CHARACTERIZATION OF SPACES $\mathcal{H}(M)$

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Throughout $\mathcal{C}$ will denote a fixed Hilbert space, called the coefficient space. By a vector we mean an element of $\mathcal{C}$, and by an operator we mean a bounded linear transformation of $\mathcal{C}$ into itself. The norm of a vector c is denoted $|c|$, and the operator norm and adjoint of an operator A are denoted $|A|$ and $\bar{A}$. If b is a vector, then $\bar{b}$ is the linear functional on $\mathcal{C}$ such that $\bar{b}a = \langle a, b\rangle$ for every vector a. We choose and fix an operator I such that iI is both isometric and selfadjoint. We choose and fix a non-empty open subset Ω of the complex plane which is symmetric about the real axis.

Consider a Hilbert space $\mathcal{H}$ whose elements are vector valued analytic functions defined in Ω. We say that $\mathcal{H}$ is a space $\mathcal{H}(M)$ if it admits a kernel function of the form

$$K(w,z) = [M(z)I\bar{M}(w) - I]/[2\pi(z-\bar{w})],$$

where $M(z)$ is an operator valued analytic function defined in Ω such that

$$M(\bar{w})I\bar{M}(w) = \bar{M}(w)IM(\bar{w}) = I$$

for all w in Ω. Spaces of this type arise in the theory of non-selfadjoint transformations. See L. de Branges [1, 2, 3] where, however, an additional hypothesis is made. This hypothesis states that the boundary of Ω in the extended complex plane contains at least one real point or the point at infinity, and the functions in the space have continuity properties at this point. In [1] the boundary point of continuity is taken at the origin, while in [2, 3] it is the point at infinity. We show that the extra hypothesis is really not needed to obtain a meaningful theory. The main difficulty is that a complete characterization of the spaces $\mathcal{H}(M)$ has been lacking.

Theorem. *Let $\mathcal{H}$ be a Hilbert space whose elements are vector valued analytic functions defined in Ω. In order that $\mathcal{H}$ be a space $\mathcal{H}(M)$ it is necessary and sufficient that*

(I) *for each w in Ω, the transformation $F(z) \to F(w)$ of $\mathcal{H}$ into $\mathcal{C}$ is continuous,*

(II) *for each w in Ω, $[F(z) - F(w)]/(z-w)$ belongs to $\mathcal{H}$ as a function of z for every $F(z)$ in $\mathcal{H}$,*

(III) *the identity*

$$\langle F(t), [G(t) - G(\beta)]/(t-\beta)\rangle - \langle [F(t) - F(\alpha)]/(t-\alpha), G(t)\rangle$$
$$+(\alpha - \bar{\beta})\,\langle [F(t) - F(\alpha)]/(t-\alpha), [G(t) - G(\beta)]/(t-\beta)\rangle = 2\pi\bar{G}(\beta)IF(\alpha)$$

holds for all functions $F(z)$ and $G(z)$ in $\mathcal{H}$ and all numbers α and β in Ω.

The proof uses two simple lemmas. We write $\mathcal{C}_+$ and $\mathcal{C}_-$ for the kernels of $I+i$ and $I-i$ respectively. These subspaces are orthogonal and span the coefficient space.

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(1) This is a transcription with minor corrections of an unpublished paper that was written in 1968. Some more recent related papers are listed at the end. The author thanks Sylvio Levy for suggesting that the paper should be put on the web.

Lemma 1. *Let $\mathcal{N}_+, \mathcal{N}_-, \mathcal{K}_+, \mathcal{K}_-$ be closed subspaces of $\mathcal{C}$ such that $\mathcal{N}_+ + \mathcal{N}_-$ and $\mathcal{K}_+ + \mathcal{K}_-$ are dense in $\mathcal{C}$. Suppose that for some operator Y, $\bar{c}Yc > 0$ for every nonzero vector c in $\mathcal{N}_+$ or $\mathcal{K}_+$, and $\bar{c}Yc < 0$ for every nonzero vector c in $\mathcal{N}_-$ or $\mathcal{K}_-$. Then* $\dim \mathcal{N}_+ = \dim \mathcal{K}_+$ *and* $\dim \mathcal{N}_- = \dim \mathcal{K}_-$

This is a standard result for finite dimensional spaces. The method of proof in the general case is the same.

Lemma 2. *Let X be an invertible operator such that iX is selfadjoint. Write $X = X_+ - X_-$ where X_+ and X_- are operators such that iX_+ and iX_- are non-negative and $X_+X_- = X_-X_+ = 0$. In order that X admit a representation*

$$X = \bar{A}IA$$

where A is an invertible operator, it is necessary and sufficient that $\dim \overline{\mathcal{R}(X_+)} = \dim \mathcal{C}_+$ *and* $\dim \overline{\mathcal{R}(X_-)} = \dim \mathcal{C}_-$.

Proof of Lemma 1. The conditions imply that $\mathcal{N}_+ \cap \mathcal{K}_- = \mathcal{K}_+ \cap \mathcal{K}_- = (0)$. Since $\mathcal{K}_+ + \mathcal{K}_-$ is dense in $\mathcal{C}$, it follows that $\dim \mathcal{N}_+ \leq \dim \mathcal{K}_+$. (For non-separable spaces this takes a bit of argument.) Likewise $\mathcal{N}_+ \cap \mathcal{N}_- = (0)$ and $\mathcal{N}_+ + \mathcal{N}_-$ is dense in $\mathcal{C}$, so $\dim \mathcal{K}_- \leq \dim \mathcal{N}_-$. The reverse inequalities are obtained in a similar way from the relations $\mathcal{N}_- \cap \mathcal{K}_+ = (0)$. □

Proof of Lemma 2. *Necessity.* We have $\bar{c}\,iXc > 0$ for every nonzero vector c in $A^{-1}\mathcal{C}_+$, and $\bar{c}\,iXc < 0$ for every nonzero vector c in $A^{-1}\mathcal{C}_-$. Similar inequalities hold with respect to the subspaces $\overline{\mathcal{R}(X_+)}$ and $\overline{\mathcal{R}(X_-)}$. Therefore the necessity of the conditions follows from Lemma 1.

Sufficiency. If the conditions are satisfied, then there exists a unitary operator U which maps $\overline{\mathcal{R}(X_+)}$ onto $\mathcal{C}_+$ and $\overline{\mathcal{R}(X_-)}$ onto $\mathcal{C}_-$. We obtain the desired representation with $A = U[(iX_+)^{1/2} + (iX_-)^{1/2}]$. □

Proof of Theorem. Necessity. Condition (I) follows from the existence of a kernel function. Conditions (II) and (III) are verified by a direct calculation for functions of the form $F(z) = K(u,z)a, G(z) = K(v,z)b$ where u and v are numbers in Ω and a and b are vectors. The general cases in (II) and (III) follow by linearity and continuity. The calculations are somewhat tedious but they are not difficult.

Sufficiency. Condition (I) implies the existence of a kernel function $K(w,z)$ for the space. In the proof we fix a number w in Ω with $i(\bar{w} - w) > 0$. Consider arbitrary numbers α and β in Ω and vectors a and b. Set $F(z) = K(u,z)a, G(z) = K(v,z)b$. By condition (II) the functions $[F(z) - F(w)]/(z-w)$ and $[G(z) - G(\bar{w})]/(z - \bar{w})$ belong to $\mathcal{H}$. By condition (III)

$$\langle F(t), [G(t) - G(\bar{w})]/(t - \bar{w})\rangle - \langle [F(t) - F(w)]/(t-w), G(t)\rangle = 2\pi\bar{G}(\bar{w})IF(w).$$

The inner products in this identity can be evaluated, and by the arbitrariness of the vectors a and b we obtain an operator equation which is equivalent to

$$\begin{aligned} (1) \qquad & [2\pi(\beta - w)K(\bar{w},\beta)]\, I\, [2\pi(\bar{\alpha} - w)\bar{K}(w,\alpha)] \\ & \quad = 2\pi(\beta - w)[K(\alpha,\beta) - K(\bar{w},\beta)] - 2\pi(\bar{\alpha} - w)[K(\alpha,\beta) - K(\alpha,w)]. \end{aligned}$$

Set

$$P(\alpha,\beta) = I + 2\pi(\beta - \bar{\alpha})K(\alpha,\beta)$$

for all α and β in Ω. To complete the proof we must show that $P(\alpha,\beta) = M(\beta)I\bar{M}(\alpha)$ for some operator valued function $M(z)$ which is defined in Ω and satisfies $\bar{M}(z)IM(\bar{z}) = M(\bar{z})I\bar{M}(z) = I$ for all z in Ω.

By (1), $P(\alpha,\beta) = P(\bar{w},\beta)I\bar{P}(w,\alpha)$. This can be written

$$P(\alpha,\beta) = M_1(\beta)I\bar{M}_2(\alpha),$$

where

$$M_1(\beta) = P(\bar{w},\beta) \qquad \text{and} \qquad M_2(\alpha) = P(w,\alpha)$$

for arbitrary α and β in Ω. For each α and β in Ω the operators $M_1(\beta)$ and $M_2(\alpha)$ are invertible. To prove this it is sufficient to show that

(2) $$P(\alpha,\beta)IP(\bar{\beta},\bar{\alpha})I = IP(\bar{\beta},\bar{\alpha})IP(\alpha,\beta) = 1.$$

The identity

$$IP(\bar{\beta},\bar{\alpha})IP(\alpha,\beta) = 1$$

is equivalent to

$$2\pi(\bar{\alpha}-\beta)K(\bar{\beta},\bar{\alpha})\,I\,2\pi(\beta-\bar{\alpha})\bar{K}(\beta,\alpha) = 2\pi(\beta-\bar{\alpha})K(\alpha,\beta) + 2\pi(\bar{\alpha}-\beta)K(\bar{\beta},\bar{\alpha}).$$

This follows from (1) on replacing β by $\bar{\alpha}$ and w by β. The equality

$$P(\alpha,\beta)IP(\bar{\beta},\bar{\alpha})I = 1$$

is a consequence of $IP(\bar{\beta},\bar{\alpha})IP(\alpha,\beta) = 1$ (replace α by $\bar{\beta}$, β by $\bar{\alpha}$, and multiply both sides by I). This proves (2) and hence the assertion that for each α and β in Ω, $M_1(\beta)$ and $M_2(\alpha)$ are invertible.

Now from $P(\alpha,\beta) + \bar{P}(\beta,\alpha) = 0$ we obtain

$$M_1(\beta)I\bar{M}_2(\alpha) = M_2(\beta)I\bar{M}_1(\alpha).$$

Therefore

$$X = M_2(\beta)^{-1}M_1(\beta)I = I\bar{M}_1(\alpha)\bar{M}_2(\alpha)^{-1}$$

is a constant operator. Suppose we have shown that $X = \bar{A}IA$ for some invertible operator A. Let us see how the theorem follows. On setting

(3) $$M(z) = M_1(z)IA^{-1} = M_2(z)\bar{A}I$$

for all z in Ω, we obtain

$$M(\beta)^{-1}M_1(\beta)I = A = I\bar{M}(\alpha)\bar{M}_2(\alpha)^{-1},$$

so

$$P(\alpha,\beta) = M_1(\beta)I\bar{M}_2(\alpha) = M(\beta)I\bar{M}(\alpha).$$

By the definition of $P(\alpha,\beta)$ this gives

$$2\pi(\beta-\bar{\alpha})K(\alpha,\beta) = M(\beta)I\bar{M}(\alpha) - I.$$

Setting $\beta = \bar{\alpha}$ we get $M(\bar{\alpha})I\bar{M}(\alpha) = I$. But $M(\alpha)$ is invertible and so this implies that $\bar{M}(\alpha)IM(\bar{\alpha}) = I$. Then $\mathcal{H}$ is a space $\mathcal{H}(M)$.

To complete the proof we must show that $X = \bar{A}IA$ for some invertible operator A. We have

$$iX = iM_2(\beta)^{-1}M_1(\beta)I = iP(w,\beta)^{-1}P(\bar{w},\beta)I = iIP(\bar{\beta},\bar{w})IP(\bar{w},\beta)I$$

for every β in Ω. Choosing $\beta = w$ we get $iX = -iIP(\bar{w},\bar{w})I$, or

(4) $$iX = iI + 2\pi i(\bar{w}-w)iIK(\bar{w},\bar{w})iI.$$

Also $(iX)^{-1} = iI^{-1}P(\bar{w}, \bar{w})^{-1}I^{-1} = iP(w, w)$ and so

$$(iX)^{-1} = iI - 2\pi i(\bar{w} - w)K(w, w). \tag{5}$$

Set $\mathcal{K}_+ = \mathcal{C}_+$ and $\mathcal{K}_- = (iX)^{-1}\mathcal{C}_-$. If c is a nonzero vector in $\mathcal{K}_+$, then $iIc = c$ and by (4) we have $\bar{c}\, iXc > 0$. If c is a nonzero vector in $\mathcal{K}_-$, then $c = (iX)^{-1}a$ where a is a vector such that $iIa = -a$. By (5),

$$\bar{c}\, iXc = \bar{a}\,(iX)^{-1}a < 0.$$

Write $X = X_+ - X_-$ as in Lemma 2. Set $\mathcal{N}_+ = \overline{\mathcal{R}(iX_+)}$ and $\mathcal{N}_- = \overline{\mathcal{R}(iX_-)}$. If c is a nonzero vector in $\mathcal{N}_+$ then $\bar{c}\, iXc > 0$. If c is a nonzero vector in $\mathcal{N}_-$ then $\bar{c}\, iXc < 0$. We show that $\mathcal{K}_+ + \mathcal{K}_-$ is dense in $\mathcal{C}$. Let c be a vector orthogonal to $\mathcal{K}_+ + \mathcal{K}_-$. Then c belongs to $\mathcal{C}_-$ and $iIc = -c$. Since c is also orthogonal to $\mathcal{K}_- = (iX)^{-1}\mathcal{C}_-$,

$$\bar{c}\,(iX)^{-1}c = 0.$$

But c belongs to $\mathcal{C}_-$. Therefore $(iX)^{-1}c$ belongs to $\mathcal{K}_-$ and

$$((iX)^{-1}c)^-\, iX((iX)^{-1}c) = \bar{c}\,(iX)^{-1}c = 0.$$

It follows that $(iX)^{-1}c = 0$ and so $c = 0$. Hence $\mathcal{K}_+ + \mathcal{K}_-$ is dense in $\mathcal{C}$. Finally $\mathcal{N}_+ + \mathcal{N}_- = \mathcal{C}$ by the definitions of $\mathcal{N}_+$ and $\mathcal{N}_-$. By Lemma 1,

$$\dim \mathcal{N}_+ = \dim \mathcal{K}_+ = \dim \mathcal{C}_+$$

and

$$\dim \mathcal{N}_- = \dim \mathcal{K}_- = \dim \mathcal{C}_-.$$

By Lemma 2, $X = \bar{A}IA$ for some invertible operator A. We have already indicated how this implies the theorem. □

References

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Supplementary References

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