# AN EXTENSION PROBLEM FOR THE COEFFICIENTS OF RIEMANN MAPPINGS

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By a **Riemann mapping** we mean an analytic function which is defined and univalent on the unit disk. Such a function is said to be **normalized** if it has value zero and positive derivative at the origin.

We are concerned with normalized Riemann mappings of the unit disk into itself. If $B(z)$ is such a function, the operator $f(z)$ into $f(B(z))$ maps the Dirichlet space contractively into itself. Elementary examples show, however, that other functions also have this property [**4**], and therefore the property cannot be used to characterize the class.

More generally, if $B(z)$ is a normalized Riemann mapping of the unit disk into itself, the operator $f(z)$ into $f(B(z))$ acts as a contraction on a family of spaces which generalize the Dirichlet space. A theorem of de Branges [**3**] shows that this property is characteristic of the class of normalized Riemann mappings of the unit disk into itself. We study de Branges' theorem and its proof for possible generalization to the problem of characterizing initial segments of coefficients of a normalized Riemann mapping of the unit disk into itself. Such a characterization is not known at this time, but we shall pose a problem which we hope may lead in this direction.

Given any real number $\nu$, let $\mathfrak{D}^\nu$ be the Kreĭn space of generalized power series $f(z) = \sum_{n=1}^\infty a_n z^{\nu+n}$ with finite self-product

$$\langle f(z), f(z) \rangle_{\mathfrak{D}^\nu} = \sum_{n=1}^\infty (\nu+n)|a_n|^2.$$

Constant terms, if present, make no contribution to the sum and are identified to zero. The operator $f(z)$ into $f(B(z))$ is an everywhere defined contraction of $\mathfrak{D}^\nu$ into itself for every real number $\nu$ [**3**]. This means that $f(B(z))$ belongs to $\mathfrak{D}^\nu$ whenever $f(z)$ is in $\mathfrak{D}^\nu$, and then

$$\langle f(B(z)), f(B(z)) \rangle_{\mathfrak{D}^\nu} \le \langle f(z), f(z) \rangle_{\mathfrak{D}^\nu}.$$

This contractive substitution property is characteristic of the class of normalized Riemann mappings of the unit disk into itself.

THEOREM (DE BRANGES [**3**]). *Let $B(z) = B_1 z + B_2 z^2 + \cdots$ $(B_1 > 0)$ be a formal power series such that for every nonpositive integer $\nu$, the operator $f(z)$ into $f(B(z))$ is an everywhere defined and contractive mapping of $\mathfrak{D}^\nu$ into itself. Then $B(z)$ represents a normalized Riemann mapping of the unit disk into itself.*

The proof given in [**3**] makes an interesting link between geometric function theory and operator theory. We first review some preliminary notions.

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{P}$ be a Hilbert space which is contained contractively in $\mathcal{H}$. Let $P$ be the selfadjoint operator on $\mathcal{H}$ which coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. Then there is a unique Hilbert space $\mathcal{Q}$ which is contained contractively in $\mathcal{H}$

such that the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ coincides with $Q = 1 - P$. We call $\mathcal{Q}$ the **complement** of $\mathcal{P}$ in $\mathcal{H}$. The space $\mathcal{Q}$ is characterized as the set of elements $g$ of $\mathcal{H}$ such that

$$\sup_{f \in \mathcal{P}} [\, \|g + f\|_{\mathcal{H}}^2 - \|f\|_{\mathcal{P}}^2] < \infty,$$

and in this situation the value of the supremum is $\|g\|_{\mathcal{Q}}^2$. If $h = f + g$ with $f \in \mathcal{P}$ and $g \in \mathcal{Q}$, then $\|h\|_{\mathcal{H}}^2 \leq \|f\|_{\mathcal{P}}^2 + \|q\|_{\mathcal{Q}}^2$. Every $h$ in $\mathcal{H}$ has a unique **minimal decomposition** of this form for which equality holds. It is obtained with $f = Ph$ and $g = Qh$.

In our applications, $\mathcal{H}, \mathcal{P}, \mathcal{Q}$ are Hilbert spaces of power series which converge in the unit disk and have reproducing kernels $K_{\mathcal{H}}(w, z), K_{\mathcal{P}}(w, z), K_{\mathcal{Q}}(w, z)$. The kernel functions have the property that for each fixed $w$ in the unit disk,

$$P : K_{\mathcal{H}}(w, z) \to K_{\mathcal{P}}(w, z) \qquad \text{and} \qquad Q : K_{\mathcal{H}}(w, z) \to K_{\mathcal{Q}}(w, z)$$

as functions of $z$, and therefore $K_{\mathcal{H}}(w, z) = K_{\mathcal{P}}(w, z) + K_{\mathcal{Q}}(w, z)$.

Proof of Theorem. Let $T$ be the operator $T : f(z) \to f(B(z))$ on the Dirichlet space $\mathcal{D}$. Initially we view $\mathcal{D}$ as a space of formal power series and define substitution by $B(z)$ by formal algebraic operations.

We show that $B(z)$ represents an analytic function which is bounded by 1 in the unit disk. Since $f(z) = z$ belongs to $\mathcal{D}$, so does $f(B(z)) = B(z)$. Therefore $B(z)$ represents an analytic function in the unit disk. Suppose that there is an $\alpha$ with $|\alpha| < 1$ such that $|B(\alpha)| > 1$. Then

$$\|z^n / B(\alpha)^n\|_{\mathcal{D}}^2 = n / |B(\alpha)|^{2n} \to 0$$

as $n \to \infty$. Since $\|B(z)^n / B(\alpha)^n\|_{\mathcal{D}} \leq \|z^n / B(\alpha)^n\|_{\mathcal{D}}$ for every n, $\|B(z)^n / B(\alpha)^n\|_{\mathcal{D}} \to 0$. This is impossible because the value of $B(z)^n / B(\alpha)^n$ for $z = \alpha$ is 1 for every $n$, and convergence in the norm of the Dirichlet space implies pointwise convergence in the unit disk. Therefore $B(z)$ represents an analytic function which is bounded by 1 in the unit disk.

Notice that we may alternatively view the Dirichlet space $\mathcal{D}$ as a space of formal power series or analytic functions defined on the unit disk. The functional point of view is convenient in showing that $B(z)$ is a Riemann mapping.

Let $\mathfrak{M}(B)$ be the range of $T$ viewed as a Hilbert space in the inner product which makes $T$ an isomorphism from $\mathcal{D}$ onto $\mathfrak{M}(B)$. Notice that $T$ has zero kernel, and so $\mathfrak{M}(B)$ is well defined. Since $T$ is a contraction by hypothesis, $\mathfrak{M}(B)$ is a Hilbert space which is contained contractively in $\mathcal{D}$. The adjoint of the inclusion operator from $\mathfrak{M}(B)$ into $\mathcal{D}$ coincides with $TT^*$. Let $\mathfrak{G}(B)$ be the complement of $\mathfrak{M}(B)$ in $\mathcal{D}$. Then $\mathfrak{G}(B)$ is a Hilbert space which is contained contractively in $\mathcal{D}$ such that the adjoint of the inclusion of $\mathfrak{G}(B)$ in $\mathcal{D}$ coincides with $1 - TT^*$.

The Dirichlet space $\mathcal{D}$ has reproducing kernel

$$\log \frac{1}{1 - \bar{w}z} = \bar{w}z + \frac{1}{2}\bar{w}^2 z^2 + \frac{1}{3}\bar{w}^3 z^3 + \cdots .$$

The reproducing kernel of $\mathfrak{M}(B)$ is easily seen to be

$$\log \frac{1}{1 - \bar{B}(w)B(z)}.$$

Therefore $\mathfrak{G}(B)$ has reproducing kernel

$$\log \frac{1-\bar{B}(w)B(z)}{1-\bar{w}z} = \log \frac{1}{1-\bar{w}z} - \log \frac{1}{1-\bar{B}(w)B(z)}.$$

In each case, the reproducing kernel may be interpreted either in the sense of formal power series or in the functional sense using the principal branch of the logarithm. For if $z$ and $w$ are points in the unit disk, $1-\bar{w}z$ and $1-\bar{B}(w)B(z)$ lie in the interior of the circle of radius 1 and center 1, and their quotient has argument in $(-\pi, \pi)$.

The idea of the proof is to use the hypotheses to construct a contraction operator $G$ on $\mathfrak{G}(B)$ to $\mathcal{D}$ with the property that

$$G : \log \frac{1-\bar{B}(w)B(z)}{1-\bar{w}z} \to \log \frac{B'(0)/B^*(z) - B'(0)/\bar{B}(w)}{1/z - 1/\bar{w}}$$

for every $w$ in the unit disk, where $B^*(z) = \bar{B}_1 z + \bar{B}_2 z^2 + \cdots$. Analyticity on the right hand side then forces $B(z)$ to be a Riemann mapping.

To begin the construction we define $G$ on elements of $\mathfrak{G}(B)$ of the form $h(z) = (1 - TT^*)f(z)$ where $f(z) = \sum_{n=1}^{r} a_n z^n$ is a polynomial. In this case, $T^*$ applied to $f(z)$ is the unique polynomial $g(z) = \sum_{n=1}^{r} b_n z^n$ such that the coefficients of $1/z^r, \ldots, 1/z, 1$ in the formal expansion

$$f(1/z) - g(1/B^*(z)) = \sum_{n=1}^{r} a_n z^{-n} - \sum_{n=1}^{r} b_n B^*(z)^{-n}$$

vanish, that is,

$$f(1/z) - g(1/B^*(z)) = \text{const.} + k(z),$$

where $k(z) = \sum_{n=1}^{\infty} c_n z^n$ is a power series with constant term zero. See [**3**] or [**5, Th. 3.6**]. Since $B^*(z)$ satisfies the hypotheses of the theorem along with $B(z)$,

$$\langle g(1/B^*(z)), g(1/B^*(z))\rangle_{\mathcal{D}^{-r-1}} \le \langle g(1/z), g(1/z)\rangle_{\mathcal{D}^{-r-1}}.$$

Substituting

$$g(1/B^*(z)) = f(1/z) - \text{const.} - k(z),$$

we obtain

$$\langle f(1/z) - \text{const.} - k(z), f(1/z) - \text{const.} - k(z)\rangle_{\mathcal{D}^{-r-1}} \le \langle g(1/z), g(1/z)\rangle_{\mathcal{D}^{-r-1}}.$$

In expanded form, the last inequality asserts that

$$\begin{aligned}(-r-1+1)|a_r|^2 + (-r-1+2)|a_{r-1}|^2 + \cdots + (-r-1+r)|a_1|^2 + |c_1|^2 + 2|c_2|^2 + 3|c_3|^2 + \cdots \\ \le (-r-1+1)|b_r|^2 + (-r-1+2)|b_{r-1}|^2 + \cdots + (-r-1+r)|b_1|^2,\end{aligned}$$

and hence

$$\begin{aligned}\|k(z)\|_{\mathcal{D}}^2 &\le \|f(z)\|_{\mathcal{D}}^2 - \|g(z)\|_{\mathcal{D}}^2 \\ &= \langle (1-TT^*)f(z), f(z)\rangle_{\mathcal{D}} \\ &= \|h(z)\|_{\mathfrak{G}(B)}^2.\end{aligned}$$

Since elements of the form $h(z) = (1 - TT^*)f(z)$, $f(z)$ a polynomial, are dense in $\mathfrak{G}(B)$, there is a unique contraction $G$ from $\mathfrak{G}(B)$ to $\mathcal{D}$ such that in the preceding notation,

$$G : h(z) \to k(z) \tag{1}$$

.

We compute the action of $G$ on kernel functions for $\mathfrak{G}(B)$. For fixed $w$ in the unit disk, let

$$G : \log \frac{1 - \bar{B}(w)B(z)}{1 - \bar{w}z} \to \varphi(w, z).$$

For any positive integer $n$, let

$$1 - TT^* : \frac{1}{n}\, \bar{w}^n\, z^n \to h_n(z),$$
$$G : h_n(z) \to k_n(z).$$

Then

$$\log \frac{1 - \bar{B}(w)B(z)}{1 - \bar{w}z} = (1 - TT^*)\{\sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, z^n\}$$
$$= \sum_{n=1}^{\infty} h_n(z),$$

with convergence in the metric of $\mathfrak{G}(B)$, and

$$\varphi(w, z) = \sum_{n=1}^{\infty} k_n(z)$$

in the metric of $\mathcal{D}$. For each $n$, let $S_n(z)$ be the polynomial of degree $n$ with constant term zero such that $T^* : z^n \to S_n(z)$. Then

$$h_n(z) = \frac{1}{n}\, \bar{w}^n\, [z^n - S_n(B(z))]$$

and

$$k_n(z) = \frac{1}{n}\, \bar{w}^n\, [1/z^n - S_n(1/B^*(z)) + c_n],$$

where the constant $c_n$ is chosen to make the expression a power series with constant term zero. Therefore

$$\log \frac{1 - \bar{B}(w)B(z)}{1 - \bar{w}z} = \sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, [z^n - S_n(B(z))], \tag{2}$$

$$\varphi(w, z) = \sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, [1/z^n - S_n(1/B^*(z)) + c_n]. \tag{3}$$

These series converge in the metrics of $\mathfrak{G}(B)$ and $\mathcal{D}$, respectively, as well as pointwise for $w$ and $z$ in the unit disk.

We use information from (2) to evaluate (3). By (2), for $w$ and $z$ in the unit disk,

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, S_n(B(z)) = \log \frac{1}{1-\bar{w}z} - \log \frac{1-\bar{B}(w)B(z)}{1-\bar{w}z}$$
$$= \log \frac{1}{1-\bar{B}(w)B(z)}.$$

Consider positive numbers $\epsilon$ and $\delta$ such that $\epsilon(1+\delta) < 1$. If $|\bar{w}| < \epsilon$ and $|\zeta| < 1+\delta$ then $|\bar{B}(w)\zeta| \le \epsilon(1+\delta) < 1$ by Schwarz's lemma. Hence $\log 1/[1-\bar{B}(w)\zeta]$ is analytic in $\bar{w}$ and $z$ for $|\bar{w}| < \epsilon$ and $|\zeta| < 1+\delta$. In this domain, the function has a power series expansion

$$\log \frac{1}{1-\bar{B}(w)\zeta} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\bar{w}^n\zeta^m,$$

which converges absolutely and uniformly on compact sets. Since the expansion

$$\log \frac{1}{1-\bar{B}(w)\zeta} = \sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, S_n(\zeta)$$

is valid for $\zeta = B(z)$ with $z$ in the unit disk, it holds for all $\bar{w}$ and $\zeta$ satisfying $|\bar{w}| < \epsilon$ and $|\zeta| < 1+\delta$.

Now let $z$ be a fixed nonzero number in the unit disk. For $w$ in some neighborhood of the origin depending on $z$,

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, /z^n = \log \frac{1}{1-\bar{w}/z},$$
$$\sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, S_n(1/B^*(z)) = \log \frac{1}{1-\bar{B}(w)/B^*(z)},$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, [1/z^n - S_n(1/B^*(z))] = \log \frac{1-\bar{B}(w)/B^*(z)}{1-\bar{w}/z}$$
$$= \log \frac{z}{B^*(z)} \frac{B^*(z)-\bar{B}(w)}{z-\bar{w}}.$$

The series

$$q(\bar{w}) = \sum_{n=1}^{\infty} \frac{1}{n}\, \bar{w}^n\, c_n$$

is therefore convergent in a neighborhood of the origin, and by (3),

$$\varphi(w,z) = \log \frac{z}{B^*(z)} \frac{B^*(z)-\bar{B}(w)}{z-\bar{w}} + q(\bar{w}). \tag{4}$$

The preceding relations are derived initially for fixed nonzero $z$ in the unit disk and $w$ in a neighborhood of the origin depending on $z$, but they hold more generally. Observe that $\bar{w} \to \varphi(w,z)$ is analytic as a vector-valued mapping from the unit disk into the Dirichlet space $\mathcal{D}$, and therefore for fixed $z$ in the unit disk, $\varphi(w,z)$ is analytic as a function of $\bar{w}$ in the unit disk. Since $B'(0) > 0$, we may choose $\eta > 0$ such that the function

$$\frac{z}{B^*(z)} \frac{B^*(z) - \bar{B}(w)}{z - \bar{w}}$$

takes its values in an arbitrarily small neighborhood of 1 for $|\bar{w}| < \eta$ and $|z| < \eta$. We may further suppose that $q(\bar{w})$ is analytic for $|\bar{w}| < \eta$. Then the right side of (4) is analytic in $\bar{w}$ and $z$ for $|\bar{w}| < \eta$ and $|z| < \eta$. Since (4) holds for fixed nonzero $z$ and $w$ in a neighborhood of the origin, it holds whenever $|\bar{w}| < \eta$ and $0 < |z| < \eta$. By continuity, this extends to $|\bar{w}| < \eta$ and $|z| < \eta$. Setting $z = 0$, we get

$$0 = \log \frac{1}{B'(0)} \frac{\bar{B}(w)}{\bar{w}} + q(\bar{w})$$

for $|\bar{w}| < \eta$. Therefore for $|\bar{w}| < \eta$ and $|z| < \eta$,

$$\varphi(w,z) = \log \frac{z}{B^*(z)} \frac{B^*(z) - \bar{B}(w)}{z - \bar{w}} - \log \frac{1}{B'(0)} \frac{\bar{B}(w)}{\bar{w}},$$

and hence

$$e^{\varphi(w,z)} = B'(0) \frac{z}{B^*(z)} \frac{B^*(z) - \bar{B}(w)}{z - \bar{w}} \frac{\bar{w}}{\bar{B}(w)}$$

and

$$\frac{B^*(z)}{z} \frac{\bar{B}(w)}{\bar{w}} e^{\varphi(w,z)} = B'(0) \frac{B^*(z) - \bar{B}(w)}{z - \bar{w}}.$$

By analyticity, the last identity holds for all $w$ and $z$ in the unit disk. The identity implies that $B(z)$ is a Riemann mapping. ■

The proof is an expanded form of the argument in de Branges [**3**]. A proof of the theorem has also been constructed by N. K. Nikol'skiĭ and V. I. Vasyunin (private communication).

The contractive substitution property has a local version which depends only on initial segments of the coefficients of a normalized Riemann mapping $B(z) = B_1 z + B_2 z^2 + \cdots$ of the unit disk into itself. For any real number $\nu$ and positive integer $r$, let $\mathfrak{D}_r^\nu$ be the finite-dimensional Kreĭn space of generalized power series $f(z) = \sum_{n=1}^\infty a_n z^{\nu+n}$ such that

$$\langle f(z), f(z) \rangle_{\mathfrak{D}_r^\nu} = \sum_{n=1}^{r} (\nu + n)|a_n|^2$$

and terms with $n > r$ or $\nu + n = 0$ are identified to zero. The inequality

$$\langle f(B(z)), f(B(z)) \rangle_{\mathfrak{D}_r^\nu} \le \langle f(z), f(z) \rangle_{\mathfrak{D}_r^\nu} \tag{5}$$

holds for every $f(z)$ in $\mathfrak{D}_r^\nu$ [**1**,**5**]. See [**4**] for a variant on the proof of the inequality and a symmetry involving $\nu$. The inequality depends only on the coefficients $B_1, \ldots, B_r$, and the question arises if this property is characteristic of initial segments of coefficients [**1**]?

An analogy with the Carathéodory-Fejér interpolation problem [**2**] is suggestive. However, it is difficult to see how to proceed in parallel fashion because of the dependence of the condition (5) on $\nu$. It may be possible instead to construct the operator $G$ defined by (1) by an extension process that uses initial segments of coefficients to define finite-dimensional approximations of the operator.

PROBLEM: *Study approximations of the operator $G$ defined by (1) that depend only on initial segments of the coefficients of $B(z)$.*

The point is to try to isolate conditions needed to make a step-by-step construction of the operator from local data.

## REFERENCES


1. L. de Branges, " Unitary linear systems whose transfer functions are Riemann mapping functions," *Integral Equations and Operator Theory* 19 (1986), 105-124.
2. L. de Branges, " Underlying concepts in the proof of the Bieberbach conjecture," *Proceedings of the International Congress of Mathematicians* 1986, pp.25-42, Berkeley, California, 1986.
3. L. de Branges, *Square Summable Power Series*, Springer-Verlag, in preparation.
4. Kin Y. Li and James Rovnyak, "On the coefficients of Riemann mappings of the unit disk into itself," preprint, 1991.
5. J. Rovnyak, "Coefficient estimates for Riemann mapping functions," J. Analyse Math. 52 (1989), 53-93.