kuzhel' [**48**], but the treatment here is different.

Theorems 2.3.3 and 2.3.4 first appear in Dritschel [**30**] in the present generality but were derived in special cases in earlier works. In fact, for most of the extension theorems in §2.2–2.4, there have been versions worked out by Arsene, Constantinescu, and Ghéondea [**8**]. The present work was done independently of the recent papers by Constantinescu and Ghéondea [**22**]. These papers contain a great amount of detailed information on related extension problems which we do not treat. The problems are generally of a more precise nature and require stronger hypotheses. While there is overlap with the present work in some cases, there is no inclusion of one theory in the other. In the bicontractive case, forerunners of the results in §2.2–2.4 also appear in Dritschel [**29**].

The method of proof of Theorem 2.4.1 is adapted from the account of the Hilbert space result in Pták and Vrbová [**56**]. In [**17**], de Branges uses complementation theory to prove a version of Theorem 2.4.1 when both extension spaces are Hilbert spaces. Two-by-two extension theorems in the Hilbert space case are due to Arsene and Ghéondea [**7**], Davis [**24**], Davis, Kahan, and Weinberger [**26**], and Shmul'yan and Yanovskaya [**65**]. Related papers are Davis [**25**] and Parrott [**54**].

The question of uniqueness of Julia operators is partially settled in Theorem 2.4.5 and Appendix B. We feel that uniqueness will fail in general, but we do not know an example.

Chapter 3: Commutant Lifting of Contraction Operators

3.1 Dilation Theory

The dilation properties of Kreĭn space operators are similar to the Hilbert space case. The main problem is to finesse the fact that a densely defined Kreĭn space isometry may fail to have a continuous extension: in some cases, this causes difficulty with uniqueness. It turns out that minimal isometric dilations of contractions are unique and have the familiar properties of the Hilbert space case. The same is true of minimal unitary dilations of bicontractions. For general operators, minimal isometric and minimal unitary dilations always exist with the properties of the Hilbert space case, but we cannot assert uniqueness nor that all minimal isometric and unitary dilations have desirable additional properties.

Definition 3.1.1. *Let $T \in \mathbf{B}(\mathcal{H})$, where $\mathcal{H}$ is a Kreĭn space. An* **isometric dilation** *of T is an isometry $U \in \mathbf{B}(\tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is a Kreĭn space containing $\mathcal{H}$ isometrically as a regular subspace, such that*

$$T^n = \mathrm{Pr}_{\mathcal{H}}\, U^n|\mathcal{H}, \qquad n = 1, 2, 3, \ldots.$$

An isometric dilation $U \in \mathbf{B}(\tilde{\mathcal{H}})$ of T is **minimal** *if*

$$\bigvee_{n=0}^{\infty} U^n \mathcal{H} = \tilde{\mathcal{H}}.$$

Minimal isometric dilations always exist.

Theorem 3.1.2. *Let $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{H}$ a Kreĭn space, and let $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ be a defect operator for T. Define $\tilde{\mathcal{H}} = \mathcal{H} \oplus \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}} \oplus \cdots$ using fixed choices of fundamental decompositions for $\mathcal{H}$ and $\tilde{\mathcal{D}}$. Then the matrix*

$$U = \begin{pmatrix} T & 0 & 0 & 0 & \cdots \\ \tilde{D}^* & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix}$$

acts as an everywhere defined and continuous operator on $\tilde{\mathcal{H}}$, and U is a minimal isometric dilation of T.

Proof. The infinite matrix determines an element U of $\mathbf{B}(\mathcal{H})$ because the shift operator is everywhere defined and continuous on $\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}} \oplus \cdots$. If $f = (f_0, f_1, \dots)^t$ is in $\tilde{\mathcal{H}}$, then

$$\begin{aligned}\langle Uf, Uf\rangle_{\tilde{\mathcal{H}}} &= \left\langle (T^*T + \tilde{D}\tilde{D}^*)f_0, f_0 \right\rangle_{\mathcal{H}} + \langle f_1, f_1\rangle_{\tilde{\mathcal{D}}} + \langle f_2, f_2\rangle_{\tilde{\mathcal{D}}} + \cdots \\ &= \langle f, f\rangle_{\tilde{\mathcal{H}}}\end{aligned}$$

because $T^*T + \tilde{D}\tilde{D}^* = 1$. Thus U is isometric. An inductive argument shows that for any element of $\tilde{\mathcal{H}}$ of the form $f = (f_0, 0, 0, \dots)^t$ and any $n = 1, 2, 3, \dots$,

$$U^n f = \begin{pmatrix} T^n f_0 \\ \tilde{D}^* T^{n-1} f_0 \\ \tilde{D}^* T^{n-2} f_0 \\ \vdots \\ \tilde{D}^* f_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Therefore $T^n f_0 = \mathrm{Pr}_{\mathcal{H}}\, U^n f_0$ for any $f_0 \in \mathcal{H}$. The same identity implies that for every positive integer n,

$$\bigvee_{j=0}^{n} U^j \mathcal{H} = \mathcal{H} \oplus \tilde{\mathcal{D}} \oplus \cdots \oplus \tilde{\mathcal{D}}$$

with n copies of $\tilde{\mathcal{D}}$. Hence U is a minimal isometric dilation of T. ■

The minimal isometric dilation of a Kreĭn space operator $T \in \mathbf{B}(\mathcal{H})$ constructed in Theorem 3.1.2 has special properties:

(a) $\mathcal{H}$ is invariant under U^* and $U^*|\mathcal{H} = T^*$;

(b) the closure $\mathcal{L}$ of $\{Uh - Th : h \in \mathcal{H}\}$ is a regular subspace of $\tilde{\mathcal{H}}$, and

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus \tilde{\mathcal{L}} \oplus U\tilde{\mathcal{L}} \oplus U^2\tilde{\mathcal{L}} \oplus \cdots;$$

(c) the positive and negative indices of $\mathcal{L}$ coincide with $h_\pm(1 - T^*T)$.

Proofs are immediate from the construction in Theorem 3.1.2.

Let $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{H}$ a Kreĭn space, and let $U \in \mathbf{B}(\tilde{\mathcal{H}})$ be a minimal isometric dilation of T. If $\tilde{\mathcal{H}}'$ is another Kreĭn space containing $\mathcal{H}$ as a regular subspace, and if $U' = WUW^{-1}$ where $W \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}')$ is a unitary operator which coincides with the identity on $\mathcal{H}$, then the operator $U' \in \mathbf{B}(\tilde{\mathcal{H}}')$ is also a minimal isometric dilation of T. Two minimal isometric dilations of T related in this way are said to be *isomorphic*. Isomorphic minimal isometric dilations are abstractly indistinguishable. In particular, properties (a) - (c) are invariant under isomorphism.

The next result shows that minimal isometric dilations of contractions are unique up to isomorphism, and therefore all have the properties (a)–(c).

THEOREM 3.1.3. *Let $\mathcal{H}$ be a Kreĭn space, and assume that $T \in \mathbf{B}(\mathcal{H})$ is a contraction.*

(i) Any two minimal isometric dilations of T are isomorphic.

(ii) If $U \in \mathbf{B}(\tilde{\mathcal{H}})$ is a minimal isometric dilation of T, then $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is uniformly positive, and hence any maximal negative subspace of $\mathcal{H}$ is maximal negative in $\tilde{\mathcal{H}}$.

(iii) The operator T is a bicontraction if and only if any minimal isometric dilation of T is a bicontraction.

Proof. To prove (i), it is sufficient to show that if $U \in \mathbf{B}(\tilde{\mathcal{H}})$ is a minimal isometric dilation of T of the form constructed in Theorem 3.1.2, and if $U' \in \mathbf{B}(\tilde{\mathcal{H}}')$ is any other minimal isometric dilation of T, then U and U' are isomorphic.

For any elements $f_0, \dots, f_n$ and $g_0, \dots, g_m$ of $\mathcal{H}$,

$$\left\langle \sum_{j=0}^{n} U^j f_j, \sum_{k=0}^{m} U^k g_k \right\rangle_{\tilde{\mathcal{H}}} = \left\langle \sum_{j=0}^{n} {U'}^j f_j, \sum_{k=0}^{m} {U'}^k g_k \right\rangle_{\tilde{\mathcal{H}}'}$$

by the definition of an isometric dilation. Since both dilations are minimal, in a routine way we obtain a well defined and densely defined isometry W_0 on $\tilde{\mathcal{H}}$, having dense range in $\tilde{\mathcal{H}}'$, such that

$$W_0 \left\{ \sum_{j=0}^{n} U^j f_j \right\} = \sum_{j=0}^{n} {U'}^j f_j$$

for any nonnegative integer n and vectors $f_0, \dots, f_n$ in $\mathcal{H}$. Both the domain and range of W_0 include $\mathcal{H}$, and W_0 coincides with the identity on $\mathcal{H}$. The domain of W_0 is invariant under U, and $W_0 U g = U' W_0 g$ for all g in the domain of W_0.

Since T is a contraction, $\tilde{\mathcal{D}}$ is a Hilbert space by Corollary 1.4.2. Therefore $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is uniformly positive. We show that $\tilde{\mathcal{H}}' \ominus \mathcal{H}$ is also uniformly positive. Since $\operatorname{ran} W_0$ is dense in $\tilde{\mathcal{H}}'$ and includes $\mathcal{H}$, the intersection of $\operatorname{ran} W_0$ and $\tilde{\mathcal{H}}' \ominus \mathcal{H}$ is dense in $\tilde{\mathcal{H}}' \ominus \mathcal{H}$. In a similar way, the intersection of $\operatorname{dom} W_0$ and $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is dense in $\tilde{\mathcal{H}} \ominus \mathcal{H}$. Since $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is positive and W_0 is isometric, $\tilde{\mathcal{H}}' \ominus \mathcal{H}$ is positive and therefore uniformly positive.

If $\mathcal{M}$ is a maximal negative subspace of $\mathcal{H}$, by what we just proved $\mathcal{M}$ is maximal negative in both $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}'$. Now $\mathcal{M}$ is in the domain of W_0 and $W_0 \mathcal{M} = \mathcal{M}$. Hence by Theorem 1.4.4, W_0 has an extension to an operator $W \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}')$. It is clear that W is unitary, W coincides with the identity on $\mathcal{H}$, and $U' = W U W^{-1}$. Therefore U and U' are isomorphic, and (i) is proved. A proof of (ii) is obtained incidentally from the argument.

For (iii), consider any minimal isometric dilation $U \in \mathbf{B}(\tilde{\mathcal{H}})$ of T. By part (ii) of the theorem, $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is uniformly positive. Hence any maximal uniformly negative subspace $\mathcal{H}_-$ of $\mathcal{H}$ is maximal negative in $\tilde{\mathcal{H}}$ as well. Both T and U are contractions, and $U\mathcal{H}_-$ and $T\mathcal{H}_-$ have the same projection onto $\mathcal{H}_-$ by the definition of an isometric dilation. By Theorem 1.3.6, U and T are simultaneously bicontractions or not. ■

Parallel results hold for unitary dilations.

Definition 3.1.4. *Let $T \in \mathbf{B}(\mathcal{H})$, where $\mathcal{H}$ is a Kreĭn space. A* **unitary dilation** *of T is a unitary operator $U \in \mathbf{B}(\tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is a Kreĭn space containing $\mathcal{H}$ isometrically as a regular subspace, such that*

$$T^n = \mathrm{Pr}_{\mathcal{H}}\, U^n|\mathcal{H} \qquad \text{and} \qquad T^{*n} = \mathrm{Pr}_{\mathcal{H}}\, U^{-n}|\mathcal{H}$$

for all $n = 1, 2, 3, \ldots$. A unitary dilation $U \in \mathbf{B}(\tilde{\mathcal{H}})$ of T is **minimal** *if*

$$\bigvee_{n=-\infty}^{\infty} U^n\mathcal{H} = \mathcal{H}.$$

We give two methods of constructing minimal unitary dilations. The first iterates the construction of minimal isometric dilations.

Theorem 3.1.5. *Let $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{H}$ a Kreĭn space. Let $V \in \mathbf{B}(\mathcal{H}')$ be any minimal isometric dilation of T, and let $U^* \in \mathbf{B}(\tilde{\mathcal{H}})$ be a minimal isometric dilation of V^* of the form constructed in Theorem 3.1.2. Then U is a minimal unitary dilation of T.*

Proof. By assumption, U^* has the form

$$U^* = \begin{pmatrix} V^* & 0 & 0 & 0 & \cdots \\ \tilde{E}^* & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix} \in \mathbf{B}(\mathcal{H}' \oplus \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}} \oplus \cdots),$$

where $\tilde{E} \in \mathbf{B}(\tilde{\mathcal{E}}, \mathcal{H}')$ is a defect operator for $V^* \in \mathbf{B}(\mathcal{H}')$. Since V is an isometry, $\tilde{E}$ is an isometry with range equal to $\ker V^*$. To see this, note that $1 - VV^* = \tilde{E}\tilde{E}^*$ is a projection operator, and $\ker \tilde{E}^* = \ker \tilde{E}\tilde{E}^* = \ker(1 - VV^*) = V\mathcal{H}'$. Then apply Theorem 1.1.6 to obtain the assertion. It follows that

$$V^*\tilde{E} = 0 \qquad \text{and} \qquad \tilde{E}^*\tilde{E} = 1.$$

A matrix multiplication shows that $U^*U = 1$. Since U^* is an isometry by construction, U is unitary. For any $n = 1, 2, 3, \ldots$,

$$\mathrm{Pr}_{\mathcal{H}}\, V^n|\mathcal{H} = T^n \qquad \text{and} \qquad \mathrm{Pr}_{\mathcal{H}}\, V^{*n}|\mathcal{H} = T^{*n}$$

and

$$\Pr_{\mathcal{H}'} U^{*n}|\mathcal{H}' = V^{*n} \qquad \text{and} \qquad \Pr_{\mathcal{H}'} U^{n}|\mathcal{H}' = V^{n}$$

because V and U^* are isometric dilations of T and V^*. The second pair of relations can be further restricted to $\mathcal{H}$. Applying the projection onto $\mathcal{H}$ and using the first pair of relations, we then see that U is a unitary dilation of T.

It remains to show that U is minimal. From the matrix form of U, we have $U|\mathcal{H} = V|\mathcal{H}$, and so $\vee_0^n U^j\mathcal{H} = \vee_0^n V^j\mathcal{H}$. Since V is a minimal isometric dilation of T by assumption, $\vee_0^\infty U^j\mathcal{H} = \mathcal{H}'$. As in the proof of Theorem 3.1.2, an inductive argument shows that for any element of $\tilde{\mathcal{H}}$ of the form $f = (f_0, 0, 0, \ldots)^t$, $f_0 \in \mathcal{H}'$, and any $n = 1, 2, 3, \ldots$,

$$U^{*n} f = \begin{pmatrix} V^{*n} f_0 \\ \tilde{E}^* V^{*\,n-1} f_0 \\ \tilde{E}^* V^{*\,n-2} f_0 \\ \vdots \\ \tilde{E}^* f_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

The identity implies that $\vee_0^\infty U^{*j}\mathcal{H}' = \mathcal{H}' \oplus \tilde{\mathcal{E}} \oplus \cdots \oplus \tilde{\mathcal{E}}$ with n copies of $\tilde{\mathcal{E}}$. Since $\vee_0^\infty U^j\mathcal{H} = \mathcal{H}'$, this shows that U is a minimal unitary dilation of T. ■

The preceding method of constructing unitary dilations is useful for applications, but it does not yield other desirable properties similar to the Hilbert space case. These properties are obtained from a different construction.

THEOREM 3.1.6. *Let $\mathcal{H}$ be a Kreĭn space, and let $T \in \mathbf{B}(\mathcal{H})$ have Julia operator*

$$\begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus D, \mathcal{H} \oplus \tilde{\mathcal{D}}),$$

Define $\tilde{\mathcal{H}} = \cdots \oplus \mathcal{D} \oplus \mathcal{D} \oplus \mathcal{H} \oplus \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}} \oplus \cdots$ using fixed choices of fundamental decompositions in the summands. Then the matrix

$$U = \begin{pmatrix} & & & & \cdots & & & & \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & D & \boxed{T} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & L & \tilde{D}^* & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ & & & & \cdots & & & & \end{pmatrix}$$

acts as an everywhere defined and continuous operator on $\tilde{\mathcal{H}}$, and U is a minimal unitary dilation of T.

Proof. The continuity of shift operators implies that U exists as an element of $\mathbf{B}(\tilde{\mathcal{H}})$. For any element

$$h = (\ldots, d_2, d_1, f, \tilde{d}_1, \tilde{d}_2, \ldots)^t$$

of $\tilde{\mathcal{H}}$, with f as the element of $\mathcal{H}$, we have

$$Uh = \begin{pmatrix} \vdots \\ d_3 \\ d_2 \\ Dd_1 + Tf \\ Ld_1 + \tilde{D}^* f \\ \tilde{d}_1 \\ \tilde{d}_2 \\ \vdots \end{pmatrix} \qquad \text{and} \qquad U^* h = \begin{pmatrix} \vdots \\ d_2 \\ d_1 \\ D^* f + L^* \tilde{d}_1 \\ T^* f + \tilde{D} \tilde{d}_1 \\ \tilde{d}_2 \\ \tilde{d}_3 \\ \vdots \end{pmatrix}.$$

A straightforward calculation yields

$$\langle Uh, Uh \rangle_{\tilde{\mathcal{H}}} = \langle U^* h, U^* h \rangle_{\tilde{\mathcal{H}}} = \langle h, h \rangle_{\tilde{\mathcal{H}}},$$

and so U is unitary. If $h = (\ldots, 0, 0, f, 0, 0, \ldots)^t$, where $f \in \mathcal{H}$, then

$$U^n h = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ T^n f \\ \tilde{D}^* T^{n-1} f \\ \tilde{D}^* T^{n-2} f \\ \vdots \\ \tilde{D}^* f \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \text{and} \qquad U^{*\,n} h = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ D^* f \\ \vdots \\ D^* T^{*\,n-2} f \\ D^* T^{*\,n-1} f \\ T^{*\,n} f \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

The identities yield

$$\bigvee_{j=0}^{n} U^j \mathcal{H} = \mathcal{H} \oplus \tilde{\mathcal{D}} \oplus \cdots \oplus \tilde{\mathcal{D}},$$

$$\bigvee_{j=0}^{n} U^{*\,j} \mathcal{H} = \mathcal{D} \oplus \cdots \oplus \mathcal{D} \oplus \mathcal{H},$$

with $n+1$ summands on the right side in each case. It follows that U is a minimal unitary dilation of T. ■

As in the case of isometric dilations, the construction of a minimal unitary dilation in Theorem 3.1.6 for a given operator $T \in \mathbf{B}(\mathcal{H})$ immediately yields additional properties:

(a′) the closures $\tilde{\mathcal{L}}$ and $\mathcal{L}$ of $\{Uh - Th : h \in \mathcal{H}\}$ and $\{U^*h - T^*h : h \in \mathcal{H}\}$ are regular subspaces of $\tilde{\mathcal{H}}$, and

$$\tilde{\mathcal{H}} = \cdots \oplus U^{*2}\mathcal{L} \oplus U^*\mathcal{L} \oplus \mathcal{L} \oplus \mathcal{H} \oplus \tilde{\mathcal{L}} \oplus U\tilde{\mathcal{L}} \oplus U^2\tilde{\mathcal{L}} \oplus \cdots;$$

(b′) the positive and negative indices of $\tilde{\mathcal{L}}$ are equal to $h_\pm(1-T^*T)$, and those of $\mathcal{L}$ are equal to $h_\pm(1-TT^*)$.

Let $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{H}$ a Kreĭn space, and let $U \in \mathbf{B}(\tilde{\mathcal{H}})$ be a minimal unitary dilation of T. If $\tilde{\mathcal{H}}'$ is another Kreĭn space containing $\mathcal{H}$ as a regular subspace, and if $U' = WUW^{-1}$ where $W \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}')$ is a unitary operator which coincides with the identity on $\mathcal{H}$, then the operator $U' \in \mathbf{B}(\tilde{\mathcal{H}}')$ is a minimal unitary dilation of T. In this situation, we say that U and U' are *isomorphic*. The properties (a′) and (b′) are invariant under isomorphism.

Minimal unitary dilations of bicontractions are unique up to isomorphism.

THEOREM 3.1.7. *Let $\mathcal{H}$ be a Kreĭn space, and let $T \in \mathbf{B}(\mathcal{H})$ be a bicontraction.*

(i) Any two minimal unitary dilations of T are isomorphic.

(ii) If $U \in \mathbf{B}(\tilde{\mathcal{H}})$ is a minimal unitary dilation of T, then $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is uniformly positive, and hence any maximal negative subspace of $\mathcal{H}$ is maximal negative in $\tilde{\mathcal{H}}$.

Proof. The argument is similar to the proof of Theorem 3.1.3, parts (i) and (ii). In place of Theorem 3.1.2 we use Theorem 3.1.6. Since T is a bicontraction, the spaces $\mathcal{D}$ and $\tilde{\mathcal{D}}$ in Theorem 3.1.6 are Hilbert spaces by Corollary 1.4.3. Otherwise we proceed as before, except that in place of sums of the form $\sum_{j=0}^{n} U^j f_j$ with $f_0, \ldots, f_n \in \mathcal{H}$ we now use sums of the form $\sum_{j=-n}^{n} U^j f_j$ with $f_{-n}, \ldots, f_n \in \mathcal{H}$. ■

Theorem 3.1.3 (iii) has a variant form which does not require that T be a contraction. If $T \in \mathbf{B}(\mathcal{H})$, where $\mathcal{H}$ is a Kreĭn space, and if T^* is a contraction, then any minimal isometric dilation of T of the form constructed in Theorem 3.1.2 is a bicontraction. Conversely, if such a dilation is a bicontraction, then T^* is a contraction. For U is a bicontraction if and only if U^* is a contraction, and the condition for this is that $[T^* \quad \tilde{D}]$ be a contraction. By Theorem 1.4.2, this holds if and only if T^* is a contraction.

We give an example. Let $\mathcal{H}$ be the anti-space of a Hilbert space, and let $T = -2V^*$, where $V \in \mathbf{B}(\mathcal{H})$ is an isometry whose range is not all of $\mathcal{H}$. Then T^* is a contraction, but T is not a contraction because its kernel is not uniformly positive. Let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be $\mathcal{H}$ as a vector space, with

$$\begin{aligned} \langle f, g\rangle_{\mathcal{D}} &= -\langle f, g\rangle_{\mathcal{H}}, \\ \langle f, g\rangle_{\tilde{\mathcal{D}}} &= -\langle Pf, Pg\rangle_{\mathcal{H}} + \langle (1-P)f, (1-P)g\rangle_{\mathcal{H}}, \end{aligned}$$

for all $f, g \in \mathcal{H}$, where P is the projection on $V\mathcal{H}$. A Julia operator for T is given by

$$\begin{pmatrix} T & \sqrt{3} \\ 1 - P - \sqrt{3}P & 2V \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{H} \oplus \tilde{\mathcal{D}}).$$

Its adjoint

$$\begin{pmatrix} T^* & 1 - P + \sqrt{3}P \\ -\sqrt{3} & 2V^* \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \tilde{\mathcal{D}}, \mathcal{H} \oplus \mathcal{D})$$

is a Julia operator for T^*. Since T^* is a contraction, the minimal isometric dilation of T constructed by Theorem 3.1.2 is a bicontraction by the previous remarks.

3.2 Commutant Lifting

The background is now in place for a commutant lifting theorem for contraction operators on Kreĭn spaces.

THEOREM 3.2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{K})$ be contractions with minimal isometric dilations $U_1 \in \mathbf{B}(\tilde{\mathcal{H}})$ and $U_2 \in \mathbf{B}(\tilde{\mathcal{K}})$, respectively. Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction satisfying*

$$AT_1 = T_2 A.$$

Then there exists a contraction $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that

$$\tilde{A}U_1 = U_2 \tilde{A}$$

and $A = \operatorname{Pr}_{\mathcal{K}} \tilde{A}|\mathcal{H}$.

Proof. Since any two minimal isometric dilations of a contraction are isomorphic by Theorem 3.1.3, we may assume that the minimal isometric dilations of T_1 and T_2 have the form constructed in Theorem 3.1.2. In other words, we may take

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus \tilde{\mathcal{D}}_1 \oplus \tilde{\mathcal{D}}_1 \oplus \cdots \quad \text{and} \quad \tilde{\mathcal{K}} = \mathcal{K} \oplus \tilde{\mathcal{D}}_2 \oplus \tilde{\mathcal{D}}_2 \oplus \cdots$$

where $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ are Hilbert spaces,

$$U_j = \begin{pmatrix} T_j & 0 & 0 & 0 & \cdots \\ \tilde{D}_j^* & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix}, \qquad j = 1, 2,$$

and $\tilde{D}_1 \in \mathbf{B}(\tilde{\mathcal{D}}_1, \mathcal{H})$, $\tilde{D}_2 \in \mathbf{B}(\tilde{\mathcal{D}}_2, \mathcal{K})$ are defect operators for T_1, T_2. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ be fundamental decompositions, used, for example, to induce Hilbert space norms on $\mathcal{H}$, $\tilde{\mathcal{H}}$, $\mathcal{K}$, $\tilde{\mathcal{K}}$. Set $\mathcal{K}_0 = \mathcal{K}$ and

$$\mathcal{K}_n = \mathcal{K} \oplus \tilde{\mathcal{D}}_2 \oplus \cdots \oplus \tilde{\mathcal{D}}_2, \qquad n = 1, 2, \ldots,$$

with n copies of $\tilde{\mathcal{D}}_2$ in the direct sum on the right. We construct contraction operators

$$\tilde{A}_n : \tilde{\mathcal{H}} \to \mathcal{K}_n, \qquad n = 0, 1, 2, \ldots,$$

such that $A = \mathrm{Pr}_{\mathcal{K}} \tilde{A}_n | \mathcal{H}$ for all nonnegative integers n and

(α) $\tilde{A}_n U_1 = U_2 \tilde{A}_{n-1}$,
(β) $\tilde{A}_{n-1} = \mathrm{Pr}_{\mathcal{K}_{n-1}} \tilde{A}_n$

if $n \geq 1$.

Let $\tilde{A}_0$ map any element $(f, g_1, g_2, \ldots)^t$ of $\tilde{\mathcal{H}}$ to the element Af of $\mathcal{K}_0$. Since A is a contraction and $\tilde{\mathcal{D}}_1$ is a Hilbert space, $\tilde{A}_0$ is a contraction. By construction, $A = \mathrm{Pr}_{\mathcal{K}} \tilde{A}_0 | \mathcal{H}$.

Assume that $\tilde{A}_0, \ldots, \tilde{A}_r$ have the required properties. We determine $\tilde{A}_{r+1}$ as a matrix

$$\tilde{A}_{r+1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

relative to the decompositions

$$\tilde{\mathcal{H}} = U\tilde{\mathcal{H}} \oplus (\tilde{\mathcal{H}} \ominus U\mathcal{H}) \qquad \text{and} \qquad \mathcal{K}_{r+1} = \mathcal{K}_r \oplus \tilde{\mathcal{D}}_2.$$

The conditions (α) and (β) with $n = r + 1$ require that

$$\begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} U_1 = U_2 \tilde{A}_r \qquad \text{and} \qquad \begin{pmatrix} C_{11} & C_{12} \end{pmatrix} = \tilde{A}_r$$

Define C_{11}, C_{12}, C_{21} by these equations. The two definitions of C_{11} are identical in the case $r = 0$, because

$$\tilde{A}_0 U_1 = \mathrm{Pr}_{\mathcal{K}_0} U_2 \tilde{A}_0.$$

In fact, this identity reduces to $AT_1 = T_2 A$, which holds by hypothesis. In the case $r \geq 1$, the two definitions of C_{11} are identical because

$$\tilde{A}_r U_1 = U_2 \tilde{A}_{r-1} = U_2 \mathrm{Pr}_{\mathcal{K}_{r-1}} \tilde{A}_r = \mathrm{Pr}_{\mathcal{K}_r} U_2 \tilde{A}_r.$$

Here the first two equalities are by (α) and (β) with $n = r$, and the third is an elementary property of the minimal isometric dilation.

Since $\tilde{A}_r$ is a contraction and U_1 and U_2 are isometries, the operators

$$\begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} C_{11} & C_{12} \end{pmatrix}$$

are contractions. By Theorem 2.4.4, there exists an operator C_{22} such that

$$\tilde{A}_{r+1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is a contraction. By construction, (α) and (β) hold with $n = r + 1$. Finally,

$$\mathrm{Pr}_{\mathcal{K}}\, \tilde{A}_{r+1} | \mathcal{H} = \mathrm{Pr}_{\mathcal{K}}\, \mathrm{Pr}_{\mathcal{K}_r} \tilde{A}_{r+1} | \mathcal{H} = \mathrm{Pr}_{\mathcal{K}}\, \tilde{A}_r | \mathcal{H} = A.$$

This completes the inductive construction of operators $\tilde{A}_0$, $\tilde{A}_1$, ... with the stated properties.

We show that $\|\tilde{A}_n\|^2 \leq 1 + 2\|A\|^2$ for all n. This is true for $n = 0$ because $\|\tilde{A}_0\|^2 = \|A\|^2$ by the definition of $\tilde{A}_0$. For $n \geq 1$, we obtain $\tilde{A}_0 = \mathrm{Pr}_{\mathcal{K}}\, \tilde{A}_n$ by repeated application of (β). Hence we may write

$$\tilde{A}_n = \begin{pmatrix} \tilde{A}_0 \\ X \end{pmatrix} \in \mathbf{B}(\tilde{\mathcal{H}}, \mathcal{K} \oplus (\mathcal{K}_n \ominus \mathcal{K})),$$

where $X \in \mathbf{B}(\tilde{\mathcal{H}}, \mathcal{K}_n \ominus \mathcal{K})$. Since $\tilde{A}_0$ and $\tilde{A}_n$ are contractions and $\mathcal{K}_n \ominus \mathcal{K}$ is a Hilbert space, Corollary 2.2.3 yields

$$\|\tilde{A}_n\|^2 \leq 1 + 2\|\tilde{A}_0\|^2 = 1 + 2\|A\|^2,$$

which proves the assertion.

By (β), $\tilde{A}^*_{n-1} = \tilde{A}^*_n | \mathcal{K}_{n-1}$ for all $n \geq 1$. Hence there is an operator $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that $\|\tilde{A}\|^2 \leq 1 + 2\|A\|^2$ and $\tilde{A}^*_n = \tilde{A}^* | \mathcal{K}_n$, equivalently, $\tilde{A}_n = \mathrm{Pr}_{\mathcal{K}_n} \tilde{A}$ for every nonnegative integer n. It follows that $\lim_{n \to \infty} \tilde{A}_n f = \tilde{A} f, \quad f \in \tilde{\mathcal{H}}$, and from this we verify without difficulty that $\tilde{A}$ has the required properties. ■

The norm estimate obtained in the proof of Theorem 3.2.1, namely,

$$\|\tilde{A}\|^2 \le 1 + 2\|A\|^2,$$

holds for any operator $\tilde{A}$ having the properties stated in the theorem (it is not special to the construction in the proof). Moreover, arbitrary fundamental decompositions of $\mathcal{H}$ and $\mathcal{K}$ may be used in the computation of norms. To see this, recall that by Theorem 3.1.3 (ii),

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus (\tilde{\mathcal{H}} \ominus \mathcal{H}) \qquad \text{and} \qquad \tilde{\mathcal{K}} = \mathcal{K} \oplus (\tilde{\mathcal{K}} \ominus \mathcal{K}),$$

where $\tilde{\mathcal{H}} \ominus \mathcal{H}$ and $\tilde{\mathcal{K}} \ominus \mathcal{K}$ are Hilbert spaces. Thus fundamental decompositions of $\mathcal{H}$ and $\mathcal{K}$ determine fundamental decompositions of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$. The norm estimate follows from Corollary 2.2.3 by an argument used in the proof of Theorem 3.2.1.

Theorem 3.2.1 has a companion for bicontractions, which is deduced as an immediate consequence.

THEOREM 3.2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{K})$ be contractions with minimal isometric dilations $U_1 \in \mathbf{B}(\tilde{\mathcal{H}})$ and $U_2 \in \mathbf{B}(\tilde{\mathcal{K}})$, respectively. Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a bicontraction satisfying*

$$AT_1 = T_2 A.$$

Then there exists a bicontraction $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that

$$\tilde{A} U_1 = U_2 \tilde{A}$$

and $A = \mathrm{Pr}_{\mathcal{K}} \tilde{A}|\mathcal{H}$.

Proof. Let $\mathcal{N}_1$ be a maximal negative subspace of $\mathcal{H}$, and let $\mathcal{N}_2$ be a maximal uniformly negative subspace of $\mathcal{K}$. Since A is a bicontraction, $A\mathcal{N}_1$ is a maximal negative subspace of $\mathcal{K}$ by Theorem 1.3.6, and so

$$\mathrm{Pr}_{\mathcal{N}_2} A\mathcal{N}_1 = \mathcal{N}_2.$$

By Theorem 3.1.3 (ii), $\mathcal{N}_1$ is a maximal negative subspace of $\tilde{\mathcal{H}}$ and $\mathcal{N}_2$ is a maximal uniformly negative subspace of $\tilde{\mathcal{K}}$. Since $\tilde{A}$ is a contraction, $\tilde{A}\mathcal{N}_1$ is a negative subspace of $\tilde{\mathcal{K}}$. It is maximal negative in $\tilde{\mathcal{K}}$ because

$$\mathrm{Pr}_{\mathcal{N}_2} \tilde{A}\mathcal{N}_1 = \mathrm{Pr}_{\mathcal{N}_2} \mathrm{Pr}_{\mathcal{K}} \tilde{A}\mathcal{N}_1 = \mathrm{Pr}_{\mathcal{N}_2} A\mathcal{N}_1 = \mathcal{N}_2.$$

Therefore $\tilde{A}$ is a bicontraction by Theorem 1.3.6. ■

With stronger hypotheses, it is also possible to lift an intertwining relation to minimal unitary dilations.

Theorem 3.2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{K})$ be bicontractions with minimal unitary dilations $U_1 \in \mathbf{B}(\tilde{\mathcal{H}})$ and $U_2 \in \mathbf{B}(\tilde{\mathcal{K}})$, respectively. Let $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a bicontraction satisfying*

$$AT_1 = T_2 A.$$

Then there exists a bicontraction $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that

$$\tilde{A} U_1 = U_2 \tilde{A}$$

and $A = \mathrm{Pr}_{\mathcal{K}} \tilde{A}|\mathcal{H}$.

Proof. By Theorem 3.1.7, minimal unitary dilations of bicontractions are unique up to isomorphism. Therefore we may assume that U_1, U_2 are constructed as in Theorem 3.1.5. Let $V_1 \in \mathbf{B}(\mathcal{H}')$, $V_2 \in \mathbf{B}(\mathcal{K}')$ be minimal isometric dilations of T_1, T_2. We take U_1^*, U_2^* to be minimal isometric dilations of V_1^*, V_2^*.

By Theorem 3.2.2, there exists a bicontraction $A' \in \mathbf{B}(\mathcal{H}', \mathcal{K}')$ such that

$$A'V_1 = V_2 A'$$

and $A = \mathrm{Pr}_{\mathcal{K}} A'\mathcal{H}$. The operators

$$T_1' = V_2^* \in \mathbf{B}(\mathcal{K}') \qquad \text{and} \qquad T_2' = V_1^* \in \mathbf{B}(\mathcal{H}')$$

are contractions by Theorem 3.1.3 (iii), and $A'^* \in \mathbf{B}(\mathcal{K}', \mathcal{H}')$ is a bicontraction satisfying

$$A'^* T_1' = T_2' A'^*.$$

Since U_2^* and U_1^* are minimal isometric dilations of T_1' and T_2', respectively, a second application of Theorem 3.2.2 produces a bicontraction $\tilde{A}^* \in \mathbf{B}(\tilde{\mathcal{K}}, \tilde{\mathcal{H}})$ such that

$$\tilde{A}^* U_2^* = U_1^* \tilde{A}^*$$

and $A'^* = \mathrm{Pr}_{\mathcal{H}'} \tilde{A}^*|\mathcal{K}'$. Thus $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ is a bicontraction satisfying $\tilde{A}U_1 = U_2\tilde{A}$. Finally, we have $A' = \mathrm{Pr}_{\mathcal{K}'} \tilde{A}|\mathcal{H}'$, and restricting this relation further to $\mathcal{H}$, we obtain

$$A = \mathrm{Pr}_{\mathcal{K}} A'|\mathcal{H} = \mathrm{Pr}_{\mathcal{K}} \tilde{A}|\mathcal{H}.$$

Thus A has the required properties. ■

3.3 Characterization of Extensions

We do not solve the problem of labeling all operators produced by the commutant lifting theorem, but we give some information in this direction.

Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{K})$ be contractions with minimal isometric dilations $U_1 \in \mathbf{B}(\tilde{\mathcal{H}})$ and $U_2 \in \mathbf{B}(\tilde{\mathcal{K}})$. Assume that $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction such that

$$AT_1 = T_2 A.$$

By Theorem 3.1.3, $\mathcal{H}^\perp = \tilde{\mathcal{H}} \ominus \mathcal{H}$ and $\mathcal{K}^\perp = \tilde{\mathcal{K}} \ominus \mathcal{K}$ are uniformly positive in the Kreĭn spaces $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$.

LEMMA 3.3.1. *If $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$, $\tilde{A}U_1 = U_2\tilde{A}$, and $\mathrm{Pr}_{\mathcal{K}}\, \tilde{A}|\mathcal{H} = A$, then $\tilde{A}\mathcal{H}^\perp \subset \mathcal{K}^\perp$ and therefore $\tilde{A}^*|\mathcal{K} = A^*$.*

Proof. By Theorem 3.1.3, we may choose U_1 and U_2 as in Theorem 3.1.2. Thus

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus \tilde{\mathcal{D}}_1 \oplus \tilde{\mathcal{D}}_1 \oplus \cdots \quad \text{and} \quad \tilde{\mathcal{K}} = \mathcal{K} \oplus \tilde{\mathcal{D}}_2 \oplus \tilde{\mathcal{D}}_2 \oplus \cdots$$

where $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ are Hilbert spaces,

$$U_j = \begin{pmatrix} T_j & 0 & 0 & 0 & \cdots \\ \tilde{D}_j^* & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix}, \qquad j = 1, 2,$$

and $\tilde{D}_1 \in \mathbf{B}(\tilde{\mathcal{D}}_1, \mathcal{H})$, $\tilde{D}_2 \in \mathbf{B}(\tilde{\mathcal{D}}_2, \mathcal{K})$ are defect operators for T_1, T_2. Writing $\tilde{A} = (A_{jk})_{j,k=1}^\infty$ relative to the same decompositions and comparing entries in the relation $\tilde{A}U_1 = U_2\tilde{A}$, we find that $A_{11} = A$ and $A_{1n} = 0$ for $n \geq 1$. In particular, $\tilde{A}\mathcal{H}^\perp \subset \mathcal{K}^\perp$. ■

Let $\mathcal{L}$ be the direct sum of the anti-space of $\mathcal{H}$ together with $\mathcal{K}$, so $\mathcal{L}$ is the space of pairs $\binom{f}{g}$ with f in $\mathcal{H}$ and g in $\mathcal{K}$, and

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{L}} = -\langle f, f \rangle_{\mathcal{H}} + \langle g, g \rangle_{\mathcal{K}}, \qquad f \in \mathcal{H}, g \in \mathcal{K}.$$

Since $A \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is a contraction, its graph

$$\mathcal{G}(A) = \left\{ \begin{pmatrix} f \\ Af \end{pmatrix} : \ f \in \mathcal{H} \right\}$$

is a closed negative subspace of $\mathcal{L}$.

Given a contraction $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$, we define $\tilde{\mathcal{L}}$ and $\mathcal{G}(\tilde{A})$ in a parallel way, so $\tilde{\mathcal{L}}$ is the space of pairs $\binom{f}{g}$ with f in $\tilde{\mathcal{H}}$ and g in $\tilde{\mathcal{K}}$, and

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\tilde{\mathcal{L}}} = -\langle f, f\rangle_{\tilde{\mathcal{H}}} + \langle g, g\rangle_{\tilde{\mathcal{K}}}, \qquad f \in \tilde{\mathcal{H}}, g \in \tilde{\mathcal{K}}.$$

The graph

$$\mathcal{G}(\tilde{A}) = \left\{ \begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} : \ f \in \tilde{\mathcal{H}} \right\}$$

of $\tilde{A}$ is a closed negative subspace of $\tilde{\mathcal{L}}$. If

$$\tilde{U} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in \mathbf{B}(\tilde{\mathcal{L}}),$$

the relation $\tilde{A}U_1 = U_2\tilde{A}$ holds if and only if $\tilde{U}\mathcal{G}(\tilde{A}) \subset \mathcal{G}(\tilde{A})$.

Choose fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_+ \oplus \tilde{\mathcal{L}}_-$ in a consistent way such that

$$\mathcal{L}_+ = \mathcal{K}_+ \oplus \mathcal{H}_-, \qquad \mathcal{L}_- = \mathcal{H}_+ \oplus \mathcal{K}_-,$$

and

$$\tilde{\mathcal{L}}_+ = \mathcal{L}_+ \oplus \mathcal{K}^\perp = \mathcal{K}_+ \oplus \mathcal{H}_- \oplus \mathcal{K}^\perp,$$
$$\tilde{\mathcal{L}}_- = \mathcal{L}_- \oplus \mathcal{H}^\perp = \mathcal{H}_+ \oplus \mathcal{K}_- \oplus \mathcal{H}^\perp.$$

Notice that $\mathcal{L}^\perp = \tilde{\mathcal{L}} \ominus \mathcal{L}$ is given by $\mathcal{L}^\perp = \mathcal{H}^\perp \oplus \mathcal{K}^\perp$.

THEOREM 3.3.2. *In the preceding situation, the relation*

$$\mathcal{M} = \mathcal{G}(\tilde{A})$$

establishes a one-to-one correspondence between all contractions $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ *such that* $\tilde{A}U_1 = U_2\tilde{A}$ *and* $\mathrm{Pr}_{\mathcal{K}}\, \tilde{A}|\mathcal{H} = A$ *and all closed negative* $\tilde{U}$*-invariant subspaces* $\mathcal{M}$ *of* $\tilde{\mathcal{L}}$ *satisfying*

(i) $\mathcal{M} \subset \mathcal{G}(A) + \mathcal{L}^\perp$, *and*
(ii) $\mathrm{Pr}_{\tilde{\mathcal{L}}_-} \mathcal{M} = \mathrm{Pr}_{\tilde{\mathcal{L}}_-} \left[\mathcal{G}(A) + \mathcal{L}^\perp\right]$.

Proof. Assume that $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ is a contraction, $\tilde{A}U_1 = U_2\tilde{A}$, and $\mathrm{Pr}_{\mathcal{K}}\, \tilde{A}|\mathcal{H} = A$, and define $\mathcal{M} = \mathcal{G}(\tilde{A})$. Clearly $\mathcal{M}$ is a closed negative $\tilde{U}$-invariant subspace of $\tilde{\mathcal{L}}$. For any $f \in \tilde{\mathcal{H}}$,

$$\begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} = \begin{pmatrix} (\mathrm{Pr}_{\mathcal{H}} + \mathrm{Pr}_{\mathcal{H}^\perp})f \\ (\mathrm{Pr}_{\mathcal{K}} + \mathrm{Pr}_{\mathcal{K}^\perp})\tilde{A}(\mathrm{Pr}_{\mathcal{H}} + \mathrm{Pr}_{\mathcal{H}^\perp})f \end{pmatrix} = \begin{pmatrix} \mathrm{Pr}_{\mathcal{H}}f \\ A\mathrm{Pr}_{\mathcal{H}}f \end{pmatrix} + \begin{pmatrix} \mathrm{Pr}_{\mathcal{H}^\perp}f \\ \mathrm{Pr}_{\mathcal{K}^\perp}\tilde{A}f \end{pmatrix}$$

by Lemma 3.3.1, and (i) holds.

Let $P_\pm$ be the projections of $\mathcal{H}$ onto $\mathcal{H}_\pm$, $Q_\pm$ the projections of $\mathcal{K}$ onto $\mathcal{K}_\pm$. For any $f \in \tilde{\mathcal{H}}$,

$$\mathrm{Pr}_{\tilde{\mathcal{L}}_-} \begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} = \begin{pmatrix} P_+ \mathrm{Pr}_{\mathcal{H}} f + \mathrm{Pr}_{\mathcal{H}^\perp} f \\ Q_- A \mathrm{Pr}_{\mathcal{H}} f \end{pmatrix} = (P_+ + Q_- A)\mathrm{Pr}_{\mathcal{H}} f + \mathrm{Pr}_{\mathcal{H}^\perp} f$$

by Lemma 3.3.1. Therefore

$$\mathrm{Pr}_{\tilde{\mathcal{L}}_-} \mathcal{M} = \mathrm{ran}\,(P_+ + Q_- A) + \mathcal{H}^\perp.$$

In a similar way,

$$\begin{aligned} \mathrm{Pr}_{\tilde{\mathcal{L}}_-} \left[\mathcal{G}(A) + \mathcal{L}^\perp\right] &= \mathrm{Pr}_{\mathcal{H}_+ \oplus \mathcal{K}_-} \left\{ \begin{pmatrix} f \\ Af \end{pmatrix} : \ f \in \mathcal{H} \right\} + \mathcal{H}^\perp \\ &= \mathrm{ran}\,(P_+ + Q_- A) + \mathcal{H}^\perp, \end{aligned}$$

and (ii) follows.

Conversely, let $\mathcal{M}$ be a closed negative $\tilde{U}$-invariant subspace of $\tilde{\mathcal{L}}$ satisfying (i) and (ii). If $g \in \tilde{\mathcal{K}}$ and $\binom{0}{g} \in \mathcal{M}$, then by (i) we can write

$$\begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} u \\ Au \end{pmatrix} + h + k$$

with $u \in \mathcal{H}$, $h \in \mathcal{H}^\perp$, $k \in \mathcal{K}^\perp$. Projecting onto $\mathcal{H}$ and $\mathcal{H}^\perp$, we find that $u = 0$ and $h = 0$. Therefore $g = k \in \mathcal{K}^\perp$. But $\mathcal{M}$ is a negative subspace of $\tilde{\mathcal{L}}$, and $\mathcal{K}^\perp$ is uniformly positive in $\tilde{\mathcal{L}}$, so $g = 0$. It follows that $\mathcal{M} = \mathcal{G}(\tilde{A})$ is the graph of a linear transformation $\tilde{A}$ with domain in $\tilde{\mathcal{H}}$ and range in $\tilde{\mathcal{K}}$.

We use conditions (i) and (ii) to show that $\mathrm{dom}\,\tilde{A} = \tilde{\mathcal{H}}$. For any $f \in \mathrm{dom}\,\tilde{A}$, by (i) there exist $u \in \mathcal{H}$, $h \in \mathcal{H}^\perp$, $k \in \mathcal{K}^\perp$ such that

$$\begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} = \begin{pmatrix} u \\ Au \end{pmatrix} + h + k.$$

Projecting onto $\mathcal{H}$, $\mathcal{H}^\perp$, $\mathcal{K}^\perp$, we find that $u = \mathrm{Pr}_{\mathcal{H}} f$, $h = \mathrm{Pr}_{\mathcal{H}^\perp} f$, $k = \mathrm{Pr}_{\mathcal{K}^\perp} \tilde{A} f$, and so

$$\begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} = \begin{pmatrix} \mathrm{Pr}_{\mathcal{H}} f \\ A\mathrm{Pr}_{\mathcal{H}} f \end{pmatrix} + \mathrm{Pr}_{\mathcal{H}^\perp} f + \mathrm{Pr}_{\mathcal{K}^\perp} \tilde{A} f \tag{3.3.1}$$

and

$$\begin{aligned} \mathrm{Pr}_{\tilde{\mathcal{L}}_-} \begin{pmatrix} f \\ \tilde{A}f \end{pmatrix} &= \mathrm{Pr}_{\mathcal{H}_+ \oplus \mathcal{K}_- \oplus \mathcal{H}^\perp} \left\{ \begin{pmatrix} \mathrm{Pr}_{\mathcal{H}} f \\ A\mathrm{Pr}_{\mathcal{H}} f \end{pmatrix} + \mathrm{Pr}_{\mathcal{H}^\perp} f + \mathrm{Pr}_{\mathcal{K}^\perp} \tilde{A} f \right\} \\ &= (P_+ + Q_- A)\mathrm{Pr}_{\mathcal{H}} f + \mathrm{Pr}_{\mathcal{H}^\perp} f. \end{aligned}$$

But since

$$\Pr_{\tilde{\mathcal{L}}_-}\mathcal{M} = \Pr_{\tilde{\mathcal{L}}_-}\left[\mathcal{G}(A) + \mathcal{L}^\perp\right] = \operatorname{ran}(P_+ + Q_-A) + \mathcal{H}^\perp$$

by (ii), it follows that $\operatorname{dom}\tilde{A} = \tilde{\mathcal{H}}$.

Since $\mathcal{M}$ is closed, $\tilde{A}$ is continuous by the closed graph theorem. That is, $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$. Since $\mathcal{M}$ is negative, $\tilde{A}$ is a contraction. The invariance of $\mathcal{M}$ under $\tilde{A}$ implies that $\tilde{A}U_1 = U_2\tilde{A}$. The identity (3.3.1) implies that $\Pr_{\mathcal{K}}\tilde{A}|\mathcal{H} = A$.

The correspondence between operators and their graphs is obviously one-to-one, and so the result is proved. ∎

The result takes a simpler form when A is a bicontraction. As above, $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{K})$ are contractions with minimal isometric dilations $U_1 \in \mathbf{B}(\tilde{\mathcal{H}})$ and $U_2 \in \mathbf{B}(\tilde{\mathcal{K}})$.

THEOREM 3.3.3. *Assume that A is a bicontraction. Then the relation*

$$\mathcal{M} = \mathcal{G}(\tilde{A})$$

establishes a one-to-one correspondence between all bicontractions $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that $\tilde{A}U_1 = U_2\tilde{A}$ and $\Pr_{\mathcal{K}}\tilde{A}|\mathcal{H} = A$ and all maximal negative $\tilde{U}$-invariant subspaces $\mathcal{M}$ of $\tilde{\mathcal{L}}$ satisfying $\mathcal{M} \subset \mathcal{G}(A) + \mathcal{L}^\perp$.

Proof. Let $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ be a bicontraction such that $\tilde{A}U_1 = U_2\tilde{A}$ and $\Pr_{\mathcal{K}}\tilde{A}|\mathcal{H} = A$. By Theorem 3.3.2, $\mathcal{G}(\tilde{A})$ is a negative $\tilde{U}$-invariant subspace of $\mathcal{L}$ satisfying $\mathcal{M} \subset \mathcal{G}(A) + \mathcal{L}^\perp$. In addition,

$$\Pr_{\tilde{\mathcal{L}}_-}\mathcal{M} = \Pr_{\tilde{\mathcal{L}}_-}\left[\mathcal{G}(A) + \mathcal{L}^\perp\right] = \operatorname{ran}(P_+ + Q_-A) + \mathcal{H}^\perp,$$

where the last equality follows as in the proof of Theorem 3.3.2. Since A is a bicontraction, $\operatorname{ran}(P_+ + Q_-A) = \mathcal{L}_-$ by Theorems 1.3.3 and 1.3.4. Therefore $\Pr_{\tilde{\mathcal{L}}_-}\mathcal{M} = \mathcal{L}_- + \mathcal{H}^\perp = \tilde{\mathcal{L}}_-$, and $\mathcal{M}$ is maximal negative.

Conversely, let $\mathcal{M}$ be a maximal negative $\tilde{U}$-invariant subspace of $\tilde{\mathcal{L}}$ satisfying $\mathcal{M} \subset \mathcal{G}(A) + \mathcal{L}^\perp$. Then $\mathcal{M}$ is closed and

$$\Pr_{\tilde{\mathcal{L}}_-}\mathcal{M} = \tilde{\mathcal{L}}_- = \operatorname{ran}(P_+ + Q_-A) + \mathcal{H}^\perp = \Pr_{\tilde{\mathcal{L}}_-}\left[\mathcal{G}(A) + \mathcal{L}^\perp\right].$$

By Theorem 3.3.2, $\mathcal{M} = \mathcal{G}(\tilde{A})$ is the graph of a contraction $\tilde{A} \in \mathbf{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ such that $\tilde{A}U_1 = U_2\tilde{A}$ and $\Pr_{\mathcal{K}}\tilde{A}|\mathcal{H} = A$. But A is a bicontraction, and so

$$\Pr_{\mathcal{K}_-}\tilde{A}\mathcal{H}_- = \Pr_{\mathcal{K}_-}A\mathcal{H}_- = \mathcal{K}_-.$$

Since $\mathcal{K}_-$ is maximal uniformly negative in $\tilde{\mathcal{K}}$, the operator $\tilde{A}$ is a bicontraction by Theorem 1.3.6. ∎

3.4 Abstract Leech Theorem

We use a theorem of Shmul'yan on the factorization of bicontractions.

THEOREM 3.4.1. *If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are Kreĭn spaces and $A \in \mathbf{B}(\mathcal{A},\mathcal{C})$ and $B \in \mathbf{B}(\mathcal{B},\mathcal{C})$ are bicontractions, then $A = BC$ for some bicontraction $C \in \mathbf{B}(\mathcal{A},\mathcal{B})$ if and only if $AA^* \le BB^*$.*

Proof. The condition is necessary: if $A = BC$ with C a bicontraction, then $AA^* = BCC^*B^* \le BB^*$.

Conversely, assume $AA^* \le BB^*$. We show that $\ker B^* \subset \ker A^*$. Suppose $f \in \ker B^*$ and $f \ne 0$. Let $\mathcal{C}_-$ be a maximal uniformly negative subspace of $\mathcal{C}$. Then $f \notin \mathcal{C}_-$ because $\ker B^*$ is a uniformly positive subspace of $\mathcal{C}$ by Theorem 1.3.1. Let $\tilde{\mathcal{C}}_-$ be the span of f and $\mathcal{C}_-$, so $\mathcal{C}_-$ is properly contained in $\tilde{\mathcal{C}}_-$. Note that $A^*\tilde{\mathcal{C}}_-$ is a negative subspace of $\mathcal{A}$. For if $g = \alpha f + h$ where α is a complex number and h is in $\mathcal{C}_-$, then since $AA^* \le BB^*$ by assumption,

$$\langle A^*g, A^*g\rangle_{\mathcal{A}} \le \langle B^*g, B^*g\rangle_{\mathcal{B}} = \langle B^*h, B^*h\rangle_{\mathcal{B}} \le 0. \tag{3.4.1}$$

Since $A^*\tilde{\mathcal{C}}_- \supset A^*\mathcal{C}_-$ and $A^*\mathcal{C}_-$ is maximal negative by Theorem 1.3.6, $A^*\tilde{\mathcal{C}}_- = A^*\mathcal{C}_-$. Therefore A^* annihilates some nonzero element $g = \alpha f + h$ of $\tilde{\mathcal{C}}_-$, where α is a complex number and $h \in \mathcal{C}_-$. For such g, equality holds in (3.4.1), and so $h = 0$ and $\alpha \ne 0$. Hence $f \in \ker A^*$. It follows that $\ker B^* \subset \ker A^*$.

The inclusion on kernels allows us to construct a linear transformation X_0 on $\operatorname{ran} B^* \oplus \ker B$ into $\mathcal{A}$ such that $A^* = X_0B^*$ and X_0 annihilates $\ker B$. Since $AA^* \le BB^*$ and $\ker B$ is uniformly positive by Theorem 1.3.1, we obtain

$$\langle X_0u, X_0u\rangle_{\mathcal{A}} \le \langle u, u\rangle_{\mathcal{B}}, \quad u \in \operatorname{dom} X_0.$$

The domain of X_0 is dense in $\mathcal{B}$ and contains $B^*\mathcal{C}_-$, which is maximal uniformly negative in $\mathcal{B}$ by Theorem 1.3.6. Moreover, X_0 maps $B^*\mathcal{C}_-$ onto $A^*\mathcal{C}_-$, which is maximal uniformly negative in $\mathcal{A}$. By Theorem 1.4.4, X_0 extends to a bicontraction $C^* \in \mathbf{B}(\mathcal{B},\mathcal{A})$. Then $C \in \mathbf{B}(\mathcal{A},\mathcal{B})$ is a bicontraction and $A = BC$. ■

Leech's theorem is a structured form of Shmul'yan's theorem.

THEOREM 3.4.2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be Kreĭn spaces, $S_{\mathcal{A}} \in \mathbf{B}(\mathcal{A})$, $S_{\mathcal{B}} \in \mathbf{B}(\mathcal{B})$, $S_{\mathcal{C}} \in \mathbf{B}(\mathcal{C})$ isometries, and $A \in \mathbf{B}(\mathcal{A},\mathcal{C})$, $B \in \mathbf{B}(\mathcal{B}, \mathcal{C})$ bicontractions. Assume that*

$$AS_{\mathcal{A}} = S_{\mathcal{C}}A \qquad \text{and} \qquad BS_{\mathcal{B}} = S_{\mathcal{C}}B.$$

Then $A = BC$ for some bicontraction $C \in \mathbf{B}(\mathcal{A},\mathcal{B})$ satisfying

$$CS_{\mathcal{A}} = S_{\mathcal{B}}C$$

if and only if $AA^* \le BB^*$.

The case in which $S_\mathcal{A}$, $S_\mathcal{B}$, $S_\mathcal{C}$ coincide with the identity operators is Shmul'yan's theorem.

Proof. Necessity follows from Theorem 3.4.1.

Conversely, assume $AA^* \le BB^*$. Set $\mathcal{H} = \overline{A^*\mathcal{C}} \subset \mathcal{A}$ and $\mathcal{K} = \overline{B^*\mathcal{C}} \subset \mathcal{B}$. Since A and B are contractions, $\mathcal{H}^\perp = \ker A$ and $\mathcal{K}^\perp = \ker B$ are uniformly positive by Theorem 1.3.1. In particular, $\mathcal{H}$ and $\mathcal{K}$ are regular subspaces of $\mathcal{A}$ and $\mathcal{B}$ and may be viewed as Kreĭn spaces in the scalar products of $\mathcal{A}$ and $\mathcal{B}$. By Shmul'yan's theorem there is a bicontraction $X \in \mathbf{B}(\mathcal{A},\mathcal{B})$ such that $A = BX$. Define

$$T_1 = \mathrm{Pr}_\mathcal{H}\, S_\mathcal{A}|\mathcal{H} \in \mathbf{B}(\mathcal{H}), \qquad T_2 = \mathrm{Pr}_\mathcal{K}\, S_\mathcal{B}|\mathcal{K} \in \mathbf{B}(\mathcal{K}),$$

and

$$Y = \mathrm{Pr}_\mathcal{K}\, X|\mathcal{H} \in \mathbf{B}(\mathcal{H},\mathcal{K}).$$

Then T_1 and T_2 are contractions, and Y is a bicontraction. Since $S_\mathcal{A}^*\mathcal{H} \subset \mathcal{H}$ and $S_\mathcal{B}^*\mathcal{K} \subset \mathcal{K}$, we have $T_1^* = S_\mathcal{A}^*|\mathcal{H}$ and $T_2^* = S_\mathcal{B}^*|\mathcal{K}$. We also have $X^*|\mathcal{K} = Y^*$ because $X^*B^* = A^*$, and so

$$\begin{aligned} Y^*T_2^*B^* &= X^*S_\mathcal{B}^*B^* = X^*B^*S_\mathcal{C}^* \\ &= A^*S_\mathcal{C}^* = S_\mathcal{A}^*A^* = S_\mathcal{A}^*X^*B^* = T_1^*Y^*B^*. \end{aligned}$$

Therefore $YT_1 = T_2Y$.

Next note that

$$\tilde{\mathcal{H}} = \bigvee_0^\infty S_\mathcal{A}^n\mathcal{H} \qquad \text{and} \qquad \tilde{\mathcal{K}} = \bigvee_0^\infty S_\mathcal{B}^n\mathcal{K}$$

are regular subspaces of $\mathcal{A}$ and $\mathcal{B}$ and hence Kreĭn spaces in the scalar products of $\mathcal{A}$ and $\mathcal{B}$. For example, we have

$$\mathcal{H} \subset \mathcal{H} \vee S_\mathcal{A}\mathcal{H} \subset \mathcal{H} \vee S_\mathcal{A}\mathcal{H} \vee S_\mathcal{A}^2\mathcal{H} \subset \cdots,$$

where at each stage the extension is obtained by forming a direct sum with a Hilbert space. Therefore each subspace in the chain is regular. By Lemma 1.1.9, $\tilde{\mathcal{H}}$ is a regular subspace of $\mathcal{A}$. In a similar way, $\tilde{\mathcal{K}}$ is a regular subspace of $\mathcal{B}$. Note that $\tilde{\mathcal{H}}$ is invariant under $S_\mathcal{A}$ and $S_\mathcal{A}^*$, and $\tilde{\mathcal{K}}$ is invariant under $S_\mathcal{B}$ and $S_\mathcal{B}^*$. Therefore $U_1 = S_\mathcal{A}|\tilde{\mathcal{H}}$ and $U_2 = S_\mathcal{B}|\tilde{\mathcal{K}}$ are minimal isometric dilations of T_1 and T_2.

By Theorem 3.2.2, there is a bicontraction $\tilde{Y} \in \mathbf{B}(\tilde{\mathcal{H}},\tilde{\mathcal{K}})$ such that $\tilde{Y}U_1 = U_2\tilde{Y}$ and $\mathrm{Pr}_\mathcal{K}\,\tilde{Y}|\mathcal{H} = Y$. By Lemma 3.3.1, $\tilde{Y}^*|\mathcal{K} = Y^*$. Set

$$C = \tilde{Y}\mathrm{Pr}_{\tilde{\mathcal{H}}} \in \mathbf{B}(\mathcal{A},\mathcal{B}).$$

Then $C = \mathrm{Pr}_{\tilde{\mathcal{K}}}\tilde{Y}\mathrm{Pr}_{\tilde{\mathcal{H}}}$, $C^* = \mathrm{Pr}_{\tilde{\mathcal{H}}}\tilde{Y}^*\mathrm{Pr}_{\tilde{\mathcal{K}}}$, and

$$X^*|\mathcal{K} = Y^* = \tilde{Y}^*|\mathcal{K} = C^*|\mathcal{K}.$$

So from $A^* = X^*B^*$ we obtain $A^* = C^*B^*$, hence $A = BC$. Since $\tilde{Y}$ is a bicontraction and the orthogonal complements of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$ in $\mathcal{A}$ and $\mathcal{B}$ are uniformly positive, C is a bicontraction. Finally,

$$CS_{\mathcal{A}}|\tilde{\mathcal{H}} = \tilde{Y}U_1 = U_2\tilde{Y} = S_{\mathcal{B}}C|\tilde{\mathcal{H}},$$

whereas

$$CS_{\mathcal{A}}|\tilde{\mathcal{H}}^{\perp} = 0 = S_{\mathcal{B}}C|\tilde{\mathcal{H}}^{\perp}.$$

Thus $CS_{\mathcal{A}} = S_{\mathcal{B}}C$. ■

Notes on Chapter 3

The dilation theory in §3.1 follows the methods of the Hilbert space case as given in Sz.-Nagy and Foiaş [**66**], with some modifications. An extension of dilation theory to arbitrary operators was first made by Davis [**23**]; see Bognár [**12**]. In the indefinite setting, dilation theory is used by Bruinsma, Dijksma, and de Snoo [**20**], Constantinescu and Ghéondea [**21,22**], and Dritschel [**29**].

The commutant lifting theorem in the Hilbert space case is given in Sz.-Nagy and Foiaş [**66**]. It was inspired by applications in interpolation theory due to Sarason [**60**]. There have been numerous accounts and applications; references are given in Rosenblum and Rovnyak [**59**]. In the indefinite case, extensions of the commutant lifting theorem are proved in Alpay [**1**], de Branges [**14,16,17**], Constantinescu and Ghéondea [**21,22**], and Dritschel [**29,30**] in various settings. The general case of Theorem 3.2.1 first appears in Dritschel [**30**]. The corresponding result on bicontractions, Theorem 3.2.2, is due to Dritschel [**29**]; different proofs and extensions are given in Constantinescu and Ghéondea [**22**] and de Branges [**17**]. Theorem 3.2.3 is new.

The use of matrix completions in commutant lifting problems is a known way to organize calculations in the Hilbert space case. The idea seems to be due to Parrott [**54**]. It has been adopted by other authors, including Frazho [**35**] and Pták and Vrbová [**56**]. The method is also used in Dritschel [**29,30**] and Constantinescu and Ghéondea [**21,22**].

Theorems 3.3.2 and 3.3.3 appear in Dritschel [**30**] and [**29**], respectively. The graph approach is due to Ball and Helton [**11**]. Much has been written on the labeling problem in the Hilbert space case. See Arsene, Ceauşescu, and Foiaş [**6**], Foiaş and Frazho [**34**], and Helton et al. [**39**]. A comparison of methods is given in Frazho [**35**].

Leech's original theorem [**51**] concerns the factorization of power series with operator coefficients in the form $A(z) = B(z)C(z)$. Helton [**39, p. 52**] gives a version for bounded measurable matrix valued functions. Rosenblum [**58**] proves an abstract form of Leech's theorem using the commutant lifting theorem (see also [**59**]). Leech's theorem was generalized to Kreĭn spaces by de Branges [**14,16**]. The version in Theorem 3.4.2 appears in Dritschel [**30**] and extends the abstract result in Rosenblum [**58**].

Appendix A: Complementation Theory

The operator methods in this paper are related to a theory of complementation in Kreĭn spaces due to de Branges [**13,15**]. The Kreĭn space version of complementation extends Hilbert space notions which appear in de Branges and Rovnyak [**18,19**]. We state without proof some results on an operator approach to complementation, with a full account to appear in a later paper.

A Kreĭn space $\mathcal{P}$ is *contained continuously*, *contractively*, or *isometrically* in a Kreĭn space $\mathcal{H}$ if $\mathcal{P}$ is a linear subspace of $\mathcal{H}$ and the inclusion mapping is continuous, contractive, or isometric, respectively. Let $\mathcal{P}$ be a Kreĭn space which is contained continuously in a Kreĭn space $\mathcal{H}$, and let A be the inclusion mapping. We associate with $\mathcal{P}$ the selfadjoint operator $P \in \mathbf{B}(\mathcal{H})$ given by

$$P = AA^*.$$

The operator P plays the role of a generalized projection for $\mathcal{P}$. Viewed as mappings on $\mathcal{H}$, the operators P and A^* have the same action. In the terminology of de Branges, P is the selfadjoint operator on $\mathcal{H}$ which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The positive and negative indices of the Kreĭn space $\mathcal{P}$ coincide with the hermitian indices $h_\pm(P)$ of the operator P by Theorem 1.2.1. The range $\mathcal{P}_0$ of P is a scalar product space with scalar product defined by

$$\langle Pf, Pg\rangle_{\mathcal{P}_0} = \langle Pf, g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}.$$

The space $\mathcal{P}_0$ is contained in $\mathcal{P}$ as a dense subspace, and the $\mathcal{P}_0$ and $\mathcal{P}$ scalar products coincide on $\mathcal{P}_0$. Every selfadjoint operator P arises in this way.

THEOREM A1. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Write P in any way in the form*

$$P = EE^*,$$

where $E \in \mathbf{B}(\mathcal{E}, \mathcal{H})$ for some Kreĭn space $\mathcal{E}$ and E has zero kernel. Let $\mathcal{P}_E$ be the range of E viewed as a Kreĭn space in the scalar product which makes E a Kreĭn space isomorphism of $\mathcal{E}$ onto $\mathcal{P}_E$. Then $\mathcal{P}_E$ is contained continuously in $\mathcal{H}$, and the adjoint of the inclusion of $\mathcal{P}_E$ in $\mathcal{H}$ coincides with P.

Special properties hold whenever the operators P and Q for two spaces $\mathcal{P}$ and $\mathcal{Q}$ satisfy $P+Q=1$. In the special case of isometric inclusion, these properties reflect the fact that $\mathcal{P}$ and $\mathcal{Q}$ are regular subspaces of the larger Kreĭn space $\mathcal{H}$, and $\mathcal{H}$ decomposes into the orthogonal direct sum of $\mathcal{P}$ and $\mathcal{Q}$.

Theorem A2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be Kreĭn spaces which are contained continuously in a Kreĭn space $\mathcal{H}$, and let P and Q be the selfadjoint operators on $\mathcal{H}$ which coincide with the adjoints of the inclusions of $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{H}$. Assume that $P + Q = 1$.*

(i) The mapping $U : (f, g) \to f + g$ is a partial isometry from $\mathcal{P} \times \mathcal{Q}$ onto $\mathcal{H}$ with adjoint $U^* : h \to (Ph, Qh)$.

(ii) The intersection $\mathcal{L}$ of $\mathcal{P}$ and $\mathcal{Q}$ is a Kreĭn space in the scalar product defined by

$$\langle f, g\rangle_{\mathcal{L}} = \langle f, g\rangle_{\mathcal{P}} + \langle f, g\rangle_{\mathcal{Q}}, \qquad f, g \in \mathcal{L}.$$

The Kreĭn space $\mathcal{L}$ is called the **overlapping space** *for $\mathcal{P}$ and $\mathcal{Q}$. It is contained continuously in $\mathcal{H}$, and the adjoint of the inclusion coincides with PQ.*

(iii) The following conditions are equivalent: (a) $\mathcal{P}$ is contained contractively in $\mathcal{H}$, (b) $\mathcal{Q}$ is contained contractively in $\mathcal{H}$, (c) $P^2 \leq P$, (d) $Q^2 \leq Q$, (e) U is a contraction, and (f) the overlapping space $\mathcal{L}$ is a Hilbert space.

(iv) The following conditions are equivalent: (a) $\mathcal{P}$ and $\mathcal{Q}$ are contained isometrically in $\mathcal{H}$ as regular subspaces with $\mathcal{Q} = \mathcal{P}^{\perp}$, (b) $P^2 = P$, (c) $Q^2 = Q$, (d) U is an isometry, and (e) the overlapping space $\mathcal{L}$ contains no nonzero element.

The ideas in Theorem A2 go back to Schwartz [**63**], who created a theory of operator ranges both in the Hilbert space and Kreĭn space settings. The authors thank Daniel Alpay for calling their attention to Schwartz's paper; a related work is Alpay [**2**]. It is of interest to know when a unique Kreĭn space $\mathcal{P}$ is associated with a given selfadjoint operator P. The following uniqueness condition, using an extra hypothesis, is similar to one given by Schwartz [**63**].

Theorem A3. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Kreĭn spaces which are contained continuously in $\mathcal{H}$ such that the adjoints of the inclusions each coincide with P. If $\mathcal{P}_1$ is contained continuously in $\mathcal{P}_2$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

Uniqueness holds under a condition of a different nature.

Theorem A4. *Let $\mathcal{H}$ be a Kreĭn space, and let $P \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Let $\mathcal{P}_1$ be a Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{P}_1$ in $\mathcal{H}$ coincides with P. Assume that the range of P contains a subspace $\mathcal{M}$ which is maximal uniformly definite in $\mathcal{P}_1$. Then if $\mathcal{P}_2$ is any Kreĭn space which is contained continuously in $\mathcal{H}$ such that the adjoint of the inclusion of $\mathcal{P}_2$ in $\mathcal{H}$ coincides with P, $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

Corollary A5. *Let $\mathcal{P}$ be a Kreĭn space which is contained continuously in a Kreĭn space $\mathcal{H}$, and let $\mathcal{P} \in \mathbf{B}(\mathcal{H})$ be the selfadjoint operator which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. Assume that the range of P contains a subspace $\mathcal{M}$ which is maximal uniformly definite in $\mathcal{P}$. If*

$$P = EE^*,$$

where $E \in \mathbf{B}(\mathcal{E}, \mathcal{H})$ for some Kreĭn space $\mathcal{E}$ and E has zero kernel, then E is a Kreĭn space isomorphism of $\mathcal{E}$ onto $\mathcal{P}$.

The condition for uniqueness in Theorem A4 is always satisfied in the important special case of contractive inclusion. This yields a new derivation of a result of de Branges [**13**].

Theorem A6. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be Kreĭn spaces which are contained continuously and contractively in a Kreĭn space $\mathcal{H}$ such that the adjoints of the inclusions each coincide with the selfadjoint operator P on $\mathcal{H}$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal isometrically.*

Combining Theorem A6 and Corollary 5, we see that the Kreĭn spaces in de Branges' theory of complementation [**13**] are operator ranges. In the indefinite case, this was first shown by Heinz Langer (private communication, 1988) using his theory of definitizable operators and spectral functions [**49**]. In Hilbert spaces, this has been known for a long time. It was shown, for example, in seminar lectures by Marvin Rosenblum at the University of Virginia in the 1960's. In Hilbert spaces, Sarason has used the operator view in applications to function theory in a series of papers including [**61,62**].

The choice of a contraction operator leads to an example of complementation Kreĭn spaces.

Theorem A7. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ be a contraction. Define $\mathcal{M}(T)$ to be the range of T in the scalar product which makes T a partial isometry of $\mathcal{H}$ onto $\mathcal{M}(T)$. Let $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ be any defect operator for T^*. Define $\mathcal{H}(T)$ to be the range of D in the scalar product which makes D a Kreĭn space isomorphism of $\mathcal{D}$ onto $\mathcal{H}(T)$. Then $\mathcal{M}(T)$ and $\mathcal{H}(T)$ are Kreĭn spaces which are contained continuously and contractively in $\mathcal{K}$, and the adjoints of the inclusions coincide with TT^* and $1 - TT^*$, respectively. The definition of $\mathcal{H}(T)$ is independent of the choice of defect operator D for T^*. An element g of $\mathcal{K}$ belongs to $\mathcal{H}(T)$ if and only if*

$$\sup_{u \in \mathcal{H}} \left[\langle g + Tu, g + Tu \rangle_{\mathcal{K}} - \langle u, u \rangle_{\mathcal{H}} \right] < \infty,$$

in which case the value of the supremum is $\langle g, g \rangle_{\mathcal{H}(T)}$.

Appendix B: More on Julia Operators

The existence of a Julia operator was proved in §1.2 by an argument based on the factorization of any selfadjoint operator in the form AA^*, where A is an operator having zero kernel. We present an alternative proof following the original method due to Arsene, Constantinescu, and Ghéondea [**8**]. The method gives the additional information that a particular choice of Julia operator satisfies norm estimates (Theorem B3). We also show that uniqueness holds under weaker assumptions than that of Theorem 2.4.5 (Theorem B4).

Two preliminary Hilbert space results are needed. As usual, we write $*$ for Kreĭn space adjoint and $\times$ for Hilbert space adjoint. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and T is a continuous everywhere defined operator on $\mathcal{H}$ to $\mathcal{K}$, then by the *polar factorization* of T we mean the representation $T = RU$, where U is a partial isometry on $\mathcal{H}$ to $\mathcal{K}$ with kernel equal to the kernel of T and R is a nonnegative operator on $\mathcal{K}$ which is zero on the orthogonal complement of the range of U.

LEMMA B1. *Let $\mathcal{H}$ be a Kreĭn space, and let $H \in \mathbf{B}(\mathcal{H})$ be a selfadjoint operator. Assume that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a fundamental decomposition. Then there is a Kreĭn space $\mathcal{A}$ with fundamental decomposition $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ and an operator $A \in \mathbf{B}(\mathcal{A}, \mathcal{H})$ with these properties:*

(i) A has zero kernel and $H = AA^*$;
(ii) if $A = RW$ is the polar factorization of A as an operator on $\mathcal{A}_+ \oplus |\mathcal{A}_-|$ to $\mathcal{H}_+ \oplus |\mathcal{H}_-|$, then $AA^* = R^2WW^*$ and $R^2 = AA^*(WW^*)^\times$.

Proof. The operator A constructed in the proof of Theorem 1.2.2 satisfies (i). We show that A also satisfies (ii).

The representation $A = RW$ in the proof of Theorem 1.2.2 is the polar factorization of A as an operator on $\mathcal{A}_+ \oplus |\mathcal{A}_-|$ to $\mathcal{H}_+ \oplus |\mathcal{H}_-|$. The operator $WJ_{\mathcal{A}}W^\times$ is 1 on $\mathcal{M}_+$, -1 on $\mathcal{M}_-$, and 0 on the orthogonal complement of $\mathcal{M}_+ + \mathcal{M}_-$ in $\mathcal{H}_+ \oplus |\mathcal{H}_-|$. Hence $R(WJ_{\mathcal{A}}W^\times)R = R^2(WJ_{\mathcal{A}}W^\times)$, and

$$AA^* = RWW^*R^* = R(WJ_{\mathcal{A}}W^\times)RJ_{\mathcal{H}} = R^2(WJ_{\mathcal{A}}W^\times)J_{\mathcal{H}} = R^2WW^*.$$

Since $WW^*(WW^*)^\times = WW^\times$ is the projection of $\mathcal{H}_+ \oplus |\mathcal{H}_-|$ onto $\overline{\operatorname{ran} R}$,

$$AA^*(WW^*)^\times = R^2WW^*(WW^*)^\times = R^2,$$

and so (ii) holds. ■

We use a result due to Kreĭn [**41**] and Reid [**57**] and rediscovered by Lax [**50**] and Dieudonné [**27**]. We include Reid's proof for completeness.

LEMMA B2. *Let $\mathcal{H}$ be a Hilbert space and $A, X \in \mathbf{B}(\mathcal{H})$. If A is nonnegative and AX is selfadjoint, then for all $f \in \mathcal{H}$,*

$$|\langle AXf, f\rangle| \leq \|X\| \langle Af, f\rangle.$$

Proof. It is enough to give the proof when $\|X\| = 1$. Since A is nonnegative, for any f and g in $\mathcal{H}$,

$$|\langle Af, g\rangle| \leq \langle Af, f\rangle^{1/2} \langle Ag, g\rangle^{1/2} \leq \frac{1}{2} \left[\langle Af, f\rangle + \langle Ag, g\rangle\right].$$

Since $AX = X^* A$, for any positive integer n,

$$|\langle AX^n f, f\rangle| \leq \frac{1}{2} \left[\langle AX^n f, X^n f\rangle + \langle Af, f\rangle\right] = \frac{1}{2} \left[\left\langle AX^{2n} f, f\right\rangle + \langle Af, f\rangle\right].$$

Iteration of this inequality yields

$$|\langle AXf, f\rangle| \leq 2^{-n} \left\langle AX^{2^n} f, f\right\rangle + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) \langle Af, f\rangle,$$

and we obtain the result on letting n tend to ∞. ■

THEOREM B3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces with fundamental decompositions $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. For any $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, there exist Kreĭn spaces $\mathcal{D}$ and $\tilde{\mathcal{D}}$ with fundamental decompositions $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_+ \oplus \tilde{\mathcal{D}}_-$ and a Julia operator*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

such that

$$\|L\| \leq \|T\|$$

and

$$\max\left(\|D\|, \|\tilde{D}\|\right) \leq \left[1 + \|T\|^2\right]^{1/2}.$$

Proof. By Lemma B1, there exist Kreĭn spaces $\mathcal{D}$ and $\tilde{\mathcal{D}}$ with fundamental decompositions $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_+ \oplus \tilde{\mathcal{D}}_-$, and operators $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ with zero kernels, such that

$$1 - TT^* = DD^*, \qquad 1 - T^*T = \tilde{D}\tilde{D}^*, \qquad \text{(B} - 1\text{)}$$

and the polar representations $D = RW$ and $\tilde{D} = \tilde{R}\tilde{W}$ as Hilbert space operators satisfy

$$1 - TT^* = R^2 WW^*, \qquad \tilde{R}^2 = (1 - T^*T)(\tilde{W}\tilde{W}^*)^\times .$$

In particular, $\|D\|$ and $\|\tilde{D}\|$ are bounded by $\left[1 + \|T\|^2\right]^{1/2}$.

We construct $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ with the aid of Lemma B2, applied to $A = R^2$ and $X = (WW^*)T(\tilde{W}\tilde{W}^*)^\times T^\times$ viewed as operators on $\mathcal{K}_+ \oplus |\mathcal{K}_-|$. Clearly A is nonnegative, and

$$\begin{aligned} AX &= R^2(WW^*)T(\tilde{W}\tilde{W}^*)^\times T^\times = (1 - TT^*)T(\tilde{W}\tilde{W}^*)^\times T^\times \\ &= T(1 - T^*T)(\tilde{W}\tilde{W}^*)^\times T^\times = T\tilde{R}^2 T^\times \end{aligned}$$

is selfadjoint. Since $\|X\| \leq \|T\|^2$, Lemma B2 yields

$$\left|\left\langle R^2 X f, f\right\rangle_{\mathcal{K}_+ \oplus |\mathcal{K}_-|}\right| \leq \|T\|^2 \left\langle R^2 f, f\right\rangle_{\mathcal{K}_+ \oplus |\mathcal{K}_-|}, \qquad f \in \mathcal{K}.$$

In other words, $T\tilde{R}^2 T^\times \leq \|T\|^2 R^2$ as operators on $\mathcal{K}_+ \oplus |\mathcal{K}_-|$. Therefore $T\tilde{R} = RC$ where $C \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ and $\|C\| \leq \|T\|$. We can choose C so that $\operatorname{ran} C \subset \overline{\operatorname{ran} R}$, and then $T\tilde{R}\tilde{W} = RWW^\times C\tilde{W}$. We obtain

$$T\tilde{D} = -DL^*, \qquad \text{(B} - 2\text{)}$$

where $L = -(W^\times C\tilde{W})^* \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ and $\|L\| \leq \|T\|$.

It remains to prove the six identities in (1.2.1) and (1.2.2). The relations (1.2.1a) and (1.2.2a,b) hold by (B-1) and (B-2). By (1.2.2b),

$$\tilde{D}LD^* = -\tilde{D}\tilde{D}^*T^* = -(1 - T^*T)T^* = -T^*(1 - TT^*) = -T^*DD^*.$$

Since $\ker D = \{0\}$, (1.2.1b) follows. Similarly,

$$\begin{aligned} DL^*L &= -T\tilde{D}L = TT^*D = (1 - DD^*)D = D(1 - D^*D), \\ \tilde{D}LL^* &= -T^*DL^* = T^*T\tilde{D} = (1 - \tilde{D}\tilde{D}^*)\tilde{D} = \tilde{D}(1 - \tilde{D}^*\tilde{D}), \end{aligned}$$

and (1.2.1c) and (1.2.2c) hold. ■

We proved uniqueness of Julia operators in Theorem 2.4.5 under the assumption that either T or T^* is a contraction. The conclusion can be obtained with a weaker hypothesis, namely, that one of the operators $1 - T^*T$ or $1 - TT^*$ has at most a finite number of negative squares.

THEOREM B4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Kreĭn spaces, and let $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ have two Julia operators*

$$\begin{pmatrix} T & D_j \\ \tilde{D}_j^* & L_j \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}_j, \mathcal{K} \oplus \tilde{\mathcal{D}}_j), \qquad j = 1, 2.$$

If one of the indices $h_-(1 - T^*T)$, $h_-(1 - TT^*)$, $h_+(1 - T^*T)$, or $h_+(1 - TT^*)$ is finite, then there exist unitary operators $V \in \mathbf{B}(\mathcal{D}_2, \mathcal{D}_1)$ and $\tilde{V} \in \mathbf{B}(\tilde{\mathcal{D}}_2, \tilde{\mathcal{D}}_1)$ such that

$$\begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{V}^* \end{pmatrix} \begin{pmatrix} T & D_1 \\ \tilde{D}_1^* & L_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}.$$

LEMMA B5. *Any dense subspace $\mathcal{M}$ of a Pontryagin space $\mathcal{H}$ contains a maximal uniformly negative subspace.*

Proof of Lemma B5. We prove the assertion by showing that if a uniformly negative subspace $\mathcal{N}$ of $\mathcal{M}$ is not maximal, then there exists a uniformly negative subspace $\mathcal{N}'$ of $\mathcal{M}$ which properly contains $\mathcal{N}$. Granting this, then starting with $\mathcal{N} = \{0\}$ one can construct in a finite number of steps a maximal uniformly negative subspace of $\mathcal{M}$.

Assume that $\mathcal{N}$ is a uniformly negative subspace which is contained in $\mathcal{M}$ and that $\mathcal{N}$ is not maximal. Choose a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\mathcal{N} \subset \mathcal{H}_-$. Norms are to be computed in the associated Hilbert space $\mathcal{H}_+ \oplus |\mathcal{H}_-|$. Note that $\dim \mathcal{H}_- < \infty$ because $\mathcal{H}$ is a Pontryagin space.

Since $\mathcal{N}$ is not maximal, there is a unit vector $e \in \mathcal{H}_-$ which is orthogonal to $\mathcal{N}$. Since $\mathcal{M}$ is dense in $\mathcal{H}$, we can choose $\phi \in \mathcal{M}$ such that $\|e - \phi\| \leq 1/3$. Then $\|\phi\| \geq 2/3$. We show that the span $\mathcal{N}'$ of $\mathcal{N}$ and ϕ is uniformly negative. For any $g \in \mathcal{N}'$ we can write $g = f + \alpha\phi$ where $f \in \mathcal{N}$ and α is a scalar. Then

$$\begin{aligned} \langle g, g\rangle_{\mathcal{H}} &= \langle f, f\rangle_{\mathcal{H}} + |\alpha|^2 \langle \phi, \phi\rangle_{\mathcal{H}} + 2\,\mathrm{Re}\,[\overline{\alpha}\langle f, \phi\rangle_{\mathcal{H}}] \\ &= -\|f\|^2 - |\alpha|^2\|\phi\|^2 + 2\,\mathrm{Re}\,[\overline{\alpha}\langle f, \phi - e\rangle_{\mathcal{H}}] \end{aligned}$$

because $e \perp f$. Thus

$$\begin{aligned} \langle g, g\rangle_{\mathcal{H}} &\leq -\|f\|^2 - |\alpha|^2\|\phi\|^2 + 2|\alpha|\|f\|\|\phi - e\| \\ &\leq -\|f\|^2 - \frac{4}{9}|\alpha|^2 + 2\cdot\frac{1}{3}|\alpha|\|f\| \\ &= -\left[\|f\| - \frac{1}{3}|\alpha|\right]^2 - \frac{1}{3}|\alpha|^2 \\ &\leq 0 \end{aligned}$$

with equality only if $g = 0$. Thus the form $\langle g, g\rangle_{\mathcal{H}}$ is strictly negative on the unit sphere of $\mathcal{N}'$, which is a compact set. Hence $\langle g, g\rangle_{\mathcal{H}} \leq -\delta\|g\|^2$ for all $g \in \mathcal{N}'$ and some $\delta > 0$. Therefore $\mathcal{N}'$ is a subspace of $\mathcal{M}$ which is uniformly negative and properly contains $\mathcal{N}$. ■

Proof of Theorem B4. We assume that $h_-(1 - T^*T) < \infty$. For the case $h_-(1-TT^*) < \infty$, we obtain the conclusion by replacing T by T^*. The other cases are handled by easy modifications as noted below.

The hypothesis $h_-(1-T^*T) < \infty$ implies that $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ are Pontryagin spaces each with finite negative index equal to $h_-(1 - T^*T)$ (see the remarks following Definition 1.2.3).

Since $\tilde{D}_1\tilde{D}_1^* = 1 - T^*T = \tilde{D}_2\tilde{D}_2^*$, for any $f, g \in \mathcal{H}$,

$$\left\langle \tilde{D}_1^* f, \tilde{D}_1^* g \right\rangle_{\tilde{\mathcal{D}}_1} = \left\langle \tilde{D}_2^* f, \tilde{D}_2^* g \right\rangle_{\tilde{\mathcal{D}}_2}.$$

It follows that there is a well defined and densely defined isometry X from $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$ with $\operatorname{dom} X = \operatorname{ran} \tilde{D}_1^*$ and $\operatorname{ran} X = \operatorname{ran} \tilde{D}_2^*$ such that

$$X\tilde{D}_1^* f = \tilde{D}_2^* f, \qquad f \in \mathcal{H}.$$

Since $\tilde{\mathcal{D}}_1$ is a Pontryagin space, Lemma B5 implies that $\operatorname{dom} X$ contains a maximal uniformly negative subspace $\tilde{\mathcal{D}}_1^-$. But X is an isometry and $\tilde{\mathcal{D}}_2$ is a Pontryagin space with the same negative index as that of $\tilde{\mathcal{D}}_1$. So $X\tilde{\mathcal{D}}_1^-$ is a maximal uniformly negative subspace $\tilde{\mathcal{D}}_2^-$ of $\tilde{\mathcal{D}}_2$. Therefore by Theorem 1.4.4, X has an extension by continuity to a unitary operator $\tilde{V}^* \in \mathbf{B}(\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2)$. By construction, $\tilde{D}_2^* = \tilde{V}^*\tilde{D}_1^*$.

If we had assumed instead that one of the positive hermitian indices is finite, then the same conclusion could be drawn. We simply replace the defect spaces by their anti-spaces and note that the densely defined isometry X is still an isometry on the new spaces. Then we apply the argument given above to conclude that X may be extended to a unitary operator.

We show next that

$$\begin{pmatrix} D_1 \\ \tilde{V}^* L_1 \end{pmatrix} \in \mathbf{B}(\mathcal{D}_1, \mathcal{K} \oplus \tilde{\mathcal{D}}_2)$$

is a defect operator for the adjoint of $\left(\begin{smallmatrix} T \\ \tilde{D}_2^* \end{smallmatrix}\right) \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{D}}_2)$. The operator has zero kernel because D_1 has zero kernel. We have

$$\begin{aligned}
1 - \begin{pmatrix} T \\ \tilde{D}_2^* \end{pmatrix} \begin{pmatrix} T^* & \tilde{D}_2 \end{pmatrix} &= 1 - \begin{pmatrix} T \\ \tilde{V}^*\tilde{D}_1^* \end{pmatrix} \begin{pmatrix} T^* & \tilde{D}_1\tilde{V} \end{pmatrix} \\
&= \begin{pmatrix} 1 - TT^* & -T\tilde{D}_1\tilde{V} \\ -\tilde{V}^*\tilde{D}_1^*T^* & 1 - \tilde{V}^*\tilde{D}_1^*\tilde{D}_1\tilde{V} \end{pmatrix} \\
&= \begin{pmatrix} D_1D_1^* & D_1L_1^*\tilde{V} \\ \tilde{V}^*L_1D_1^* & \tilde{V}^*L_1L_1^*\tilde{V} \end{pmatrix} \\
&= \begin{pmatrix} D_1 \\ \tilde{V}^*L_1 \end{pmatrix} \begin{pmatrix} D_1^* & L_1^*\tilde{V} \end{pmatrix},
\end{aligned}$$

which proves the assertion.

Now to complete the proof, think of $\begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & L_2 \end{pmatrix}$ as $(A \quad B)$, where $A = \begin{pmatrix} T \\ \tilde{D}_2^* \end{pmatrix} \in \mathbf{B}(\mathcal{H}, \mathcal{K} \oplus \tilde{\mathcal{D}}_2)$ and $B = \begin{pmatrix} D_2 \\ L_2 \end{pmatrix} \in \mathbf{B}(\mathcal{D}_2, \mathcal{K} \oplus \tilde{\mathcal{D}}_2)$. The operator A is an isometry and hence a contraction. The operator A^* has defect operator

$$\begin{pmatrix} D_1 \\ \tilde{V}^* L_1 \end{pmatrix} \in \mathbf{B}(\mathcal{D}_1, \mathcal{K} \oplus \tilde{\mathcal{D}}_2)$$

by what was shown above. Since $(A \quad B)$ is unitary and hence a bicontraction, Theorem 2.3.3 yields a bicontraction $V \in \mathbf{B}(\mathcal{D}_2, \mathcal{D}_1)$ such that

$$\begin{pmatrix} D_2 \\ L_2 \end{pmatrix} = \begin{pmatrix} D_1 \\ \tilde{V}^* L_1 \end{pmatrix} V.$$

We see easily that V is unitary. In fact,

$$D_1 V V^* D_1^* = D_2 D_2^* = 1 - TT^* = D_1 D_1^*,$$

and since D_1 has zero kernel, this implies that $VV^* = 1$, that is, V^* is an isometry. But $D_2 = D_1 V$ and therefore V has zero kernel. By elementary properties of isometries and partial isometries described in §1.1(C), V is unitary. By construction,

$$\begin{pmatrix} T & D_2 \\ \tilde{D}_2^* & L_2 \end{pmatrix} = \begin{pmatrix} T & D_1 V \\ \tilde{V}^* \tilde{D}_1^* & \tilde{V}^* L_1 V \end{pmatrix},$$

as required. ■

After this paper was completed, the authors obtained a copy of Agnes Yang's thesis [**68**]. This includes a stronger form of Kreĭn 's theorem, Lemma B2, which is due to Dijksma, Langer, and de Snoo ("Unitary colligations in Krein spaces and their role in the extension theory of isometries and symmetric linear relations in Hilbert spaces," Functional Analysis, II (Dubrovnic 1985), pp. 1–42, Lecture Notes in Math., Vol. 1242, Springer, Berlin-New York, 1987; MR 89a:47055). The stronger form of Kreĭn 's theorem is essentially no more difficult to prove, and by appropriate choices of operators one obtains a somewhat more direct proof of the construction of Julia operators in Theorem B3.

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