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Canonical Models in Quantum Scattering Theory

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A formulation of quantum scattering theory can be given in terms of self-adjoint transformations. In this view the basic scattering situation consists of a pair of self-adjoint transformations H_+ and H_- in the same Hilbert space $\mathfrak{H}$. The transformations are assumed to be close together in the sense that $(H_+ - w)^{-1} - (H_- - w)^{-1}$ is completely continuous for some, and hence all, nonreal numbers w. We formulate the fundamental problem of scattering theory as a special case of the structure problem for transformations in Hilbert space.

By the structure problem for an everywhere defined and bounded transformation, we mean the problem of finding the invariant subspaces of the transformation and the problem of writing the transformation as an integral in terms of invariant subspaces. Consider the scattering situation associated with a pair of self-adjoint transformations H_+ and H_-. Let H be the intersection of H_+ and H_-. By this we mean the transformation whose graph is the intersection of the graphs of H_+ and H_-. Then H is a closed symmetric transformation in $\mathfrak{H}$. The adjoint H^* of H is a closed relation in $\mathfrak{H}$ which extends H_+ and H_-. We take the structure problem for H^* to be the fundamental scattering problem.

A relation is defined by its graph, which is a vector subspace of the Cartesian product $\mathfrak{H} \times \mathfrak{H}$. The graph of a relation differs from the graph of a transformation in that it may contain nonzero elements $(0, f)$ lying above the origin. Otherwise the theory of relations is analogous to the theory of transformations. By the finite spectrum of a relation T in a Hilbert space $\mathfrak{H}$, we mean the set of numbers w such that $T - w$ fails to have an everywhere defined and bounded inverse in $\mathfrak{H}$. The structure problem for a relation is similar to the structure problem for an everywhere defined and bounded transformation, once the meaning of invariant subspace is clarified. The following concept of invariance is of interest for the adjoint H^* of a symmetric transformation H which is given as the intersection of self-adjoint transformations H_+ and H_-. An invariant subspace $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}$ in which there exists a relation T with these properties:

(1) T has no finite spectrum as a relation in $\mathfrak{M}$.

(2) The graph of H^* contains the graph of T, and the graph of T contains the intersections of the graphs of H_+ and H_- with $\mathfrak{M} \times \mathfrak{M}$.

Note that the second condition is satisfied if the graph of T is the intersection of the graph of H^* with $\mathfrak{M} \times \mathfrak{M}$.

Our approach to any given transformation uses a canonical model of the transformation in a Hilbert space whose elements are analytic functions. Various canonical models are known. The choice of a model depends on the properties of the transformation and the needs of the problem. The first canonical model is appropriate for a closed symmetric transformation H such that $H - w$ has a bounded, partially defined inverse for every complex number w. A complete continuity hypothesis is also made. The canonical model is constructed in a Hilbert space whose elements are vector valued entire functions.

The following conventions are convenient in work with vector valued functions. Let $\mathcal{C}$ be a fixed Hilbert space which is used as a coefficient space. By a vector we always mean an element of this space. By an operator we mean a bounded transformation of vectors into vectors. The absolute value symbol is used for the norm of a vector and for the operator norm of an operator. If b is a vector, let $\bar{b}$ be the linear functional on vectors defined by the inner product $\bar{b}a = \langle a, b\rangle$ for every vector a. If a and b are vectors, let $a\bar{b}$ be the operator defined by $(a\bar{b})c = a(\bar{b}c)$ for every vector c. The adjoint of an operator A is denoted $\bar{A}$. Standard conventions are used for operator inequalities. A vector valued function $F(z)$ is said to be analytic in a region if the complex valued function $\bar{c}F(z)$ is analytic in the region for every vector c. A vector valued analytic function is said to be of bounded type in a region if $\bar{c}F(z)$ is of bounded type in the region for every vector c (i.e. it is the ratio of bounded analytic functions in the region). A vector valued function $F(z)$ which is analytic and of bounded type in the upper half-plane $y > 0$ is said to be of nonpositive mean type in the half-plane if $\bar{c}F(z)$ is of nonpositive mean type for every vector c, or in other words if

$$\lim_{y \to +\infty} e^{-\epsilon y} \bar{c}F(iy) = 0$$

for every $\epsilon > 0$. A vector valued function $F(z)$ which is analytic and of bounded type in the lower half-plane $y < 0$ is said to be of nonpositive mean type in the half-plane if $\bar{c}F(z)$ is of nonpositive mean type for every vector c, or in other words if

$$\lim_{y \to -\infty} e^{\epsilon y} \bar{c} F(iy) = 0$$

for every $\epsilon > 0$. Analogous conventions are made for an operator valued function $G(z)$, the related complex valued functions being $\bar{b} G(z) a$ for all choices of vectors a and b.

We begin with a construction of the spaces of entire functions.

THEOREM 1. Let $A(z)$ and $B(z)$ be operator valued entire functions such that $B(z)\bar{A}(\bar{z}) = A(z)\bar{B}(\bar{z})$, $E_+(z) = A(z) - iB(z)$ has invertible values in the upper half-plane, and $E_-(z) = A(z) + iB(z)$ has invertible values in the lower half-plane. Suppose that the values of $1 - E_+(z)$ and $1 - E_-(z)$ are completely continuous and that $K(z,z) \geq 0$ for all complex z, where

$$K(w,z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z-\bar{w})] .$$

Let $\mathcal{H}(A,B)$ be the set of vector valued entire functions $F(z)$ such that

$$\|F(t)\|^2 = \int_{-\infty}^{+\infty} |E_+(t)^{-1} F(t)|^2 \, dt < \infty ,$$

$E_+(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the upper half-plane, and $E_-(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the lower half-plane. Then $\mathcal{H}(A,B)$ is a Hilbert space which contains $K(w,z)c$ as a function of z for every vector c and every complex number w. If $F(z)$ is in $\mathcal{H}(A,B)$, then

$$\bar{c} F(w) = \langle F(t), K(w,t)c \rangle$$

for every vector c and complex number w.

Multiplication by z in $\mathcal{H}(A,B)$ is the transformation defined by $F(z) \to zF(z)$ whenever $F(z)$ and $zF(z)$ are in $\mathcal{H}(A,B)$.

THEOREM 2. If $\mathcal{H}(A,B)$ is a given space, multiplication by z in $\mathcal{H}(A,B)$ is a closed symmetric transformation. The dimension of the orthogonal complement of the range of multiplication by $z - w$ does not exceed the dimension of $\mathcal{C}$ for any w. The function $[F(z) - E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ belongs to $\mathcal{H}(A,B)$ whenever $F(z)$ belongs to $\mathcal{H}(A,B)$ and w is a point of continuity of $E_+(z)^{-1}F(z)$. The identity

$$
\begin{aligned}
&- 2\pi i\bar{G}(\beta)\bar{E}_+(\beta)^{-1}E_+(\alpha)^{-1}F(\alpha) \\
&= \langle F(t),\ [G(t)-E_+(t)E_+(\beta)^{-1}G(\beta)]/(t-\beta)\rangle \\
&- \langle [F(t)-E_+(t)E_+(\alpha)^{-1}F(\alpha)]/(t-\alpha),\ G(t)\rangle \\
&+ (\alpha-\bar{\beta})\langle [F(t)-E_+(t)E_+(\alpha)^{-1}F(\alpha)]/(t-\alpha), \\
&\qquad [G(t)-E_+(t)E_+(\beta)^{-1}G(\beta)]/(t-\beta)\rangle
\end{aligned}
$$

holds for all $F(z)$ and $G(z)$ in $\mathcal{H}(A,B)$ when α is a point of continuity of $E_+(z)^{-1}F(z)$ and β is a point of continuity of $E_+(z)^{-1}G(z)$. The transformation $F(z) \to F(w)$ of $\mathcal{H}(A,B)$ into C is completely continuous for every complex number w.

These properties characterize multiplication by z in $\mathcal{H}(A,B)$.

THEOREM 3. Let L be a closed symmetric transformation in a Hilbert space $\mathcal{H}$ such that the dimension of the orthogonal complement of the range of $L-w$ does not exceed the dimension of $\mathcal{C}$ for some number w above the real axis. Suppose that L has an extension T which is a transformation with spectrum below the real axis such that

$$i(T^*-\bar{w})^{-1} - i(T-w)^{-1} - i(\bar{w}-w)(T^*-\bar{w})^{-1}(T-w)^{-1}$$

is nonnegative and completely continuous. Suppose that the graph of L is the intersection of the graph of T and the graph of the adjoint T^* of T. Then L as a transformation in $\mathcal{H}$ is unitarily equivalent to multiplication by z in some space $\mathcal{H}(A,B)$. The unitary equivalence can be made in such a way that the transformation $(T-w)^{-1}$ in $\mathcal{H}$ corresponds to $F(z) \to [F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ in $\mathcal{H}(A,B)$.

When symmetric transformations appear in quantum scattering theory, they appear in pairs having a common extension which is a relation having no finite spectrum. The canonical models of the symmetric transformations are multiplication by z in a space $\mathcal{H}(A,B)$ and multiplication by z in a space $\mathcal{H}(C,D)$. Since the canonical models are taken of transformations in the same Hilbert space, there is an induced isometry of the space $\mathcal{H}(A,B)$ onto the space $\mathcal{H}(C,D)$. To study the situation we introduce a new space whose elements are pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ with $F_+(z)$ in $\mathcal{H}(A,B)$ and $F_-(z)$ the corresponding element of $\mathcal{H}(C,D)$, the norm being defined by

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 = \|F_+(t)\|^2_{\mathcal{H}(A,B)} + \|F_-(t)\|^2_{\mathcal{H}(C,D)} .$$

The study of such spaces involves matrix valued entire functions

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

whose entries are operator valued entire functions of z.

Let $\mathcal{C}^2$ be the space of pairs $\begin{pmatrix} u \\ v \end{pmatrix}$ of elements of $\mathcal{C}$, taken in the norm $\left| \begin{pmatrix} u \\ v \end{pmatrix} \right|^2 = |u|^2 + |v|^2$. For each pair $\begin{pmatrix} u \\ v \end{pmatrix}$, let $\begin{pmatrix} u \\ v \end{pmatrix}^- = (\bar{u} \quad \bar{v})$ be the linear functional on $\mathcal{C}^2$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} a \\ b \end{pmatrix} = (\bar{u} \quad \bar{v}) \begin{pmatrix} a \\ b \end{pmatrix} = \bar{u}a + \bar{v}b .$$

In what follows the word "matrix" is used to mean a 2×2-matrix having operator entries. Matrix conventions are determined by the action of the matrices on $\mathcal{C}^2$. The adjoint $\bar{M}$ of the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $\bar{M} = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix}$. The symbol I is used for the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Matrix valued entire functions are used to construct new Hilbert spaces whose elements are pairs of vector valued entire functions.

THEOREM 4. Let $M(z)$ be a matrix valued entire function such that the values of $M(z)I - I$ are matrices of completely continuous operators,

$$M(z)I\bar{M}(\bar{z}) = I = \bar{M}(\bar{z})IM(z)$$

and

$$[M(z)I\bar{M}(z) - I]/(z - \bar{z}) \geq 0$$

for all complex z. Then there exists a unique Hilbert space $\mathcal{H}(M)$, whose elements are pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of vector valued entire functions, with this property:

$$\frac{M(z)I\bar{M}(w) - I}{2\pi(z-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathcal{H}(M)$ as a function of z for all choices of vectors u and v and for all complex numbers w, and

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - I}{2\pi(t-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

for all $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ in $\mathcal{H}(M)$. In addition the space has these properties: the pair $\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$ belongs to the space whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to the space, for every w. The identity for difference quotients

$$2\pi \begin{pmatrix} G_+(\beta) \\ G_-(\beta) \end{pmatrix}^- I \begin{pmatrix} F_+(\alpha) \\ F_-(\alpha) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} [G_+(t)-G_+(\beta)]/(t-\beta) \\ [G_-(t)-G_-(\beta)]/(t-\beta) \end{pmatrix} \right\rangle$$
$$- \left\langle \begin{pmatrix} [F_+(t)-F_+(\alpha)]/(t-\alpha) \\ [F_-(t)-F_-(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\rangle$$
$$+ (\alpha-\bar{\beta}) \left\langle \begin{pmatrix} [F_+(t)-F_+(\alpha)]/(t-\alpha) \\ [F_-(t)-F_-(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} [G_+(t)-G_+(\beta)]/(t-\beta) \\ [G_-(t)-G_-(\beta)]/(t-\beta) \end{pmatrix} \right\rangle$$

holds for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ of the space and for all complex numbers α and β.

These properties characterize the space.

THEOREM 5. Let $\mathcal{H}$ be a Hilbert space whose elements are pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of vector valued entire functions. Suppose that

$\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$ belongs to the space whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to the space, for every w, and that the identity for difference quotients holds. If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \rightarrow \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$ is a completely continuous transformation of $\mathcal{H}$ into C^2 for some number w, then $\mathcal{H}$ is equal isometrically to a space $\mathcal{H}(M)$ as in Theorem 4.

The spaces $\mathcal{H}(M)$ are now related to the spaces $\mathcal{H}(A, B)$.

THEOREM 6. If $\mathcal{H}(M)$ is a given space,

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

then spaces $\mathcal{H}(A, B)$ and $\mathcal{H}(C, D)$ exist, $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \rightarrow \sqrt{2}F_+(z)$ is a partial isometry of $\mathcal{H}(M)$ onto $\mathcal{H}(A, B)$, and $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \rightarrow \sqrt{2}F_-(z)$ is a partial isometry of $\mathcal{H}(M)$ onto $\mathcal{H}(C, D)$. The kernels of the partial isometries are orthogonal subspaces of $\mathcal{H}(M)$ which contain only constants.

We use the space $\mathcal{H}(M)$ only in the special case that the partial isometries are isometries. This is the case if, and only if, there is no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}(M)$ such that $u = 0$ or $v = 0$. The theory of spaces $\mathcal{H}(M)$ has been developed in previous work of the first author [7-9]. For the scattering problem itself we use a canonical model introduced by the first author [6]. The following theorem is a characterization of the relevant spaces of analytic functions.

THEOREM 7. Let $\mathcal{L}$ be a Hilbert space of vector valued functions analytic in the upper half-plane and in the lower half-plane. Suppose that the transformation of $\mathcal{L}$ into $\mathcal{C}$ defined by $F(z) \to F(w)$ is continuous for every nonreal number w. Suppose that $[F(z)-F(w)]/(z-w)$ belongs to $\mathcal{L}$ whenever $F(z)$ belongs to $\mathcal{L}$, for every nonreal number w, and that the identity

$$0 = \langle F(t), [G(t)-G(\beta)]/(t-\beta)\rangle - \langle [F(t)-F(\alpha)]/(t-\alpha), G(t)\rangle$$
$$+ (\alpha-\bar{\beta})\langle [F(t)-F(\alpha)]/(t-\alpha), [G(t)-G(\beta)]/(t-\beta)\rangle$$

holds for all $F(z)$ and $G(z)$ in $\mathcal{L}$ and for all nonreal numbers α and β. Then there exists an operator valued function $\varphi(z)$, analytic in the upper half-plane and in the lower half-plane, such that $\varphi(z) = -\bar{\varphi}(\bar{z})$, $[\varphi(z)+\bar{\varphi}(w)]c/[\pi i(\bar{w}-z)]$ belongs to $\mathcal{L}$ as a function of z for every vector c and nonreal number w, and

$$\bar{c}F(w) = \langle F(t), [\varphi(t)+\bar{\varphi}(w)]c/[\pi i(\bar{w}-t)]\rangle$$

for every $F(z)$ in $\mathcal{L}$. The space $\mathcal{L}$ with these properties is uniquely determined by $\varphi(z)$..

The space is called $\mathcal{L}(\varphi)$. The theory of $\mathcal{L}(\varphi)$ spaces is analogous to the theory of $\mathcal{H}(M)$ spaces.

THEOREM 8. Let $\varphi(z)$ be an operator valued function, analytic in the upper half-plane and in the lower half-plane, such that $\varphi(z) = -\bar{\varphi}(\bar{z})$. A necessary and sufficient condition for the existence of a corresponding space $\mathcal{L}(\varphi)$ is that $\mathrm{Re}\,\varphi(z) = \frac{1}{2}[\varphi(z)+\bar{\varphi}(z)] \geq 0$ for $y > 0$. If $\mathcal{L}(\varphi)$ is a given space which contains no nonzero constant, there exists a self-adjoint transformation H in the space such that

$$(H-w)^{-1}\colon F(z) \to [F(z)-F(w)]/(z-w)$$

for each nonreal number w.

A scattering problem is obtained by mapping one such space isometrically onto another.

THEOREM 9. If $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ are given spaces such that

$$\varphi_+(z)\varphi_-(z) = 1 = \varphi_-(z)\varphi_+(z) \quad ,$$

then the transformation $F(z) \to \varphi_+(z)F(z)$ takes $\mathcal{L}(\varphi_-)$ isometrically onto $\mathcal{L}(\varphi_+)$. If these spaces contain no nonzero constant, there exist self-adjoint transformations H_+ and H_- in $\mathcal{L}(\varphi_+)$ such that

$$(H_+ - w)^{-1}: F(z) \to [F(z) - F(w)]/(z-w)$$

$$(H_- - w)^{-1}: F(z) \to [F(z) - \varphi_+(z)\varphi_-(w)F(w)]/(z-w)$$

for nonreal values of w. If w is not real, the transformation

$$1 - (H_+ - \bar{w})^{-1}(H_+ - w)(H_- - \bar{w})(H_- - w)^{-1}$$

takes $F(z)$ into $(w-\bar{w})[\varphi_+(z) - \varphi_+(\bar{w})]\varphi_-(w)F(w)/(z-\bar{w})$. The dimension of the range of $(H_+-w)^{-1} - (H_--w)^{-1}$ does not exceed the dimension of $\mathcal{C}$ for any nonreal number w. If $1-\varphi_+(z)$ and $1+\varphi_-(z)$ have completely continuous values, then $(H_+-w)^{-1} - (H_--w)^{-1}$ is completely continuous for every nonreal number w.

An equivalent scattering problem can of course be formulated in $\mathcal{L}(\varphi_-)$. We use such pairs of spaces as a canonical model of the general scattering problem.

THEOREM 10. Let H_+ and H_- be self-adjoint transformations in a Hilbert space $\mathcal{H}$. Suppose that there is no nonzero element of $\mathcal{H}$ which belongs to the kernel of every transformation $(H_+-w)^{-1} - (H_--w)^{-1}$, w not real. Suppose that the dimension of the range of $(H_+-w)^{-1} - (H_--w)^{-1}$ does not exceed the dimension of the coefficient space $\mathcal{C}$ for some nonreal number w. Then there exists an isometric transformation U_+ of $\mathcal{H}$ onto some space $\mathcal{L}(\varphi_+)$ and an isometric transformation U_- of $\mathcal{H}$ onto some space $\mathcal{L}(\varphi_-)$, $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ containing no nonzero constant and related by

$$\varphi_+(z)\varphi_-(z) = 1 = \varphi_-(z)\varphi_+(z) ,$$

with these properties:

(1) If $U_+: f \to F_+(z)$, then

$$U_+(H_+-w)^{-1}: f \to [F_+(z) - F_+(w)]/(z-w)$$

for all nonreal numbers w.

(2) If $U_-: f \to F_-(z)$, then

$$U_-(H_--w)^{-1}: f \to [F_-(z) - F_-(w)]/(z-w)$$

for all nonreal numbers w.

(3) If $U_+: f \to F_+(z)$ and if $U_-: f \to F_-(z)$, then

$$F_+(z) = \varphi_+(z)F_-(z) .$$

(4) An element f of $\mathcal{H}$ is in the kernel of $(H_+-w)^{-1}-(H_--w)^{-1}$ for a nonreal number w if, and only if, $F_+(w) = F_-(w) = 0$. If $(H_+-w)^{-1} - (H_--w)^{-1}$ is completely continuous for some nonreal number w, then $\varphi_+(z)$ and $\varphi_-(z)$ can be chosen so that $i-\varphi_+(z)$ and $i + \varphi_-(z)$ have completely continuous values.

We now determine the form of the invariant subspaces in a scattering problem.

THEOREM 11. Let $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ be given spaces, which contain no nonzero constant, such that

$$\varphi_+(z)\varphi_-(z) = 1 = \varphi_-(z)\varphi_+(z)$$

and $i-\varphi_+(z)$ and $i+\varphi_-(z)$ have completely continuous values. Let H_+ and H_- be the self-adjoint transformations in $\mathcal{L}(\varphi_+)$ such that

$$(H_+-w)^{-1}: F(z) \to [F(z)-F(w)]/(z-w)$$

$$(H_--w)^{-1}: F(z) \to [F(z)-\varphi_+(z)\varphi_-(w)F(w)]/(z-w)$$

for nonreal numbers w. Suppose that the kernel of $(H_+-w)^{-1}-(H_--w)^{-1}$ is the set of $F(z)$ in $\mathcal{L}(\varphi_+)$ such that $F(w) = 0$, for every nonreal number w. Let H^* be the adjoint of the intersection H of H_+ and H_-. Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$ in which there is given a relation T, having no finite spectrum, such that the graph of H^* contains the graph of T, and the graph of T contains the intersections of the graphs of H_+ and H_- with $\mathfrak{M}\times\mathfrak{M}$. Suppose (*) that the graph of T is the closed span of the intersections of the graphs of H_+ and H_- with $\mathfrak{M}\times\mathfrak{M}$. Then there exists a unique space $\mathcal{H}(M)$ with these properties:

(1) $$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[i\varphi_+(z)F_+(z)+F_-(z)]$$

is an isomteric transformation of $\mathcal{H}(M)$ onto $\mathfrak{M}$.

(2) $i\varphi_+(z)F_+(z)+F_-(z)$ is in the range of H_+-w for a nonreal number w if, and only if, $F_+(w) = 0$.

(3) $i\varphi_+(z)F_+(z)+F_-(z)$ is in the range of H_--w for a nonreal number w if, and only if, $F_-(w) = 0$.

The space $\mathcal{H}(M)$ contains no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ such that $u = 0$ or $v = 0$.

The starred hypothesis is conjectured to be unnecessary. Conversely such spaces $\mathcal{H}(M)$ are invariant subspaces in the scattering problem.

THEOREM 12. Let $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ be given spaces, which contain no nonzero constant, such that

$$\varphi_+(z)\varphi_-(z) = 1 = \varphi_-(z)\varphi_+(z) .$$

Let H_+ and H_- be the self-adjoint transformations in $\mathcal{L}(\varphi_+)$ such that

$$(H_+-w)^{-1}: F(z) \to [F(z)-F(w)]/(z-w)$$

$$(H_--w)^{-1}: F(z) \to [F(z)-\varphi_+(z)\varphi_-(w)F(w)]/(z-w)$$

for nonreal values of w. Suppose that the kernel of $(H_+-w)^{-1} - (H_--w)^{-1}$ is the set of $F(z)$ in $\mathcal{L}(\varphi_+)$ such that $F(w) = 0$, for every nonreal number w. Let H^* be the adjoint of the intersection H of H_+ and H_-. If $\mathcal{H}(M)$ is a given space and if $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[i\varphi_+(z)F_+(z)+F_-(z)]$ is an isometric transformation of $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$, then $\mathcal{H}(M)$ contains no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ such that $u = 0$ or $v = 0$, and the range of the transformation is a closed subspace $\mathfrak{M}$ of $\mathcal{L}(\varphi_+)$ with this invariance property: there is a relation T in $\mathfrak{M}$, having no finite spectrum, such that the graph of H^* contains the graph of T. Assume that $i\varphi_+(z)F_+(z)-F_-(z)$ is in the range of H_+-w if, and only if, $F_+(w) = 0$, and that it is in the range of H_--w if, and only if, $F_-(w) = 0$. Then the graph of T contains the intersections of the graphs of H_+ and H_- with $\mathfrak{M} \times \mathfrak{M}$.

We conclude with conditions for a space $\mathcal{H}(M)$ to be so associated with $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$.

THEOREM 13. Let $\mathcal{H}(M)$ be a given space and let $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ be a pair of spaces such that

$$\varphi_+(z)\varphi_-(z) = 1 = \varphi_-(z)\varphi_+(z) .$$

A necessary and sufficient condition that

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[i\varphi_+(z)F_+(z)+F_-(z)]$$

be a transformation of $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$ which is bounded by 1 is that

$$\varphi_+(z)\{[A(z)-iB(z)] - [A(z)+iB(z)]W(z)\}$$
$$= [D(z)+iC(z)] + [D(z)-iC(z)]W(z)$$

for some operator valued function $W(z)$, which is analytic and bounded by 1 in the upper half-plane. In this case the orthogonal complement in $\mathcal{H}(M)$ of the set on which the transformation is isometric consists of elements of the form $M(z)\begin{pmatrix} u \\ v \end{pmatrix}$ where u and v are vectors such that $\bar{u}v = \bar{v}u$.

These results reduce the scattering problem to the study of spaces $\mathcal{H}(M)$, about which much is known from previous work [7-9]. A complete solution of the problem is, however, known only in the case of a one-dimensional coefficient space. The answers in that case are given by the theory [2-5] of Hilbert spaces of entire functions. (A vector generalization of the theory is in process.) The proofs of the theorems presume a knowledge of operator valued analytic functions as given in the appendix on square summable power series. This theory has a separate interest in that it provides a simple canonical model which is instructive in approaching the more elaborate models required for quantum scattering theory. An introduction to square summable power series is available in lecture notes [10]. The notation of these notes and of the appendix is not always consistent with the notation used in the body of the paper.

Proof of Theorem 1. By Theorem 20 of the appendix, $E_+(z)$ and $E_-(z)$ have invertible values except at isolated points in the complex plane. So the integrals appearing in the definition of the $\mathcal{H}(A,B)$-norm are meaningful. The parallelogram law is used to show that $\mathcal{H}(A,B)$ is a vector space over the complex numbers. The inner product in the space which corresponds to its norm is

$$\langle F(t), G(t)\rangle = \int_{-\infty}^{+\infty} \bar{G}(t)\bar{E}_+(t)^{-1}E_+(t)^{-1}F(t)\,dt .$$

The required linearity, symmetry, and positivity of an inner product follow from this formula. Since

$$2\pi i(\bar{w}-z)K(w,z) = E_+(z)\bar{E}_+(w) - E_-(z)\bar{E}_-(w)$$

and since $E_+(z)^{-1}E_-(z)$ is bounded by 1 in the upper half-plane (because of the inequality $K(z,z) \geq 0$), the function $2\pi i(\bar{w}-z)E_+(z)^{-1}K(w,z)$ is analytic and bounded in the upper half-plane for any fixed w. Since $2\pi i(\bar{w}-z)$ is of bounded type and of zero mean type in the upper half-plane, the analytic function $E_+(z)^{-1}K(w,z)$ is of bounded type and of nonpositive mean type in the upper half-plane. For the same reasons the analytic function $E_-(z)^{-1}K(w,z)$ is of bounded type and of nonpositive mean type in the lower half-plane. It is clear that

$$\int_{-\infty}^{+\infty} |E_+(t)^{-1}K(w,t)c|^2\,dt < \infty$$

for every vector c when w is not real. The same conclusion holds when w is real, for

$$\int_{w+1}^{\infty} |E_+(t)^{-1}K(w,t)c|^2\,dt \quad \text{and} \quad \int_{-\infty}^{w-1} |E_+(t)^{-1}K(w,t)\,c|^2\,dt$$

are finite and $E_+(t)^{-1}K(w,t)c$ is a continuous function of t in the interval $(w-1, w+1)$. So $K(w,z)c$ belongs to $\mathcal{H}(A,B)$ in all cases.

If $F(z)$ belongs to $\mathcal{H}(A,B)$, $E_+(z)^{-1}F(z)$ is analytic in the upper half-plane and continuous in the closed half-plane with the possible exception of singularities at isolated points on the real axis. By the theory of functions which are analytic and of bounded type in the upper half-plane,

$$E_+(z)^{-1}F(z) = (2\pi i)^{-1}\int_{-\infty}^{+\infty} (t-z)^{-1}E_+(t)^{-1}F(t)\,dt$$

$$0 = \int_{-\infty}^{+\infty} (t-\bar{z})^{-1}E_+(t)^{-1}F(t)\,dt$$

for $y > 0$. Similarly

$$E_-(z)^{-1}F(z) = -(2\pi i)^{-1}\int_{-\infty}^{+\infty} (t-z)^{-1}E_-(t)^{-1}F(t)\,dt$$

$$0 = \int_{-\infty}^{+\infty} (t-\bar{z})^{-1}E_-(t)^{-1}F(t)\,dt$$

for $y < 0$. It follows from these four equations that

$$F(w) = \int_{-\infty}^{+\infty} \bar{K}(w,t)\bar{E}_+(t)^{-1}E_+(t)^{-1}F(t)\,dt$$

when w is not real. The formula for real w is obtained by showing that $K(w,z)c$ depends continuously on w in the metric of $\mathcal{H}(A,B)$ for every vector c. We omit this part of the proof, which depends on a straightforward estimate of integrals.

To prove completeness of $\mathcal{H}(A,B)$ consider a Cauchy sequence $(F_n(z))$ in the space. Since $\bar{c}F_n(w) = \langle F_n(t), K(w,t)c\rangle$ for every vector c, $(F_n(w))$ is a Cauchy sequence of vectors for every w. Let $F(w) = \lim F_n(w)$ as $n \to \infty$. Since $\|K(w,t)c\|^2 = \bar{c}K(w,w)c$ is bounded independently of w on every bounded set, the convergence is uniform on bounded sets, and $F(z)$ is a vector valued entire function. Since $E_+(t)^{-1}F(t) = \lim E_+(t)^{-1}F_n(t)$ as $n \to \infty$ except possibly at isolated points on the real axis,

$$\int_{-\infty}^{+\infty} |E_+(t)^{-1}F(t)|^2\,dt = \lim_{n\to\infty} \int_{-\infty}^{+\infty} |E_+(t)^{-1}F_n(t)|^2\,dt$$

$$\leq \lim_{n\to\infty} \|F_n(t)\|^2 < \infty .$$

Since

$$|\bar{c}F(w)|^2 \leq \bar{c}K(w,w)c \lim_{n\to\infty} \|F_n(t)\|^2$$

for every vector c, we obtain the operator inequality

$$F(w)\bar{F}(w) \leq K(w,w) \lim_{n\to\infty} \|F_n(t)\|^2 .$$

This implies that

$$E_+(w)^{-1}F(w)\bar{F}(w)\bar{E}_+(w)^{-1} \leq E_+(w)^{-1}K(w,w)\bar{E}_+(w)^{-1} \lim_{n\to\infty} \|F_n(t)\|^2$$

$$\leq [2\pi i(\bar{w}-w)]^{-1} \lim_{n\to\infty} \|F_n(t)\|^2$$

for $i(\bar{w}-w) > 0$. The operator inequality implies that

$$|E_+(w)^{-1}F(w)|^2 \leq [2\pi i(\bar{w}-w)]^{-1} \lim_{n\to\infty} \|F_n(t)\|^2$$

when $i(\bar{w}-w) > 0$. It follows that $E_+(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the upper half-plane. A similar argument will show that $E_-(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the lower half-plane. This completes the proof that $F(z)$ belongs to $\mathcal{H}(A,B)$. Since

$$F(t)-F_n(t) = \lim_{r\to\infty} [F_r(t)-F_n(t)]$$

for all real t,

$$\int_{-\infty}^{+\infty} |E_+(t)^{-1}[F(t)-F_n(t)]|^2 dt \leq \lim_{r\to\infty} \int_{-\infty}^{+\infty} |E_+(t)^{-1}[F_r(t)-F_n(t)]|^2 dt$$

or equivalently

$$\|F(t)-F_n(t)\| \leq \lim_{r\to\infty} \|F_r(t)-F_n(t)\| .$$

Since $(F_n(z))$ is assumed to be a Cauchy sequence, it converges to $F(z)$ in the metric of $\mathcal{H}(A,B)$. The theorem follows.

Proof of Theorem 2. To show that multiplication by z is symmetric in $\mathcal{H}(A,B)$, we must show that the identity $\langle tF(t), G(t)\rangle = \langle F(t), tG(t)\rangle$ holds whenever $F(z)$ and $G(z)$ are in the domain of multiplication by z. The identity follows from the definition of the inner product as an integral on the real axis. Symmetry implies that multiplication by $z-w$ has a bounded, partially defined inverse for every nonreal number w. To show that multiplication by z has a closed graph, we need only show that multiplication by $z-w$ has a closed graph, or what is equivalent, that multiplication by $z-w$ has a closed range. This is true because the range of multiplication by $z-w$ is the set of all $F(z)$ in $\mathcal{H}(A,B)$ which vanish at w.

If w is a nonreal number or a real number such that $E_+(w)$ has an inverse, the range of multiplication by $z-w$ in $\mathcal{H}(A,B)$ is the set of all $F(z)$ which vanish at w. It follows that the orthogonal complement of the range of multiplication by $z-w$ contains no nonzero function which vanishes at w. Since $F(z) \to F(w)$ is a continuous, one-to-one transformation of the orthogonal complement into $\mathcal{C}$, the dimension of the orthogonal complement does not exceed the dimension of $\mathcal{C}$. A different argument must be used if w is a real number such that $E_+(w)$ does not have an inverse. By Theorem 18 of the appendix, there exists a polynomial $P(z)$ with operator coefficients such that the coefficients of $1-P(z)$ are completely continuous, $P(z)$ has invertible values at all points $z \neq w$, and $P(z)^{-1}E_+(z)$ is an operator valued entire function which has an invertible value at w. Since $E_+(z)\bar{E}_+(\bar{z}) = E_-(z)\bar{E}_-(\bar{z})$, $P(z)^{-1}E_-(z)$ is an operator valued entire function. If $A_1(z) = P(z)^{-1}A(z)$ and $B_1(z) = P(z)^{-1}B(z)$, then a space $\mathcal{H}(A_1, B_1)$ exists, $A_1(w)-iB_1(w)$ is an invertible operator, and $F(z) \to P(z)F(z)$ is an isometric transformation of $\mathcal{H}(A_1,B_1)$ onto $\mathcal{H}(A,B)$. We have seen that the dimension of the orthogonal complement of the range of multiplication by $z-w$ in $\mathcal{H}(A_1,B_1)$ does not exceed the dimension of $\mathcal{C}$. It follows that the dimension of the orthogonal complement of the range of multiplication by $z-w$ in $\mathcal{H}(A,B)$ does not exceed the dimension of C.

Consider an element $F(z)$ of $\mathcal{H}(A,B)$ and a number w which is a point of continuity of $E_+(z)^{-1}F(z)$. Then $[F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ is a vector valued entire function. Since $F(z)$ belongs to $\mathcal{H}(A,B)$, $E_+(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the upper half-plane. It follows that

$$E_+(z)^{-1}[F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w) = [E_+(z)^{-1}F(z)-E_+(w)^{-1}F(w)]/(z-w)$$

is of bounded type and of nonpositive mean type in the upper half-plane. Since $F(z)$ belongs to $\mathcal{H}(A,B)$, $E_-(z)^{-1}F(z)$ is of bounded type and of nonpositive mean type in the lower half-plane. Since $E_-(z)^{-1}E_+(z)$ is bounded by 1 in the lower half-plane,

$$\begin{aligned} &E_-(z)^{-1}[F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w) \\ &\quad = [E_-(z)^{-1}F(z)-E_-(z)^{-1}E_+(z)E_+(w)^{-1}F(w)]/(z-w) \end{aligned}$$

is of bounded type and of nonpositive mean type in the lower half-plane. Since $F(z)$ belongs to $\mathcal{H}(A,B)$,

$$\int_{-\infty}^{+\infty} |E_+(t)^{-1}F(t)|^2 dt < \infty .$$

Since w is a point of continuity of $E_+(z)^{-1}F(z)$, it is a point of differentiability of the function and a point of continuity of $[E_+(z)^{-1}F(z)-E_+(w)^{-1}F(w)]/(z-w)$. So

$$\int_{-\infty}^{+\infty} |E_+(t)^{-1}[F(t)-E_+(t)E_+(w)^{-1}F(w)]/(t-w)|^2 dt < \infty .$$

This completes the proof that $[F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ belongs to $\mathcal{H}(A,B)$.

Now consider the elements $F(z)$ and $G(z)$ of $\mathcal{H}(A,B)$ and numbers α and β, α a point of continuity of $E_+(z)^{-1}F(z)$ and β a point of continuity of $E_+(z)^{-1}G(z)$. Then

$$\langle F(t), [G(t)-E_+(t)E_+(\beta)^{-1}G(\beta)]/(t-\beta)\rangle - \langle [F(t)-E_+(t)E_+(\alpha)^{-1}F(\alpha)]/(t-\alpha), G(t)\rangle$$

$$+(\alpha-\bar\beta)\langle [F(t)-E_+(t)E_+(\alpha)^{-1}F(\alpha)]/(t-\alpha), [G(t)-E_+(t)E_+(\beta)^{-1}G(\beta)]/(t-\beta)\rangle$$

$$= \int_{-\infty}^{+\infty} \frac{\bar G(t)\bar E_+(t)^{-1}-\bar G(\beta)\bar E_+(\beta)^{-1}}{t-\bar\beta} E_+(t)^{-1}F(t)\,dt$$

$$- \int_{-\infty}^{+\infty} \bar G(t)\bar E_+(t)^{-1} \frac{E_+(t)^{-1}F(t)-E_+(\alpha)^{-1}F(\alpha)}{t-\alpha}\,dt$$

$$+(\alpha-\bar\beta)\int_{-\infty}^{+\infty} \frac{\bar G(t)\bar E_+(t)^{-1}-\bar G(\beta)\bar E_+(\beta)^{-1}}{t-\bar\beta}\,\frac{E_+(t)^{-1}F(t)-E_+(\alpha)^{-1}F(\alpha)}{t-\alpha}\,dt$$

$$= -2\pi i\bar G(\beta)\bar E_+(\beta)^{-1}E_+(\alpha)^{-1}F(\alpha)$$

by Cauchy's formula.

If $J(w)$ is the transformation $F(z) \to F(w)$ of $\mathcal{H}(A,B)$ into $\mathcal{C}$ for any complex number w, the adjoint $J(w)^*$ of $J(w)$ is the transformation $c \to K(w,z)c$ of $\mathcal{C}$ into $\mathcal{H}(A,B)$. Since $J(w)J(w)^*: c \to K(w,w)c$ where $K(w,w)$ is a completely continuous operator, $J(w)$ is completely continuous.

Proof of Theorem 3. The hypotheses imply that the transformation $(T-\bar w)(T-w)^{-1}$ is everywhere defined and bounded by 1 when w lies above the real axis, and that

$$1 - [(T-\bar w)(T-w)^{-1}]^*(T-\bar w)(T-w)^{-1}$$

$$= i(\bar w-w)[i(T^*-\bar w)^{-1}-i(T-w)^{-1}-i(\bar w-w)(T^*-\bar w)^{-1}(T-w)^{-1}]$$

is completely continuous. The kernel of the transformation contains the range of $L-w$ since we assume that L is a symmetric transformation and that T extends L. By hypothesis the dimension of the orthogonal complement of the range of $L-w$ does not exceed the dimension of $\mathcal{C}$ for some w. For this w there exists a bounded transformation $J(w)$ of $\mathcal{H}$ into $\mathcal{C}$, with adjoint $J(w)^*$ taking $\mathcal{C}$ into $\mathcal{H}$, such that

$$2\pi J(w)^*J(w) = i(T^*-\bar w)^{-1}-i(T-w)^{-1}-i(\bar w-w)(T^*-\bar w)^{-1}(T-w)^{-1}\ .$$

Define $J(\alpha) = J(w)(T-w)(T-\alpha)^{-1}$ whenever α is not in the spectrum of T. Then $J(\alpha)$ is a bounded transformation of $\mathcal{H}$ into $\mathcal{C}$, with adjoint $J(\alpha)^*$ taking $\mathcal{C}$ into $\mathcal{H}$. The identity

$$2\pi J(\beta)^* J(\alpha) = i(T^*-\bar{\beta})^{-1} - i(T-\alpha)^{-1} - i(\bar{\beta}-\alpha)(T^*-\bar{\beta})^{-1}(T-\alpha)^{-1}$$

is easily verified when α and β are not in the spectrum of T.

Let $\mathfrak{M}$ be the set of elements f of $\mathcal{H}$ such that $J(w)f = 0$ whenever w is not in the spectrum of T. We show that $\mathfrak{M}$ contains no nonzero element. The resolvent identity

$$(\alpha-\beta)(T-\alpha)^{-1}(T-\beta)^{-1} = (T-\alpha)^{-1} - (T-\beta)^{-1}$$

implies that $\mathfrak{M}$ is invariant under $(T-w)^{-1}$ whenever w is not in the spectrum of T. But if f is in $\mathfrak{M}$,

$$0 = i(T^*-\bar{\beta})^{-1}f - i(T-\alpha)^{-1}f - i(\bar{\beta}-\alpha)(T^*-\bar{\beta})^{-1}(T-\alpha)^{-1}f$$

whenever α and β are not in the spectrum of T. It follows that

$$0 = i\langle f, (T-\beta)^{-1}f\rangle - i\langle (T-\alpha)^{-1}f, f\rangle - i(\bar{\beta}-\alpha)\langle (T-\alpha)^{-1}f, (T-\beta)^{-1}f\rangle .$$

But $i\langle (T-w)^{-1}f, f\rangle$ is an analytic function of w for w in the complement of the spectrum of T, which lies below the real axis by hypothesis. The last identity implies that the real part of the function,

$$\mathrm{Re}\, i\langle (T-w)^{-1}f, f\rangle = \tfrac{1}{2}i(\bar{w}-w)\langle (T-w)^{-1}f, (T-w)^{-1}f\rangle ,$$

is nonnegative in the upper half-plane and identically zero on the real axis. By the Poisson representation of functions analytic and having a nonnegative real part in the upper half-plane, $\langle (T-w)^{-1}f, (T-w)^{-1}f\rangle$ does not depend on w when w is in the upper half-plane. By the linearity of an inner product, $\langle (T-w)^{-1}f, g\rangle$ does not depend on w when f and g are in $\mathfrak{M}$ and w is in the upper half-plane. By the arbitrariness of g, $(T-w)^{-1}f$ does not depend on w when w is in the upper half-plane. Since

$$(\alpha-\beta)(T-\alpha)^{-1}(T-\beta)^{-1}f = (T-\alpha)^{-1}f - (T-\beta)^{-1}f ,$$

$(T-\alpha)^{-1}(T-\beta)^{-1}f = 0$ when α and β are in the upper half-plane. It follows that $f = 0$ and that $\mathfrak{M}$ contains no nonzero element.

Let $W(z)$ be the operator valued analytic function defined in the complement of the spectrum of T by

$$W(w) = 1 + 2\pi i w J(w) J(0)^* .$$

Then $W(z)$ has value 1 at the origin and $1-W(z)$ has completely continuous values. The identity

$$2\pi i(\bar{\beta}-\alpha) J(\alpha) J(\beta)^* =$$

$$2\pi i \bar{\beta} J(0) J(\beta)^* - 2\pi i \alpha J(\alpha) J(0)^* - 4\pi^2 \alpha\bar{\beta} J(\alpha) J(0)^* J(0) J(\beta)^*$$

is obtained on writing $J(w) = J(0) T(T-w)^{-1}$ since

$$2\pi J(0)^* J(0) = iT^{*-1} - iT^{-1} .$$

The identity can be written

$$J(\alpha) J(\beta)^* = [1-W(\alpha)\bar{W}(\beta)]/[2\pi i(\bar{\beta}-\alpha)]$$

when α and β are not in the spectrum of T. The identity implies that $W(z)$ is bounded by 1 in the upper half-plane and that $W(w)\bar{W}(\bar{w}) = 1$ whenever w and $\bar{w}$ are not in the spectrum of T. Theorems 18 and 19 of the appendix now allow us to construct an operator valued entire function $S(z)$ such that $1-S(z)$ has completely continuous values, $S(z)$ has invertible values on and above the real axis, and $S(z) W(z)$ is analytic in the complex plane. Let $A(z)$ and $B(z)$ be the operator valued entire functions defined by

$$A(z) - iB(z) = S(z) \quad \text{and} \quad A(z) + iB(z) = S(z) W(z) .$$

The hypotheses for the existence of a space $\mathcal{H}(A, B)$ follow from the stated properties of $S(z)$ and $W(z)$. We show that the transformation $U : f \to F(z)$, defined by $F(w) = S(w) J(w) f$ whenever w is not in the spectrum of T, takes $\mathcal{H}$ isometrically onto $\mathcal{H}(A, B)$.

Consider first the case in which $f = J(w)^* \bar{S}(w) c$ for some vector c. Then

$$\begin{aligned} F(z) &= S(z)[1-W(z)\bar{W}(w)]\bar{S}(w) c / [2\pi i(\bar{w}-z)] \\ &= [B(z)\bar{A}(w) - A(z)\bar{B}(w)] c / [\pi(z-\bar{w})] \\ &= K(w, z) c \end{aligned}$$

belongs to $\mathcal{H}(A, B)$ as required. If f is a finite sum of elements $J(w)^*\bar{S}(w) c$ where c is in $\mathcal{C}$ and w is not in the spectrum of T, then $F(z)$ is a corresponding finite sum of elements $K(w, z)c$. A straightforward computation will show that the norm of $F(z)$ in $\mathcal{H}(A, B)$ is equal to the norm of f in $\mathcal{H}$. Since $\mathcal{H}$ contains no nonzero element f such that $J(w)f = 0$ for all numbers w in the upper half-plane, and since $S(w)$ has invertible values in the upper half-plane, $\mathcal{H}$ contains no nonzero element f such that $S(w)J(w)f=0$ for all numbers w not in the spectrum of T. Therefore $\mathcal{H}$ is the closed span of elements $J(w)^*\overline{S}(w)c$ where c is in $\mathcal{C}$ and w is not in the spectrum of T. It follows that the transformation U takes $\mathcal{H}$ isometrically into $\mathcal{H}(A, B)$. Since the range of U is closed, and since it contains $K(w, z)c$ for every vector c and every number w above the real axis, it is all of $\mathcal{H}(A, B)$.

If f is in $\mathcal{H}$ and if w is in the upper half-plane, let $U : f \to F(z)$ and $U : (T-w)^{-1}f \to G(z)$. Then

$$\begin{aligned} (z-w)S(z)^{-1}G(z) &= (z-w)J(0)T(T-z)^{-1}(T-w)^{-1}f \\ &= J(0)T(T-z)^{-1}f - J(0)T(T-w)^{-1}f \\ &= S(z)^{-1}F(z) - S(w)^{-1}F(w) . \end{aligned}$$

Since $S(z) = E_+(z)$, $G(z) = [F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$. Thus the transformation $(T-w)^{-1}$ in $\mathcal{H}$ corresponds to the transformation $F(z) \to [F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ in $\mathcal{H}(A, B)$. It follows from Theorem 2 that the adjoint of the transformation $F(z) \to [F(z)-E_+(z)E_+(w)^{-1}F(w)]/(z-w)$ in $\mathcal{H}(A, B)$ is the transformation $F(z) \to [F(z)-E_-(z)E_-(\overline{w})^{-1}F(\overline{w})]/(z-\overline{w})$. The transformation $(T^*-w)^{-1}$ in $\mathcal{H}$ therefore corresponds to the transformation $F(z) \to [F(z)-E_-(z)E_-(w)^{-1}F(w)]/(z-w)$ in $\mathcal{H}(A, B)$ when w is not in the spectrum of T^*. Since we assume that the graph of L is the intersection of the graph of T and the graph of T^*, it follows that L corresponds under U to multiplication by z in $\mathcal{H}(A, B)$.

Proof of Theorem 4. We show that the positive-definiteness inequality

$$\sum \begin{pmatrix} u_i \\ v_i \end{pmatrix}^{-} \frac{M(w_i)I\overline{M}(w_k)-I}{2\pi(w_i-\overline{w}_k)} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \geq 0$$

holds whenever $w_1, \ldots, w_r$ is a finite set of points in the plane, and $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ are corresponding vectors. Since we assume that $\overline{M}(0)IM(0) = I$, we can write

$$\frac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})} = \frac{1-M(z)I\bar{M}(0)I}{2}\;\frac{\varphi(z)+\bar{\varphi}(w)}{\pi i(\bar{w}-z)}\;\frac{1-IM(0)I\bar{M}(w)}{2}$$

where

$$\varphi(z) = [1-M(z)I\bar{M}(0)I]^{-1}[1+M(z)I\bar{M}(0)I]iI$$

is a matrix valued function which is defined and analytic on the set where $1-M(z)I\bar{M}(0)I$ has invertible values. The set is not empty because $\frac{1}{2}[1-M(z)I\bar{M}(0)I]$ has the identity matrix for its value at the origin. Since the values of $1-\frac{1}{2}[1-M(z)I\bar{M}(0)I]$ are matrices of completely continuous operators, it follows from (the matrix analogue of) Theorem 18 of the appendix that $\frac{1}{2}[1-M(z)I\bar{M}(0)I]$ has invertible values at all except isolated points in the complex plane. The hypothesis that

$$[M(z)I\bar{M}(z)-I]/(z-\bar{z}) \geq 0$$

for all complex z implies that

$$[\varphi(z)+\bar{\varphi}(z)]/[\pi i(\bar{z}-z)] \geq 0$$

at all points where $\varphi(z)$ is defined. Since a function which is analytic and has a nonnegative real part in a neighborhood of a point cannot have a singularity at that point, $\varphi(z)$ has an analytic extension to the upper half-plane and the lower half-plane. The hypothesis that $M(z)I\bar{M}(\bar{z}) = I$ implies that $\varphi(z) = -\bar{\varphi}(\bar{z})$. Let (t_n) be an enumeration of the singularities of $\varphi(z)$, which are isolated points on the real axis. By the (matrix generalization of) the Poisson representation of a function which is analytic and has a nonnegative real part in a half-plane, there exists a nonnegative matrix P and a sequence of nonnegative matrices (P_n) such that

$$\frac{\varphi(z)+\bar{\varphi}(w)}{\pi i(\bar{w}-z)} = P + \sum \frac{P_n}{(t_n-z)(t_n-\bar{w})}$$

whenever z and w are not singularities of $\varphi(z)$. If $w_1, \dots, w_r$ are nonreal numbers and $u_1, \dots, u_r$ and $v_1, \dots, v_r$ are corresponding vectors, then

$$\sum_{i,k} \begin{pmatrix} u_i \\ v_i \end{pmatrix}^- \frac{\varphi(w_i)+\bar{\varphi}(w_k)}{\pi i(\bar{w}_k-w_i)} \begin{pmatrix} u_k \\ v_k \end{pmatrix} =$$

$$= \sum_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}^- P \sum_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} + \sum_n \sum_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}^- \frac{1}{(t_n-w_i)} P_n \sum_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} \frac{1}{(t_n-\bar{w}_k)}$$

is a sum of nonnegative terms and so is nonnegative. The desired positive-definiteness inequality for nonreal values of $w_1, \ldots, w_r$ is obtained on replacing $\begin{pmatrix} u_k \\ v_k \end{pmatrix}$ by $\frac{1}{2}[1-IM(0)\overline{IM}(w_k)]\begin{pmatrix} u_k \\ v_k \end{pmatrix}$ for every k. The general case follows by continuity.

The proof of the existence of $\mathcal{H}(M)$ follows the proof of Lemma 11 of [10], with obvious changes. Let $\mathcal{H}_0$ be the set of all finite sums of functions of the form

$$\frac{M(z)\overline{IM}(w)-I}{2\pi(z-\overline{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

for arbitrary vectors u and v and numbers w. If

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} = \sum \frac{M(z)\overline{IM}(w_k)-I}{2\pi(z-\overline{w}_k)}\begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix} = \sum \frac{M(z)\overline{IM}(w_k)-I}{2\pi(z-\overline{w}_k)}\begin{pmatrix} c_k \\ d_k \end{pmatrix}$$

are two such pairs, define

$$\left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\rangle = \sum \begin{pmatrix} c_i \\ d_i \end{pmatrix}^- \frac{M(w_i)\overline{IM}(w_k)-I}{2\pi(w_i-\overline{w}_k)}\begin{pmatrix} a_k \\ b_k \end{pmatrix}.$$

As in the proof of Lemma 11 of [10], $\mathcal{H}_0$ is a well-defined vector space with inner product. It is contained isometrically as a dense subspace of a Hilbert space $\mathcal{H}$, whose elements are pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of vector valued functions. The pair

$$\frac{M(z)\overline{IM}(w)-I}{2\pi(z-\overline{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathcal{H}$ for all choices of vectors u and v, and

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w)-I}{2\pi(t-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

for every $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ in the space. Furthermore the argument of [10] shows that the space having this property is uniquely determined by $M(z)$. We complete the proof of the theorem by showing that the elements of the space are pairs of vector valued entire functions, and that the space is properly behaved with respect to difference quotients.

To see that the elements of $\mathcal{H}$ arc pairs of continuous functions, apply the Schwarz inequality to the formula

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(\beta)-F_+(\alpha) \\ F_-(\beta)-F_-(\alpha) \end{pmatrix} =$$

$$= \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)I\bar{M}(\beta)-I}{2\pi(t-\bar{\beta})} \begin{pmatrix} u \\ v \end{pmatrix} - \frac{M(t)I\bar{M}(\alpha)-I}{2\pi(t-\bar{\alpha})} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

to obtain

$$\left| \begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(\beta)-F_+(\alpha) \\ F_-(\beta)-F_-(\alpha) \end{pmatrix} \right|^2$$

$$\leq \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 \begin{pmatrix} u \\ v \end{pmatrix}^- \left[\frac{M(\beta)I\bar{M}(\beta)-I}{2\pi(\beta-\bar{\beta})} - \frac{M(\beta)I\bar{M}(\alpha)-I}{2\pi(\beta-\bar{\alpha})} \right.$$

$$\left. - \frac{M(\alpha)I\bar{M}(\beta)-I}{2\pi(\alpha-\bar{\beta})} + \frac{M(\alpha)I\bar{M}(\alpha)-I}{2\pi(\alpha-\bar{\alpha})} \right] \begin{pmatrix} u \\ v \end{pmatrix}$$

By the arbitrariness of u and v, we obtain the matrix inequality

$$\begin{pmatrix} F_+(\beta)-F_+(\alpha) \\ F_-(\beta)-F_-(\alpha) \end{pmatrix} \begin{pmatrix} F_+(\beta)-F_+(\alpha) \\ F_-(\beta)-F_-(\alpha) \end{pmatrix}^-$$

$$\leq \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 \left[\frac{M(\beta)I\bar{M}(\beta)-I}{2\pi(\beta-\bar{\beta})} - \frac{M(\beta)I\bar{M}(\alpha)-I}{2\pi(\beta-\bar{\alpha})} - \frac{M(\alpha)I\bar{M}(\beta)-I}{2\pi(\alpha-\bar{\beta})} + \frac{M(\alpha)I\bar{M}(\alpha)-I}{2\pi(\alpha-\bar{\alpha})} \right],$$

which implies that

$$\left| \begin{pmatrix} F_+(\beta)-F_+(\alpha) \\ F_-(\beta)-F_-(\alpha) \end{pmatrix} \right|^2$$

$$\leq \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 \left| \frac{M(\beta)I\bar{M}(\beta)-I}{2\pi(\beta-\bar{\beta})} - \frac{M(\beta)I\bar{M}(\alpha)-I}{2\pi(\beta-\bar{\alpha})} - \frac{M(\alpha)I\bar{M}(\beta)-I}{2\pi(\alpha-\bar{\beta})} + \frac{M(\alpha)I\bar{M}(\alpha)-I}{2\pi(\alpha-\bar{\alpha})} \right| .$$

Since $[M(z)I\bar{M}(w)-I]/(z-\bar{w}) = [M(z)-M(\bar{w})]I\bar{M}(w)/(z-\bar{w})$ is a continuous function of z for every w, it follows that $F_+(z)$ and $F_-(z)$ are continuous functions of z.

Since $M(z)$ has invertible values in the complex plane, the same equation implies that

$$\frac{M(z)-M(w)}{z-w} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathfrak{H}$ for all vectors u and v and numbers w, and that

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \bar{M}(w)I \begin{pmatrix} F_+(\bar{w}) \\ F_-(\bar{w}) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)-M(w)}{2\pi(t-w)} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle .$$

If $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix} = \dfrac{M(z)-M(w)}{2\pi(z-w)} \begin{pmatrix} u \\ v \end{pmatrix}$ for any fixed $u, v,$ and w, then

$$(\beta - w)\begin{pmatrix}[G_+(z)-G_+(\beta)]/(z-\beta)\\ [G_-(z)-G_-(\beta)]/(z-\beta)\end{pmatrix} = \frac{M(z)-M(\beta)}{2\pi(z-\beta)}\begin{pmatrix}u\\ v\end{pmatrix} - \frac{M(z)-M(w)}{2\pi(z-w)}\begin{pmatrix}u\\ v\end{pmatrix}.$$

Therefore, $\begin{pmatrix}[G_+(z)-G_+(\beta)]/(z-\beta)\\ [G_-(z)-G_-(\beta)]/(z-\beta)\end{pmatrix}$ belongs to $\mathcal{H}$ if $\beta \neq w$. Now suppose that $\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix}$ and $\begin{pmatrix}[F_+(z)-F_+(\alpha)]/(z-\alpha)\\ [F_-(z)-F_-(\alpha)]/(z-\alpha)\end{pmatrix}$ belong to $\mathcal{H}$ for some number α. Then

$$\left\langle\begin{pmatrix}F_+(t)\\ F_-(t)\end{pmatrix}, \begin{pmatrix}[G_+(t)-G_+(\beta)]/(t-\beta)\\ [G_-(t)-G_-(\beta)]/(t-\beta)\end{pmatrix}\right\rangle - \left\langle\begin{pmatrix}[F_+(t)-F_+(\alpha)]/(t-\alpha)\\ [F_-(t)-F_-(\alpha)]/(t-\alpha)\end{pmatrix}, \begin{pmatrix}G_+(t)\\ G_-(t)\end{pmatrix}\right\rangle$$

$$+(\alpha-\bar\beta)\left\langle\begin{pmatrix}[F_+(t)-F_+(\alpha)]/(t-\alpha)\\ [F_-(t)-F_-(\alpha)]/(t-\alpha)\end{pmatrix}, \begin{pmatrix}[G_+(t)-G_+(\beta)]/(t-\beta)\\ [G_-(t)-G_-(\beta)]/(t-\beta)\end{pmatrix}\right\rangle$$

$$=\left\langle\begin{pmatrix}F_+(t)\\ F_-(t)\end{pmatrix}, \frac{1}{\beta-w}\,\frac{M(t)-M(\beta)}{2\pi(t-\beta)}\begin{pmatrix}u\\ v\end{pmatrix}\right\rangle - \left\langle\begin{pmatrix}F_+(t)\\ F_-(t)\end{pmatrix}, \frac{1}{\beta-w}\,\frac{M(t)-M(w)}{2\pi(t-w)}\begin{pmatrix}u\\ v\end{pmatrix}\right\rangle$$

$$-\left\langle\begin{pmatrix}[F_+(t)-F_+(\alpha)]/(t-\alpha)\\ [F_-(t)-F_-(\alpha)]/(t-\alpha)\end{pmatrix}, \frac{M(t)-M(w)}{2\pi(t-w)}\begin{pmatrix}u\\ v\end{pmatrix}\right\rangle$$

$$+(\alpha-\bar\beta)\left\langle\begin{pmatrix}[F_+(t)-F_+(\alpha)]/(t-\alpha)\\ [F_-(t)-F_-(\alpha)]/(t-\alpha)\end{pmatrix}, \frac{1}{\beta-w}\,\frac{M(t)-M(\beta)}{2\pi(t-\beta)}\begin{pmatrix}u\\ v\end{pmatrix}\right\rangle$$

$$-(\alpha-\bar\beta)\left\langle\begin{pmatrix}[F_+(t)-F_-(\alpha)]/(t-\alpha)\\ [F_-(t)-F_-(\alpha)]/(t-\alpha)\end{pmatrix}, \frac{1}{\beta-w}\,\frac{M(t)-M(w)}{2\pi(t-w)}\begin{pmatrix}u\\ v\end{pmatrix}\right\rangle$$

$$= \begin{pmatrix}u\\ v\end{pmatrix}^- \frac{\bar M(\beta)I}{\bar\beta-\bar w}\begin{pmatrix}F_+(\bar\beta)\\ F_-(\bar\beta)\end{pmatrix} - \begin{pmatrix}u\\ v\end{pmatrix}^- \frac{\bar M(w)I}{\bar\beta-\bar w}\begin{pmatrix}F_+(\bar w)\\ F_-(\bar w)\end{pmatrix}$$

$$-\begin{pmatrix}u\\ v\end{pmatrix}^- \bar M(w)I\begin{pmatrix}[F_+(\bar w)-F_+(\alpha)]/(\bar w-\alpha)\\ [F_-(\bar w)-F_-(\alpha)]/(\bar w-\alpha)\end{pmatrix} + (\alpha-\bar\beta)\begin{pmatrix}u\\ v\end{pmatrix}^- \frac{\bar M(\beta)I}{\bar\beta-\bar w}\begin{pmatrix}[F_+(\bar\beta)-F_+(\alpha)]/(\bar\beta-\alpha)\\ [F_-(\bar\beta)-F_-(\alpha)]/(\bar\beta-\alpha)\end{pmatrix}$$

$$-(\alpha-\bar\beta)\begin{pmatrix}u\\ v\end{pmatrix}^- \frac{\bar M(w)I}{\bar\beta-\bar w}\begin{pmatrix}[F_+(\bar w)-F_+(\alpha)]/(\bar w-\alpha)\\ [F_-(\bar w)-F_-(\alpha)]/(\bar w-\alpha)\end{pmatrix} = 2\pi\begin{pmatrix}G_+(\beta)\\ G_-(\beta)\end{pmatrix}^- \begin{pmatrix}F_+(\alpha)\\ F_-(\alpha)\end{pmatrix}$$

after obvious cancellations. The desired identity for difference quotients therefore holds when $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ are finite sums of functions $\frac{M(z)-M(w)}{z-w}\begin{pmatrix} u \\ v \end{pmatrix}$ with w different from α and β. Such sums are dense in $\mathcal{H}$ for any fixed α and β.

For any number w there is now a dense set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}$ such that $\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$ belongs to $\mathcal{H}$ such that the identity for difference quotients holds. It implies that

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} + (w-\bar{w})\begin{pmatrix} [F_+(t)-F_+(w)]/(t-w) \\ [F_-(t)-F_-(w)]/(t-w) \end{pmatrix} \right\|^2$$

$$= \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 + 2\pi(\bar{w}-w)\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}^- I \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$$

where

$$\left| \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}^- I \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} \right| \le \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 \left| \frac{M(w)I\bar{M}(w)-I}{2\pi(w-\bar{w})} \right| .$$

Since

$$|w-\bar{w}| \left\| \begin{pmatrix} [F_+(t)-F_+(w)]/(t-w) \\ [F_-(t)-F_-(w)]/(t-w) \end{pmatrix} \right\|$$

$$\le \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} + (w-\bar{w})\begin{pmatrix} [F_+(t)-F_+(w)]/(t-w) \\ [F_-(t)-F_-(w)]/(t-w) \end{pmatrix} \right\| + \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|$$

by the triangle inequality,

$$\left\| \begin{pmatrix} [F_+(t)-F_+(w)]/(t-w) \\ [F_-(t)-F_-(w)]/(t-w) \end{pmatrix} \right\|$$

$$\le |w-\bar{w}|^{-1} \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\| + |w-\bar{w}|^{-1} \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\| [1+|M(w)I\bar{M}(w)-I|]^{\frac{1}{2}} .$$

So the transformation

$$R(w): \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$$

is continuous in the metric of $\mathfrak{H}$ when w is not real. Since the transformation is known to be densely defined, an obvious approximation argument will show that it is everywhere defined when w is not real. The identity for difference quotients is obtained by continuity for nonreal values of α and β. The same conclusion follows for real values of w once we know that $R(w)$ is bounded. A bound can be obtained from Cauchy's formula in the form

$$\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} [F_+(z)-F_+(w+e^{i\theta})]/(z-w-e^{i\theta}) \\ [F_-(z)-F_-(w+e^{i\theta})]/(z-w-e^{i\theta}) \end{pmatrix} (1-e^{2i\theta})\, d\theta .$$

We need the formula only when $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is a finite sum of elements $\frac{M(z)-M(\alpha)}{z-\alpha}\begin{pmatrix} u \\ v \end{pmatrix}$ where $\alpha \neq w$ and $\alpha \neq w+e^{i\theta}$ for all real θ. It is easily verified that $\frac{M(z)-M(\alpha)}{z-\alpha}\begin{pmatrix} u \\ v \end{pmatrix}$ depends continuously on α in the metric of $\mathfrak{H}$ for each fixed u and v. The Stieltjes sums defining the above integral therefore converge in the metric of $\mathfrak{H}$. It follows that

$$\left\| \begin{pmatrix} [F_+(t)-F_+(w)]/(t-w) \\ [F_-(t)-F_-(w)]/(t-w) \end{pmatrix} \right\|$$
$$\leq \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\| \frac{1}{2\pi} \int_0^{2\pi} \{1+[1+|M(w+e^{i\theta})\, I\bar{M}(w+e^{i\theta}) - I|]^{\frac{1}{2}}\}\, d\theta .$$

The required estimate of $R(w)$ follows.

If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathcal{H}$, then $F_+(z)$ and $F_-(z)$ are continuous functions, and $[F_+(z)-F_+(w)]/(z-w)$ and $[F_-(z)-F_-(w)]/(z-w)$ are continuous functions of z for every w. This implies that $F_+(z)$ and $F_-(z)$ are entire. The theorem follows.

Proof of Theorem 5. By the proof of Theorem III of [7], there exists a matrix valued entire function $M(z)$ with these properties: $M(0)I = I$, $\dfrac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$ belongs to $\mathcal{H}$ for all vectors u and v and numbers w, and

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-}\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w)-I}{2\pi(t-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}$. Since the elements of $\mathcal{H}$ are analytic in the complex plane, $M(z)I\bar{M}(\bar{z}) = I$. The identity $\bar{M}(\bar{w})IM(w) = I$ follows at all points w where $M(w)$ has an inverse. Since this set of points is both open and closed, and since it contains the origin, it is the complex plane.

Since we assume that

$$J(w): \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$$

is a completely continuous transformation of $\mathcal{H}$ into C^2 for some number w, and since its adjoint is

$$J(w)^*: \begin{pmatrix} u \\ v \end{pmatrix} \to \frac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix},$$

the composed transformations

$$J(\alpha)J(\beta)^*: \begin{pmatrix} u \\ v \end{pmatrix} \to \frac{M(\alpha)I\bar{M}(\beta)-I}{2\pi(\alpha-\bar{\beta})}\begin{pmatrix} u \\ v \end{pmatrix}$$

are completely continuous whenever α or β is equal to w. Since the values of $M(z)$ are known to be invertible matrices, it follows that the values of

$$M(z)I - I = M(z)I - M(0)I$$

are matrices of completely continuous operators. The theorem follows.

Proof of Theorem 6. The hypotheses imply that $B(z)\bar{A}(\bar{z}) = A(z)\bar{B}(\bar{z})$ and that $[B(w)\bar{A}(w) - A(w)\bar{B}(w)]/(w-\bar{w}) \geq 0$ for all complex w. Since the values of $M(z)I-I$ are matrices of completely continuous operators, $1 - [A(z) - iB(z)]$ has completely continuous values. To show that $A(w)-iB(w)$ is an invertible operator when w lies above the real axis, we need only show that $\bar{A}(w) + i\bar{B}(w)$ has no nonzero vector c in its kernel. Since

$$[A(w)+iB(w)][\bar{A}(w) - i\bar{B}(w)] \leq [A(w)-iB(w)][\bar{A}(w)+i\bar{B}(w)] ,$$

such a vector is in the kernel of $\bar{A}(w) - i\bar{B}(w)$ and hence in the kernel of both $\bar{A}(w)$ and $\bar{B}(w)$. Since $D(\bar{w})\bar{A}(w) - C(\bar{w})\bar{B}(w) = 1$, $c = 0$. A similar argument will show that $1 - [A(z)+iB(z)]$ has completely continuous values and that $A(z)+iB(z)$ has invertible values in the lower half-plane. A space $\mathcal{H}(A,B)$ therefore exists. A similar argument will show that a space $\mathcal{H}(C,D)$ exists.

If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is a finite sum of elements $\dfrac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})}\begin{pmatrix} c \\ 0 \end{pmatrix}$ for vectors c and numbers w, $\sqrt{2}F_+(z)$ is a corresponding sum of elements $\dfrac{B(z)\bar{A}(w)-A(z)\bar{B}(w)}{\pi(z-\bar{w})}c$. It is easily verified that $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}F_+(z)$ is isometric as a transformation of $\mathcal{H}(M)$ into $\mathcal{H}(A,B)$, when restricted to such sums. It follows by continuity that the transformation is isometric on the closure of such sums. Since $F_+(z) = 0$ when $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is orthogonal to $\dfrac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})}\begin{pmatrix} c \\ 0 \end{pmatrix}$ for all vectors c and numbers w, the transformation $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}F_+(z)$ is a partial isometry of $\mathcal{H}(M)$ into $\mathcal{H}(A,B)$. Since the range of the transformation is a closed subspace of $\mathcal{H}(A,B)$ which contains

$\dfrac{B(z)\bar{A}(w)-A(z)\bar{B}(w)}{\pi(z-\bar{w})}c$ for every vector c and number w, it is all of $\mathcal{H}(A,B)$. A similar argument will show that $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}F_-(z)$ is a partial isometry of $\mathcal{H}(M)$ onto $\mathcal{H}(C,D)$.

The kernel of the last partial isometry is the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M)$ such that $F_-(z)=0$. Let $\mathfrak{M}$ be the set of all such elements, considered as a Hilbert space in the metric of $\mathcal{H}(M)$. Then $\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$ belongs to $\mathfrak{M}$ whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathfrak{M}$. The relation H in $\mathfrak{M}$, such that

$$(H-w)^{-1}: \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$$

for every complex number w, is self-adjoint. Since H has no finite spectrum, $(H-w)^{-1}$ is the zero transformation in $\mathfrak{M}$ for every w, and the elements of $\mathfrak{M}$ are constants. A similar argument will show that the kernel of the transformation $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}F_+(z)$ contains only constants.

It remains to prove orthogonality of the subspaces. Let P be the operator such that $\begin{pmatrix} Pc \\ 0 \end{pmatrix}$ belongs to $\mathcal{H}(M)$ for every vector c and

$$\bar{c}F_+(w) = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} Pc \\ 0 \end{pmatrix} \right\rangle$$

for every element $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M)$ such that $F_-(z) = 0$. Then there exists a space $\mathcal{H}(M_+)$ corresponding to

$$M_+(z) = 1 - \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} Iz .$$

The space is contained isometrically in $\mathcal{H}(M)$, and is the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M)$ such that $F_-(z) = 0$. Let Q be the operator such that $\begin{pmatrix} 0 \\ Qc \end{pmatrix}$ belongs to $\mathcal{H}(M)$ for every vector c and

$$\bar{c}F_-(w) = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} 0 \\ Qc \end{pmatrix} \right\rangle$$

for every element $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M)$ such that $F_+(z) = 0$. Then a space $\mathcal{H}(M_-)$ exists corresponding to

$$M_-(z) = 1 - \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} Iz .$$

The space is contained isometrically in $\mathcal{H}(M)$ and is the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M)$ such that $F_+(z) = 0$. Every element $\begin{pmatrix} 0 \\ c \end{pmatrix}$ of $\mathcal{H}(M_-)$ can be written in the form

$$\begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + M_+(z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

where $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is in $\mathcal{H}(M_+)$ and $M_+(z)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is an element of $\mathcal{H}(M)$ which is orthogonal to $\mathcal{H}(M_+)$. This implies that

$$0 = \frac{M_+(z) - M_+(w)}{z-w} \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} + M_+(z) \begin{pmatrix} [F_+(z) - F_+(w)]/(z-w) \\ [F_-(z) - F_-(w)]/(z-w) \end{pmatrix}$$

where the first term is in $\mathcal{H}(M_+)$ and the second term is orthogonal to $\mathcal{H}(M_+)$. Since the left side vanishes, each term on the right vanishes. In other words, $F_+(z)$ and $F_-(z)$ are constants which belong to the kernel of P. It follows that

$$M_+(z)\begin{pmatrix}F_+(z)\\ F_-(z)\end{pmatrix} = \begin{pmatrix}0\\ c\end{pmatrix}$$

and that $a = 0$. This completes the proof that $\mathcal{H}(M_+)$ and $\mathcal{H}(M_-)$ are orthogonal.

Proof of Theorem 7. By hypothesis the transformation $J(w): F(z) \to F(w)$ of $\mathcal{L}$ into C is continuous for every nonreal number w. Its adjoint is of the form

$$J(w)^* : c \to L(w, z)c$$

where $L(w, z)$ is an operator valued function of z which is analytic in the upper half-plane and in the lower half-plane. If α and β are nonreal numbers and if a and b are corresponding vectors, then

$$[L(\alpha, z) - L(\alpha, \bar{w})]a/(z-\bar{w}) \quad \text{and} \quad [L(\beta, z) - L(\beta, w)]b/(z-w)$$

belong to $\mathcal{L}$ for each nonreal number w. The hypotheses imply that the identity

$$\langle L(\alpha, t)a, [L(\beta, t) - L(\beta, w)]b/(t-w)\rangle$$

$$= \langle [L(\alpha, t) - L(\alpha, \bar{w})]a/(t-\bar{w}), L(\beta, t)b\rangle$$

holds. By the definitions of $L(\alpha, z)$ and $L(\beta, z)$, the identity becomes

$$\bar{b}[\bar{L}(\beta, \alpha) - \bar{L}(\beta, w)]a/(\bar{\alpha}-\bar{w}) = \bar{b}[L(\alpha, \beta) - L(\alpha, \bar{w})]a/(\beta-\bar{w}) .$$

Since $J(\alpha)J(\beta)^* = L(\beta, \alpha)$, $L(\beta, \alpha) = \bar{L}(\alpha, \beta)$. By the arbitrariness of a and b, the identity becomes

$$(\beta-\alpha)L(\alpha, \beta) = (\beta-\bar{w})L(w, \beta) - (\bar{\alpha}-\bar{w})L(\alpha, \bar{w}) .$$

Therefore there exists an operator valued function $\varphi(z)$, analytic in the upper half-plane and in the lower half-plane, such that

$$L(w, z) = [\varphi(z) + \bar{\varphi}(w)]/[\pi i(\bar{w}-z)] .$$

Since $L(w, z)$ is a continuous function of z for every nonreal number w, $\varphi(w) = -\bar{\varphi}(\bar{w})$. Since $L(w, w) = J(w)J(w)^* \geq 0$ for every nonreal number w, $\operatorname{Re}\varphi(w) \geq 0$ when w is in the upper half-plane. Since the space $\mathfrak{L}$ is uniquely determined by $L(w, z)$, and since $L(w, z)$ is uniquely determined by $\varphi(z)$, $\mathfrak{L}$ is uniquely determined by $\varphi(z)$.

Proof of Theorem 8. If $\mathfrak{L}(\varphi)$ exists, then $\operatorname{Re}\varphi(z) \geq 0$ for $y > 0$ by the proof of the last theorem. If on the other hand $\operatorname{Re}\varphi(z) \geq 0$ for $y > 0$, there exists a nonnegative operator P and a nondecreasing operator valued function $\mu(x)$ of real x such that

$$\operatorname{Re}\varphi(x+iy) = Py + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-x)^2+y^2}$$

for $y > 0$. It follows that

$$\frac{\varphi(z)+\bar{\varphi}(w)}{\pi i(\bar{w}-z)} = \frac{P}{\pi} + \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-z)(t-\bar{w})}$$

for all nonreal z and w. It follows that if $w_1, \ldots, w_r$ are nonreal numbers and if $c_1, \ldots, c_r$ are corresponding vectors, then

$$\sum \bar{c}_i \frac{\varphi(w_i)+\bar{\varphi}(w_k)}{\pi i(\bar{w}_k-w_i)} c_k \geq 0 .$$

As in the proof of Theorem 4, the positive-definiteness inequality implies the existence of a corresponding space $\mathfrak{L}$ of vector valued functions. The space $\mathfrak{L}$ contains $[\varphi(z)+\bar{\varphi}(w)]c/[\pi i(\bar{w}-z)]$ for every vector c and nonreal number w, and

$$\bar{c}F(w) = \langle F(t), [\varphi(t)+\bar{\varphi}(w)]c/[\pi i(\bar{w}-t)]\rangle$$

for every $F(z)$ in $\mathfrak{L}$. The proof that the elements of the space are continuous functions is essentially the same as in the proof of Theorem 4.

Let w be a nonreal number. We show that $[F(z)-F(w)]/(z-w)$ belongs to $\mathfrak{L}$ whenever $F(z)$ belongs to $\mathfrak{L}$. This is clear if

$$F(z) = \frac{\varphi(z)+\bar{\varphi}(\bar{\alpha})}{z-\alpha}c = \frac{\varphi(z)-\varphi(\alpha)}{z-\alpha}c$$

for some vector c and number $\alpha \neq w$ since then

$$(w-\alpha)\frac{F(z)-F(w)}{z-w} = \frac{\varphi(z)-\varphi(w)}{z-w}c - \frac{\varphi(z)-\varphi(\alpha)}{z-\alpha}c = \frac{\varphi(z)+\overline{\varphi}(\overline{w})}{z-w}c - \frac{\varphi(z)+\overline{\varphi}(\overline{\alpha})}{z-\alpha}c.$$

The identity

$$0 = \langle F(t), [G(t)-G(\beta)]/(t-\beta)\rangle - \langle [F(t)-F(\alpha)]/(t-\alpha), G(t)\rangle$$

$$+ (\alpha-\overline{\beta})\langle [F(t)-F(\alpha)]/(t-\alpha), [G(t)-G(\beta)]/(t-\beta)\rangle$$

is easily verified if $F(z)$ is an element of $\mathfrak{L}$ such that $[F(z)-F(\alpha)]/(z-\alpha)$ belongs to $\mathfrak{L}$ and if $G(z)$ is a finite sum of functions $[\varphi(z)+\overline{\varphi}(w)]c/[\pi i(\overline{w}-z)]$ where c is in $\mathcal{C}$ and $\overline{w}$ is different from β.

The identity implies that $F(z)+(w-\overline{w})[F(z)-F(w)]/(z-w)$ has the same norm as $F(z)$ and hence that

$$\|[F(t)-F(w)]/(t-w)\| \leq 2|w-\overline{w}|^{-1}\|F(t)\|.$$

Since the transformation $R(w): F(z) \to [F(z)-F(w)]/(z-w)$ is known to be densely defined in $\mathfrak{L}$, the transformation is everywhere defined. Since the identity

$$(\alpha-\beta)R(\alpha)R(\beta) = R(\alpha)-R(\beta)$$

holds for all nonreal numbers α and β, there exists a relation H in $\mathfrak{L}$ such that $R(w) = (H-w)^{-1}$ for nonreal numbers w. Since $R(\overline{w})$ is the adjoint of $R(w)$, the relation H is self-adjoint. If there is no nonzero constant c in $\mathfrak{L}$, $(H-w)^{-1}$ has no nonzero element in its kernel when w is not real. In this case H-w, and hence H, is a well-defined transformation.

<u>Proof of Theorem 9.</u> If

$$F(z) = \frac{\varphi_-(z)+\overline{\varphi}_-(w)}{\pi i(\overline{w}-z)}\overline{\varphi}_+(w)c$$

for some vector c and nonreal number w, then

$$\varphi_+(z)F(z) = \frac{\varphi_+(z)+\overline{\varphi}_+(w)}{\pi i(\overline{w}-z)}c$$

belongs to $\mathfrak{L}(\varphi_+)$. By linearity $\varphi_+(z)F(z)$ belongs to $\mathfrak{L}(\varphi_+)$ whenever $F(z)$ is a finite sum of such special functions. It is easily verified that $\varphi_+(z)F(z)$ has the same norm in $\mathcal{L}(\varphi_+)$ as $F(z)$ does in $\mathcal{L}(\varphi_-)$. Since such special functions are dense in $\mathcal{L}(\varphi_-)$, multiplication by $\varphi_+(z)$ is an isometric transformation of $\mathcal{L}(\varphi_-)$ into $\mathcal{L}(\varphi_+)$. For the same reasons, multiplication by $\varphi_-(z)$ is an isometric transformation of $\mathcal{L}(\varphi_+)$ into $\mathcal{L}(\varphi_-)$. Since these transformations are inverses, each is "onto".

If there is no nonzero constant in $\mathcal{L}(\varphi_+)$, the existence of the self-adjoint transformation H_+ is given by the last theorem. If there is no nonzero constant in $\mathcal{L}(\varphi_-)$, the existence of H_- is obtained by the same reasoning in view of the isometric correspondence between $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$. If w is not real, the identity

$$1-(H_+-\bar{w})^{-1}(H_+-w)(H_--\bar{w})(H_--w)^{-1}$$

$$=(w-\bar{w})(H_+-\bar{w})^{-1}-(w-\bar{w})(H_--w)^{-1}+(w-\bar{w})^2(H_+-\bar{w})^{-1}(H_--w)^{-1}$$

gives the required computation of the transformation on the left.

The result of the computation shows that the transformation $c \to [\varphi_+(z)-\varphi_+(\bar{w})]\varphi_-(w)c/(z-\bar{w})$ takes the coefficient space $\mathcal{C}$ onto a subspace of $\mathcal{L}(\varphi_+)$ which contains the range of

$$1-(H_+-\bar{w})^{-1}(H_+-w)(H_--\bar{w})(H_--w)^{-1} .$$

So the dimension of the range does not exceed the dimension of $\mathcal{C}$. Composing this transformation with $(H_--w)(H_--\bar{w})^{-1}$, we find that the transformation

$$(H_--\bar{w})^{-1}(H_--w)-(H_+-\bar{w})^{-1}(H_+-w)=(w-\bar{w})[(H_+-\bar{w})^{-1}-(H_--\bar{w})^{-1}]$$

has the same range. Since the same conclusion holds when w is replaced by $\bar{w}$, the dimension of the range of $(H_+-w)^{-1}-(H_--w)^{-1}$ does not exceed the dimension of $\mathcal{C}$. If $i-\varphi_+(z)$ and $i+\varphi_-(z)$ have completely continuous values, the operator $[\varphi_+(w)+\bar{\varphi}_+(w)]/(w-\bar{w})$ is completely continuous for each nonreal number w. If $J(w)$ is the transformation $F(z) \to F(w)$ of $\mathcal{L}(\varphi_+)$ into $\mathcal{C}$, then

$$J(w)J(w)^*: c \to [\varphi_+(w)+\bar{\varphi}_+(w)]c/[\pi i(\bar{w}-w)]$$

is completely continuous. This implies that $J(w)$ and $J(w)^*$ are completely continuous. Since

$$1-(H_+-\bar{w})^{-1}(H_+-w)(H_--\bar{w})(H_--w)^{-1}$$

can be written as a composition of bounded transformations, of which $J(w)$ and $J(w)^*$ are completely continuous, it is completely continuous. It follows that $(H_+-w)^{-1}-(H_--w)^{-1}$ is completely continuous.

Proof of Theorem 10. (Compare with the proof of Theorem 4 of [6].) Let w be a nonreal number such that the dimension of the range of $(H_+-w)^{-1} - (H_--w)^{-1}$ does not exceed the dimension of $\mathcal{C}$. Then the dimension of the range of

$$1 - (H_+-\overline{w})^{-1}(H_+-w)(H_--\overline{w})(H_--w)^{-1}$$

does not exceed the dimension of $\mathcal{C}$. It follows that there exists a bounded transformation $J_+(w)$ of $\mathcal{H}$ into $\mathcal{C}$, with adjoint $J_+(w)^*$ taking $\mathcal{C}$ into $\mathcal{H}$, such that

$$1-(H_+-\overline{w})^{-1}(H_+-w)(H_--\overline{w})(H_--w)^{-1} = \pi i(\overline{w}-w)J_+(w)^*\varphi_-(w)J_+(w)$$

for some invertible operator $\varphi_-(w)$, and such that the kernel of $J_+(w)$ is the kernel of

$$1-(H_+-\overline{w})^{-1}(H_+-w)(H_--\overline{w})(H_--w)^{-1}.$$

Let
$$J_+(z) = J_+(w)(H_+-w)(H_+-z)^{-1}$$

for nonreal values of z. If f is in $\mathcal{H}$, define a corresponding vector valued function $F_+(z)$, analytic in the upper half-plane and the lower half-plane, by $F_+(z) = J_+(z)f$. If α is a nonreal number,

$$\begin{aligned}(z-\alpha)J_+(z)(H_+-\alpha)^{-1}f &= (z-\alpha)J_+(w)(H_+-w)(H_+-z)^{-1}(H_+-\alpha)^{-1}f\\ &= J_+(w)(H_+-w)(H_+-z)^{-1}f-J_+(w)(H_+-w)(H_+-\alpha)^{-1}f\\ &= F_+(z)-F_+(\alpha).\end{aligned}$$

So we have obtained the formula

$$J_+(z)(H_+-\alpha)^{-1}f = [F_+(z)-F_+(\alpha)]/(z-\alpha)$$

when $F_+(z) = J_+(z)f$.

If f is an element of $\mathcal{H}$ such that the corresponding function $F_+(z)$ vanishes identically, then

$$[1-(H_+-\overline{w})^{-1}(H_+-w)(H_--\overline{w})(H_--w)^{-1}](H_+-w)(H_+-z)^{-1}f$$

vanishes for all nonreal z. This implies that

$$[(H_+-w)^{-1}-(H_--w)^{-1}](H_+-w)(H_+-z)^{-1}f$$

vanishes for all nonreal z. It follows that

$$(H_+-z)^{-1}f = (H_--w)^{-1}(H_+-w)(H_+-z)^{-1}f$$
$$= (H_--w)^{-1}f + (z-w)(H_--w)^{-1}(H_+-z)^{-1}f$$

and that

$$(H_--w)^{-1}f = (H_--z)(H_--w)^{-1}(H_+-z)^{-1}f .$$

From this we can conclude that

$$(H_--w)^{-1}(H_--z)^{-1}f = (H_--z)^{-1}(H_--w)^{-1}f = (H_--w)^{-1}(H_+-z)^{-1}f$$

and that $(H_+-z)^{-1}f = (H_--z)^{-1}f$ for all nonreal z. By hypothesis, $f = 0$ in this case.

If we define

$$\varphi_+(z) = \varphi_-(w)^{-1} + \pi i(w-z)J_+(z)J_+(\overline{w})^*$$

for nonreal z, we obtain $\varphi_+(z) = -\overline{\varphi}_+(\overline{z})$ and

$$\begin{aligned}\varphi_+(\alpha)-\varphi_+(\beta) &= \pi i(w-\alpha)J_+(w)(H_+-w)(H_+-\alpha)^{-1}(H_+-w)^{-1}(H_+-\overline{w})J_+(w)^* \\ &\quad - \pi i(w-\beta)J_+(w)(H_+-w)(H_+-\beta)^{-1}(H_+-w)^{-1}(H_+-\overline{w})J_+(w)^* \\ &= \pi iJ_+(w)(H_+-w)(H_+-\beta)^{-1}(H_+-\overline{w})J_+(w)^* \\ &\quad - \pi iJ_+(w)(H_+-w)(H_+-\alpha)^{-1}(H_+-\overline{w})J_+(w)^* \\ &= -\pi i(\alpha-\beta)J_+(w)(H_+-w)(H_+-\alpha)^{-1}(H_+-\beta)^{-1}(H_+-\overline{w})J_+(w)^* \\ &= -\pi i(\alpha-\beta)J_+(\alpha)J_+(\overline{\beta})^*\end{aligned}$$

for nonreal numbers α and β. Equivalently we have

$$J_+(z)J_+(w)^* = [\varphi_+(z)+\overline{\varphi}_+(w)]/[\pi i(\overline{w}-z)]$$

for nonreal z and w. Since $\varphi_+(z)$ is analytic and has a nonnegative real part in the upper half-plane, a space $\mathfrak{L}(\varphi_+)$ exists. We show that the transformation $f \to F_+(z)$ is an isometry of $\mathfrak{H}$ onto $\mathfrak{L}(\varphi_+)$.

If $f = J_+(\alpha)^*c$ for some vector c and nonreal number α, then $F_+(z) = [\varphi_+(z)+\overline{\varphi}_+(\alpha)]c/[\pi i(\overline{\alpha}-z)]$ belongs to $\mathfrak{L}(\varphi_+)$. By linearity $F_+(z)$ belongs to $\mathfrak{L}(\varphi_+)$ if f is a finite sum of elements $J_+(\alpha)^*c$ for vectors c and nonreal numbers α. An obvious computation will show that $F_+(z)$ then has the same norm in $\mathfrak{L}(\varphi_+)$ as f

does in $\mathcal{H}$. Since the span of elements $J_+(\alpha)^*c$ is dense in $\mathcal{H}$ by the first part of the proof, the same conclusion holds for every element f of $\mathcal{H}$. If f is in $\mathcal{H}$, then $F_+(z)$ belongs to $\mathcal{L}(\varphi_+)$ and has the same norm as f. Since the range of the transformation $f \to F_+(z)$ is a closed subspace of $\mathcal{L}(\varphi_+)$ which contains $[\varphi_+(z)+\bar{\varphi}_+(\alpha)]c/[\pi i(\bar{\alpha}-z)]$ for every vector c and nonreal number α, it is all of $\mathcal{L}(\varphi_+)$.

Let $J_-(w) = \varphi_-(w)J_+(w)$. Since $\varphi_+(w)$ and $\varphi_-(w)$ are inverse operators,

$$1-(H_--\bar{w})^{-1}(H_--w)(H_+-\bar{w})(H_+-w)^{-1} = \pi i(\bar{w}-w)J_-(w)^*\varphi_+(w)J_-(w)\ .$$

Define $J_-(z) = J_-(w)(H_--w)(H_--z)^{-1}$ for nonreal z. If f is in $\mathcal{H}$, define a corresponding vector valued function $F_-(z)$, analytic in the upper half-plane and in the lower half-plane, by $F_-(z) = J_-(z)f$. If we define

$$\varphi_-(z) = \varphi_-(w) + \pi i(w-z)J_-(z)J_-(\bar{w})^*$$

for nonreal z, then a space $\mathcal{L}(\varphi_-)$ exists by the first part of the proof and $f \to F_-(z)$ is an isometric transformation of $\mathcal{H}$ onto $\mathcal{L}(\varphi_-)$. If f is in $\mathcal{H}$, $F_+(z)$ is the corresponding element of $\mathcal{L}(\varphi_+)$, and $F_-(z)$ is the corresponding element of $\mathcal{L}(\varphi_-)$, we must show that $F_-(z) = \varphi_-(z)F_+(z)$. Equivalently we must show that

$$\begin{aligned}&\varphi_-(w)J_+(w)(H_--w)(H_--z)^{-1}\\ &= [\varphi_-(w)+\pi i(w-z)J_-(z)J_-(\bar{w})^*]J_+(w)(H_+-w)(H_+-z)^{-1}\ .\end{aligned}$$

Since $J_-(\bar{w}) = J_-(w)(H_--w)(H_--\bar{w})^{-1}$, we must show that

$$\begin{aligned}&\varphi_-(w)J_+(w)[(H_+-z)^{-1}-(H_--z)^{-1}]\\ &= \pi i J_-(z)(H_--\bar{w})(H_--w)^{-1}J_-(w)^*J_+(w)(H_+-w)(H_+-z)^{-1}\ .\end{aligned}$$

Since $J_-(z) = \varphi_-(w)J_+(w)(H_--w)(H_--z)^{-1}$, it is sufficient to show that

$$\begin{aligned}&(H_+-z)^{-1}-(H_--z)^{-1}\\ &= \pi i(H_--w)(H_--z)^{-1}(H_--\bar{w})(H_--w)^{-1}J_-(w)^*J_+(w)(H_+-w)(H_+-z)^{-1}\ .\end{aligned}$$

Since $J_-(w) = \varphi_-(w)J_+(w)$ and $J_-(w)^* = J_+(w)^*\bar{\varphi}_-(w)$, where

$$\pi i(\bar{w}-w)J_+(w)^*\bar{\varphi}_-(w)J_+(w) = 1-(H_--\bar{w})^{-1}(H_--w)(H_+-\bar{w})(H_+-w)^{-1}\ ,$$

it is sufficient to show that

$$(\bar{w}-w)[(H_+-z)^{-1} - (H_--z)^{-1}]$$

$$= (H_--\bar{w})(H_--z)^{-1}[1-(H_--\bar{w})^{-1}(H_--w)(H_+-\bar{w})(H_+-w)^{-1}](H_+-w)(H_+-z)^{-1},$$

which is seen to be true by a routine calculation. A similar argument will show that $F_+(z) = \varphi_+(z)F_-(z)$ whenever $F_+(z)$ is an element of $\mathcal{L}(\varphi_+)$ and $F_-(z)$ is an element of $\mathcal{L}(\varphi_-)$ which correspond to the same element of $\mathfrak{H}$.

For every vector c,

$$J_+(z)J_+(w)^* c = [\varphi_+(z)+\bar{\varphi}_+(w)]c/[\pi i(\bar{w}-z)]$$

$$J_-(z)J_-(w)^* \bar{\varphi}_+(w)c = [\varphi_-(z)+\bar{\varphi}_-(w)]\bar{\varphi}_+(w)c/[\pi i(\bar{w}-z)]$$

are elements of $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ respectively, which correspond to the same element $J_+(w)^*c = J_-(w)^*\bar{\varphi}_+(w)c$ of $\mathfrak{H}$. As we have seen, this relationship implies that

$$[\varphi_+(z)+\bar{\varphi}_+(w)]c/[\pi i(\bar{w}-z)] = \varphi_+(z)[\varphi_-(z)+\bar{\varphi}_-(w)]\bar{\varphi}_+(w)c/[\pi i(\bar{w}-z)]$$

and hence that $\varphi_+(z)\varphi_-(z) = 1$. A similar argument with $\mathcal{L}(\varphi_+)$ and $\mathcal{L}(\varphi_-)$ interchanged will show that $\varphi_-(z)\varphi_+(z) = 1$.

If $(H_+-w)^{-1} - (H_--w)^{-1}$ is completely continuous, then $J_+(w)$ can be chosen so that $i+\varphi_-(w)$ is completely continuous. We omit the details of the construction, which is essentially the same as the proof of Lemma 6 of [6]. When $J_+(w)$ is so chosen, it then follows that $i+\varphi_-(z)$ and $i-\varphi_+(z)$ have completely continuous values.

Proof of Theorem 11. The proof uses the identity

$$\pi i\bar{g}(\beta)\bar{\varphi}_-(\beta)f(\alpha)$$

$$= \langle f(t), [g(t)-\varphi_+(t)\varphi_-(\beta)g(\beta)]/(t-\beta)\rangle$$

$$- \langle [f(t)-f(\alpha)]/(t-\alpha), g(t)\rangle$$

$$+(\alpha-\bar{\beta})\langle [f(t)-f(\alpha)]/(t-\alpha), [g(t)-\varphi_+(t)\varphi_-(\beta)g(\beta)]/(t-\beta)\rangle ,$$

which holds in $\mathcal{L}(\varphi_+)$ for all elements $f(z)$ and $g(z)$ of the space when α and β are nonreal numbers. The proof of the identity uses Theorem 7. By that theorem,

$$0 = \langle f(t), [g(t)-g(\beta)]/(t-\beta)\rangle - \langle [f(t)-f(\alpha)]/(t-\alpha), g(t)\rangle$$

$$+ (\alpha-\overline{\beta})\langle [f(t)-f(\alpha)]/(t-\alpha), [g(t)-g(\beta)]/(t-\beta)\rangle.$$

To prove the identity we need only show that

$$\pi i\overline{g}(\beta)\overline{\varphi}_-(\beta)f(\alpha)$$

$$= -\langle f(t), [\varphi_+(t)-\varphi_+(\beta)]\varphi_-(\beta)g(\beta)/(t-\beta)\rangle$$

$$-(\alpha-\overline{\beta})\langle [f(t)-f(\alpha)]/(t-\alpha), [\varphi_+(t)-\varphi_+(\beta)]\varphi_-(\beta)g(\beta)/(t-\beta)\rangle .$$

This is true because the right side is

$$\pi i\overline{g}(\beta)\overline{\varphi}_-(\beta)f(\overline{\beta}) + \pi i(\alpha-\overline{\beta})\overline{g}(\beta)\overline{\varphi}_-(\beta)[f(\overline{\beta})-f(\alpha)]/(\overline{\beta}-\alpha)$$

by Theorem 7.

Let L_+ and L_- be the transformations in $\mathfrak{M}$ whose graphs are the intersections of the graphs of H_+ and H_- with $\mathfrak{M}\times\mathfrak{M}$. If w is a nonreal number, there exist unique continuous transformations $J_+(w)$ and $J_-(w)$ of $\mathcal{L}(\varphi_+)$ into $\mathbf{C}$ such that the range of L_+-w is the kernel of $J_+(w)$, the range of L_--w is the kernel of $J_-(w)$, and

$$f(w) = \sqrt{2}[i\varphi_+(w)J_+(w)f + J_-(w)f]$$

for every element f of $\mathfrak{M}$. To see this, note that the requirements are consistent on the intersection of the ranges of L_+-w and L_--w. For if f belongs to the intersection,

$$(H_+-w)^{-1}f = (L_+-w)^{-1}f = (T-w)^{-1}f = (L_--w)^{-1}f = (H_--w)^{-1}f$$

and $f(w) = 0$ by hypothesis. We can and must define $J_+(w)$ and $J_-(w)$ so that $J_+(w)f = -\sqrt{2}i\varphi_-(w)f(w)$ and $J_-(w)f = 0$ when f is in the range of $L_- - w$, and so that $J_+(w)f = 0$ and $J_-(w)f = \sqrt{2}f(w)$ when f is in the range of $L_+ - w$. The transformations so defined can and must be extended linearly to the vector span of the ranges of L_+-w and L_--w. The transformations can and must be extended by continuity to the closed span of the ranges of L_+-w and L_--w. Since we assume that T is the least common closed extension of L_+ and L_-, the vector span of the ranges of L_+-w and L_--w is dense in $\mathfrak{M}$. So there exist unique transformations $J_+(w)$ and $J_-(w)$ having the required properties.

If α and β are nonreal numbers, we show that

$$J_+(\beta) = J_+(\alpha)(T-\alpha)(T-\beta)^{-1} \quad\text{and}\quad J_-(\beta) = J_-(\alpha)(T-\alpha)(T-\beta)^{-1} .$$

If f is in the range of $L_+ - \beta$,

$$J_+(\alpha)(T-\alpha)(T-\beta)^{-1}f = J_+(\alpha)(L_+-\alpha)(L_+-\beta)^{-1}f = 0 .$$

If f is in the range of $L_- - \beta$,

$$J_+(\alpha)(T-\alpha)(T-\beta)^{-1}f = J_+(\alpha)(L_- -\alpha)(L_- -\beta)^{-1}f$$

$$= -\sqrt{2i}\varphi_-(\alpha)\{f(\alpha)+(\beta-\alpha)[f(\alpha)-\varphi_+(\alpha)\varphi_-(\beta)f(\beta)]/(\alpha-\beta)\}$$

$$= -\sqrt{2i}\varphi_-(\beta)f(\beta) .$$

So $J_+(\beta)$ and $J_+(\alpha)(T-\alpha)(T-\beta)^{-1}$ agree on the ranges of $L_+-\beta$ and $L_--\beta$. Since both transformations are continuous and since the closed span of the ranges of $L_+-\beta$ and $L_--\beta$ is all of $\mathfrak{M}$, the transformations are identical. A similar argument will show that the transformations $J_-(\beta)$ and $J_-(\alpha)(T-\alpha)(T-\beta)^{-1}$ are identical.

The identity

$$2\pi[J_+(\beta)g]^- J_-(\alpha)f - 2\pi[J_-(\beta)g]^- J_+(\alpha)f$$
$$= <f, (T-\beta)^{-1}g> - <(T-\alpha)^{-1}f, g> + (\alpha-\bar{\beta})<(T-\alpha)^{-1}f, (T-\beta)^{-1}g>$$

can be verified when f and g are in $\mathfrak{M}$ and α and β are nonreal numbers. When f is in the range of $L_+-\alpha$ and g is in the range of $L_+-\beta$, or when f is in the range of $L_--\alpha$ and g is in the range of $L_--\beta$, the identity holds because both sides are zero by Theorem 7. When f is in the range of $L_+-\alpha$ and g is in the range of $L_--\beta$, or when f is in the range of $L_--\alpha$ and g is in the range of $L_+-\beta$, the identity follows from the identity given at the start of the proof. The desired identity now follows by linearity and continuity for all elements f and g of $\mathfrak{M}$ when α and β are not real. It holds without restriction on α and β if we define $J_+(w)$ and $J_-(w)$ for real values of w in the unique way such that the formulas

$$J_+(\beta) = J_+(\alpha)(T-\alpha)(T-\beta)^{-1} \quad\text{and}\quad J_-(\beta) = J_-(\alpha)(T-\alpha)(T-\beta)^{-1}$$

remain valid. This definition is possible because we assume that T has no finite spectrum as a transformation in $\mathfrak{M}$.

Corresponding to every element f of $\mathfrak{M}$, we define a pair of vector valued entire functions $F_+(z)$ and $F_-(z)$ by

$$F_+(w) = J_+(w)f \quad\text{and}\quad F_-(w) = J_-(w)f$$

for all complex w. Since $f(w) = i\varphi_+(w)F_+(w)+F_-(w)$ when w is

not real, these functions cannot both vanish identically unless $f = 0$.
The set $\mathcal{H}$ of pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of vector valued entire functions so obtained is a vector space over the complex numbers. There is a unique way to define an inner product in $\mathcal{H}$ so as to make the transformation $f(z) \to \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ an isometry of $\mathfrak{M}$ onto $\mathcal{H}$. Since $\mathfrak{M}$ is a closed subspace of $\mathcal{L}(\varphi_+)$, $\mathcal{H}$ is complete in its metric and so is a well defined Hilbert space. If f is in $\mathfrak{M}$ and if $g = (T-\alpha)^{-1}f$ for some complex number α, then

$$\begin{aligned}(w-\alpha)G_+(w) &= (w-\alpha)J_+(w)(T-\alpha)^{-1}f \\ &= J_+(w)f - J_+(\alpha)f \\ &= F_+(w) - F_+(\alpha)\end{aligned}$$

for every complex number w. It follows that $G_+(z) = [F_+(z)-F_+(\alpha)]/(z-\alpha)$. For the same reasons $G_-(z) = [F_-(z)-F_-(\alpha)]/(z-\alpha)$. So the pair $\begin{pmatrix} [F_+(z)-F_+(\alpha)]/(z-\alpha) \\ [F_-(z)-F_-(\alpha)]/(z-\alpha) \end{pmatrix}$ belongs to $\mathcal{H}$ whenever the pair $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathcal{H}$, and the transformation $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} [F_+(z)-F_+(\alpha)]/(z-\alpha) \\ [F_-(z)-F_-(\alpha)]/(z-\alpha) \end{pmatrix}$ in $\mathcal{H}$ corresponds to the transformation $f \to (T-\alpha)^{-1}f$ in $\mathfrak{M}$. Since we assume that $i-\varphi_+(z)$ and $i+\varphi_-(z)$ have completely continuous values, the transformation $f(z) \to f(w)$ of $\mathcal{L}(\varphi_+)$ into $\mathcal{C}$ is completely continuous for every nonreal number w. It follows that $J_+(w)$ and $J_-(w)$ are completely continuous transformations of $\mathfrak{M}$ into $\mathcal{C}$. The hypotheses of Theorem 5 are now routinely verified. By the theorem, the space $\mathcal{H}$ is equal isometrically to a space $\mathcal{H}(M)$. The construction has been made in such a way that the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[i\varphi_+(z)F_+(z)+F_-(z)]$$

takes $\mathcal{H}(M)$ isometrically onto $\mathfrak{M}$.

If $\begin{pmatrix} u \\ v \end{pmatrix}$ is a constant in $\mathcal{H}(M)$ such that $v = 0$, then

$i\varphi_+(z)u$ belongs to $\mathcal{L}(\varphi_+)$ and u belongs to $\mathcal{L}(\varphi_-)$. Since we assume that $\mathcal{L}(\varphi_-)$ contains no nonzero constant, $u = 0$. A similar argument will show that $\mathcal{H}(M)$ contains no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ with $u = 0$.

<u>Proof of Theorem 12.</u> The proof that $\mathcal{H}(M)$ contains no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ such that $u = 0$ or $v = 0$ is the same as in the proof of Theorem 11. Since the transformation of $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$ is assumed to be an isometry, its range $\mathfrak{M}$ is a closed subspace of $\mathcal{L}(\varphi_+)$. For any fixed complex number w, let $R(w)$ be the transformation of $\mathfrak{M}$ into itself which takes $i\varphi_+(z)F_+(z)+F_-(z)$ into

$$i\varphi_+(z)\frac{F_+(z)-F_+(w)}{z-w} + \frac{F_-(z)-F_-(w)}{z-w} .$$

Since the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$$

is bounded in $\mathcal{H}(M)$ by the proof of Theorem 4, $R(w)$ is a bounded transformation in $\mathfrak{M}$. The identity

$$(\alpha-\beta)R(\alpha)R(\beta) = R(\alpha) - R(\beta)$$

is obtained by a straightforward calculation of difference quotients in $\mathcal{H}(M)$. It implies the existence of a relation T in $\mathfrak{M}$ such that $R(w) = (T-w)^{-1}$ for all complex numbers w. By this construction T has no finite spectrum as a relation in $\mathfrak{M}$. We show that it has the required properties.

To show that the graph of H^* contains the graph of T, we need only show that the graph of $(H^*-w)^{-1}$ contains the graph of $(T-w)^{-1}$. Since $(H^*-w)^{-1}$ is the adjoint of $(H-\bar{w})^{-1}$, we need only show that the identity

$$\left\langle i\varphi_+(t)\frac{F_+(t)-F_+(w)}{t-w} + \frac{F_-(t)-F_-(w)}{t-w}, g(t)\right\rangle$$
$$= \left\langle i\varphi_+(t)F_+(t)+F_-(t), \frac{g(t)-g(\bar{w})}{t-\bar{w}}\right\rangle$$

holds in $\mathcal{L}(\varphi_+)$ whenever $f(z) = i\varphi_+(z)F_+(z)+F_-(z)$ is in $\mathfrak{M}$ and $g(z)$ is an element of $\mathcal{L}(\varphi_+)$ such that

$$[g(z)-g(\bar{w})]/(z-\bar{w}) = [g(z)-\varphi_+(z)\varphi_-(\bar{w})g(\bar{w})]/(z-\bar{w}) .$$

By hypothesis this condition implies that $g(\bar{w}) = 0$. The desired identity now follows from Theorem 7 since

$$i\varphi_+(z)\frac{F_+(z)-F_+(w)}{z-w} + \frac{F_-(z)-F_-(w)}{z-w}$$

$$= \frac{f(z)-f(w)}{z-w} - i\,\frac{\varphi_+(z)-\varphi_+(w)}{z-w}\,F_+(w) \quad .$$

If $i\varphi_+(z)F_+(z)+F_-(z)$ is an element of $\mathfrak{M}$ in the range of H_+-w for some nonreal number w, then $F_+(w) = 0$ by hypothesis, and $R(w): f(z) \to [f(z)-f(w)]/(z-w)$. It follows that the graph of T contains the intersection of the graph of H_+ with $\mathfrak{M}\times\mathfrak{M}$. A similar argument will show that the graph of T contains the intersection of the graph of H_- with $\mathfrak{M}\times\mathfrak{M}$.

Proof of Theorem 13. Assume that the transformation

$$T: \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[i\varphi_+(z)F_+(z)+F_-(z)]$$

takes $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$ and is bounded by 1. Then the adjoint T^* of T takes $\mathcal{L}(\varphi_+)$ into $\mathcal{H}(M)$ and is bounded by 1. If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathcal{H}(M)$, then

$$\langle \sqrt{2}[i\varphi_+(t)F_+(t)+F_-(t)], [\varphi_+(t)+\bar{\varphi}_+(w)]c/[\pi i(\bar{w}-t)]\rangle_{\mathcal{L}(\varphi_+)}$$

$$= \sqrt{2}\bar{c}[i\varphi_+(w)F_+(w)+F_-(w)]$$

for every vector c and nonreal number w by Theorem 7. By Theorem 4,

$$\sqrt{2}\bar{c}[i\varphi_+(w)F_+(w)+F_-(w)] = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \sqrt{2}\,\frac{M(t)I\bar{M}(w)-I}{2\pi(t-\bar{w})}\begin{pmatrix} -i\bar{\varphi}_+(w)c \\ c \end{pmatrix} \right\rangle_{\mathcal{H}(M)} .$$

By the definition of the adjoint,

$$T^*: \frac{\varphi_+(z)+\bar{\varphi}_+(w)}{\pi i(\bar{w}-z)}c \to \sqrt{2}\,\frac{M(z)I\bar{M}(w)-I}{2\pi(z-\bar{w})}\begin{pmatrix}-i\bar{\varphi}_+(w)c\\ c\end{pmatrix}$$

for every vector c and nonreal number w. Since T is bounded by 1,

$$\begin{aligned}\bar{c}\,\frac{\varphi_+(w)+\bar{\varphi}_+(w)}{\pi i(\bar{w}-w)}c &= \left\|\frac{\varphi_+(t)+\bar{\varphi}_+(w)}{\pi i(\bar{w}-t)}c\right\|^2_{\mathfrak{L}(\varphi_+)}\\
&\geq \left\|\sqrt{2}\,\frac{M(t)I\bar{M}(w)-I}{2\pi(t-\bar{w})}\begin{pmatrix}-i\varphi_+(w)c\\ c\end{pmatrix}\right\|^2_{\mathcal{H}(M)}\\
&\geq (i\bar{c}\varphi_+(w)\quad \bar{c})\,\frac{M(w)I\bar{M}(w)-I}{\pi(w-\bar{w})}\begin{pmatrix}-i\bar{\varphi}_+(w)c\\ c\end{pmatrix}\end{aligned}$$

for every vector c and nonreal number w. We use the inequality only when w is in the upper half-plane. By the arbitrariness of c, the inequality reduces to an operator inequality which we use in the form

$$\begin{aligned}&\{\varphi_+(w)[A(w)+iB(w)] + [D(w)-iC(w)]\}\\
&\qquad\times\{[\bar{A}(w)-i\bar{B}(w)]\bar{\varphi}_+(w) + [\bar{D}(w)+i\bar{C}(w)]\}\\
&\geq \{\varphi_+(w)[A(w)-iB(w)] - [D(w)+iC(w)]\}\\
&\qquad\times\{[\bar{A}(w)+i\bar{B}(w)]\bar{\varphi}_+(w) - [\bar{D}(w)-i\bar{C}(w)]\}\ .\end{aligned}$$

Since we assume that the values of $M(z)I-I$ are matrices of completely continuous operators, and since $A(z)+iB(z)$ has invertible values except at isolated points, the function

$$[D(z)-iC(z)][A(z)+iB(z)]^{-1}-1$$

is analytic except at isolated points and it has completely continuous values. Since $\operatorname{Re}\varphi_+(z)\geq 0$ for $y>0$, $1+\varphi_+(z)$ has invertible values in the upper half-plane. The function

$$[1+\varphi_+(z)]^{-1}\{[D(z)-iC(z)][A(z)+iB(z)]^{-1}-1\}$$

is analytic except at isolated points in the upper half-plane, and it has completely continuous values. By Theorem 18 of the appendix, the function

$$1+[1+\varphi_+(z)]^{-1}\{[D(z)-iC(z)][A(z)+iB(z)]^{-1}-1\}$$

has invertible values except at isolated points if it has an invertible value. It follows that

$$\varphi_+(z) + [D(z)-iC(z)][A(z)+iB(z)]^{-1}$$

has invertible values except at isolated points if it has an invertible value. Since the defining inequalities for $\mathfrak{H}(M)$ imply that

$$\operatorname{Re}[D(z)-iC(z)][A(z)+iB(z)]^{-1} \geq 1$$

for real z, the inequality

$$\operatorname{Re}[D(z)-iC(z)][A(z)+iB(z)]^{-1} \geq \tfrac{1}{2}$$

holds in a neighborhood of the real axis. Since $\operatorname{Re}\varphi_+(z) \geq 0$ for $y > 0$, we obtain

$$\operatorname{Re}\{\varphi_+(z) + [D(z)-iC(z)][A(z)+iB(z)]^{-1}\} \geq \tfrac{1}{2}$$

when z is above and sufficiently near the real axis. The inequality implies that

$$\varphi_+(z) + [D(z)-iC(z)][A(z)+iB(z)]^{-1}$$

has invertible values when z is above and sufficiently near the real axis. It follows that the function has invertible values except at isolated points in the upper half-plane. So the function

$$\varphi_+(z)[A(z)+iB(z)] + [D(z)-iC(z)]$$

has invertible values except at isolated points in the upper half-plane. If we define

$$\begin{aligned} W(z) &= \{\varphi_+(z)[A(z)+iB(z)]+[D(z)-iC(z)]\}^{-1} \\ &\quad \times \{\varphi_+(z)[A(z)-iB(z)] - [D(z)+iC(z)]\} \end{aligned}$$

at points in the upper half-plane where the inverse exists, we obtain the operator inequality

$$1 \geq W(w)\bar{W}(w)$$

for w in this set. Since an analytic function cannot remain bounded in the neighborhood of an isolated singularity, $W(z)$ has an analytic extension in the upper half-plane. The function has been defined in such a way that

$$\{\varphi_+(z)[A(z)+iB(z)]+[D(z)-iC(z)]\}W(z) = \varphi_+(z)[A(z)-iB(z)]-[D(z)+iC(z)] .$$

The required form of $\varphi_+(z)$ follows.

Conversely, suppose that $\varphi_+(z)$ is of this form for some operator valued function $W(z)$ which is analytic and bounded by 1 in the upper half-plane. We show the existence of a Hilbert space $\mathfrak{F}(W)$, whose elements are vector valued analytic functions in the upper half-plane, with this property: the function

$$\frac{1-W(z)\bar{W}(w)}{2\pi i(\bar{w}-z)}c$$

belongs to $\mathfrak{F}(W)$ for every vector c when $i(\bar{w}-w) > 0$, and

$$\bar{c}F(w) = \left\langle F(t), \frac{1-W(t)\bar{W}(w)}{2\pi i(\bar{w}-t)}c\right\rangle$$

in $\mathfrak{F}(W)$ for every element $F(z)$ of the space. The existence of this space is best seen in the notation of the appendix. Since $z \to i(1+z)/(1-z)$ is a one-to-one mapping of the unit disk onto the upper half-plane, $B(z) = W(i(1+z)/(1-z))$ is analytic and bounded by 1 in the unit disk. A space $\mathfrak{H}(B)$ therefore exists. Let $\mathfrak{F}(W)$ be the set of vector valued functions $F(z)$ such that $\sqrt{4\pi}(1-z)^{-1}F(i(1+z)/(1-z))$ belongs to $\mathfrak{H}(B)$. Define an inner product in $\mathfrak{F}(W)$ so that

$$F(z) \to \sqrt{4\pi}(1-z)^{-1}F(i(1+z)/(1-z))$$

is an isometry of $\mathfrak{F}(W)$ onto $\mathfrak{H}(B)$. Then $\mathfrak{F}(W)$ is a Hilbert space which contains

$$\sqrt{4\pi}\,\frac{1-W(z)\bar{W}(i(1+w)/(1-w))}{2\pi i[-i(1+\bar{w})/(1-\bar{w})-z]}c$$

for every vector c when $|w| < 1$. Since

$$4\pi(1-z)^{-1}\frac{1-W(i(1+z)/(1-z))\,\overline{W}(i(1+w)/(1-w))}{2\pi i[-i(1+\overline{w})/(1-\overline{w})-i(1+z)/(1-z)]}(1-\overline{w})^{-1}c=\frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c\,,$$

we obtain

$$\left\langle F(t),\ \sqrt{4\pi}\,\frac{1-W(t)\,\overline{W}(i(1+w)/(1-w))}{2\pi i[-i(1+\overline{w})/(1-\overline{w})-t]}c\right\rangle_{\mathfrak{F}(W)}$$

$$=\left\langle \sqrt{4\pi}\,(1-z)^{-1}F(i(1+z)/(1-z))(1-w),\ \frac{1-B(z)\overline{B}(w)}{1-z\overline{w}}c\right\rangle_{\mathfrak{H}(B)}$$

$$=\sqrt{4\pi}\,\overline{c}F(i(1+w)/(1-w))\ .$$

It follows that $\mathfrak{F}(W)$ has the required properties.

The existence of a space $\mathfrak{F}(W)$ implies the positive-definiteness inequality

$$\sum \overline{c}_i\,\frac{1-W(w_i)\,\overline{W}(w_k)}{2\pi i(\overline{w}_k-w_i)}c_k \geq 0$$

for all finite sets of points $w_1,\ldots,w_r$ in the upper half-plane and corresponding vectors $c_1,\ldots,c_r$. The inequality holds because the number on the left is the inner product of

$$\sum \frac{1-W(z)\,\overline{W}(w_k)}{2\pi i(\overline{w}_k-z)}c_k$$

with itself in $\mathfrak{F}(W)$. We use the inequality to show that the required transformation T exists. We define T to be the transformation having a given adjoint T^*, taking $\mathfrak{L}(\varphi_+)$ into $\mathfrak{H}(M)$. Consider an element of $\mathfrak{L}(\varphi_+)$ of the form

$$\sum \frac{\varphi_+(z)+\overline{\varphi}_+(w_k)}{\pi i(\overline{w}_k-z)}c_k$$

and an element of $\mathfrak{H}(M)$ of the form

$$\sum \sqrt{2}\,\frac{M(z)I\overline{M}(w_k)-I}{2\pi(z-\overline{w}_k)}\begin{pmatrix}-i\overline{\varphi}_+(w_k)c_k\\ c_k\end{pmatrix}$$

where $w_1, \ldots, w_r$ are points in the upper half-plane and $c_1, \ldots, c_r$ are corresponding vectors. A straightforward use of the above positive-definiteness inequality will show that the norm of the element of $\mathcal{L}(\varphi_+)$ is no less than the norm of the element of $\mathcal{H}(M)$, the norms being taken in the respective spaces. Since such special elements of $\mathcal{L}(\varphi_+)$ are dense in $\mathcal{L}(\varphi_+)$, there exists a unique transformation T^* of $\mathcal{L}(\varphi_+)$ into $\mathcal{H}(M)$ which is bounded by 1, such that

$$T^*: \frac{\varphi_+(z)+\bar{\varphi}_+(w)}{\pi i(\bar{w}-z)} c \to \sqrt{2}\, \frac{M(z) I \bar{M}(w) - I}{2\pi(z-\bar{w})} \begin{pmatrix} -i\bar{\varphi}_+(w)c \\ c \end{pmatrix}$$

for every vector c when w is in the upper half-plane. The adjoint transformation T takes $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$ and is bounded by 1. If

$$T: \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to f(z)$$

then, by the definition of the adjoint,

$$\bar{c} f(w) = \left\langle f(t), \frac{\varphi_+(t)+\bar{\varphi}_+(w)}{\pi i(\bar{w}-t)} c \right\rangle_{\mathcal{L}(\varphi_+)}$$

$$= \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \sqrt{2}\, \frac{M(t) I \bar{M}(w) - I}{2\pi(t-\bar{w})} \begin{pmatrix} -i\bar{\varphi}_+(w)c \\ c \end{pmatrix} \right\rangle_{\mathcal{H}(M)}$$

$$= \sqrt{2}\, \bar{c}\, [\, i\varphi_+(w) F_+(w) + F_-(w)]$$

for every vector c when w is in the upper half-plane. By the arbitrariness of c and w,

$$f(z) = \sqrt{2}\, [\, i\varphi_+(z) F_+(z) + F_-(z)] \,.$$

So the required transformation of $\mathcal{H}(M)$ into $\mathcal{L}(\varphi_+)$ exists and is bounded by 1. We now determine the set on which the transformation is isometric.

Consider the elements of the space $\mathcal{H}(M) = \mathcal{H}(M(a))$ in a new norm, the b-norm, defined by

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_b^2 = \tfrac{1}{2} \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a))}^2 + \| i\varphi_+(t)F_+(t)+F_-(t)\|^2_{\mathcal{L}(\varphi_+)} .$$

Since we know that

$$\tfrac{1}{2} \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a))}^2 \geq \| i\varphi_+(t)F_+(t)+F_-(t)\|^2_{\mathcal{L}(\varphi_+)}$$

for every element of $\mathcal{H}(M(a))$,

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a))}^2 \geq \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_b^2 \geq \tfrac{1}{2} \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a))}^2 .$$

It follows that $\mathcal{H}(M(a))$ is a Hilbert space when considered in the b-norm. Since the transformation $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$ of $\mathcal{H}(M(a))$ into $\mathbb{C}^2$ is continuous in the metric of $\mathcal{H}(M(a))$ for every number w, it is continuous in the b-metric. The identity for difference quotients given in Theorem 4, which is known to hold in the norm of $\mathcal{H}(M(a))$, is obtained for the b-norm by a straightforward calculation. By Theorem 5 the space $\mathcal{H}(M(a))$ is a space $\mathcal{H}(M(b))$ when it is considered in the b-norm. By construction the space $\mathcal{H}(M(a))$ is contained in the space $\mathcal{H}(M(b))$ and the inclusion does not increase norms. The adjoint of the inclusion of $\mathcal{H}(M(a))$ in $\mathcal{H}(M(b))$ takes

$$\frac{M(b,z)I\bar{M}(b,w)-I}{2\pi(z-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \text{ into } \frac{M(a,z)I\bar{M}(a,w)-I}{2\pi(z-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix}$$

for every pair of vectors u and v and for every complex number w. Since the inclusion transformation is bounded by 1, its adjoint is bounded by 1. A computation of norms yields the inequality

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \frac{M(b,w)I\bar{M}(b,w)-I}{2\pi(w-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} u \\ v \end{pmatrix}^- \frac{M(a,w)I\bar{M}(a,w)-I}{2\pi(w-\bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} .$$

By the arbitrariness of u and v, we obtain the matrix inequality

$$\frac{M(b,w)I\overline{M}(b,w)-I}{2\pi(w-\overline{w})} \geq \frac{M(a,w)I\overline{M}(a,w)-I}{2\pi(w-\overline{w})} .$$

If we define $M(a,b,z) = -I\overline{M}(a,\overline{z})IM(b,z)$, then $M(a,b,z)$ is a matrix valued entire function such that

$$M(a,b,z)I\overline{M}(a,b,\overline{z}) = I = \overline{M}(a,b,\overline{z})IM(a,b,z)$$

and such that the matrix inequality

$$\frac{M(a,b,w)I\overline{M}(a,b,w)-I}{2\pi(w-\overline{w})} \geq 0$$

holds for all complex w. A space $\mathcal{H}(M(a,b))$ exists by Theorem 4 and $M(b,z) = M(a,z)M(a,b,z)$ by construction.

Let $\mathfrak{L}$ be the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M(a,b))$ such that $M(a,z)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathcal{H}(M(a))$ in the norm

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathfrak{L}}^2 = \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a,b))}^2 + \left\| M(a,t)\begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a))}^2 .$$

It is easily verified that $\mathfrak{L}$ is a Hilbert space in this norm, that $\begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$ belongs to $\mathfrak{L}$ whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathfrak{L}$, and that the identity

$$0 = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} [G_+(t)-G_+(\beta)]/(t-\beta) \\ [G_-(t)-G_-(\beta)]/(t-\beta) \end{pmatrix} \right\rangle_{\mathfrak{L}} - \left\langle \begin{pmatrix} [F_+(t)-F_+(\alpha)]/(t-\alpha) \\ [F_-(t)-F_-(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\rangle_{\mathfrak{L}}$$
$$+ (\alpha-\overline{\beta}) \left\langle \begin{pmatrix} [F_+(t)-F_+(\alpha)]/(t-\alpha) \\ [F_-(t)-F_-(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} [G_+(t)-G_+(\beta)]/(t-\beta) \\ [G_-(t)-G_-(\beta)]/(t-\beta) \end{pmatrix} \right\rangle_{\mathfrak{L}}$$

holds for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ of $\mathfrak{L}$ and for all complex numbers α and β. This implies the existence of a self-adjoint relation H in $\mathfrak{L}$, having no finite spectrum, such that

$$(H-w)^{-1}: \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \begin{pmatrix} [F_+(z)-F_+(w)]/(z-w) \\ [F_-(z)-F_-(w)]/(z-w) \end{pmatrix}$$

for every complex number w. By the spectral theorem for self-adjoint relations, $(H-w)^{-1}$ is identically zero for every w. So the elements of $\mathfrak{L}$ are constants $\begin{pmatrix} u \\ v \end{pmatrix}$. By the identity for difference quotients in $\mathcal{H}(M(a,b))$, given by Theorem 4, the identity $\bar{u}v = \bar{v}u$ holds for any such constant. The theorem follows since the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M) = \mathcal{H}(M(a))$ such that

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2_{\mathcal{H}(M)} = 2 \left\| i\varphi_+(t)F_+(t)+F_-(t) \right\|^2_{\mathfrak{L}(\varphi_+)}$$

is just the orthogonal complement in $\mathcal{H}(M)$ of the elements $M(z)\begin{pmatrix} u \\ v \end{pmatrix}$ where $\begin{pmatrix} u \\ v \end{pmatrix}$ is in $\mathfrak{L}$.

APPENDIX ON SQUARE SUMMABLE POWER SERIES

In this appendix we are concerned with the use of square summable power series to study the invariant subspaces of an everywhere defined transformation T which has bound 1. Let $\mathcal{C}(z)$ be the space of square summable power series $f(z) = \Sigma a_n z^n$ with vector coefficients, $\|f(z)\|^2 = \Sigma |a_n|^2 < \infty$. We are also concerned with factorization properties of power series $F(z) = \Sigma F_n z^n$, having operator coefficients, which represent bounded functions in the unit disk. In this case formal multiplication by $F(z)$ is a bounded transformation in $\mathcal{C}(z)$. The range of multiplication by $F(z)$ is a vector subspace of $\mathcal{C}(z)$ which contains $zF(z)$ whenever it contains $F(z)$. The orthogonal complement of multiplication by $F(z)$ is a closed subspace of $\mathcal{C}(z)$ which contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$. The theory of such spaces has been studied by Beurling [1] in the case of complex coefficients and by Lax [12] in the case of vector coefficients. The following work is an expanded and simplified version of the second author's dissertation [16].

If $B(z)$ is a power series with operator coefficients such that multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, let $\mathcal{H}(B)$ be the orthogonal complement in $\mathcal{C}(z)$ of the range of multiplication by $B(z)$. Then $\mathcal{H}(B)$ is a closed subspace of $\mathcal{C}(z)$ which contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$. Every closed subspace of $\mathcal{C}(z)$ which contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$ is of the form $\mathcal{H}(B)$ for some such $B(z)$. If we regard $\mathcal{H}(B)$ as a Hilbert space in the metric of $\mathcal{C}(z)$, then the transformation $R(0): f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}(B)$ is bounded by 1 and $\lim_{n\to\infty} \|R(0)^n f\| = 0$ for every $f(z)$ in the space. These properties characterize the transformation.

THEOREM 1. Let T be a transformation of a Hilbert space $\mathcal{H}$ into itself which is bounded by 1, such that the dimension of the range of $1-T^*T$ does not exceed the dimension of the coefficient space $\mathcal{C}$. If there is no nonzero element f of $\mathcal{H}$ such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, 3, \ldots$, then T is unitarily equivalent to the transformation $R(0): f(z) \to [f(z)-f(0)]/z$ in some space $\mathcal{H}_0$ of formal power series with vector coefficients such that $[f(z)-f(0)]/z$ belongs to $\mathcal{H}_0$ whenever $f(z)$ belongs to $\mathcal{H}_0$ and

$$(1) \qquad \|[f(z)-f(0)]/z\|_0^2 = \|f(z)\|_0^2 - |f(0)|^2$$

for every $f(z)$ in $\mathcal{H}_0$. If $\lim_{n\to\infty} \|T^n f\| = 0$ for every element f of

$\mathcal{H}_0$, then $\mathcal{H}_0$ is contained isometrically in $\mathcal{C}(z)$ and is a space $\mathcal{H}(B)$ for some power series $B(z)$ with operator coefficients such that multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$.

To determine the invariant subspaces of such a transformation, we must determine the spaces $\mathcal{H}(A)$ which are contained in a given space $\mathcal{H}(B)$. The solution of the problem involves a large class of spaces, analogous to the ones just defined, but which are not necessarily contained isometrically in $\mathcal{C}(z)$. Let $B(z)$ be a power series with operator coefficients such that $B(z)f(z)$ belongs to $\mathcal{C}(z)$ whenever $f(z)$ belongs to $\mathcal{C}(z)$ and $\|B(z)f(z)\| \leq \|f(z)\|$ for every $f(z)$. If $f(z)$ is in $\mathcal{C}(z)$, let its B-norm be defined by

$$\|f(z)\|_B^2 = \sup [\, \|f(z)+B(z)g(z)\|^2 - \|g(z)\|^2]$$

where the supremum is taken over all elements $g(z)$ of $\mathcal{C}(z)$. By $\mathcal{H}(B)$ we now mean the set of all power series in $\mathcal{C}(z)$ which have finite B-norm. (The $\mathcal{H}(B)$ terminology is used in this wider sense throughout the appendix.) Note that $\|f(z)\|_B \geq \|f(z)\|$ since we can choose $g(z) = 0$ in the supremum.

THEOREM 2. Let $B(z)$ be a power series with operator coefficients such that $B(z)f(z)$ belongs to $\mathcal{C}(z)$ whenever $f(z)$ belongs to $\mathcal{C}(z)$ and such that $\|B(z)f(z)\| \leq \|f(z)\|$ for every $f(z)$ in $\mathcal{C}(z)$. Then $\mathcal{H}(B)$ is a Hilbert space in the B-norm. If $f(z)$ is in $\mathcal{H}(B)$, then $[f(z)-f(0)]/z$ is in $\mathcal{H}(B)$ and

$$\|[f(z)-f(0)]/z\|_B^2 \leq \|f(z)\|_B^2 - |f(0)|^2 \tag{2}$$

for every $f(z)$ in $\mathcal{H}(B)$. If multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, then $\mathcal{H}(B)$ is contained isometrically in $\mathcal{C}(z)$ and coincides with the orthogonal complement of the range of multiplication by $B(z)$.

Note that when $B(z) = z$, $\mathcal{H}(B)$ is equal isometrically to the coefficient space $\mathcal{C}$, and that when $B(z) = 0$, $\mathcal{H}(B)$ is equal isometrically to $\mathcal{C}(z)$. We use two characterizations of the series $B(z)$ for which $\mathcal{H}(B)$ exists.

LEMMA 1. A transformation T of $\mathcal{C}(z)$ into itself which is bounded by 1 and which commutes with multiplication by z is multiplication by $B(z)$ for some space $\mathcal{H}(B)$.

LEMMA 2. If $B(z)$ is a power series with operator coefficients, a necessary and sufficient condition for the existence of $\mathcal{H}(B)$ is that $B(z)$ converge in the unit disk and that $|B(w)| \leq 1$ for $|w| < 1$.

We use the relation of $\mathcal{H}(B)$ to the range of multiplication by $B(z)$ when this multiplication is not a partially isometric transformation in $\mathcal{C}(z)$.

LEMMA 3. Let $\mathcal{H}(B)$ be a given space and let $h(z)$ be an element of $\mathcal{C}(z)$. A necessary and sufficient condition that $h(z)$ be of the form $h(z) = B(z)g(z)$, where $g(z)$ is in $\mathcal{C}(z)$, is that

$$\sup [\|h(z)+f(z)\|^2 - \|f(z)\|_B^2] < \infty$$

where the supremum is taken over all $f(z)$ in $\mathcal{H}(B)$. In this case $g(z)$ can be chosen so that the supremum is equal to $\|g(z)\|^2$.

The formal relation between factorization and invariant subspaces is stated as an inclusion theorem for spaces $\mathcal{H}(B)$.

THEOREM 3. Let $\mathcal{H}(A)$ and $\mathcal{H}(B)$ be given spaces. A sufficient condition that $\mathcal{H}(A)$ be contained in $\mathcal{H}(B)$ and that the inclusion not increase norms is that $B(z) = A(z)C(z)$ for some space $\mathcal{H}(C)$. If multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$, the condition is also necessary.

The relation is only formal since the inclusion need not be isometric. It is necessary to study the relation of the subspace to the full space when the inclusion is not isometric.

THEOREM 4. Let $\mathcal{H}(A)$, $\mathcal{H}(B)$, and $\mathcal{H}(C)$ be spaces such that $B(z) = A(z)C(z)$.

(A) If $f(z)$ is in $\mathcal{H}(A)$ and if $g(z)$ is in $\mathcal{H}(C)$, then $h(z) = f(z) + A(z)g(z)$ is in $\mathcal{H}(B)$ and

$$\|h(z)\|_B^2 \leq \|f(z)\|_A^2 + \|g(z)\|_C^2 .$$

(B) Every element $h(z)$ of $\mathcal{H}(B)$ has a unique minimal decomposition, $h(z) = f(z) + A(z)g(z)$, with $f(z)$ in $\mathcal{H}(A)$, $g(z)$ in $\mathcal{H}(C)$, and

$$\|h(z)\|_B^2 = \|f(z)\|_A^2 + \|g(z)\|_C^2 .$$

(C) If $f(z)$ is in $\mathcal{H}(A)$ and $g(z)$ is in $\mathcal{H}(C)$, a necessary and sufficient condition that the decomposition $h(z) = f(z)+A(z)g(z)$ be minimal is that

$$0 = \langle f(z), f_0(z)\rangle_A + \langle g(z), g_0(z)\rangle_C$$

whenever $0 = f_0(z)+A(z)g_0(z)$ is a representation of zero with $f_0(z)$ in $\mathcal{H}(A)$ and $g_0(z)$ in $\mathcal{H}(C)$.

(D) Let $h_k(z) = f_k(z)+A(z)g_k(z)$ be a minimal decomposition of $h_k(z)$ in $\mathcal{H}(B)$ with $f_k(z)$ in $\mathcal{H}(A)$ and $g_k(z)$ in $\mathcal{H}(C)$, $k = 1, 2$. If w_1 and w_2 are complex numbers, then $h(z) = f(z) + A(z)g(z)$ is a minimal decomposition of $h(z) = w_1h_1(z)+w_2h_2(z)$ in $\mathcal{H}(B)$ into $f(z) = w_1f_1(z)+w_2f_2(z)$ in $\mathcal{H}(A)$ and $g(z) = w_1g_1(z)+w_2g_2(z)$ in $\mathcal{H}(C)$.

(E) If $h_k(z) = f_k(z)+A(z)g_k(z)$ is a decomposition of $h_k(z)$ in $\mathcal{H}(B)$ with $f_k(z)$ in $\mathcal{H}(A)$ and $g_k(z)$ in $\mathcal{H}(B)$, $k=1, 2$, and if at least one of the decompositions is minimal, then

$$<h_1(z), h_2(z)>_B = <f_1(z), f_2(z)>_A + <g_1(z), g_2(z)>_C .$$

(F) A necessary and sufficient condition that $\mathcal{H}(A)$ be contained isometrically in $\mathcal{H}(B)$ is that $\mathcal{H}(A)$ contain no nonzero element of the form $A(z)g(z)$ with $g(z)$ in $\mathcal{H}(C)$.

A necessary condition that a space $\mathcal{H}(B)$ be contained isometrically in $\mathcal{C}(z)$ is that (1) hold in the B-norm since the identity holds in the metric of $\mathcal{C}(z)$. We need to know when the identity holds if we are to determine what spaces are contained isometrically in $\mathcal{C}(z)$.

LEMMA 4. A necessary and sufficient condition that (1) hold in the B-norm of a space $\mathcal{H}(B)$ is that $\mathcal{H}(B)$ contain no nonzero element of the form $B(z)c$ where c is in $\mathcal{C}$.

The study of the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}(B)$ cannot be separated from the study of its adjoint, which is related to the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}(B^*)$ where $B^*(z) = \Sigma \bar{B}_n z^n$ if $B(z) = \Sigma B_n z^n$.

LEMMA 5. Let $\mathcal{H}(B)$ be a given space. If c is a vector and if $|w| < 1$, then $[1-B(z)\bar{B}(w)]c/(1-z\bar{w})$ belongs to $\mathcal{H}(B)$ and

$$\bar{c}f(w) = <f(z), [1-B(z)\bar{B}(w)]c/(1-z\bar{w})>_B$$

for every $f(z)$ in $\mathcal{H}(B)$.

THEOREM 5. If $\mathcal{H}(B)$ is a given space, then $[B(z)-B(w)]c/(z-w)$ belongs to $\mathcal{H}(B)$ for every vector c when $|w| < 1$. There exists a transformation $f(z) \to \tilde{f}(z)$ of $\mathcal{H}(B)$ into $\mathcal{H}(B^*)$, which is bounded by 1, such that

$$\bar{c}\tilde{f}(w) = <f(z), [B(z)-B(\bar{w})]c/(z-\bar{w})>_B$$

for every vector c when $|w| < 1$. The adjoint of the transformation $R(w): f(z) \to [f(z)-f(w)]/(z-w)$ in $\mathcal{H}(B)$ is

$$R(w)^*: f(z) \to [zf(z)-B(z)\tilde{f}(\bar{w})]/(1-z\bar{w}) .$$

If $R(w)^*: f(z) \to g(z)$, then $\tilde{g}(z) = [\tilde{f}(z)-\tilde{f}(\bar{w})]/(z-\bar{w})$. If $g(z)$ is in $\mathcal{H}(B^*)$ and if $h(z)$ is the element of $\mathcal{H}(B)$ such that

$$\bar{c}h(w) = \langle g(z), [B^*(z)-\bar{B}(w)]c/(z-\bar{w})\rangle_{B^*}$$

for every vector c when $|w| < 1$, then

$$\langle h(z), f(z)\rangle_B = \langle g(z), \tilde{f}(z)\rangle_{B^*}$$

for every $f(z)$ in $\mathcal{H}(B)$.

The space $\mathcal{H}(B)$ can be characterized by its relation to the difference-quotient transformation.

THEOREM 6. Let $\mathcal{H}_0$ be a Hilbert space of formal power series with vector coefficients such that $[f(z)-f(0)]/z$ belongs to $\mathcal{H}_0$ whenever $f(z)$ belongs to $\mathcal{H}_0$ and the identity (1) holds in the zero-norm. Let $R(0)$ be the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}_0$ and let $R(0)^*$ be its adjoint in $\mathcal{H}_0$. Let d be the dimension of $\mathcal{C}$, let d_0 be the dimension of the space of vectors orthogonal to all coefficients of elements of $\mathcal{H}_0$, and let d_1 be the dimension of the closure of the range of $1 - R(0)R(0)^*$. If $d \geq d_0 + d_1$, then $\mathcal{H}_0$ is equal isometrically to some space $\mathcal{H}(B)$. If $d = d_0 + d_1$ then $B(z)$ can be chosen so that there is no nonzero vector c such that $B(z)c$ belongs to $\mathcal{H}(B)$.

If a space $\mathcal{H}(B)$ is not contained isometrically in $\mathcal{C}(z)$, then it contains a nonzero element of the form $B(z)L(z)$ where $L(z)$ is in $\mathcal{C}(z)$. We now construct a new space, the overlapping space, from the series which cause the trouble.

THEOREM 7. If $\mathcal{H}(B)$ is a given space, let $\mathcal{L} = \mathcal{L}_B$ be the set of elements $L(z)$ of $\mathcal{C}(z)$ such that $B(z)L(z)$ belongs to $\mathcal{H}(B)$. Then $\mathcal{L}$ is a Hilbert space in the norm

$$\|L(z)\|_{\mathcal{L}}^2 = \|B(z)L(z)\|_B^2 + \|L(z)\|^2 .$$

The series $[L(z)-L(0)]/z$ belongs to $\mathcal{L}$ whenever $L(z)$ belongs to $\mathcal{L}$ and $\|[L(z)-L(0)]/z\|_{\mathcal{L}} \leq \|L(z)\|_{\mathcal{L}}$ for every $L(z)$ in $\mathcal{L}$. The transformation $L(z) \to [L(z) - L(0)]/z$ has an isometric adjoint in $\mathcal{L}$.

Overlapping spaces are examples of $\mathcal{L}(\varphi)$ spaces. Since these spaces have a simple structure, they are useful in studying $\mathcal{H}(B)$ spaces. If A is an operator we write $\operatorname{Re} A$ for $\frac{1}{2}(A+\bar{A})$.

THEOREM 8. Let $\mathcal{L}$ be a Hilbert space whose elements are power series with vector coefficients. Suppose that $f(z) \to f(0)$ is a continuous transformation of $\mathcal{L}$ into $\mathcal{C}$. Suppose that the transformation $f(z) \to [f(z)-f(0)]/z$ is everywhere defined and bounded and has an isometric adjoint in $\mathcal{L}$. Then every series in $\mathcal{L}$ converges in the unit disk. There exists a power series $\varphi(z)$ with operator coefficients, which converges in the unit disk, such that $\frac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w})$ belongs to $\mathcal{L}$ for every vector c when $|w| < 1$, and

$$\bar{c}\,L(w) = \langle L(z), \tfrac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w})\rangle_{\mathcal{L}}$$

for every $L(z)$ in $\mathcal{L}$. The sum of the series $\varphi(z)$ satisfies the inequality $\operatorname{Re}\varphi(w) \geq 0$ for $|w| < 1$ and the space $\mathcal{L} = \mathcal{L}(\varphi)$ is uniquely determined by $\varphi(z)$.

A space $\mathcal{L}(\varphi)$ exists corresponding to any such power series $\varphi(z)$.

LEMMA 6. Let $\mathcal{H}(B)$ be a given space such that $1+B(0)$ has an operator inverse. Then the transformation

$$U: f(z) \to [f(z)-f(0)]/z - [B(z)-B(0)][1+B(0)]^{-1}f(0)/z$$

is everywhere defined and bounded in $\mathcal{H}(B)$, and it has an isometric adjoint.

THEOREM 9. Let $\varphi(z)$ be a power series with operator coefficients which converges to a function having a nonnegative real part in the unit disk. Then there exists a space $\mathcal{L}(\varphi)$ which is associated with $\varphi(z)$ as in Theorem 8. There exists a space $\mathcal{H}(B)$ such that $1+B(0)$ has an operator inverse and

$$\varphi(z) = [1-B(z)]/[1+B(z)]$$

as formal power series. The transformation

$$f(z) \to \tfrac{1}{2}[1+\varphi(z)]f(z)$$

takes $\mathcal{H}(B)$ isometrically onto $\mathcal{L}(\varphi)$.

The duality theory of $\mathcal{L}(\varphi)$ spaces is analogous to the duality theory of $\mathcal{H}(B)$ spaces.

THEOREM 10. If $\mathcal{L}(\varphi)$ is a given space, then $\frac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w)$ belongs to $\mathcal{L}(\varphi)$ for every vector c when $|w| < 1$. There exists a transformation $f(z) \to \tilde{f}(z)$ of $\mathcal{L}(\varphi)$ into $\mathcal{L}(\varphi^*)$, which is bounded by 1, such that

$$\bar{c}\tilde{f}(w) = \langle f(z), \tfrac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w})\rangle_{\mathcal{L}(\varphi)}$$

for every vector c when $|w| < 1$. If $R(w)$ is the transformation $f(z) \to [f(z)-f(w)]/(z-w)$ in $\mathcal{L}(\varphi)$ when $|w| < 1$, then the adjoint $R(w)^*$ of $R(w)$ has the form

$$R(w)^* : f(z) \to [zf(z)+\tilde{f}(\bar{w})]/(1-z\bar{w}) .$$

If $R(w)^* : f(z) \to g(z)$, then $\tilde{g}(z) = [\tilde{f}(z)-\tilde{f}(\bar{w})]/(z-\bar{w})$. If $g(z)$ is in $\mathcal{L}(\varphi^*)$ and if $h(z)$ is the element of $\mathcal{L}(\varphi)$ such that

$$\bar{c}h(w) = \langle g(z), \tfrac{1}{2}[\varphi^*(z)-\bar{\varphi}(w)]c/(z-\bar{w})\rangle_{\mathcal{L}(\varphi^*)}$$

for every vector c when $|w| < 1$, then

$$\langle h(z), f(z)\rangle_{\mathcal{L}(\varphi)} = \langle g(z), \tilde{f}(z)\rangle_{\mathcal{L}(\varphi^*)}$$

for every $f(z)$ in $\mathcal{L}(\varphi)$.

The duality theory of $\mathcal{L}(\varphi)$ spaces is properly related to that of $\mathcal{H}(B)$ spaces.

THEOREM 11. Let $\mathcal{H}(B)$ be a given space with overlapping space $\mathcal{L}(\varphi)$. Then $\mathcal{L}(\varphi^*)$ is contained in $\mathcal{H}(B^*)$ and the inclusion does not increase norms. For every element $L(z)$ of $\mathcal{L}(\varphi)$, let $\tilde{L}(z)$ be the element of $\mathcal{L}(\varphi^*)$ such that

$$\bar{c}\tilde{L}(w) = \langle L(z), \tfrac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w})\rangle_{\mathcal{L}(\varphi)}$$

for every vector c when $|w| < 1$. Then

$$\bar{c}\tilde{L}(w) = -\langle B(z)L(z), [B(z)-B(\bar{w})]c/(z-\bar{w})\rangle_{B}$$

for every vector c when $|w| < 1$.

Of particular interest in applications are spaces $\mathcal{H}(B)$ such that multiplication by $B(z)$ or by $B^*(z)$ is isometric in $\mathcal{C}(z)$. The following theorem is a characterization of such spaces.

THEOREM 12. Let $\mathcal{H}(B)$ be a given space, let $R(0)$ be the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}(B)$, and let $R(0)^*$ be its adjoint in $\mathcal{H}(B)$. If $\lim \|R(0)^{*n} f\|_B = 0$ as $n \to \infty$ for every $f(z)$ in $\mathcal{H}(B)$, then multiplication by $B^*(z)$ is isometric in $\mathcal{C}(z)$.

It is often necessary to know when a space $\mathcal{H}(B)$ is properly related to its adjoint space $\mathcal{H}(B^*)$.

THEOREM 13. Let $\mathcal{H}(B)$ be a given space. A necessary and sufficient condition that the transformation of $\mathcal{H}(B)$ into $\mathcal{H}(B^*)$ defined by Theorem 5 be an isometry is that there exist no nonzero vector c such that $B^*(z)c$ belongs to $\mathcal{H}(B^*)$.

When the identity (1) does not hold in a space $\mathcal{H}(B)$, it can be replaced by an equivalent space in which the identity does hold.

THEOREM 14. Let $\mathcal{H}(B)$ be a given space such that multiplication by $B^*(z)$ is isometric in $\mathcal{C}(z)$. Then $B(z) = CA(z)$ where $\mathcal{H}(A)$ exists, multiplication by $A^*(z)$ is isometric in $\mathcal{C}(z)$, there is no nonzero vector c such that $A(z)c$ belongs to $\mathcal{H}(A)$, C is an operator having an isometric adjoint, and the transformation $f(z) \to Cf(z)$ takes $\mathcal{H}(A)$ isometrically onto $\mathcal{H}(B)$.

The $\mathcal{H}(B)$ theory can be used to study the factorization properties of operator valued analytic functions $F(z)$ which are bounded by 1 in the unit disk. The results obtained are of particular interest if $1-F(z)$ has completely continuous values. The following theorem is concerned with complete continuity in factorization.

THEOREM 15. Let $F(z)$ be a power series with operator coefficients which converges in a disk of radius $a > 1$ about the origin, such that the coefficients of $1-F(z)$ are completely continuous and such that the operator $F(w)$ has a dense range for some number w, $|w| < 1$. Let $B(z)$ be a power series with operator coefficients such that multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$. If the range of multiplication by $B(z)$ contains the range of multiplication by $F(z)$ in $\mathcal{C}(z)$, then $1-B(0)\bar{B}(0)$ and the coefficients of $B(z)-B(0)$ are completely continuous.

Some awkwardness can arise in factorizations when multiplication by $B(z)$ is not isometric. In certain cases it is possible to refactor in such a way as to obtain an isometric multiplication.

THEOREM 16. Let $F(z)$ and $B(z)$ be power series with operator coefficients such that multiplication by $F(z)$ is a bounded transformation in $\mathcal{C}(z)$ and multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$. If the range of multiplication by $B(z)$ in $\mathcal{C}(z)$ contains the range of multiplication by $F(z)$ in $\mathcal{C}(z)$, and if there exists a number w, $|w| < 1$, such that the operator $F(w)$ has a dense range in $\mathcal{C}$, then there exists an isometric operator S such that $B(z) = B(z)S\bar{S}$ and multiplication by $B(z)S$ is isometric in $\mathcal{C}(z)$.

We now study the factorization properties of $B(z)$.

THEOREM 17. Let $B(z)$ be a power series with operator coefficients such that multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$ and $1-B(0)\bar{B}(0)$ is completely continuous. If there is some number w, $|w| < 1$, such that $B(w)$ has a dense range in $\mathcal{C}$, and if the polynomials which belong to $\mathcal{H}(B)$ are dense in $\mathcal{H}(B)$, then there exist projections $P_1, \ldots, P_r$ of finite dimensional range such that

$$B(z) = (1-P_1+P_1z)\ldots(1-P_r+P_rz)\,U$$

for some unitary operator U.

A fundamental theorem of complex function theory states that the zeros of a nonzero analytic function are isolated. Various analogues of zeros are available for an operator valued function $F(z)$. We may mean a point w where $F(w)$ is equal to zero, or a point w where $F(w)$ has a nonzero kernel, or a point w where $F(w)$ fails to have an everywhere defined and bounded inverse. In general such points are not isolated. For this reason the analytic function theory of operator valued functions is essentially different from the analytic function theory of complex valued functions. Yet there are situations in which operator valued functions behave like complex valued functions, apart from noncommutativity of multiplication. It is important to know when the analogy between complex valued functions and operator valued functions is sustained.

THEOREM 18. Let $F(z)$ be a power series with operator coefficients, which converges in a disk of radius $a > 1$ about the origin, such that the coefficients of $1-F(z)$ are completely continuous and the operator $F(w)$ has a dense range for some number w, $|w| < 1$. Then there exist projections $P_0, \ldots, P_r$ of finite dimensional range such that

$$F(z) = (1-P_0+P_0z)\ldots(1-P_r+P_rz)\,G(z)$$

for some power series $G(z)$ with operator coefficients such that $G(0)$ has an operator inverse.

The convergence properties of products with operator coefficients are analogous to those of products with complex coefficients.

THEOREM 19. Let (Q_n) be a sequence of projection operators having finite dimensional ranges and let (w_n) be a sequence of nonzero numbers such that $\lim |w_n| = \infty$ as $n \to \infty$. Then

$$P(z) = \lim_{n\to\infty} (1 - Q_1 z/w_1)\exp(Q_1 z/w_1)\ldots$$
$$(1 - Q_n z/w_n)\exp(Q_n z/w_n + \ldots + \frac{1}{n} Q_n z^n/w_n{}^n)$$

converges in the operator norm, uniformly for z in any bounded set. The limit $P(z)$ is an operator valued entire function such that $1-P(z)$ has completely continuous values, and $P(w)$ has an operator inverse when $w \neq w_n$ for every n.

We now characterize the products so obtained. The result generalizes the Weierstrass factorization of complex valued entire functions.

THEOREM 20. Let $F(z)$ be an operator valued entire function such that $1-F(z)$ has completely continuous values and $F(0)$ has an operator inverse. Then $F(z) = P(z)G(z)$ for some operator valued entire function $P(z)$ as in Theorem 19 and some operator valued entire function $G(z)$ such that $1-G(w)$ is a completely continuous operator and $G(w)$ has an operator inverse for every w.

Proof of Theorem 1. The proof generalizes the proof of Theorem 12 of [10]. Let $\mathcal{C}_0$ be the range of $1-T^*T$, but with a new inner product. If $a = (1-T^*T)f$ and $b = (1-T^*T)g$ are in $\mathcal{C}_0$, let $\langle a,b\rangle_0 = \langle(1-T^*T)f, g\rangle$. It is easily verified that this definition does not depend on the choice of f and g and that $\mathcal{C}_0$ has a well-defined inner product. By the dimension hypothesis we can assume without loss of generality that $\mathcal{C}_0$ is contained isometrically in the coefficient space $\mathcal{C}$. If f is in $\mathcal{H}$, define a power series $f(z) = \Sigma\, a_n z^n$ with vector coefficients $a_n = (1-T^*T)T^n f$, $n = 0, 1, 2, \ldots$. Then $|a_n|^2 = \|T^n f\|^2 - \|T^{n+1}f\|^2$. Since we assume that there is no nonzero element f of $\mathcal{H}$ such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, 3, \ldots$, there is no nonzero element f of $\mathcal{H}$ for which the corresponding power series $f(z)$ vanishes identically. The set $\mathcal{H}_0$ of all power series $f(z)$ corresponding to elements f of $\mathcal{H}$ is a Hilbert space in the unique inner product which makes the correspondence $U: f \to f(z)$ an isometry. If $U: f \to f(z)$, then $U: Tf \to [f(z) - f(0)]/z$. The identity (1) follows from the identity $|a_0|^2 = \|f\|^2 - \|Tf\|^2$. If $\lim \|T^n f\| = 0$ as $n \to \infty$ for every f in $\mathcal{H}$ then

$$\|f(z)\|_0^2 = \|f\|^2 = \sum_0^\infty (\|T^n f\|^2 - \|T^{n+1}f\|^2) = \sum_0^\infty |a_n|^2 ,$$

and $\mathcal{H}_0$ is contained isometrically in $\mathcal{C}(z)$. The theorem follows because $\mathcal{H}_0$ is a closed subspace of $\mathcal{C}(z)$ which contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$.

Proof of Theorem 2. The proof that $\mathcal{H}(B)$ is a Hilbert space is the same as for Theorem 7 of [10]. If $f(z)$ is in $\mathcal{H}(B)$ and if $g(z)$ is in $\mathcal{C}(z)$,

$$\begin{aligned} &\|[f(z)-f(0)]/z + B(z)g(z)\|^2 - \|g(z)\|^2 \\ &= \|f(z)-f(0)+zB(z)g(z)\|^2 - \|g(z)\|^2 \\ &= \|f(z)+B(z)zg(z)\|^2 - \|zg(z)\|^2 - |f(0)|^2 \\ &\le \|f(z)\|_B^2 - |f(0)|^2 . \end{aligned}$$

The inequality (2) follows from the arbitrariness of $g(z)$.

If multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, let $f(z) = B(z)h(z)$ be in the range of multiplication by $B(z)$ with $h(z)$ chosen so that $\|f(z)\| = \|h(z)\|$. Let $g(z) = wh(z)$ for a positive number w. Then

$$\|f(z)+B(z)g(z)\|^2 - \|g(z)\|^2 = (1+2w)\|h(z)\|^2 .$$

Since w is arbitrary, $\|f(z)\|_B = \infty$ unless $f(z)$ vanishes identically. If on the other hand $f(z)$ is orthogonal to the range of multiplication by $B(z)$ in $\mathcal{C}(z)$, then

$$\|f(z)+B(z)g(z)\|^2 - \|g(z)\|^2 = \|f(z)\|^2 + \|B(z)g(z)\|^2 - \|g(z)\|^2 \le \|f(z)\|^2$$

for every $g(z)$ in $\mathcal{C}(z)$. It follows that $\|f(z)\|_B \le \|f(z)\|$. Since any B-norm majorizes the norm of $\mathcal{C}(z)$, $\mathcal{H}(B)$ is contained isometrically in $\mathcal{C}(z)$.

Proof of Lemma 1. Regard $\mathcal{C}$ as a vector subspace of $\mathcal{C}(z)$. The restriction of T to $\mathcal{C}$ can be written in the form $T: c \to B(z)c$ where $B(z) = \Sigma B_n z^n$ is a power series whose coefficients are transformations of $\mathcal{C}$ into itself. Since $\|T\| \le 1$, $\Sigma |B_n c|^2 \le |c|^2$ for every vector c, and each B_n is an operator which is bounded by 1. Since T commutes with multiplication by z, $T: f(z) \to B(z)f(z)$ for every polynomial $f(z)$. Since T is bounded and since the polynomials are dense in $\mathcal{C}(z)$, the same formula evaluates the action of T for every $f(z)$ in $\mathcal{C}(z)$.

Proof of Lemma 2. For the sufficiency suppose that $B(z)$ converges to a function which is bounded by 1 in the unit disk. To show that a space $\mathcal{H}(B)$ exists we must show that $\|B(z)f(z)\| \le \|f(z)\|$ for every $f(z)$ in $\mathcal{C}(z)$. If $f(z) = \Sigma a_n z^n$, then

$$(2\pi)^{-1}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum r^{2n}|a_n|^2$$

when $0 < r < 1$. The product $g(w) = B(w)f(w)$ is a vector valued analytic function of w in the unit disk and so is represented by a power series $g(w) = \Sigma b_n w^n$ with vector coefficients. Since we assume that $|B(w)| \le 1$ for $|w| < 1$, we have $|g(w)| \le |f(w)|$ and

$$\sum r^{2n}|b_n|^2 = (2\pi)^{-1}\int_0^{2\pi} |g(re^{i\theta})|^2 d\theta$$

$$\le (2\pi)^{-1}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum r^{2n}|a_n|^2 .$$

By the arbitrariness of r, $\Sigma |b_n|^2 \le \Sigma |a_n|^2$. Since the product of the formal power series is consistent with the product of function values, $\|B(z)f(z)\| \le \|f(z)\|$ for every $f(z)$ and the existence of $\mathcal{H}(B)$ follows.

For the necessity assume that $\mathcal{H}(B)$ exists, so that $\|B(z)f(z)\| \le \|f(z)\|$ for every $f(z)$ in $\mathcal{C}(z)$. We must show that $B(z)$ converges to a function which is bounded by 1 in the unit disk. If $B(z) = \Sigma B_n z^n$, the hypotheses imply that $\|B(z)c\| \le |c|$ for every vector c, and hence that $|B_n c| \le |c|$ for all indices n. By the arbitrariness of c, each operator B_n is bounded by 1. It follows that the series $B(w) = \Sigma B_n w^n$ converges in the operator norm when $|w| < 1$. If $f(z) = \Sigma a_n z^n$ is in $\mathcal{C}(z)$, the series $f(w) = \Sigma a_n w^n$ converges in the metric of $\mathcal{C}$ when $|w| < 1$. For every vector c, $(1-z\bar{w})^{-1}c$ belongs to $\mathcal{C}(z)$ and

$$\bar{c}f(w) = \langle f(z), (1-z\bar{w})^{-1}c\rangle .$$

If we choose $f(z) = B(z)(1-z\bar{w})^{-1}a$ for some vector a and apply the Schwarz inequality, we obtain

$$|\bar{c}B(w)(1-w\bar{w})^{-1}a|^2 \le \|f(z)\|^2(1-w\bar{w})^{-1}|c|^2$$

$$\le (1-w\bar{w})^{-1}|a|^2(1-w\bar{w})^{-1}|c|^2$$

since we assume that $\|B(z)(1-z\bar{w})^{-1}a\| \le \|(1-z\bar{w})^{-1}a\|$. It follows that $|\bar{c}B(w)a| \leq |a||c|$. By the arbitrariness of a and c, $|B(w)| \leq 1$.

Proof of Lemma 3. The argument is the same as for Theorem 9 of [10].

Proof of Theorem 3. The sufficiency is obtained in an obvious way from the definition of the B-norm. For the necessity suppose that $\mathcal{H}(A)$ is contained in $\mathcal{H}(B)$ and that the inclusion does not increase norms. If $f(z)$ is in $\mathcal{C}(z)$ and if $g(z)$ is in $\mathcal{H}(A)$, then

$$\|f(z)\|^2 \geq \|B(z)f(z)+g(z)\|^2 - \|g(z)\|_B^2$$

$$\geq \|B(z)f(z)+g(z)\|^2 - \|g(z)\|_A^2 .$$

By Lemma 3, $B(z)f(z) = A(z)h(z)$ for some series $h(z)$ in $\mathcal{C}(z)$. If $h(z)$ is chosen of smallest norm, it is uniquely determined by $f(z)$. The transformation $T: f(z) \to h(z)$ so defined in $\mathcal{C}(z)$ is linear and bounded by 1. The theorem follows from Lemma 1 once it is shown that T commutes with multiplication by z in $\mathcal{C}(z)$. For if Lemma 1 applies, T is multiplication by $C(z)$ for some space $\mathcal{H}(C)$. Since $B(z)f(z) = A(z)C(z)f(z)$ for every $f(z)$ in $\mathcal{C}(z)$, $B(z) = A(z)C(z)$.

It remains to show that T commutes with multiplication by z in $\mathcal{C}(z)$. Since multiplication by $B(z)$ is assumed to be isometric in $\mathcal{C}(z)$,

$$\|f(z)\| = \|B(z)f(z)\| = \|A(z)h(z)\| \leq \|h(z)\| \leq \|f(z)\|$$

whenever $T: f(z) \to h(z)$. It follows that T is an isometry and that multiplication by $A(z)$ is isometric on the range of T. Now suppose that $T: f(z) \to h(z)$ and that $T: zf(z) \to k(z)$. Then $A(z)[zh(z)-k(z)] = 0$ where $\|A(z)zh(z)\| = \|zh(z)\|$ and $\|A(z)k(z)\| = \|k(z)\|$. Since multiplication by $A(z)$ is bounded by 1 in $\mathcal{C}(z)$, it follows that

$$\|A(z)[zh(z)-k(z)]\| = \|zh(z)-k(z)\| = 0 .$$

So $k(z) = zh(z)$ and T commutes with multiplication by z.

Proof of Theorem 4. The arguments here are the same as for square summable power series with complex coefficients. See Problems 48 and 52-56 of [10].

Proof of Lemma 4. See the proof of Theorem 16 of [10].

Proof of Lemma 5. If c is a vector and if w is a number, $|w| < 1$, $(1-z\bar{w})^{-1}c$ is a power series with vector coefficients which belongs to $\mathcal{C}(z)$ and $\bar{c}f(w) = \langle f(z), (1-z\bar{w})^{-1}c \rangle$ for every $f(z)$ in $\mathcal{C}(z)$. Since

$$\langle B(z)u(z), (1-z\bar{w})^{-1}c \rangle = \bar{c}B(w)u(w) = \langle u(z), (1-z\bar{w})^{-1}\bar{B}(w)c \rangle$$

for every $u(z)$ in $\mathcal{C}(z)$,

$$K(w,z)c = (1-z\bar{w})^{-1}c - B(z)\bar{B}(w)(1-z\bar{w})^{-1}c$$

belongs to $\mathcal{H}(B)$ by the proof of Theorem 8 of [10] and

$$(1-z\bar{w})^{-1}c = K(w,z)c + B(z)\bar{B}(w)(1-z\bar{w})^{-1}c$$

is a minimal decomposition of $(1-z\bar{w})^{-1}c$ in $\mathcal{C}(z)$ with $K(w,z)c$ in $\mathcal{H}(B)$ and $\bar{B}(w)(1-z\bar{w})^{-1}c$ in $\mathcal{C}(z)$. If $f(z)$ is in $\mathcal{H}(B)$, then

$$\langle f(z), K(w,z)c \rangle_B = \langle f(z), (1-z\bar{w})^{-1}c \rangle = \bar{c}f(w)$$

by Theorem 4E.

Proof of Theorem 5. We verify the stated form of $R(0)^*$. Observe that $[B(z)-B(0)]c/z$ belongs to $\mathcal{H}(B)$ for every vector c. For if $g(z)$ is in $\mathcal{C}(z)$,

$$\begin{aligned} &\|[B(z)-B(0)]c/z + B(z)g(z)\|^2 - \|g(z)\|^2 \\ &= \|B(z)c+B(z)zg(z)\|^2 - \|zg(z)\|^2 - |B(0)c|^2 \\ &\leq \|c+zg(z)\|^2 - \|zg(z)\|^2 - |B(0)c|^2 = |c|^2 - |B(0)c|^2 . \end{aligned}$$

By the arbitrariness of $g(z)$,

$$\|[B(z)-B(0)]c/z\|_B^2 \leq |c|^2 - |B(0)c|^2 .$$

It follows that there exists a unique vector $\tilde{f}(0)$ for every $f(z)$ in $\mathcal{H}(B)$ such that

$$\bar{c}\tilde{f}(0) = \langle f(z), [B(z)-B(0)]c/z \rangle_B$$

for every vector c. Let $f(z)$ be in $\mathcal{H}(B)$ and let $R(0)^*: f(z) \to g(z)$. Then for every vector c and number w, $|w| < 1$,

$$\begin{aligned}\bar{c}g(w) &= \langle g(z), [1-B(z)\bar{B}(w)]c/(1-z\bar{w})\rangle_B \\ &= \langle f(z), \{[1-B(z)\bar{B}(w)]c/(1-z\bar{w}) - [1-B(0)\bar{B}(w)]c\}/z\rangle_B \\ &= \langle f(z), \bar{w}[1-B(z)\bar{B}(w)]c/(1-z\bar{w})\rangle_B - \langle f(z), [B(z)-B(0)]\bar{B}(w)c/z\rangle_B \\ &= \bar{c}[wf(w)-B(w)\tilde{f}(0)]\end{aligned}$$

by Lemma 5 and the definition of the adjoint. Since c and w are arbitrary, $g(z) = zf(z)-B(z)\tilde{f}(0)$, which is the required form of $R(0)^*$. Note that

$$\begin{aligned}\|zf(z)-B(z)\tilde{f}(0)\|_B^2 &= \langle zf(z)-B(z)\tilde{f}(0), zf(z)-B(z)\tilde{f}(0)\rangle_B \\ &= \langle f(z), f(z) - [B(z)-B(0)]\tilde{f}(0)/z\rangle_B \\ &= \|f(z)\|_B^2 - |\tilde{f}(0)|^2\end{aligned}$$

by the definition of the adjoint.

The form of $R(0)^{*n}$ is obtained from the form of $R(0)^*$ by iteration. If $f(z)$ is in $\mathcal{H}(B)$, let $(f_n(z))$ be the sequence of elements of $\mathcal{H}(B)$ defined inductively by $f_0(z) = f(z)$ and $R(0)^*: f_n(z) \to f_{n+1}(z)$. If a_n is the unique vector such that

$$\bar{c}a_n = \langle f_n(z), [B(z)-B(0)]c/z\rangle_B$$

for every vector c, then $f_{n+1}(z) = zf_n(z)-B(z)a_n$ and

$$\|f_{n+1}(z)\|_B^2 = \|f_n(z)\|_B^2 - |a_n|^2 .$$

It follows that $R(0)^{*n}: f(z) \to f_n(z)$ where

$$f_n(z) = z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1})$$

and $$\|f_n(z)\|_B^2 = \|f(z)\|_B^2 - |a_0|^2 - \ldots - |a_{n-1}|^2 .$$

These properties of $R(0)^*$ are best understood by constructing a new Hilbert space $\mathcal{D}(B)$ whose elements are pairs $(f(z), g(z))$ of power series with vector coefficients. By definition $(f(z), g(z))$ belongs to $\mathcal{D}(B)$ if $f(z)$ belongs to $\mathcal{H}(B)$ and if $g(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ where

$$z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1})$$

belongs to $\mathcal{H}(B)$ for every $n = 1, 2, 3, \ldots$, and the sequence

$$\| z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}) \|_B^2 + |a_0|^2 + \ldots + |a_{n-1}|^2$$

is bounded. Note that the sequence is nondecreasing. For if $g(z)$ is in $\mathcal{C}(z)$,

$$\| z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}) + B(z) g(z) \|^2 - \| g(z) \|^2$$

$$= \| z^{n+1} f(z) - B(z)(a_0 z^n + \ldots + a_n) + B(z)(a_n + z g(z)) \|^2 - \| a_n + z g(z) \|^2 + |a_n|^2$$

$$\leq \| z^{n+1} f(z) - B(z)(a_0 z^n + \ldots + a_n) \|_B^2 + |a_n|^2 .$$

By the arbitrariness of $g(z)$,

$$\| z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}) \|_B^2$$

$$\leq \| z^{n+1} f(z) - B(z)(a_0 z^n + \ldots + a_n) \|_B^2 + |a_n|^2 .$$

If we define

$$\| (f(z), g(z)) \|^2_{\mathcal{D}(B)}$$

$$= \lim_{n \to \infty} [\| z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}) \|_B^2 + |a_0|^2 + \ldots + |a_{n-1}|^2] ,$$

we then have $\| f(z) \|_B \leq \| (f(z), g(z)) \|_{\mathcal{D}(B)}$ and $\| g(z) \| \leq \| (f(z), g(z)) \|_{\mathcal{D}(B)}$. It is easily verified that $\mathcal{D}(B)$ is a Hilbert space in the inner product corresponding to this norm. There exists an isometry $f(z) \to (f(z), \tilde{f}(z))$ of $\mathcal{H}(B)$ into $\mathcal{D}(B)$ such that

$$R(0)^{*n} : f(z) \to z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1})$$

for every $n = 1, 2, 3, \ldots$, where $\tilde{f}(z) = \Sigma a_n z^n$.

Consider any element of $\mathcal{D}(B)$ of the form $(0, g(z))$ where $g(z) = \Sigma b_n z^n$. If $f(z)$ is in $\mathcal{H}(B)$ and if $\tilde{f}(z) = \Sigma a_n z^n$, we show inductively that

$$\langle z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}),\ B(z)(b_0 z^{n-1} + \ldots + b_{n-1}) \rangle_B$$

$$= \bar{b}_0 a_0 + \ldots + \bar{b}_{n-1} a_{n-1}$$

for every $n = 1, 2, 3, \ldots$. When $n = 1$ the formula is obtained by the definition of the adjoint since

$$\langle z f(z) - B(z) a_0, B(z) b_0 \rangle_B = \langle f(z), [B(z) - B(0)] b_0 / z \rangle_B = \bar{b}_0 a_0 \ .$$

On the other hand if the formula is known for some n, the definition of the adjoint yields

$$\langle z^{n+1} f(z) - B(z)(a_0 z^n + \ldots + a_n),\ B(z)(b_0 z^n + \ldots + b_n) \rangle_B$$

$$= \langle z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}),\ B(z)(b_0 z^{n-1} + \ldots + b_{n-1}) \rangle_B$$

$$+ \langle z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}),\ [B(z) - B(0)] b_n / z \rangle_B$$

$$= \bar{b}_0 a_0 + \ldots + \bar{b}_{n-1} a_{n-1} + \bar{b}_n a_n \ .$$

The formula follows for all n. By the definition of the inner product in $\mathcal{D}(B)$,

$$\langle (f(z), \tilde{f}(z)),\ (0, g(z)) \rangle_{\mathcal{D}(B)}$$

$$= \lim_{n \to \infty} [\, \langle z^n f(z) - B(z)(a_0 z^{n-1} + \ldots + a_{n-1}),\ -B(z)(b_0 z^{n-1} + \ldots + b_{n-1}) \rangle_B$$

$$+ \bar{b}_0 a_0 + \ldots + \bar{b}_{n-1} a_{n-1}] = 0 \ .$$

So $(0, g(z))$ is orthogonal to the range of the transformation $f(z) \to (f(z), \tilde{f}(z))$. Note that every element $(f(z), g(z))$ of $\mathcal{D}(B)$ can be written

$$(f(z), g(z)) = (f(z), \tilde{f}(z)) + (0, g(z) - \tilde{f}(z))$$

where $(0, g(z) - \tilde{f}(z))$ is in $\mathcal{D}(B)$. The orthogonal complement in $\mathcal{D}(B)$ of the set of elements of the form $(f(z), \tilde{f}(z))$ where $f(z)$ is in $\mathcal{H}(B)$ is therefore the set of elements of the form $(0, g(z))$.

We show that $[B(z) - B(w)]c/(z-w)$ belongs to $\mathcal{H}(B)$ for every vector c when $|w| < 1$. Define a sequence $(g_n(z))$ of elements of $\mathcal{H}(B)$ inductively by $g_0(z) = [B(z)-B(0)]c/z$ and $g_{n+1}(z) = [g_n(z)-g_n(0)]/z$. Since $R(0)$ is bounded by 1, the expansion

$$[B(z)-B(w)]c/(z-w) = g_0(z) + wg_1(z) + w^2 g_2(z) + \ldots$$

is valid in the metric of $\mathcal{H}(B)$. The sum therefore belongs to $\mathcal{H}(B)$. If $f(z)$ is in $\mathcal{H}(B)$, let $(f_n(z))$ be the sequence of elements of $\mathcal{H}(B)$ defined inductively by $f_0(z) = f(z)$ and $R(0)^*: f_n(z) \to f_{n+1}(z)$. If $\tilde{f}(z) = \Sigma a_n z^n$, then

$$\bar{c}a_n = \langle f_n(z), [B(z)-B(0)]c/z\rangle_B = \langle f(z), g_n(z)\rangle_B .$$

It follows that

$$\bar{c}\tilde{f}(\bar{w}) = \langle f(z), \sum_0^\infty w^n g_n(z)\rangle_B = \langle f(z), [B(z)-B(w)]c/(z-w)\rangle_B .$$

Since $|w| < 1$, the expansion

$$R(w)^* = R(0)^* + \bar{w}R(0)^{*2} + \bar{w}^2 R(0)^{*3} + \ldots$$

converges in the norm of $\mathcal{H}(B)$ when $|w| < 1$. Since the expansion

$$\sum_0^\infty \bar{w}^n (a_0 z^n + \ldots + a_n) = \tilde{f}(\bar{w})/(1-z\bar{w})$$

is valid in the metric of $\mathcal{C}(z)$ when $|w| < 1$,

$$R(w)^*: f(z) \to [zf(z) - B(z)\tilde{f}(\bar{w})]/(1-z\bar{w})$$

when $|w| < 1$. These formulas allow us to determine $\tilde{f}(z)$ when $f(z) = [1-B(z)\bar{B}(\alpha)]a/(1-z\bar{\alpha})$ for some vector a and number α, $|\alpha| < 1$. By Lemma 5, $f(z)$ belongs to $\mathcal{H}(B)$ and

$$\begin{aligned}
\bar{c}\tilde{f}(\bar{w}) &= \langle f(z), [B(z)-B(w)]c/(z-w)\rangle_B \\
&= \overline{\langle [B(z)-B(w)]c/(z-w), [1-B(z)\bar{B}(\alpha)]a/(1-z\bar{\alpha})\rangle_B} \\
&= \{\bar{a}[B(\alpha)-B(w)]c/(\alpha-w)\}^- \\
&= \bar{c}[\bar{B}(\alpha)-\bar{B}(w)]a/(\bar{\alpha}-\bar{w}) .
\end{aligned}$$

By the arbitrariness of c and w, $\tilde{f}(z) = [B^*(z)-\bar{B}(\alpha)]a/(z-\bar{\alpha})$. It follows that the pair

$$([1-B(z)\bar{B}(\alpha)]a/(1-z\bar{\alpha}), [B^*(z)-\bar{B}(\alpha)]/(z-\bar{\alpha}))$$

belongs to $\mathcal{D}(B)$ and that

$$\bar{a}f(\alpha) = \langle (f(z),g(z)), ([1-B(z)\bar{B}(\alpha)]a/(1-z\bar{\alpha}), [B^*(z)-\bar{B}(\alpha)]a/(z-\bar{\alpha}))\rangle_{\mathcal{D}(B)}$$

for every element $(f(z),g(z))$ of $\mathcal{D}(B)$.

The definition of the space $\mathcal{D}(B)$ has been made in such a way that $(zf(z)-B(z)g(0), [g(z)-g(0)]/z)$ belongs to $\mathcal{D}(B)$ whenever $(f(z),g(z))$ belongs to $\mathcal{D}(B)$. The norm in $\mathcal{D}(B)$ is such that

$$\|(zf(z)-B(z)g(0), [g(z)-g(0)]/z)\|^2_{\mathcal{D}(B)} = \|(f(z),g(z))\|^2_{\mathcal{D}(B)} - |g(0)|^2 .$$

We show that $([f(z)-f(0)]/z, zg(z)-B^*(z)f(0))$ belongs to $\mathcal{D}(B)$ whenever $(f(z),g(z))$ belongs to $\mathcal{D}(B)$. Let $f_1(z)=[f(z)-f(0)]/z$ and $g_1(z) = zg(z)-B^*(z)f(0)$. Then

$$zf_1(z)-B(z)g_1(0) = f(z) - [1-B(z)\bar{B}(0)]f(0)$$

$$[g_1(z)-g_1(0)]/z = g(z) - [B^*(z)-B^*(0)]f(0)/z .$$

Since $(f(z),g(z))$ and $([1-B(z)\bar{B}(0)]f(0), [B^*(z)-B^*(0)]f(0)/z)$ belong to $\mathcal{D}(B)$, $(zf_1(z)-B(z)g_1(0), [g_1(z)-g_1(0)]/z)$ belongs to $\mathcal{D}(B)$. It follows from the definition of $\mathcal{D}(B)$ that $(f_1(z),g_1(z))$ belongs to $\mathcal{D}(B)$ and that

$$\|(zf_1(z)-B(z)g_1(0), [g_1(z)-g_1(0)]/z)\|^2_{\mathcal{D}(B)} = \|(f_1(z),g_1(z))\|^2_{\mathcal{D}(B)} - |g_1(0)|^2 .$$

On expressing these series in terms of $f(z)$ and $g(z)$ and expanding the left side as a self product, we obtain

$$\|(f(z),g(z))\|^2_{\mathcal{D}(B)} - |f(0)|^2 = \|([f(z)-f(0)]/z, zg(z)-B^*(z)f(0))\|^2_{\mathcal{D}(B)} .$$

A similar argument will obtain the identity

$$\langle ([f(z)-f(0)]/z, zg(z)-B^*(z)f(0)), (u(z),v(z))\rangle_{\mathcal{D}(B)}$$

$$= \langle (f(z),g(z)), (zu(z)-B(z)v(0), [v(z)-v(0)]/z)\rangle_{\mathcal{D}(B)}$$

for all elements $(f(z),g(z))$ and $(u(z),v(z))$ of $\mathcal{D}(B)$.

If $(f(z), g(z))$ is in $\mathcal{D}(B)$, define a sequence of elements $(f_n(z), g_n(z))$ of $\mathcal{D}(B)$ inductively by

$$(f_0(z), g_0(z)) = (f(z), g(z))$$

$$(f_{n+1}(z), g_{n+1}(z)) = (zf_n(z) - B(z)g_n(0), [g_n(z) - g_n(0)]/z).$$

Since $\|(f_n(z), g_n(z))\|_{\mathcal{D}(B)} \leq \|(f(z), g(z))\|_{\mathcal{D}(B)}$ for every n, the expansion

$$([zf(z) - B(z)g(w)]/(1-zw),\ [g(z) - g(w)]/(z-w))$$

$$= (f_1(z), g_1(z)) + w(f_2(z), g_2(z)) + w^2(f_3(z), g_3(z)) + \ldots$$

is valid in the metric of $\mathcal{D}(B)$ when $|w| < 1$. The sum therefore belongs to $\mathcal{D}(B)$. If $(f(z), g(z))$ and $(u(z), v(z))$ are in $\mathcal{D}(B)$ and if α and β are numbers, $|\alpha| < 1$ and $|\beta| < 1$, the identity

$$(1-\alpha\bar{\beta})\langle([zf(z) - B(z)g(\alpha)]/(1-z\alpha),\ [g(z) - g(\alpha)]/(z-\alpha)),$$

$$([zu(z) - B(z)v(\beta)]/(1-z\beta), [v(z) - v(\beta)]/(z-\beta))\rangle_{\mathcal{D}(B)}$$

$$-\alpha\langle([zf(z) - B(z)g(\alpha)]/(1-z\alpha), [g(z) - g(\alpha)]/(z-\alpha)), (u(z), v(z))\rangle_{\mathcal{D}(B)}$$

$$-\bar{\beta}\langle(f(z), g(z)), ([zu(z) - B(z)v(\beta)]/(1-z\beta), [v(z) - v(\beta)]/(z-\beta))\rangle_{\mathcal{D}(B)}$$

$$= \langle(f(z), g(z)), (u(z), v(z))\rangle_{\mathcal{D}(B)} - \bar{v}(\beta)g(\alpha)$$

is obtained by a straightforward calculation. See for example the analogous Problem 87 of [10]. A similar argument will show that

$$([f(z) - f(w)]/(z-w),\ [zg(z) - B^*(z)f(w)]/(1-zw))$$

belongs to $\mathcal{D}(B)$ whenever $(f(z), g(z))$ belongs to $\mathcal{D}(B)$ if $|w| < 1$. If $(f(z), g(z))$ and $(u(z), v(z))$ are in $\mathcal{D}(B)$ and if α and β are numbers, $|\alpha| < 1$ and $|\beta| < 1$, then

$$(1-\alpha\bar{\beta})<([f(z)-f(\alpha)]/(z-\alpha),[zg(z)-B^*(z)f(\alpha)]/(1-z\alpha)),$$
$$([u(z)-u(\beta)]/(z-\beta),[zv(z)-B^*(z)u(\beta)]/(1-z\beta))>_{\mathcal{D}(B)}$$
$$-\alpha<([f(z)-f(\alpha)]/(z-\alpha),[zg(z)-B^*(z)f(\alpha)]/(1-z\alpha)),(u(z),v(z))>_{\mathcal{D}(B)}$$
$$-\bar{\beta}<(f(z),g(z)),([u(z)-u(\beta)]/(z-\beta),[zv(z)-B^*(z)u(\beta)]/(1-z\beta))>_{\mathcal{D}(B)}$$
$$=<(f(z),g(z)),(u(z),v(z))>_{\mathcal{D}(B)}-\bar{u}(\beta)f(\alpha).$$

If $(f(z),g(z))$ and $(u(z),v(z))$ are in $\mathcal{D}(B)$ and if w is a number, $|w|<1$, then

$$<([f(z)-f(w)]/(z-w),[zg(z)-B^*(z)f(w)]/(1-zw)),(u(z),v(z))>_{\mathcal{D}(B)}$$
$$=<(f(z),g(z)),([zu(z)-B(z)v(\bar{w})]/(1-z\bar{w}),[v(z)-v(\bar{w})]/(z-\bar{w}))>_{\mathcal{D}(B)}.$$

If c is a vector and if $|w|<1$, then $([B(z)-B(\bar{w})]c/(z-\bar{w}),$ $[1-B^*(z)B(\bar{w})]c/(1-z\bar{w}))$ belongs to $\mathcal{D}(B)$ and the identity

$$\bar{c}g(w)=<(f(z),g(z)),([B(z)-B(\bar{w})]c/(z-\bar{w}),[1-B^*(z)B(\bar{w})]c/(1-z\bar{w}))>_{\mathcal{D}(B)}$$

holds for every element $(f(z),g(z))$ of $\mathcal{D}(B)$. This follows directly from the definition of the space when $w=0$. The general case is obtained by a straightforward use of the previous identities. (A similar calculation appears in detail in the proof of Theorem 10.)

We show that the transformation $(f(z),g(z))\to(g(z),f(z))$ takes the space $\mathcal{D}(B)$ associated with $\mathcal{H}(B)$ isometrically onto the space $\mathcal{D}(B^*)$ associated with $\mathcal{H}(B^*)$. If $(f(z),g(z))$ is a finite sum of elements of the form

$$([1-B(z)\bar{B}(w)]u/(1-z\bar{w}),[B^*(z)-\bar{B}(w)]u/(z-\bar{w}))$$
$$([B(z)-B(\bar{w})]v/(z-\bar{w}),[1-B^*(z)B(\bar{w})]v/(1-z\bar{w}))$$

where u and v are vectors and $|w|<1$, then $(g(z),f(z))$ is a finite sum of elements of the same form with $B(z)$ replaced by $B^*(z)$. It is easily verified that

$$\|(f(z),g(z))\|_{\mathcal{D}(B)}=\|(g(z),f(z))\|_{\mathcal{D}(B^*)}$$

for such sums. Since such sums $(f(z), g(z))$ are dense in $\mathcal{D}(B)$ and since the corresponding sums $(g(z), f(z))$ are dense in $\mathcal{D}(B^*)$, it follows that the transformation $(f(z), g(z)) \to (g(z), f(z))$ takes $\mathcal{D}(B)$ isometrically onto $\mathcal{D}(B^*)$. The theorem follows on recalling the relation of $\mathcal{D}(B)$ to $\mathcal{H}(B)$ and of $\mathcal{D}(B^*)$ to $\mathcal{H}(B^*)$.

Proof of Theorem 6. Let $\mathcal{H}_1$ be the space of power series $f(z)$ such that $[f(z)-f(0)]/z$ belongs to $\mathcal{H}_0$, in the norm

$$\|f(z)\|_1^2 = \|[f(z)-f(0)]/z\|_0^2 + |f(0)|^2 .$$

It is easily verified that $\mathcal{H}_1$ is a Hilbert space in this metric. Since (1) holds in $\mathcal{H}_0$, the space $\mathcal{H}_1$ contains $\mathcal{H}_0$ isometrically. Let $S(0)$ be the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}_1$ and let $S(0)^*$ be its adjoint in $\mathcal{H}_1$. A straightforward computation will show that $S(0)^*: f(z) \to zf(z)$ whenever $f(z)$ is in $\mathcal{H}_0$ and that $S(0)^*$ annihilates the orthogonal complement of $\mathcal{H}_0$. The action of $R(0)^*$ on an element $f(z)$ of $\mathcal{H}_0$ is the projection of $zf(z)$ in $\mathcal{H}_0$. It follows that $f(z)$ is in the kernel of $1-R(0)R(0)^*$ if, and only if, $zf(z)$ belongs to $\mathcal{H}_0$. This happens if, and only if, $f(z)$ is orthogonal to all series $[g(z)-g(0)]/z$ where $g(z)$ belongs to the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}_1$. It follows that the transformation $f(z) \to [f(z)-f(0)]/z$ takes the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}_1$ into the closure of the range of $1-R(0)R(0)^*$, and that the range of the transformation is dense in the closure of the range of $1-R(0)R(0)^*$. The kernel of the transformation is the set of all constants orthogonal to $\mathcal{H}_0$, which is the same as the set of all vectors orthogonal to coefficients of $\mathcal{H}_0$. If $d \geq d_0 + d_1$, there exists a partially isometric transformation U which takes $\mathcal{C}$ onto the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}_1$. If $d = d_0 + d_1$, we can choose U to be an isometry.

If $f(z) = \Sigma a_n z^n$ is in $\mathcal{H}_0$, define a sequence of elements $f_n(z)$ of $\mathcal{H}_0$ inductively by $f_0(z) = f(z)$ and $f_{n+1}(z) = [f_n(z) - f_n(0)]/z$. Since $f_n(0) = a_n$, the identity (1) implies that

$$\|f_{n+1}(z)\|_0^2 = \|f_n(z)\|_0^2 - |a_n|^2 .$$

It follows on summation that

$$\|f_{n+1}(z)\|_0^2 = \|f(z)\|_0^2 - |a_0|^2 - \ldots - |a_n|^2 .$$

By the arbitrariness of n, $\Sigma_0^\infty |a_n|^2 \leq \|f(z)\|_0^2$. So the space $\mathcal{H}_0$ is contained in $\mathcal{C}(z)$ and the inclusion does not increase norms. By the proof of Lemma 1, the transformation U is of the form $U: c \to B(z)c$ for some power series $B(z)$ with operator coefficients. As there the coefficients of $B(z)$ are bounded by 1 and $B(z)$ converges in the unit disk. If $f(z)$ is in $\mathcal{H}_1$ and is orthogonal to $\mathcal{H}_0$, it is of the form $f(z) = B(z)a$ for some vector a such that $\|f(z)\|_1 = |a|$. It follows that

$$\bar{c}f(w) = \bar{c}B(w)a = \langle B(z)a, B(z)\bar{B}(w)c\rangle_1 = \langle f(z), B(z)\bar{B}(w)c\rangle_1 .$$

Since $\mathcal{H}_0$ is contained in $\mathcal{C}(z)$ and since the inclusion does not increase norms, the transformation $f(z) \to f(w)$ of $\mathcal{H}_0$ into $\mathcal{C}$ is continuous for every number w, $|w| < 1$. The adjoint transformation of $\mathcal{C}$ into $\mathcal{H}_0$ is of the form $c \to K(w,z)c$ for some power series $K(w,z)$ with operator coefficients. If $f(z)$ is in $\mathcal{H}_0$, then

$$\bar{c}f(w) = \langle f(z), K(w,z)c\rangle_0$$

for every $f(z)$ in $\mathcal{H}_0$. We now compute the form of $K(w,z)$.

Note that $K(w,z)c+B(z)\bar{B}(w)c$ belongs to $\mathcal{H}_1$ for every vector c when $|w| < 1$ and

$$\bar{c}f(w) = \langle f(z), K(w,z)c+B(z)\bar{B}(w)c\rangle_1$$

for every $f(z)$ in $\mathcal{H}_1$. But by the definition of $\mathcal{H}_1$, $c+z\bar{w}K(w,z)c$ belongs to $\mathcal{H}_1$ and

$$\begin{aligned}
&\langle f(z), c+z\bar{w}K(w,z)c\rangle_1 \\
&= \bar{c}f(0) + w\langle [f(z)-f(0)]/z, K(w,z)c\rangle_0 \\
&= \bar{c}f(0) + \bar{c}\,[f(w)-f(0)] = \bar{c}f(w)
\end{aligned}$$

for every $f(z)$ in $\mathcal{H}_1$. By the arbitrariness of $f(z)$ and c,

$$1 + z\bar{w}K(w,z) = K(w,z) + B(z)\bar{B}(w) .$$

It follows that $K(w,z) = [1-B(z)\bar{B}(w)]/(1-z\bar{w})$. Since

$$\bar{c}K(w,w)c = \|K(w,z)c\|_0^2 \geq 0$$

for every vector c, $|\bar{B}(w)c| \leq |c|$ for every vector c when $|w| < 1$. By the arbitrariness of c, $|B(w)| \leq 1$ for $|w| < 1$. A space $\mathcal{H}(B)$ exists by Lemma 2. The finite sums of elements of the form $K(w,z)c$, c in $\mathcal{C}$ and $|w| < 1$, are dense in $\mathcal{H}_0$ and $\mathcal{H}(B)$, and the norm of such functions is the same in $\mathcal{H}_0$ as in $\mathcal{H}(B)$ since it is an expression in terms of $B(z)$. It follows that $\mathcal{H}_0$ is equal isometrically to $\mathcal{H}(B)$. If $d = d_0+d_1$, we have chosen $B(z)$ so that $B(z)c = 0$ for no nonzero vector c. Since $\mathcal{H}(B)$ contains no nonzero element of the form $B(z)c$, where c is in $\mathcal{C}$, there is no nonzero vector c such that $B(z)c$ belongs to $\mathcal{H}(B)$.

Proof of Theorem 7. We omit the routine verification that $\mathcal{L}$ is a Hilbert space. If $L(z)$ is in $\mathcal{L}$, the inequality

$$\|B(z)[L(z)-L(0)]/z\|_B^2 \le \|B(z)L(z)\|_B^2 + |L(0)|^2$$

is obtained by a straightforward computation of the B-norm on the left. It follows that $[L(z)-L(0)]/z$ is in $\mathcal{L}$ whenever $L(z)$ is in $\mathcal{L}$ and that $\|[L(z)-L(0)]/z\|_{\mathcal{L}} \le \|L(z)\|_{\mathcal{L}}$. The main problem is to show that the transformation $T: L(z) \to [L(z)-L(0)]/z$ in $\mathcal{L}$ has an isometric adjoint. The action of T^* can be determined for special elements of $\mathcal{L}$ obtained through minimal decompositions. If $f(z)$ is in $\mathcal{C}(z)$, the minimal decomposition of $B(z)f(z)$ is of the form

$$B(z)f(z) = B(z)g(z) + B(z)[f(z)-g(z)]$$

where $g(z)$ is a uniquely determined element of $\mathcal{L}$. By Theorem 4C this element is characterized by the identity

$$0 = \langle B(z)g(z), B(z)L(z)\rangle_B + \langle f(z)-g(z), -L(z)\rangle$$

for every element $L(z)$ of $\mathcal{L}$. By the definition of the inner product in $\mathcal{L}$, this condition is

$$\langle f(z), L(z)\rangle = \langle g(z), L(z)\rangle_{\mathcal{L}} .$$

In other words the transformation $f(z) \to g(z)$ of $\mathcal{C}(z)$ into $\mathcal{L}$ defined by minimal decompositions is the adjoint of the inclusion of $\mathcal{L}$ in $\mathcal{C}(z)$. In particular when $g(z) = L(z)$ we obtain

$$\|g(z)\|_{\mathcal{L}}^2 = \langle f(z), g(z)\rangle$$

where by Theorem 4E

$$\langle B(z)f(z), B(z)f(z)\rangle = \langle f(z), f(z)-g(z)\rangle.$$

It follows that

$$\|g(z)\|_{\mathcal{L}}^2 = \|f(z)\|^2 - \|B(z)f(z)\|^2 .$$

But the inclusion of $\mathcal{L}$ in $\mathcal{C}(z)$ commutes with the difference-quotient transformation. It follows that under the adjoint transformation of $\mathcal{C}(z)$ into $\mathcal{L}$, the adjoint of the difference-quotient transformation in $\mathcal{C}(z)$ corresponds to the adjoint of the difference-quotient transformation in $\mathcal{L}$. Explicitly $T^*: g(z) \to h(z)$ where $h(z)$ is the unique element of $\mathcal{L}$ such that

$$B(z)zf(z) = B(z)h(z) + B(z)[zf(z) - h(z)]$$

is a minimal decomposition. Since

$$\|h(z)\|_{\mathcal{L}}^2 = \|zf(z)\|^2 - \|B(z)zf(z)\|^2$$

and since multiplication by z is isometric in $\mathcal{C}(z)$, $\|h(z)\|_{\mathcal{L}} = \|g(z)\|_{\mathcal{L}}$. We have shown that T^* is isometric on special elements of $\mathcal{L}$ obtained by minimal decomposition. These elements of $\mathcal{L}$ are those in the range of the adjoint of the inclusion of $\mathcal{L}$ in $\mathcal{C}(z)$. Since the inclusion is one-to-one, the adjoint has a dense range in $\mathcal{L}$. Since T^* is continuous, it is isometric in $\mathcal{L}$.

Proof of Theorem 8. Let $R(0)$ be the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{L}$. Since $R(0)$ is bounded by 1, the series $\Sigma_0^\infty w^n R(0)^n f$ converges in the metric of $\mathcal{L}$ if $f(z)$ is in $\mathcal{L}$ and $|w| < 1$. Since $J: f(z) \to f(0)$ is a continuous transformation of $\mathcal{L}$ into $\mathcal{C}$, the expansion $f(w) = \Sigma_0^\infty w^n JR(0)^n f$ is valid in the metric of $\mathcal{C}$ when $|w| < 1$, and $f(z) \to f(w)$ is a continuous transformation of $\mathcal{L}$ into $\mathcal{C}$. The adjoint transformation, which takes $\mathcal{C}$ into $\mathcal{L}$, is of the form $c \to L(w,z)c$ where $L(w,z)$ is a power series with operator coefficients. It follows that $L(w,z)c$ belongs to $\mathcal{L}$ for every vector c and that

$$\bar{c}f(w) = \langle f(z), L(w,z)c\rangle_{\mathcal{L}}$$

for every $f(z)$ in $\mathcal{L}$. The series $L(w,z)$ converges in the unit disk and $L(\alpha,\beta) = \bar{L}(\beta,\alpha)$. Then $[L(w,z)-L(0,z)]c/\bar{w}$ belongs to $\mathcal{L}$ for every vector c when $0 < |w| < 1$, and

$$\langle f(z), [L(w,z)-L(0,z)]c/\bar{w}\rangle_{\mathcal{L}} = \bar{c}[f(w)-f(0)]/w = \langle [f(z)-f(0)]/z, L(w,z)c\rangle_{\mathcal{L}}$$

for every $f(z)$ in $\mathcal{L}$. By the definition of the adjoint,

$$R(0)^*: L(w,z)c \to [L(w,z)-L(0,z)]c/\bar{w}.$$

Since $R(0)^*$ is isometric by hypothesis,

$$<[L(\alpha,z)-L(0,z)]a/\bar{\alpha}, [L(\beta,z)-L(0,z)]b/\bar{\beta}>_{\mathcal{L}} = <L(\alpha,z)a, L(\beta,z)b>_{\mathcal{L}}$$

for all vectors a and b when $0 < |\alpha| < 1$ and $0 < |\beta| < 1$. By the definition of $L(w,z)$ the identity can be written

$$\bar{b}[L(\alpha,\beta) - L(\alpha,0) - L(0,\beta) + L(0,0)]a/(\bar{\alpha}\beta) = \bar{b}L(\alpha,\beta)a \ .$$

Since a and b are arbitrary vectors,

$$(1-\bar{\alpha}\beta)L(\alpha,\beta) = L(\alpha,0)+L(0,\beta)-L(0,0) \ .$$

In terms of $\varphi(z) = 2L(0,z)-L(0,0)$ the identity reads

$$L(w,z) = \tfrac{1}{2}[\varphi(z)+\bar{\varphi}(w)]/(1-z\bar{w}) \ .$$

Since $\|L(w,z)c\|_{\mathcal{L}}^2 \geq 0$ for every vector c when $|w| < 1$,

$$\tfrac{1}{2}\bar{c}[\varphi(w)+\bar{\varphi}(w)]c/(1-w\bar{w}) \geq 0 \ .$$

By the arbitrariness of c, Re $\varphi(w) \geq 0$. That $\mathcal{L}$ is uniquely determined by $\varphi(z)$ follows from the fact that the finite linear combinations of the series $\frac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w})$ are dense in $\mathcal{L}$ and the inner product of any two such series is determined solely from a knowledge of $\varphi(z)$.

Proof of Lemma 6. By the proof of Theorem 5, $[B(z)-B(0)]c/z$ belongs to $\mathcal{H}(B)$ for every vector c and $c \to [B(z)-B(0)]c/z$ is a bounded transformation of $\mathcal{C}$ into $\mathcal{H}(B)$. It follows that U is an everywhere defined and bounded transformation in $\mathcal{H}(B)$. To show that the adjoint U^* of U is isometric in $\mathcal{H}(B)$ we need only show that UU^* is the identity on every element of $\mathcal{H}(B)$ of the form $K(w,z)c = [1-B(z)\bar{B}(w)]c/(1-z\bar{w})$, since the finite sums of such elements are dense in $\mathcal{H}(B)$. The definition of the adjoint can be used directly to verify that U^* takes $K(w,z)c$ into

$$[K(w,z)-K(0,z)]c/\bar{w} - K(0,z)[1+\bar{B}(0)]^{-1}[\bar{B}(w)-\bar{B}(0)]c/\bar{w} \ .$$

The action of UU^* on $K(w,z)c$ is now seen to be the identity by an obvious computation.

Proof of Theorem 9. Since Re $\varphi(w) \geq 0$ for $|w| < 1$, $1+\varphi(w)$ has an operator inverse and

$$B(z) = [1-\varphi(z)]/[1+\varphi(z)]$$

is a well-defined power series which converges to a function which is bounded by 1 in the unit disk. A space $\mathcal{H}(B)$ exists by Lemma 2. Let $\mathcal{L}$ be the set of all power series of the form $F(z) = \frac{1}{2}[1+\varphi(z)]f(z)$ for some corresponding element $f(z)$ of $\mathcal{H}(B)$. Then $\mathcal{L}$ is a Hilbert space in the norm which makes the transformation $f(z) \to F(z)$ an isometry of $\mathcal{H}(B)$ onto $\mathcal{L}$. If this transformation takes $f(z)$ into $F(z)$, it takes

$$[f(z)-f(0)]/z-[B(z)-B(0)][1+B(0)]^{-1}f(0)/z$$

into $[F(z)-F(0)]/z$. By Lemma 6, $F(z) \to [F(z)-F(0)]/z$ is an everywhere defined and bounded transformation in $\mathcal{L}$ which has an isometric adjoint. The transformation $F(z) \to F(0)$ of $\mathcal{L}$ into $\mathcal{C}$ is continuous since it is the composition of the continuous transformations $F(z) \to f(z)$, $f(z) \to f(0)$, and $c \to \frac{1}{2}[1+\varphi(0)]c$. It remains to show that if c is any vector and if w is a number, $|w| < 1$, then

$$L(w, z) = \tfrac{1}{2}[\varphi(z)+\overline{\varphi}(w)]c/(1-z\overline{w})$$

belongs to $\mathcal{L}$ and that

$$\overline{c}F(w) = \langle F(z), L(w, z)c\rangle_{\mathcal{L}}$$

for every $F(z)$ in $\mathcal{L}$. Let $F(z) = \frac{1}{2}[1+\varphi(z)]f(z)$ where $f(z)$ is in $\mathcal{H}(B)$. Since

$$L(w, z)c = \tfrac{1}{2}[1+\varphi(z)]K(w, z)\tfrac{1}{2}[1+\overline{\varphi}(w)]c ,$$

it belongs to $\mathcal{L}$ and

$$\langle F(z), L(w, z)c\rangle_{\mathcal{L}} = \langle f(z), K(w, z)\tfrac{1}{2}[1+\overline{\varphi}(w)]c\rangle_B$$

$$= \tfrac{1}{2}\overline{c}[1+\varphi(w)]f(w) = \overline{c}F(w) .$$

<u>Proof of Theorem 10.</u> Since $R(0)^*$ is isometric, $R(0)R(0)^*$ is the identity transformation. If $R(0)^*: f(z) \to g(z)$, then $g(z) = zf(z)+a_0$ for some vector a_0. Continuing inductively we obtain

$$R(0)^{*n}: f(z) \to z^n f(z) + a_0 z^{n-1} + \ldots + a_{n-1}$$

for some vectors (a_n). Let $\tilde{f}(z)$ be the formal power series $\tilde{f}(z) = \Sigma a_n z^n$. Since $R(0)^*$ is isometric and since the transformation $L(z) \to L(0)$ of $\mathcal{L}(\varphi)$ into $\mathcal{C}$ is bounded, the coefficients of $\tilde{f}(z)$ are bounded. The series $\tilde{f}(z)$ converges in the unit disk and the transformation $f(z) \to \tilde{f}(w)$ of $\mathcal{L}(\varphi)$ into $\mathcal{C}$ is bounded when $|w| < 1$. Since

$$[1-wR(0)]^{-1} = 1 + wR(0) + w^2R(0)^2 + \ldots$$

when $|w| < 1$, a straightforward computation yields

$$R(w)^*: f(z) \to [zf(z) + \tilde{f}(\bar{w})]/(1-z\bar{w})$$

for every $f(z)$ in $\mathcal{L}(\varphi)$ when $|w| < 1$. But for every vector c when $|\alpha| < 1$,

$$\bar{c}[\alpha f(\alpha) + \tilde{f}(\bar{w})]/(1-\alpha\bar{w})$$

$$= \langle [zf(z)+\tilde{f}(\bar{w})]/(1-z\bar{w}),\ \tfrac{1}{2}[\varphi(z)+\bar{\varphi}(\alpha)]c/(1-z\bar{\alpha})\rangle_{\mathcal{L}(\varphi)}$$

$$= \langle f(z), \{\tfrac{1}{2}[\varphi(z)+\bar{\varphi}(\alpha)]c/(1-z\bar{\alpha}) - \tfrac{1}{2}[\varphi(w)+\bar{\varphi}(\alpha)]c/(1-w\bar{\alpha})\}/(z-w)\rangle_{\mathcal{L}(\varphi)}$$

$$= \langle f(z), \bar{\alpha}(1-w\bar{\alpha})^{-1}\tfrac{1}{2}[\varphi(z)+\bar{\varphi}(\alpha)]c/(1-z\bar{\alpha})\rangle_{\mathcal{L}(\varphi)}$$

$$+ \langle f(z), (1-w\bar{\alpha})^{-1}\tfrac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w)\rangle_{\mathcal{L}(\varphi)}$$

$$= \bar{c}\alpha f(\alpha)/(1-\alpha\bar{w}) + (1-\alpha\bar{w})^{-1}\langle f(z), \tfrac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w)\rangle_{\mathcal{L}(\varphi)}$$

and hence

$$\bar{c}\tilde{f}(\bar{w}) = \langle f(z), \tfrac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w)\rangle_{\mathcal{L}(\varphi)} .$$

In order to relate $\mathcal{L}(\varphi)$ and $\mathcal{L}(\varphi^*)$, we construct a new Hilbert space $\mathcal{E}(\varphi)$ whose elements are pairs $(f(z), g(z))$ of power series with vector coefficients. By definition, $(f(z), g(z))$ belongs to $\mathcal{E}(\varphi)$ if $f(z)$ belongs to $\mathcal{L}(\varphi)$ and if $g(z) = \Sigma a_n z^n$ where

$$z^n f(z) + a_0 z^{n-1} + \ldots + a_{n-1}$$

belongs to $\mathcal{L}(\varphi)$ for every $n = 1, 2, 3, \ldots$, and the sequence

$$\| z^n f(z) + a_0 z^{n-1} + \ldots + a_{n-1} \|_{\mathcal{L}(\varphi)}$$

is bounded. This sequence is nondecreasing because the difference-quotient transformation in $\mathcal{L}(\varphi)$ is bounded by 1. It is easily verified that $\mathcal{E}(\varphi)$ is a Hilbert space in the norm

$$\|(f(z), g(z))\|_{\mathcal{E}(\varphi)} = \lim_{n\to\infty} \| z^n f(z) + a_0 z^{n-1} + \ldots + a_{n-1} \|_{\mathcal{L}(\varphi)} .$$

By the definition of the space,

$$(f(z), g(z)) \to ([f(z)-f(0)]/z, zg(z) + f(0))$$

and

$$(f(z), g(z)) \to (zf(z) + g(0), [g(z)-g(0)]/z)$$

are everywhere defined isometries in $\mathcal{E}(\varphi)$. Since $R(0)^*$ is isometric, $f(z) \to (f(z), \tilde{f}(z))$ takes $\mathcal{L}(\varphi)$ isometrically into $\mathcal{E}(\varphi)$. The orthogonal complement of the range of the transformation $f(z) \to (f(z), \tilde{f}(z))$ is the set of elements of $\mathcal{E}(\varphi)$ of the form $(0, g(z))$.

If $f_0(z) = \frac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w})$ for some vector c where $|w| < 1$, then $\tilde{f}_0(z) = \frac{1}{2}[\varphi^*(z)-\bar{\varphi}(w)]c/(z-\bar{w})$. Therefore $(\frac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w}), \frac{1}{2}[\varphi^*(z)-\bar{\varphi}(w)]c/(z-\bar{w}))$ belongs to $\mathcal{E}(\varphi)$ and the identity

$$\bar{c}f(w) = \langle (f(z),g(z)), (\tfrac{1}{2}[\varphi(z)+\bar{\varphi}(w)]c/(1-z\bar{w}), \tfrac{1}{2}[\varphi^*(z)-\bar{\varphi}(w)]c/(z-\bar{w}))\rangle_{\mathcal{E}(\varphi)}$$

holds for every pair $(f(z), g(z))$ in $\mathcal{E}(\varphi)$. We will show also that $(\frac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w}), \frac{1}{2}[\varphi^*(z)+\varphi(\bar{w})]c/(1-z\bar{w}))$ belongs to $\mathcal{E}(\varphi)$ for every vector c when $|w| < 1$, and

$$\bar{c}g(w) = \langle (f(z),g(z)), (\tfrac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w}), \tfrac{1}{2}[\varphi^*(z)+\varphi(\bar{w})]c/(1-z\bar{w}))\rangle_{\mathcal{E}(\varphi)}$$

for every pair $(f(z), g(z))$ in $\mathcal{E}(\varphi)$. Suppose first that $w = 0$. Then if $\varphi(z) = \Sigma \varphi_n z^n$,

$$\tfrac{1}{2}[\varphi^*(z)+\varphi(0)]c = \tfrac{1}{2}[\bar{\varphi}(0)+\varphi(0)]c + \tfrac{1}{2}\bar{\varphi}_1 cz + \tfrac{1}{2}\bar{\varphi}_2 cz^2 + \ldots$$

and

$$z^n \tfrac{1}{2}[\varphi(z)-\varphi(0)]c/z + \tfrac{1}{2}[\bar{\varphi}(0)+\varphi(0)]cz^{n-1} + \tfrac{1}{2}\bar{\varphi}_1 cz^{n-2} + \ldots + \tfrac{1}{2}\bar{\varphi}_{n-1}c$$
$$= z^{n-1}\tfrac{1}{2}[\varphi(z)+\bar{\varphi}(0)]c + \tfrac{1}{2}\bar{\varphi}_1 cz^{n-2} + \ldots + \tfrac{1}{2}\bar{\varphi}_{n-1}c .$$

Since $(\frac{1}{2}[\varphi(z)+\bar{\varphi}(0)]c, \frac{1}{2}[\varphi^*(z)-\bar{\varphi}(0)]c/z)$ belongs to $\mathcal{E}(\varphi)$, $(\frac{1}{2}[\varphi(z)-\varphi(0)]c/z, \frac{1}{2}[\varphi^*(z)+\varphi(0)]c)$ belongs to $\mathcal{E}(\varphi)$. If $(f(z), g(z))$ is any element of $\mathcal{E}(\varphi)$ and if $g(z) = \Sigma a_n z^n$, then

$$\langle (f(z), g(z)), (\tfrac{1}{2}[\varphi(z) - \varphi(0)]c/z, \tfrac{1}{2}[\varphi^*(z) + \varphi(0)]c) \rangle_{\mathcal{E}(\varphi)}$$

$$= \lim_{n\to\infty} \langle z^n f(z) + a_0 z^{n-1} + \ldots + a_{n-1}, z^n \tfrac{1}{2}[\varphi(z) - \varphi(0)]c/z$$

$$+ \tfrac{1}{2}[\overline{\varphi}(0) + \varphi(0)]cz^{n-1} + \tfrac{1}{2}\overline{\varphi}_1 cz^{n-2} + \ldots + \tfrac{1}{2}\overline{\varphi}_{n-1} c \rangle_{\mathcal{L}(\varphi)}$$

$$= \lim_{n\to\infty} \langle z^{n-1}[zf(z) + g(0)] + a_1 z^{n-2} + \ldots + a_{n-1},$$

$$z^{n-1}\tfrac{1}{2}[\varphi(z) + \overline{\varphi}(0)]c + \tfrac{1}{2}\overline{\varphi}_1 cz^{n-2} + \ldots + \tfrac{1}{2}\overline{\varphi}_{n-1} c \rangle_{\mathcal{L}(\varphi)}$$

$$= \langle (zf(z) + g(0), [g(z) - g(0)]/z),$$

$$(\tfrac{1}{2}[\varphi(z) + \overline{\varphi}(0)]c, \tfrac{1}{2}[\varphi^*(z) - \overline{\varphi}(0)]c/z) \rangle_{\mathcal{E}(\varphi)}$$

$$= \bar{c}\, g(0) \;.$$

Now consider an arbitrary value of w, $|w| < 1$. By an iteration procedure, as in the proof of Theorem 5, $([f(z) - f(w)]/(z-w),\ [zg(z) + f(w)]/(1-zw))$ belongs to $\mathcal{E}(\varphi)$ whenever $(f(z), g(z))$ belongs to $\mathcal{E}(\varphi)$ and the identity

$$(1-\alpha\bar{\beta}) \langle ([f(z) - f(\alpha)]/(z-\alpha), [zg(z) + f(\alpha)]/(1-z\alpha)),$$

$$([u(z) - u(\beta)]/(z-\beta), [zv(z) + u(\beta)]/(1-z\beta)) \rangle_{\mathcal{E}(\varphi)}$$

$$- \alpha \langle ([f(z) - f(\alpha)]/(z-\alpha), [zg(z) + f(\alpha)]/(1-z\alpha)), (u(z), v(z)) \rangle_{\mathcal{E}(\varphi)}$$

$$- \bar{\beta} \langle (f(z), g(z)), ([u(z) - u(\beta)]/(z-\beta), [zv(z) + u(\beta)]/(1-z\beta)) \rangle_{\mathcal{E}(\varphi)}$$

$$= \langle (f(z), g(z)), (u(z), v(z)) \rangle_{\mathcal{E}(\varphi)}$$

holds whenever $(f(z), g(z))$ and $(u(z), v(z))$ belong to $\mathcal{E}(\varphi)$, $|\alpha| < 1$ and $|\beta| < 1$. The transformation

$$(f(z), g(z)) \to (f(z), g(z)) + \bar{w}([f(z) - f(\bar{w})]/(z-\bar{w}), [zg(z) + f(\bar{w})]/(1-z\bar{w}))$$

takes $(\tfrac{1}{2}[\varphi(z) - \varphi(0)]c/z, \tfrac{1}{2}[\varphi^*(z) + \varphi(0)]c)$ into $(\tfrac{1}{2}[\varphi(z) - \varphi(\bar{w})]c/(z-\bar{w})$ $\tfrac{1}{2}[\varphi^*(z) + \varphi(\bar{w})]c/(1-z\bar{w}))$. The adjoint of this transformation is

$$(f(z), g(z)) \to (f(z), g(z)) + w([zf(z) + g(w)]/(1-zw), [g(z) - g(w)]/(z-w)) \;.$$

Therefore $(\frac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w}), \frac{1}{2}[\varphi^*(z)+\varphi(\bar{w})]c/(1-z\bar{w}))$ belongs to $\mathcal{E}(\varphi)$, and if $(f(z),g(z))$ is any element of $\mathcal{E}(\varphi)$, then

$$<(f(z),g(z)),(\tfrac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w}), \tfrac{1}{2}[\varphi^*(z)+\varphi(\bar{w})]c/(1-z\bar{w}))>_{\mathcal{E}(\varphi)}$$

$$= <(f(z),g(z))+w([zf(z)+g(w)]/(1-zw),[g(z)-g(w)]/(z-w)),$$

$$(\tfrac{1}{2}[\varphi(z)-\varphi(0)]c/z, \tfrac{1}{2}[\varphi^*(z)+\varphi(0)]c)>_{\mathcal{E}(\varphi)}$$

$$= \bar{c}\,\{g(0) - w[g(0)-g(w)]/w\}$$

$$= \bar{c}g(w)\ .$$

As in the proof of Theorem 5, it follows that $(f(z),g(z)) \to (g(z),f(z))$ is an isometric transformation of $\mathcal{E}(\varphi)$ onto $\mathcal{E}(\varphi^*)$. The theorem follows by noting that $f(z) \to \tilde{f}(z)$ is the composition of the natural mappings of $\mathcal{L}(\varphi)$ into $\mathcal{E}(\varphi)$, $\mathcal{E}(\varphi)$ into $\mathcal{E}(\varphi^*)$, and $\mathcal{E}(\varphi^*)$ into $\mathcal{L}(\varphi^*)$.

Proof of Theorem 11. Let $h(z) = f(z) + B(z)g(z)$ be the minimal decomposition of any element $h(z)$ of $\mathcal{C}(z)$ with $f(z)$ in $\mathcal{H}(B)$ and $g(z)$ in $\mathcal{C}(z)$. By the proof of Lemma 5 of [10],

$$[h(z)-h(0)]/z = [f(z)-f(0)]/z + [B(z)-B(0)]g(0)/z + B(z)[g(z)-g(0)]/z$$

is a minimal decomposition of $[h(z)-h(0)]/z$ in $\mathcal{C}(z)$ with $[f(z)-f(0)]/z + [B(z)-B(0)]g(0)/z$ in $\mathcal{H}(B)$ and $[g(z)-g(0)]/z$ in $\mathcal{C}(z)$. Define a sequence of minimal decompositions $h_n(z) = f_n(z)+B(z)g_n(z)$ inductively by $h_0(z) = h(z)$, $f_0(z) = f(z)$, $g_0(z) = g(z)$, $h_{n+1}(z) = [h_n(z)-h_n(0)]/z$, $f_{n+1}(z) = [f_n(z)-f_n(0)]/z + [B(z)-B(0)]g_n(0)/z$, and $g_{n+1}(z) = [g_n(z)-g_n(0)]/z$. Since the difference-quotient transformation is bounded by 1 in $\mathcal{C}(z)$, the expansion

$$[h(z)-h(w)]/(z-w) = h_1(z) + wh_2(z) + w^2h_3(z) + \ldots$$

is valid in the metric of $\mathcal{C}(z)$ when $|w| < 1$, and similarly for $[g(z)-g(w)]/(z-w)$. It follows that

$$[h(z)-h(w)]/(z-w) = [f(z)-f(w)]/(z-w)+[B(z)-B(w)]g(w)/(z-w)$$

$$+ B(z)[g(z)-g(w)]/(z-w)$$

is a minimal decomposition of $[h(z)-h(w)]/(z-w)$ in $\mathcal{C}(z)$ with

$[f(z)-f(w)]/(z-w) + [B(z)-B(w)]g(w)/(z-w)$ in $\mathcal{H}(B)$ and $[g(z)-g(w)]/(z-w)$ in $\mathcal{C}(z)$.

For any vector c and number α, $|\alpha| < 1$, consider the minimal decomposition of $h(z) = f(z)+B(z)g(z)$ of $h(z)=B(z)c/(1-z\bar{\alpha})$ in $\mathcal{C}(z)$. By the proof of Theorem 7, it is obtained with

$$g(z) = c/(1-z\bar{\alpha}) - \tfrac{1}{2}[\varphi(z)+\bar{\varphi}(\alpha)]c/(1-z\bar{\alpha}) .$$

If w is a number, $|w| < 1$, then

$$(1-w\bar{\alpha})[h(z)-h(w)]/(z-w) = \bar{\alpha}h(z) + [B(z)-B(w)]c/(z-w)$$

$$(1-w\bar{\alpha})[g(z)-g(w)]/(z-w) = \bar{\alpha}g(z) - \tfrac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w) .$$

By Theorem 4D the minimal decomposition of $[B(z)-B(w)]c/(z-w)$ is obtained with

$$[B(z)-B(w)]c/(z-w) + \tfrac{1}{2}B(z)[\varphi(z)-\varphi(w)]c/(z-w)$$

as the term in $\mathcal{H}(B)$ and $-\tfrac{1}{2}[\varphi(z)-\varphi(w)]c/(z-w)$ as the term in $\mathcal{C}(z)$.

If $L(z)$ is in $\mathcal{L}(\varphi)$, let $\tilde{L}(z)$ be the corresponding element of $\mathcal{L}(\varphi^*)$ such that

$$\bar{c}\tilde{L}(w) = \langle L(z), \tfrac{1}{2}[\varphi(z)-\varphi(\bar{w})]c/(z-\bar{w})\rangle_{\mathcal{L}(\varphi)}$$

for every vector c when $|w| < 1$. Let $f(z) = B(z)L(z)$ be in $\mathcal{H}(B)$ and let $g(z)$ be the element of $\mathcal{H}(B^*)$ such that

$$\bar{c}g(w) = \langle f(z), [B(z)-B(\bar{w})]c/(z-\bar{w})\rangle_B$$

for every vector c when $|w| < 1$. By Theorem 4E, $\bar{c}g(w) = -\bar{c}\tilde{L}(w)$. By the arbitrariness of c and w, $g(z) = -\tilde{L}(z)$.

If $(f(z), g(z))$ belongs to $\mathcal{E}(\varphi)$ and if $g(z) = \Sigma a_n z^n$, then

$$z^n f(z) + z^{n-1}a_0 + \dots + a_{n-1}$$

belongs to $\mathcal{L}(\varphi)$ for every $n = 1, 2, 3, \dots$, and

$$\|z^n f(z) + z^{n-1}a_0 + \dots + a_{n-1}\|_{\mathcal{L}(\varphi)} \le \|(f(z), g(z))\|_{\mathcal{E}(\varphi)} .$$

Since $z^n B(z)f(z) + B(z)(z^{n-1}a_0 + \dots + a_{n-1})$ is in $\mathcal{H}(B)$ and since

$$\|z^n B(z)f(z) + B(z)(z^{n-1}a_0 + \ldots + a_{n-1})\|_B^2 + |a_0|^2 + \ldots + |a_{n-1}|^2$$
$$\leq \|z^n f(z) + z^{n-1}a_0 + \ldots + a_{n-1}\|_{\mathfrak{L}(\varphi)}^2$$

for every $n = 1, 2, 3, \ldots$, $(B(z)f(z), -g(z))$ is in $\mathfrak{D}(B)$ and $\|(B(z)f(z), -g(z))\|_{\mathfrak{D}(B)} \leq \|(f(z), g(z))\|_{\mathcal{E}(\varphi)}$. It follows from the proofs of Theorems 5 and 10 that $\mathfrak{L}(\varphi^*)$ is contained in $\mathcal{H}(B^*)$ and that the inclusion does not increase norms.

Proof of Theorem 12. If the space $\mathfrak{D}(B)$ is defined for $\mathcal{H}(B)$ as in the proof of Theorem 5, the hypotheses imply that the formula $\|(f(z), g(z))\|_{\mathfrak{D}(B)} = \|g(z)\|$ holds for every $f(z)$ in $\mathcal{H}(B)$ if $g(z) = \tilde{f}(z)$ is the corresponding element of $\mathcal{H}(B^*)$ defined by Theorem 5. The same formula holds also when $(f(z), g(z))$ is an element of $\mathfrak{D}(B)$ such that $f(z) = 0$ and multiplication by $B(z)$ annihilates every coefficient of $g(z)$. Since the set of elements $(f(z), g(z))$ for which the formula holds contains $([f(z)-f(0)]/z, zg(z)-B^*(z)f(0))$ whenever it contains $(f(z), g(z))$, and since it is closed in the metric of $\mathfrak{D}(B)$, it contains every element of $\mathfrak{D}(B)$. It follows that $\mathcal{H}(B^*)$ is contained isometrically in $\mathcal{C}(z)$. If $h(z)$ is in $\mathcal{C}(z)$, the minimal decomposition $B^*(z)h(z) = f(z) + B^*(z)g(z)$ of $B^*(z)h(z)$ in $\mathcal{C}(z)$ with $f(z)$ in $\mathcal{H}(B^*)$ and $g(z)$ in $\mathcal{C}(z)$ is obtained with $f(z) = 0$ and $\|B^*(z)h(z)\| = \|g(z)\|$. It follows that multiplication by $B^*(z)$ is a partially isometric transformation in $\mathcal{C}(z)$. There is no nonzero vector c such that $B^*(z)c = 0$ since then $z^n c$ belongs to $\mathcal{H}(B)$ for every n, $R(0)^*$: $z^n c \to z^{n+1}c$, and $\|z^n c\|_B = \|z^{n+1}c\|_B$. The hypotheses now imply that $c = 0$.

Since multiplication by $B^*(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, the kernel of multiplication by $B^*(z)$ is the orthogonal complement of the set of elements $g(z)$ in $\mathcal{C}(z)$ such that $\|B^*(z)g(z)\| = \|g(z)\|$. Since $zg(z)$ belongs to this set whenever $g(z)$ belongs to it, $[f(z)-f(0)]/z$ belongs to the kernel of multiplication by $B^*(z)$ whenever $f(z)$ belongs to the kernel of multiplication by $B^*(z)$. Since multiplication by $B^*(z)$ annihilates no nonzero constant, it annihilates no nonzero element of $\mathcal{C}(z)$. Since multiplication by $B^*(z)$ is a partial isometry which has no nonzero element in its kernel, it is an isometry.

Proof of Theorem 13. If there is no nonzero vector c such that $B^*(z)c$ belongs to $\mathcal{H}(B^*)$, then the space $\mathfrak{D}(B)$ in the proof of Theorem 5 contains no element $(f(z), g(z))$ such that $g(z) = B^*(z)c$ for a nonzero vector c. Since $([f(z)-f(0)]/z, zg(z)-B^*(z)f(0))$

belongs to $\mathcal{D}(B)$ whenever $(f(z),g(z))$ belongs to $\mathcal{D}(B)$, $\mathcal{D}(B)$ contains no nonzero element $(f(z),g(z))$ such that $g(z) = 0$. By the proof of Theorem 5, the transformation $(f(z),g(z)) \to g(z)$ is an isometry of $\mathcal{D}(B)$ onto $\mathcal{H}(B^*)$. Since the transformation $f(z) \to (f(z),\tilde{f}(z))$ is an isometry of $\mathcal{H}(B)$ into $\mathcal{D}(B)$, the composed transformation $f(z) \to \tilde{f}(z)$ is an isometry of $\mathcal{H}(B)$ into $\mathcal{H}(B^*)$.

If on the other hand the transformation of $\mathcal{H}(B)$ into $\mathcal{H}(B^*)$ is an isometry, then the range of the transformation $f(z) \to (f(z),\tilde{f}(z))$ of $\mathcal{H}(B)$ into $\mathcal{D}(B)$ is orthogonal to all elements $(u(z),v(z))$ of $\mathcal{D}(B)$ such that $v(z) = 0$. Since the orthogonal complement in $\mathcal{D}(B)$ of elements $(f(z),\tilde{f}(z))$ where $f(z)$ is in $\mathcal{H}(B)$ contains only elements of the form $(u(z),v(z))$ with $u(z) = 0$, there is no nonzero element $(u(z),v(z))$ of $\mathcal{D}(B)$ such that $v(z) = 0$. If c is a vector such that $B^*(z)c$ belongs to $\mathcal{H}(B^*)$, then there is an element $f(z)$ of $\mathcal{H}(B)$ such that $(f(z),B^*(z)c)$ belongs to $\mathcal{D}(B)$. By the proof of Theorem 5, $(B^*(z)c, f(z))$ belongs to $\mathcal{D}(B^*)$. It follows from the definition of $\mathcal{D}(B^*)$ that $(0, -c+zf(z))$ belongs to the space. Since this implies that $(-c+zf(z),0)$ belongs to $\mathcal{D}(B)$, $c = 0$.

<ins>Proof of Theorem 14.</ins> Since the transformation $f(z) \to \tilde{f}(z)$ of $\mathcal{H}(B)$ into $\mathcal{H}(B^*)$ is isometric by Theorem 13 and since $\mathcal{H}(B^*)$ is contained isometrically in $\mathcal{C}(z)$, the range of the transformation is a closed subspace of $\mathcal{C}(z)$. By Theorem 5, the range of the transformation contains $[f(z)-f(0)]/z$ whenever it contains $f(z)$. It follows that the range of the transformation is a space $\mathcal{H}(A^*)$ for some power series $A^*(z)$ with operator coefficients such that multiplication by $A^*(z)$ is a partially isometric transformation in $\mathcal{C}(z)$. Since $\mathcal{H}(A^*)$ is contained isometrically in $\mathcal{H}(B^*)$ and since multiplication by $B^*(z)$ is isometric in $\mathcal{C}(z)$, $B^*(z) = A^*(z)C^*(z)$ for some space $\mathcal{H}(C^*)$ by Theorem 3. Since $[B^*(z)-B^*(w)]c/(z-w)$ is in the range of the transformation $f(z) \to \tilde{f}(z)$ for every vector c when $|w| < 1$, by the proof of Theorem 5, it belongs to $\mathcal{H}(A^*)$. Since $[A^*(z)-A^*(w)]C^*(w)c/(z-w)$ belongs to $\mathcal{H}(A^*)$ by Theorem 5,

$$A^*(z)[C^*(z)-C^*(w)]c/(z-w)$$

$$= [B^*(z)-B^*(w)]c/(z-w) - [A^*(z)-A^*(w)]C^*(w)c/(z-w)$$

belongs to $\mathcal{H}(A^*)$. Since $[C^*(z)-C^*(w)]c/(z-w)$ belongs to $\mathcal{H}(C^*)$ by Theorem 5 and since $\mathcal{H}(A^*)$ is contained isometrically in $\mathcal{H}(B^*)$, $A^*(z)[C^*(z)-C^*(w)]c/(z-w) = 0$ by Theorem 4F. Since c is arbitrary, $B^*(z) = A^*(z)C^*(z) = A^*(z)C^*(w)$ when $|w| < 1$. Since multiplication by $B^*(z)$ is isometric in $\mathcal{C}(z)$ and since multiplication by $A^*(z)$ does not increase norms, $C^*(w)$ is an isometric operator. By Lemma 5,

$$\|[1-C(z)\bar{C}(w)]c/(1-z\bar{w})\|_C^2 = \bar{c}[1-C(w)\bar{C}(w)]c = 0$$

for every vector c and

$$\bar{c}f(w) = \langle f(z), [1-C(z)\bar{C}(w)]c/(1-z\bar{w})\rangle_C = 0$$

for every $f(z)$ in $\mathcal{H}(C)$. By the arbitrariness of c and w, $\mathcal{H}(C)$ contains no nonzero element. Since $[C(z)-C(w)]c/(z-w)$ belongs to $\mathcal{H}(C)$ for every vector c when $|w| < 1$, $C(z) = C$ is a constant.

The set $\mathfrak{h}$ of vectors c such that $A^*(z)c = 0$ is a closed vector subspace of $\mathcal{C}$. By the proof of Theorem 3, $\mathfrak{h}$ is orthogonal to the range of $\bar{C}$. Since $\bar{C}$ is an isometric operator, the dimension of the orthogonal complement of $\mathfrak{h}$ in $\mathcal{C}$ is equal to the dimension of $\mathcal{C}$. Therefore there exists an isometric operator S whose range is the orthogonal complement of $\mathfrak{h}$ in $\mathcal{C}$. Multiplication by $A^*(z)S$ is isometric in $\mathcal{C}(z)$ and $\mathcal{H}(A^*S)$ is equal isometrically to $\mathcal{H}(A^*)$. Since we can replace $A^*(z)$ by $A^*(z)S$ without changing the associated space, we can always choose $A^*(z)$ so that multiplication by $A^*(z)$ is isometric in $\mathcal{C}(z)$.

Since $\mathcal{H}(B)$ is the closed span of elements $[1-B(z)\bar{B}(w)]c/(1-z\bar{w})$ and since the transformation $f(z) \to \tilde{f}(z)$ takes $[1-B(z)\bar{B}(w)]c/(1-z\bar{w})$ into $[B^*(z)-\bar{B}(w)]c/(z-\bar{w})$, the space $\mathcal{H}(A^*)$ is the closed span of elements

$$[B^*(z)-\bar{B}(w)]c/(z-\bar{w}) = [A^*(z)-\bar{A}(w)]\bar{C}c/(z-\bar{w}).$$

Since $[A^*(z)-\bar{A}(w)]c/(z-\bar{w})$ belongs to $\mathcal{H}(A^*)$ for every vector c when $|w| < 1$, $\mathcal{H}(A^*)$ is the closed span of elements of this form. Let $f(z) \to \hat{f}(z)$ be the transformation of $\mathcal{H}(A)$ into $\mathcal{H}(A^*)$ defined by

$$\bar{c}\hat{f}(w) = \langle f(z), [A(z)-A(\bar{w})]c/(z-\bar{w})\rangle_A$$

for every vector c when $|w| < 1$. Since the transformation is an isometry by Lemma 9, its range is a closed subspace of $\mathcal{H}(A^*)$, and so it is the full space. It follows that the adjoint transformation, which takes $\mathcal{H}(A^*)$ into $\mathcal{H}(A)$, is an isometry. By Theorem 13, there is no nonzero vector c such that $A(z)c$ belongs to $\mathcal{H}(A)$.

If $f(z)$ is in $\mathcal{H}(A)$, there is a unique element $g(z)$ of $\mathcal{H}(B)$ such that $\hat{f}(z) = \tilde{g}(z)$, and $\|f(z)\|_A = \|g(z)\|_B$. By Theorem 5,

$$\bar{c}g(w) = \langle \tilde{g}(z), [B^*(z) - \bar{B}(w)]c/(z-\bar{w})\rangle_{B^*}$$

$$= \langle \hat{f}(z), [A^*(z) - \bar{A}(w)]\bar{C}c/(z-\bar{w})\rangle_{A^*}$$

$$= \bar{c}Cf(w) \;.$$

By the arbitrariness of c and w, $g(z) = Cf(z)$. We have shown that the transformation $f(z) \to Cf(z)$ is an isometry of $\mathcal{H}(A)$ into $\mathcal{H}(B)$. Since $\mathcal{H}(C)$ contains no nonzero element, it follows from Theorem 4 that the transformation takes $\mathcal{H}(A)$ onto $\mathcal{H}(B)$.

Proof of Theorem 15. We give an explicit proof only in the case $F(0)$ has a dense range. The general case in which $F(w)$ has a dense range, $|w| < 1$, follows by a linear fractional substitution: the transformation

$$f(z) \to g(z) = \sqrt{(1-w\bar{w})}\, f((w-z)/(1-z\bar{w}))/(1-z\bar{w})$$

takes $\mathcal{C}(z)$ isometrically onto itself in such a way that the value of $g(z)$ at 0 is determined by the value of $f(z)$ at w.

If $h(z)$ is in $\mathcal{C}(z)$, the formal product $F(z)h(z)$ belongs to $\mathcal{C}(z)$ and $F(z)h(z)$ has a minimal decomposition, $F(z)h(z) = f(z) + B(z)g(z)$, with $f(z)$ in $\mathcal{H}(B)$ and $g(z)$ in $\mathcal{C}(z)$. Since we assume that the range of multiplication by $B(z)$ in $\mathcal{C}(z)$ contains the range of multiplication by $F(z)$ in $\mathcal{C}(z)$, $f(z)$ belongs to the range of multiplication by $B(z)$ in $\mathcal{C}(z)$. Since $f(z)$ belongs to $\mathcal{H}(B)$ and since multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, $f(z) = 0$. Let T be the transformation of $\mathcal{C}(z)$ into itself defined by $T\colon h(z) \to g(z)$ where $F(z)h(z) = 0 + B(z)g(z)$ is the minimal decomposition of $F(z)h(z)$ in $\mathcal{C}(z)$ with 0 in $\mathcal{H}(B)$ and $g(z)$ in $\mathcal{C}(z)$. Since multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$, the minimal decomposition is obtained with a series $g(z)$ such that $\|B(z)g(z)\| = \|g(z)\|$. Since multiplication by $F(z)$ is a bounded transformation in $\mathcal{C}(z)$, T is a bounded transformation. On the other hand, if $T\colon h(z) \to g(z)$, then $T\colon zh(z) \to zg(z)$ since $F(z)zh(z) = 0 + B(z)zg(z)$ where

$$\|B(z)zg(z)\| = \|B(z)g(z)\| = \|g(z)\| = \|zg(z)\| \;.$$

By the proof of Lemma 1, T is formal multiplication by a power series $G(z)$ with operator coefficients. The definition of T now implies that $F(z) = B(z)G(z)$.

If c is a vector, consider the minimal decomposition $c = f(z) + B(z)g(z)$ of c as an element of $\mathcal{C}(z)$ with $f(z)$ in $\mathcal{H}(B)$ and $g(z)$ in $\mathcal{C}(z)$. Since difference quotients are well-behaved under minimal decompositions,

$$0 = [f(z)-f(0)]/z+[B(z)-B(0)]g(0)/z+ B(z)[g(z)-g(0)]/z$$

is a minimal decomposition of 0 as an element of $\mathcal{C}(z)$ with $[f(z)-f(0)]/z+[B(z)-B(0)]g(0)/z$ in $\mathcal{H}(B)$ and $[g(z)-g(0)]/z$ in $\mathcal{C}(z)$. It follows that $[g(z)-g(0)]/z = 0$ and that $g(z) = g(0)$ is a constant. Since $|g(0)| = \|g(z)\| \leq \|c\| = |c|$, there exists an operator U such that $g(0) = Uc$ for every vector c. If c is in $\mathcal{C}$, $c = c - B(z)Uc+B(z)Uc$ is a minimal decomposition of c as an element of $\mathcal{C}(z)$ with $c-B(z)Uc$ in $\mathcal{H}(B)$ and Uc in $\mathcal{C}(z)$. Since $|Uc| = \|Uc\| \leq \|c\| = |c|$ for every vector c, the operator U is bounded by 1.

If c is in $\mathcal{C}$, $F(z)c = 0+B(z)G(z)c$ is a minimal decomposition of $F(z)c$ as an element of $\mathcal{C}(z)$ with 0 in $\mathcal{H}(B)$ and $G(z)c$ in $\mathcal{C}(z)$. Since

$$F(0)c = F(0)c - B(z)UF(0)c + B(z)UF(0)c$$

is a minimal decomposition of $F(0)c$ in $\mathcal{C}(z)$ with $F(0)c-B(z)UF(0)c$ in $\mathcal{H}(B)$ and $UF(0)c$ in $\mathcal{C}(z)$, it follows that

$$[F(z)-F(0)]c = -F(0)c+B(z)UF(0)c+B(z)[G(z)-UF(0)]c$$

is a minimal decomposition of $[F(z)-F(0)]c$ in $\mathcal{C}(z)$ with $-[1-B(z)U]F(0)c$ in $\mathcal{H}(B)$ and $[G(z)-UF(0)]c$ in $\mathcal{C}(z)$.

The n-th partial sum $F_n(z)$ of $F(z)$ is a polynomial of degree n with operator coefficients which converges formally to $F(z)$ in the operator norm for coefficients. Since we assume that $F(z)$ converges in a disk of radius $a > 1$, the bound of multiplication by $F(z)-F_n(z)$ in $\mathcal{C}(z)$ goes to zero as $n \to \infty$. Since we assume that $1-F(z)$ has completely continuous coefficients, $F(z)-F(0)$ has completely continuous coefficients and $F_n(z)-F_n(0)$ has completely continuous coefficients for every n. Since $F_n(z)-F_n(0)$ is a polynomial of degree n, multiplication by $F_n(z)-F_n(0)$ is a completely continuous transformation of $\mathcal{C}$ into $\mathcal{C}(z)$ for every n. Since the bound of multiplication by $[F(z)-F(0)]-[F_n(z)-F_n(0)]$ in $\mathcal{C}(z)$ goes to zero as $n \to \infty$, multiplication by $F(z)-F(0)$ is a completely continuous transformation of $\mathcal{C}$ into $\mathcal{C}(z)$. But multiplication by $G(z)-UF(0)$ as a transformation of $\mathcal{C}$ into $\mathcal{C}(z)$ is the composition of multiplication by $F(z)-F(0)$, as a transformation of $\mathcal{C}$ into $\mathcal{C}(z)$, with a continuous transformation of $\mathcal{C}(z)$ into $\mathcal{C}(z)$: the transformation $h(z) \to g(z)$ such that

$$h(z) = [h(z) - B(z)g(z)] + B(z)g(z)$$

is the minimal decomposition of $h(z)$ in $\mathcal{C}(z)$ with $h(z)-B(z)g(z)$ in $\mathcal{H}(B)$ and $g(z)$ in $\mathcal{C}(z)$. Since the composition of a continuous transformation and a completely continuous transformation is completely continuous, multiplication by $G(z)-UF(0)$ is a completely continuous transformation of $\mathcal{C}$ into $\mathcal{C}(z)$. It follows that $G(z)-UF(0)$ has completely continuous coefficients. Therefore $B(z)[G(z)-UF(0)]$ has completely continuous coefficients. Since $F(z)-F(0)$ has completely continuous coefficients, $[1-B(z)U]F(0)$ has completely continuous coefficients. Since we assume that $1-F(0)$ is completely continuous and that $F(0)$ has a dense range in $\mathcal{C}$, $F(0)$ has an operator inverse. So we can conclude that $1-B(z)U$ is bounded by 1, $B(0)U\bar{U}\bar{B}(0) \leq B(0)\bar{B}(0) \leq 1$ and

$$0 \leq 1-B(0)\bar{B}(0) \leq 1-B(0)U\bar{U}\bar{B}(0)$$

where $1-B(0)U$, and hence $1-B(0)U\bar{U}\bar{B}(0)$, is completely continuous. It follows that $1-B(0)\bar{B}(0)$ is completely continuous.

Proof of Theorem 16. By hypothesis there is a number w, $|w| < 1$, such that $F(w)$ has a dense range in $\mathcal{C}$. By the proof of Lemma 1, $F(z) = B(z)G(z)$ for some power series $G(z)$ with operator coefficients such that multiplication by $G(z)$ is a bounded transformation in $\mathcal{C}(z)$. Since $F(w) = B(w)G(w)$, the operator $B(w)$ has a dense range in $\mathcal{C}$ and the adjoint operator $\bar{B}(w)$ has zero kernel. If P is the nonnegative operator such that $P^2 = B(w)\bar{B}(w)$, then P has zero kernel and dense range. But if $c = Pa$ is in the range of P, $|\bar{B}(w)a| = |Pa|$. So there exists an isometric operator U such that $\bar{B}(w) = UP$. Since the range of U is orthogonal to the kernel of $B(w)$, it is contained in the orthogonal complement $\mathfrak{M}$ of those vectors c such that $B(z)c = 0$. The dimension of $\mathfrak{M}$ is therefore equal to the dimension of $\mathcal{C}$. If S is the choice of an isometric operator whose range is $\mathfrak{M}$, the range of $1-S\bar{S}$ is contained in the kernel of multiplication by $B(z)$ and the theorem follows.

Proof of Theorem 17. If $\mathcal{H}(B)$ contains no nonzero element, then $B(z)$ is a constant because $[B(z)-B(0)]c/z$ belongs to $\mathcal{H}(B)$ for every vector c. Since $[1-B(z)\bar{B}(w)]c/(1-z\bar{w})$ belongs to $\mathcal{H}(B)$ for every vector c when $|w| < 1$, $1-B(0)\bar{B}(0)$ must vanish identically and $\bar{B}(0)$ is an isometric operator. Since multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$, $B(0)$ is an isometric, and hence a unitary, operator. The theorem follows in this case.

If $\mathcal{H}(B)$ contains a nonzero element, it contains a nonzero polynomial since the polynomials are assumed dense in $\mathcal{H}(B)$.

Since $[f(z)-f(0)]/z$ belongs to $\mathcal{H}(B)$ whenever $f(z)$ belongs to $\mathcal{H}(B)$, $\mathcal{H}(B)$ contains a nonzero constant c. Since c is orthogonal to the range of multiplication by $B(z)$ in $\mathcal{C}(z)$, it is orthogonal to the range of the operator $B(0)$. Thus the range of $B(0)$ is not dense in $\mathcal{C}$ if $\mathcal{H}(B)$ contains a nonzero element.

Let $R(0)$ be the transformation $f(z) \to [f(z)-f(0)]/z$ in $\mathcal{H}(B)$ and let $R(0)^*$ be its adjoint in $\mathcal{H}(B)$. We show that there is no nonzero element in the kernel of $R(0)^*-\bar{w}$ if $0 < |w| < 1$. For if $f(z)$ is in the kernel of $R(0)^*-\bar{w}$, then

$$<[g(z)-g(0)]/z, f(z)> = w<g(z), f(z)>$$

for every element $g(z)$ of $\mathcal{H}(B)$. Since $w \neq 0$, we find that $f(z)$ is orthogonal to $g(z)$ whenever $g(z)$ is a constant in $\mathcal{H}(B)$. It follows inductively from the same formula that $f(z)$ is orthogonal to the polynomials of degree n for every $n = 1, 2, 3, \ldots$. Since we assume that the polynomials are dense in $\mathcal{H}(B)$, $f(z) = 0$.

We show that $J(0): f(z) \to f(0)$ is a completely continuous transformation of $\mathcal{H}(B)$ into $\mathcal{C}$. Since $[1-B(z)\bar{B}(0)]c$ belongs to $\mathcal{H}(B)$ for every vector c, and since

$$\bar{c}f(0) = <f(z), [1-B(z)\bar{B}(0)]c>$$

for every $f(z)$ in $\mathcal{H}(B)$, the adjoint $J(0)^*$ of $J(0)$ is the transformation of $\mathcal{C}$ into $\mathcal{H}(B)$ which takes c into $[1-B(z)\bar{B}(0)]c$ for every vector c. Since we assume that $J(0)J(0)^*: c \to [1-B(0)\bar{B}(0)]c$ is a completely continuous transformation of $\mathcal{C}$ into $\mathcal{C}$, $J(0)^*$ is a completely continuous transformation of $\mathcal{C}$ into $\mathcal{H}(B)$ and $J(0)$ is a completely continuous transformation of $\mathcal{H}(B)$ into $\mathcal{C}$.

The identity

$$<[f(z)-f(0)]/z, [g(z)-g(0)]/z> = <f(z), g(z)> - \bar{g}(0)f(0),$$

which holds for all $f(z)$ and $g(z)$ in $\mathcal{H}(B)$, can be written schematically

$$J(0)^*J(0) = 1-R(0)^*R(0).$$

Since $J(0)$ and $J(0)^*$ are completely continuous, $1-R(0)^*R(0)$ is a completely continuous transformation in $\mathcal{H}(B)$. Since $R(0)$ is bounded by 1, $1-wR(0)$ has an everywhere defined and bounded inverse in $\mathcal{H}(B)$ when $|w| < 1$. Since

$$[1-wR(0)^*][1-\bar{w}R(0)] - [R(0)^*-\bar{w}][R(0)-w] = (1-w\bar{w})[1-R(0)^*R(0)]$$

is completely continuous,

$$1-[1-wR(0)^*]^{-1}[R(0)^*-\bar{w}][R(0)-w][1-\bar{w}R(0)]^{-1}$$

is a completely continuous transformation when $|w| < 1$. The transformation

$$[1-wR(0)^*]^{-1}[R(0)^*-\bar{w}][R(0)-w][1-\bar{w}R(0)]^{-1}$$

therefore has a finite dimensional kernel and a closed range when $|w| < 1$. Equivalently the transformation $R(0)-w$ has a finite dimensional kernel and a closed range when $|w| < 1$. Since $R(0)^*-\bar{w}$ has no nonzero element in its kernel when $w \neq 0$, the range of $R(0)-w$ is dense in $\mathcal{H}(B)$, and hence all of $\mathcal{H}(B)$, when $0 < |w| < 1$. It follows that $R(0)^*-w$ has a bounded, partially defined inverse in $\mathcal{H}(B)$ when $0 < |w| < 1$. The bound $\|(R(0)^*-w)^{-1}\|$ of the inverse is a continuous function of w in the punctured disk. Since we assume that $B(w)$ has a dense range in $\mathcal{H}(B)$ for some w, $|w| < 1$, and since $w \neq 0$ when $\mathcal{H}(B)$ contains a nonzero element, $\bar{B}(w)$ has kernel zero for a nonzero value of w. For this value of w there is no nonzero vector c such that $c/(1-z\bar{w})$ belongs to $\mathcal{H}(B)$. It follows that $R(0)-\bar{w}$ has no nonzero element in its kernel and that the transformation has an everywhere defined and bounded inverse in $\mathcal{H}(B)$. We have shown that the set $\mathcal{S}$ of points w, $0 < |w| < 1$, such that $R(0)-w$ has an everywhere defined and bounded inverse is not empty. Since the set is both open and closed in the punctured disk, it contains all points w, $0 < |w| < 1$.

If $f(z)$ is a polynomial which belongs to $\mathcal{H}(B)$ and if $|w| < 1$, then

$$J(0)[1-wR(0)]^{-1}: f(z) \to f(w)$$

and
$$|f(w)| \le \|f(z)\| \, \|J(0)\| \, \|[1-wR(0)]^{-1}\|$$

where $\|[1-wR(0)]^{-1}\|$ is a continuous function of w in the region $|w| > 1$. Since we assume that the polynomials are dense in $\mathcal{H}(B)$, we can conclude that all elements of $\mathcal{H}(B)$ are series which converge in the complex plane, and that $f(z) \to f(w)$ is a continuous transformation of $\mathcal{H}(B)$ into $\mathbb{C}$ for every complex number w. It follows that there exists a power series $K(w,z)$ with operator coefficients such that $K(w,z)c$ belongs to $\mathcal{H}(B)$ for every vector c and

$$\bar{c}f(w) = \langle f(z), K(w,z)c\rangle$$

for every $f(z)$ in $\mathcal{H}(B)$. Since $[B(z)-B(0)]c/z$ belongs to $\mathcal{H}(B)$ for every vector c and since $\|[B(z)-B(0)]c/z\| \le |c|$, $B(z)$

converges in the complex plane. Since $K(w,z) = [1-B(z)\bar{B}(w)]c/(1-z\bar{w})$ when $|w| < 1$, the same formula holds by analytic continuation for all values of w.

The space of constants which belong to $\mathcal{H}(B)$ is a space $\mathcal{H}(A_0)$ in the metric of $\mathcal{C}(z)$, where $A_0(z) = 1-P_0+P_0z$ for some projection operator P_0. Since $\mathcal{H}(A_0)$ is contained in $\mathcal{H}(B)$, $B(z) = A_0(z)C_0(z)$ for some space $\mathcal{H}(C_0)$, which satisfies the hypotheses of the theorem. Since $1-A_0(0)\bar{A}_0(0) \leq 1-B(0)\bar{B}(0)$ is completely continuous, P_0 has finite dimensional range. Continuing inductively we obtain a sequence (P_n) of projections of finite dimensional range such that $B(z) = A_n(z)C_n(z)$ for every n, where

$$A_n(z) = (1-P_0+P_0z)\ldots(1-P_n+P_nz)$$

and $\mathcal{H}(C_n)$ is a space which satisfies the hypotheses of the theorem. The space $\mathcal{H}(A_n)$ is contained isometrically in $\mathcal{H}(B)$ and is the space of polynomials of degree at most n which belong to $\mathcal{H}(B)$. Since the polynomials which belong to $\mathcal{H}(B)$ are dense in $\mathcal{H}(B)$ by hypothesis, the union of the spaces $\mathcal{H}(A_n)$ is dense in $\mathcal{H}(B)$. It follows that

$$[1-B(z)\bar{B}(w)]c/(1-z\bar{w}) = \lim_{n\to\infty}[1-A_n(z)\bar{A}_n(w)]c/(1-z\bar{w})$$

in the metric of $\mathcal{C}(z)$ for every vector c and complex number w. Since

$$[1-A_n(w)\bar{A}_n(w)]/(1-w\bar{w}) \leq [1-B(w)\bar{B}(w)]/(1-w\bar{w})$$

where $[1-B(w)\bar{B}(w)]/(1-w\bar{w})$ is a completely continuous operator,

$$[1-B(w)\bar{B}(w)]/(1-w\bar{w}) = \lim_{n\to\infty}[1-A_n(w)\bar{A}_n(w)]/(1-w\bar{w})$$

in the operator norm for every complex number w. We use this fact when $w = 1$. For large values of n, the operator norm of the difference is less than 1. When n is this large, the operator norm of $[1-C_n(w)\bar{C}_n(w)]/(1-w\bar{w})$ is less than 1. This implies that $\mathcal{H}(C_n)$ contains no nonzero element. By the first part of the proof, $C_n(z) = C_n(0)$ is then a unitary operator and the theorem follows.

Proof of Theorem 18. Since the hypotheses imply that $F(z)$ represents a bounded function in the unit disk, multiplication by $F(z)$ is a bounded transformation in $\mathcal{C}(z)$. Let $\mathcal{H}$ be the closure in $\mathcal{C}(z)$ of the polynomials which are orthogonal to the range of multiplication by $F(z)$ in $\mathcal{C}(z)$. Since the set of such polynomials is invariant under the transformation $f(z)\to[f(z)-f(0)]/z$, $[f(z)-f(0)]/z$

belongs to $\mathcal{H}$ whenever $f(z)$ belongs to $\mathcal{H}$. The space $\mathcal{H}$ is therefore a space $\mathcal{H}(B)$ in the metric of $\mathcal{C}(z)$, and multiplication by $B(z)$ is a partially isometric transformation in $\mathcal{C}(z)$. By Theorem 16 our hypotheses allow us to choose $B(z)$ so that multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$. Since $\mathcal{H}(B)$ is the orthogonal complement of the range of multiplication by $B(z)$ in $\mathcal{C}(z)$, the range of multiplication by $B(z)$ in $\mathcal{C}(z)$ contains the range of multiplication by $F(z)$ in $\mathcal{C}(z)$. By the proof of Theorem 15, $F(z) = B(z)G(z)$ for some power series $G(z)$ with operator coefficients. Since we assume that $F(w)$ has a dense range in $\mathcal{C}$ for some number w, $|w| < 1$, $B(w)$ has a dense range in $\mathcal{C}$ for some such w. By Theorem 15, the hypotheses imply that $1-B(0)\bar{B}(0)$ is completely continuous. By Theorem 17 there exist projections $P_0, \ldots, P_r$ of finite dimensional range such that

$$B(z) = (1-P_0 + P_0 z) \ldots (1 - P_r + P_r z)U$$

for some unitary operator U. Since we can multiply $B(z)$ on the right by a unitary operator without changing the associated space, we can suppose $B(z)$ chosen so that $U = 1$. Then $B(z)$ is a polynomial with operator coefficients, $1-B(z)$ has completely continuous values, and $B(w)$ has an inverse when $w \neq 0$. Since we assume that $1-F(z)$ has completely continuous values, $1-G(z)$ has completely continuous values.

We show that the operator $G(0)$ has a dense range in $\mathcal{C}$. If c is a vector orthogonal to the range of the operator $G(0)$, then c is orthogonal to the range of multiplication by $G(z)$ in $\mathcal{C}(z)$. Since multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$, $B(z)c$ is orthogonal to the range of multiplication by $F(z) = B(z)G(z)$ in $\mathcal{C}(z)$. Since $B(z)c$ is a polynomial, it belongs to $\mathcal{H}(B)$ by construction. Since multiplication by $B(z)$ is isometric in $\mathcal{C}(z)$, $c = 0$. Density of the range of the operator $G(0)$ follows. Since $1-G(0)$ is completely continuous, $G(0)$ has an operator inverse and the theorem follows.

<u>Proof of Theorem 19.</u> Since the derivative of the entire function

$$1 - (1-z)\exp(z + \tfrac{1}{2}z^2 + \ldots + \tfrac{1}{n}z^n)$$

is

$$z^n \exp(z + \tfrac{1}{2}z^2 + \ldots + \tfrac{1}{n}z^n),$$

the coefficients of its power series expansion about the origin are nonnegative numbers. Since

$$1 - (1-x)\exp(x + \tfrac{1}{2}x^2 + \ldots + \tfrac{1}{n}x^n) \leq 1 \text{ when } x > 0,$$

$$0 \leq (1-x)\exp(x+\tfrac{1}{2}x^2+\ldots+\tfrac{1}{n}x^n) \leq 1$$

when $0 \leq x \leq 1$. On the other hand, for these values of x,

$$-\log[(1-x)\exp(x+\tfrac{1}{2}x^2+\ldots+\tfrac{1}{n}x^n)]$$

$$=\frac{1}{n+1}x^{n+1}+\frac{1}{n+2}x^{n+2}+\ldots \leq x^{n+1}/(1-x)$$

and

$$1-(1-x)\exp(x+\tfrac{1}{2}x^2+\ldots+\tfrac{1}{n}x^n) \leq 1-\exp[-x^{n+1}/(1-x)] \leq \exp[x^{n+1}/(1-x)]-1.$$

Apply this inequality with x replaced by $|w/w_n|$ when $|w| < |w_n|$. The result is

$$|1-(1-Q_n w/w_n)\exp(Q_n w/w_n+\tfrac{1}{2}Q_n w^2/w_n{}^2+\ldots+\tfrac{1}{n}Q_n w^n/w_n{}^n)|$$

$$\leq 1-(1-|w/w_n|)\exp(|w/w_n|+\tfrac{1}{2}|w/w_n|^2+\ldots+\tfrac{1}{n}|w/w_n|^n)$$

$$\leq \exp[|w/w_n|^{n+1}/(1-|w/w_n|)]-1.$$

If we write

$$P_n(z)=(1-Q_1 z/w_1)\exp(Q_1 z/w_1)\ldots(1-Q_n z/w_n)\exp(Q_n z/w_n+\ldots+\tfrac{1}{n}Q_n z^n/w_n^n)$$

and make use of the norm inequality

$$1+|AB-1| \leq [1+|A-1|][1+|B-1|],$$

we obtain

$$|P_r(w)-P_s(w)| \leq |P_n(w)|\exp\left[\sum_{k=n+1}^{s}|w/w_k|^{k+1}/(1-|w/w_k|)\right]$$

$$-|P_n(w)|\exp\left[\sum_{k=n+1}^{r}|w/w_k|^{k+1}/(1-|w/w_k|)\right]$$

when $n \leq r \leq s$ with n so large (for any given w) that $|w/w_k|<1$ when $k \geq n$. Since $\lim |w/w_k| = 0$ as $n \to \infty$ uniformly for w in any bounded set, $P(w) = \lim P_n(w)$ as $n \to \infty$ in the operator norm uniformly for w in any bounded set. Since a uniform limit of analytic functions is analytic, $P(z)$ is defined and analytic in the complex plane. Since $1-P_n(z)$ has completely continuous values for every

n, and since a norm limit of completely continuous operators is completely continuous, $1-P(z)$ has completely continuous values. It is easily verified that $P_n(z)$ has invertible values except at the points $w_1, \ldots, w_n$. If $w \neq w_k$ for every k, $|P_n(w)^{-1}P(w) - 1| < 1$ for sufficiently large values of n (depending on w). This implies that $P_n(w)^{-1}P(w)$ and hence $P(w)$ is an invertible operator.

Proof of Theorem 20. Since an operator valued entire function is continuous and since the invertible operators are an open subset of the operator algebra, the set of complex numbers w such that $F(w)$ is invertible is open. The set contains the origin by hypothesis. If the set is the whole plane, the theorem follows with $Q_n = 0$ for every n and $G(z) = F(z)$. Otherwise let w_1 be the choice of a point nearest the origin such that $F(w_1)$ fails to have an operator inverse. Since $1-F(w_1)$ is completely continuous, the range of $F(w_1)$ is not dense in $\mathcal{C}$. Let Q_1 be the projection operator whose range is the orthogonal complement of the range of $F(w_1)$. Then

$$(1-Q_1z/w_1)^{-1} = (z-w_1)^{-1}(z-w_1-Q_1z)$$

is an operator valued function which is defined and analytic in the complex plane except at w_1 and

$$(1-Q_1z/w_1)^{-1}F(z) = F(w_1) + (z-w_1-Q_1z)[F(z)-F(w_1)]/(z-w_1)$$

is an operator valued entire function of z. So $F(z) = P_1(z)F_2(z)$ where $F_2(z)$ is an operator valued entire function of z. If $F_2(z)$ has invertible values at all points in the complex plane, let $P(z) = P_1(z)$ and $G(z) = F_2(z)$. Otherwise let w_2 be a number nearest the origin such that $F_2(w_2)$ fails to have an operator inverse, and continue inductively in the obvious way. At the n-th stage $F_n(z)$ is an operator valued entire function, $1-F_n(z)$ has completely continuous values, and $F_n(0)$ is an invertible operator. Let w_n be a number nearest the origin such that $F_n(w_n)$ fails to have an operator inverse. Let Q_n be the projection operator whose range is the orthogonal complement of the range of $F_n(w_n)$, and let $F_{n+1}(z)$ be the operator valued entire function such that $F_n(z) = P_n(z)F_{n+1}(z)$. If $F_{n+1}(z)$ has invertible values in the complex plane, let $P(z) = P_n(z)$ and $F_{n+1}(z) = G(z)$. Otherwise go on to $F_{n+1}(z)$.

In the worst case $F_n(z)$ is defined for every n. It follows from Theorem 18 that Q_n has finite dimensional range for every n and that $\lim |w_n| = \infty$ as $n \to \infty$. By Theorem 19, $P(z) = \lim P_n(z)$ converges in the operator norm, uniformly on bounded sets. The limit is an operator valued entire function such that $1-P(z)$ has completely continuous values and $P(z)$ has invertible values except at isolated points. It follows that $G(z) = \lim F_{n+1}(z)$ exists as $n \to \infty$ in the operator norm, uniformly on bounded sets. So $G(z)$ is an operator valued entire function and $1-G(z)$ has completely continuous values. Since $F(z) = P_n(z)F_{n+1}(z)$ for every n, $F(z) = P(z)G(z)$. By construction $F_n(z)$ has invertible values in the disk of radius $|w_n|$. Since $F_n(z) = P_{n+1}(z)^{-1}P(z)G(z)$ where $P_{n+1}(z)^{-1}P(z)$ has invertible values in the disk, $G(z)$ has invertible values in the disk. Since $\lim |w_n| = \infty$ as $n \to \infty$, $G(z)$ has invertible values in the complex plane.

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