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Weakly Commensurable Groups, with Applications to Differential Geometry

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Abstract

This article is an expanded version of the talk delivered by the second-named author at the GAAGTA conference. We have included a discussion of very recent results and conjectures on absolutely almost simple algebraic groups having the same maximal tori and finite-dimensional division algebras having the same maximal subfields; in particular Theorem 5.1 contains a finiteness result in this direction that appears in print for the first time.

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Contents

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1 Introduction

1.1. Let M be a Riemannian manifold. In differential geometry one considers the following sets of data associated with M:

- $\mathcal{E}(M)$, the *spectrum of the Laplace-Beltrami operator* (for the purposes of this article, the spectrum is the collection of eigenvalues with their multiplicities);
- $\mathcal{L}(M)$, the *length spectrum*, i.e. the collection of lengths of all closed geodesics in M *with* multiplicities;
- $L(M)$, the *weak length spectrum*, i.e. the collection of lengths of all closed geodesics *without* multiplicities.

(Of course, in order to ensure that the multiplicities in the definition of $\mathcal{E}(M)$ and $\mathcal{L}(M)$ are finite, one needs to impose some additional conditions on M; however we will not discuss these technicalities here particularly because in the case of compact locally symmetric spaces, which are the most important classes of manifolds to be considered in this article, problems of this nature do not arise.) Two Riemannian manifolds M_1 and M_2 are called *commensurable* if they admit a common finite-sheeted cover M, i.e. if there is a diagram

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

in which M is a Riemannian manifold and π_1, π_2 are finite-sheeted locally isometric covering maps.

We can now formulate the following question that has attracted the attention of mathematicians working in different areas of analysis and geometry for quite some time:

Let M_1 and M_2 be Riemannian manifolds. Are M_1 and M_2 necessarily isometric/commensurable if:

(1) $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, *i.e. M_1 and M_2 are* isospectral*;*

(2) $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ *(or $L(M_1) = L(M_2)$), i.e. M_1 and M_2 are* iso-length spectral*;*

(3) $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$, *i.e. M_1 and M_2 are* length-commensurable.

Among conditions (1)–(3), the condition of isospectrality is definitely the most famous one. In fact, the question of whether (or when) isospectrality implies isometricity is best known in its informal formulation due to Mark Kac [16] (1966), which is "*Can you hear the shape of a drum*?" For historical accuracy, we should point out that the question itself was analyzed in various forms long before [16], and this analysis had provided substantial evidence in favor of the affirmative answer. In particular, H. Weyl in 1911 proved a result, which was subsequently sharpened and generalized by various authors and is currently known as *Weyl's Law.* It states that if M is an n-dimensional compact Riemannian manifold, and $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ is the sequence of the eigenvalues of the Laplacian of M, then

$$N(\lambda) = \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\lambda^n + o(\lambda^n),$$

where

$$N(\lambda) = \#\{j \mid \sqrt{\lambda_j} \leqslant \lambda\}$$

and $\Gamma(s)$ is the Γ-function (see [3], [25] for a discussion of Weyl's Law, its history and applications in mathematics and physics). This implies that the distribution of the eigenvalues alone allows one to recover such invariants of M as its dimension and volume, and therefore these invariants are shared by all isospectral Riemannian manifolds. Moreover, isospectral compact Riemannian manifolds share the heat kernel invariants (see [39] and references therein). These powerful analytic techniques and results led one to believe that isospectral manifolds should indeed be isometric. In 1964, however, Milnor [24] gave an example of two isospectral, but not isometric, flat 16-dimensional tori. Then in 1980, M.-F. Vignéras [43] used the arithmetic of quaternion algebras to construct nonisometric isospectral Riemann surfaces. A few years later, Sunada [40] found a different method for constructing isospectral and iso-length spectral, but not isometric, Riemannian manifolds. Sunada's method relied on rather simple group-theoretic properties of the fundamental group, which made it applicable in many situations. In fact, since its discovery, this method has been implemented in a variety of situations to produce examples of nonisometric manifolds for which various geometric invariants are equal (cf., for example, [20]). Notice, however, that the constructions of Vignéras and Sunada *always* produce commensurable manifolds. So, it appears that the "right question" regarding isospectral manifolds should be *whether two isospectral (compact Riemannian) manifolds are necessarily commensurable.* While the answer to this question is still negative in the general case (cf. Lubotzky et al. [21]), our results [30], [31], [32] show that the answer is in the affirmative for many compact arithmetically defined locally symmetric spaces (cf. Theorem 6.8). Before our work, such results were available only for arithmetically defined hyperbolic 2- and 3-manifolds, cf. [9] and [36].

Next, we turn to the isometricity question formulated in terms of condition (2) of iso-length spectrality. It is worth pointing out that this question is dictated even by (naïve) geometric intuition. Indeed, if we take M_i to be the 2-dimensional Euclidean sphere of radius r_i for $i = 1, 2$, then $L(M_i) = \{2\pi r_i\}$. So, in this case the condition of iso-length spectrality $L(M_1) = L(M_2)$ does imply the isometricity of M_1 and M_2, and it is natural to ask if this sort of conclusion can be drawn in a more general situation. Superficially, this question does not seem to be connected with isospectrality, but in fact using the trace formula one proves that if M_1 and

M_2 are compact locally symmetric spaces of nonpositive curvature then

$$\mathcal{E}(M_1) = \mathcal{E}(M_2) \quad \Rightarrow \quad L(M_1) = L(M_2)$$

(cf. [30, Theorem 10.1]). It follows that nonisometric isospectral locally symmetric spaces (in particular, those constructed by Vignéras, Sunada and Lubotzky et al.) *automatically* provide examples of nonisometric iso-length spectral manifolds, and again one should ask about commensurability rather than isometricity of iso-length spectral manifolds.

While there are important open questions about commensurability expressed in terms of the classical conditions of isospectrality and iso-length spectrality (the most famous one being whether two isospectral Riemann surfaces are necessarily commensurable), it seems natural to suggest that a systematic study of commensurability should involve (or even be based upon) conditions that are invariant under passing to a commensurable manifold. From this perspective, one needs to point out that conditions (1) and (2), of isospectrality and iso-length spectrality, respectively, do not possess this property, while condition (3) of length-commensurability does. Note that (3) is formulated in terms of the set $\mathbb{Q} \cdot L(M)$ which is sometimes called the *rational length spectrum*. While this set is not as closely related to the geometry of M as $L(M)$, it nevertheless has several very convenient features. First, it indeed does not change if M is replaced by a commensurable manifold. Second, unlike $L(M)$, which has been completely identified in very few situations, $\mathbb{Q} \cdot L(M)$ can be described in more case. Here is one example.

1.2. Example. Let $\mathbb{H} = \{x+iy \in \mathbb{C} | y > 0\}$ be the upper half-plane with the standard hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. It is well-known that the standard isometric action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by fractional linear transformations allows us to identify $\mathbb{H}$ with the symmetric space $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$. Let $\pi \colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical projection. Given a discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ containing $\{\pm 1\}$ with torsion-free image $\pi(\Gamma)$, the quotient $M = \mathbb{H}/\Gamma$ is a Riemann surface. It is well-known that closed geodesics in M correspond to nontrivial semi-simple elements in Γ (cf. [33, 2.1]); the precise nature of this correspondence is not important for us as we only need information about the length. One shows that if c_γ is the closed geodesic in M corresponding to a semi-simple element $\gamma \in \Gamma$, $\gamma \neq \pm 1$, then its length is given by

$$\ell(c_\gamma) = \frac{2}{n_\gamma} \cdot |\log |t_\gamma|| \tag{1}$$

where t_γ is an eigenvalue of γ (note that since $\pi(\Gamma)$ is discrete and torsion-free, any semi-simple $\gamma \in \Gamma$ is automatically hyperbolic, i.e. $t_\gamma \in \mathbb{R}$), and $n_\gamma \geqslant 1$ is an integer which geometrically is the winding number and algebraically is the index $[C_\Gamma(\gamma) : \{\pm 1\} \cdot \langle \gamma \rangle]$, where $C_\Gamma(\gamma)$ is the centralizer of γ in Γ (in other words, $C_\Gamma(\gamma) = T \cap \Gamma$ where T is the maximal $\mathbb{R}$-torus of SL_2 containing γ). So, the length spectrum $L(M)$ consists of the values $\ell(c_\gamma)$ where γ runs over semi-simple elements in $\Gamma \setminus \{\pm 1\}$ with the property that $C_\Gamma(\gamma) = \{\pm 1\} \cdot \langle \gamma \rangle$ (primitive semi-simple elements), while the rational length spectrum $\mathbb{Q} \cdot L(M)$ is the union of the sets $\mathbb{Q} \cdot \log |t_\gamma|$, where γ runs over all semi-simple $\gamma \in \Gamma \setminus \{\pm 1\}$, and in fact it suffices to take just one element out of every class of elements having the same centralizer in Γ, i.e. one element in $(T \cap \Gamma) \setminus \{\pm 1\}$ for every maximal $\mathbb{R}$-torus T of SL_2 such that the latter set is non-empty. Now, let us recall the following example which demonstrates the well-known fact that the problem of identifying *primitive* semi-simple elements is extremely difficult.

Let D be a quaternion division over $\mathbb{Q}$ that splits over $\mathbb{R}$, and let Γ be a torsion-free arithmetic subgroup of $G(\mathbb{Q})$ where $G = \mathrm{SL}_{1,D}$. One can view Γ as a discrete subgroup of $G(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})$. Set $M = \mathbb{H}/\Gamma$. It is well-known that the maximal subfields of D are of the form $K = \mathbb{Q}(\sqrt{d})$ with d satisfying $d \notin {\mathbb{Q}_p^{\times}}^2$ for primes p where D is ramified (thus, the relevant values of d can be easily characterized in terms of congruences). Clearly, the problem of describing primitive semi-simple elements in Γ contains the problem of identifying the fundamental unit $\varepsilon(d)$ in every such subfield with $d > 0$ (or more precisely, the smallest unit with norm 1), which is beyond our reach. On the other hand, there is a well-known formula that gives *some* unit in a real quadratic field $\mathbb{Q}(\sqrt{d})$ (assuming that d is square-free)

$$\eta(d) = \prod_{r=1}^{d-1} \left[\sin\left(\frac{\pi r}{d}\right) \right]^{-\left(\frac{d}{r}\right)}, \tag{2}$$

where $\left(\dfrac{d}{r}\right)$ is the Kronecker symbol (or the character associated with the quadratic extension), cf. [6, Ch. V, §4, Theorem 2]. (Recall that $\varepsilon(d)$ and $\eta(d)$ are related by the equation $\eta(d) = \varepsilon(d)^{2h(d)}$, where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, indicating that a systematic description of $\varepsilon(d)$ for a sufficiently general infinite sequence of d's is nearly impossible.) So, in this case the rational length spectrum can be described as the set of all rational multiples of $\log \eta(d)$, where $\eta(d)$ is given by (2) and d runs through positive square-free integers described by certain congruences. (It would be interesting to see if one can give a similar description of the rational length spectrum for other arithmetically defined locally symmetric spaces — see [30, Proposition 8.5] regarding the formula for the length of a closed geodesic in the general case. Obviously, this will require an intrinsic construction of (sufficiently many) units in a $\mathbb{Q}$-torus, which has not been offered so far.)

This example suggests that at least in some cases the rational length spectrum $\mathbb{Q} \cdot L(M)$ may be more tractable than the length spectrum $L(M)$ or the spectrum $\mathcal{E}(M)$ of the Laplace-Beltrami operator. But then the question arises if the rational length spectrum retains enough information to characterize the commensurability class of M. So, we would like to point out that our entire work that resolved many questions about isospectral and iso-length spectral arithmetically defined locally symmetric spaces of absolutely simple real algebraic groups is based on an analysis of the rational length spectrum and length-commensurability. In fact, with just one exception, length-commensurability and the (much) stronger condition of isospectrality lead to the same new results about isospectral locally symmetric spaces (see Theorem 6.8 and the subsequent discussion). So, we hope that the analysis of the rational length-spectrum and the associated notion of length-commensurability will become a standard tool in the investigation of locally symmetric spaces. Furthermore, the notion of length-commensurability has an algebraic counterpart, which we termed *weak commensurability* (of Zariski-dense subgroup) — see §2. What is interesting is that the study of the geometric problems mentioned earlier has led to a number of algebraic problems of considerable independent interest such as characterization of absolutely almost simple algebraic K-groups in terms of the isomorphism classes of their maximal K-tori, and in particular, characterizing finite-dimensional division K-algebras in terms of the isomorphism classes of their maximal subfields. These questions have been completely resolved for algebraic groups over number fields and their arithmetic subgroups, and we will review these results in §§3–4, but remain an area of active research over general fields, with some important results obtained very recently (cf. §4). Thus, in broad terms, our project can be described as the analysis of the consequences of length-commensurability for locally symmetric spaces and of related algebraic problems involving classification of algebraic groups over general (finitely-generated) fields and the investigation of their Zariski-dense subgroups.

1.3. Hyperbolic manifolds. Cumulatively, our papers [30], [31], [32] and the results of Garibaldi [12] and Garibaldi-Rapinchuk [13] answer the key questions about length-commensurable *arithmetically defined* locally symmetric spaces of absolutely simple real algebraic groups of all types. In particular, we know when length-commensurability implies commensurability (the answer depends on the Killing-Cartan type of the group), and that in all cases the arithmetically defined locally symmetric spaces that are length-commensurable to a given arithmetically defined locally symmetric space form finitely many commensurability classes. We will postpone the technical formulations of these results until §6, and instead showcase the consequences of these results for real hyperbolic manifolds.

Let $\mathbb{H}^d$ be the real hyperbolic d-space. The isometry group of $\mathbb{H}^d$ is $\mathcal{G} = \mathrm{PO}(d,1)$, and by an arithmetic hyperbolic d-manifold we mean the quotient $M = \mathbb{H}^d/\Gamma$ by an *arithmetic* subgroup Γ of $\mathcal{G}$ (see §3 regarding the notion of arithmeticity). Previously, results about iso-length spectral arithmetically defined hyperbolic d-manifolds were available only for $d = 2$ (Reid [36]) and $d = 3$ (Reid et al. [9]). We obtained the following for length-commensurable (hence, isospectral) arithmetic hyperbolic manifolds of any dimension $d \neq 3$.

Theorem 1.4. *Let M_1 and M_2 be arithmetically defined hyperbolic d-manifolds.*

(1) *Suppose d is either even or $\equiv 3 (\mathrm{mod}\ 4)$. If M_1 and M_2 are not commensurable, then after a possible interchange of M_1 and M_2, there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio λ_2/λ_1 is transcendental; in particular, M_1 and M_2 are not length-commensurable. Thus, in this case length-commensurability implies commensurability.*
(2) *For any $d \equiv 1 (\mathrm{mod}\ 4)$ there exist length-commensurable, but not commensurable, arithmetic hyperbolic d-manifolds.*

Furthermore, one can ask about *how different are* $L(M_1)$ *and* $L(M_2)$ (*or* $\mathbb{Q} \cdot L(M_1)$ *and* $\mathbb{Q} \cdot L(M_2)$) *given the fact that M_1 and M_2 are* not *length-commensurable.* For example, can the symmetric difference $L(M_1) \vartriangle L(M_2)$ be finite? Under some minor additional assumptions, we proved in [32] that if M_1 and M_2 are non-length commensurable arithmetically defined hyperbolic d-manifolds

($d \neq 3$), and $\mathscr{F}_i$ is the subfield of $\mathbb{R}$ generated by $L(M_i)$ ($i = 1, 2$), then the compositum $\mathscr{F}_1\mathscr{F}_2$ has infinite transcendence degree over at least one of the fields $\mathscr{F}_1$ or $\mathscr{F}_2$. (Informally, this means that if M_1 and M_2 are not length-commensurable then the sets $L(M_1)$ and $L(M_2)$ are *very* different.) In fact, the same conclusion holds true for quotients $M_i = \mathbb{H}^{d_i}/\Gamma_i$ ($i = 1, 2$) by *any* Zariski-dense subgroups Γ_i of $\mathrm{PO}(d_i, 1)$ if $d_1 \neq d_2$ (assuming that $d_1, d_2 \neq 3$). We have similar results for *complex* and *quaternionic* hyperbolic spaces.

2 Weakly commensurable Zariski-dense subgroups

2.1. The method developed for studying the consequences of the length-commensurability of two locally symmetric spaces is based on translating the problem into a study of the implications of weak commensurability of their fundamental groups. To motivate the formal definition, let us return for a moment to the case of Riemann surfaces which we considered in Example 1.2. Let $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ be two Riemann surfaces where $\Gamma_1, \Gamma_2 \subset \mathrm{SL}_2(\mathbb{R})$ are discrete subgroups with torsion-free images in $\mathrm{PSL}_2(\mathbb{R})$. For $i = 1, 2$, let c_{γ_i} be a closed geodesic in M_i corresponding to a semi-simple element $\gamma_i \in \Gamma_i \setminus \{\pm 1\}$. Then it follows from (1) that

$$\ell_{\Gamma_1}(\gamma_1)/\ell_{\Gamma_2}(\gamma_2) \in \mathbb{Q} \quad \Leftrightarrow \quad \exists\, m, n \in \mathbb{N} \ \text{ such that } \ t_{\gamma_1}^m = t_{\gamma_2}^n,$$

or equivalently, the subgroups generated by the eigenvalues have nontrivial intersection. This leads us to the following.

2.2. Definition. Let $G_1 \subset \mathrm{GL}_{N_1}$ and $G_2 \subset \mathrm{GL}_{N_2}$ be two semi-simple algebraic groups defined over a field F of characteristic zero.

(a) Semi-simple elements $\gamma_1 \in G_1(F)$ and $\gamma_2 \in G_2(F)$ are said to be *weakly commensurable* if the subgroups of $\overline{F}^{\times}$ generated by their eigenvalues intersect nontrivially.

(b) (Zariski-dense) subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

It should be noted that in [30] we gave a more technical, but *equivalent*, definition of weakly commensurable elements, viz. we required the existence of maximal F-tori T_i of G_i for $i = 1, 2$ such that $\gamma_i \in T_i(F)$ and for some characters $\chi_i \in X(T_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This reformulation of (a) demonstrates that the notion of weak commensurability does not depend on the choice of matrix realizations of the G_i's (and is in fact more convenient in our proofs).

The above discussion of Riemann surfaces implies that if two Riemann surfaces $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ are length-commensurable, then the corresponding fundamental groups Γ_1 and Γ_2 are weakly commensurable. As we will see later, the same conclusion remains valid for general locally symmetric spaces of finite volume (cf. Theorem 6.2), but now we would like to depart from geometry and discuss some algebraic aspects of weak commensurability, along with a few problems of independent interest that its analysis leads to.

From a purely algebraic point of view, the investigation of weakly commensurable Zariski-dense subgroups fits into the classical framework of characterizing linear groups in terms of the spectra (eigenvalues) of its elements. However in the set-up described in Definition 2.2 it is not obvious at all how one should match the eigenvalues of (semi-simple) elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ as generally speaking γ_1 has N_1 eigenvalues and γ_2 has N_2. In the theory of complex representations of finite groups, one combines the eigenvalues of elements into the character values, and organizes the information about eigenvalues into the character table which involves *all* representations. This approach appears problematic for Zariski-dense subgroups of semi-simple algebraic groups as the ambient groups G_1 and G_2 have infinitely many inequivalent representations, so matching somehow their representations and requiring two elements to have the same eigenvalues in *all* respective representations is not practical, to say the least. On the other hand, instead of considering all representations, one could try to match the eigenvalues in a "canonical" matrix realization of the ambient group, but unfortunately it is not clear which matrix realization should be considered canonical. A reasonable alternative to these two extreme approaches would be to match the eigenvalues of $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ in *some* two representations of G_1 and G_2, respectively. This, however, actually brings us back to the notion of weak commensurability. Indeed, given an algebraic F-group $G \subset \mathrm{GL}_N$ and a (semi-simple) element $\gamma \in G(F)$ with eigenvalues $\lambda_1, \ldots, \lambda_N \in \overline{F}^{\times}$, for any rational representation $\rho\colon G \to \mathrm{GL}_{N'}$, every eigenvalue of $\rho(\gamma)$ is of the form $\lambda_1^{m_1} \cdots \lambda_N^{m_N}$ (where $m_1, \ldots, m_N$ are some integers), in other words, it is an element of the subgroup of $\overline{F}^{\times}$ generated by the eigenvalues of γ in the original representation. Conversely, any element of this subgroup can be realized as an eigenvalue of $\rho(\gamma)$ in *some* rational representation ρ. Thus, in the above notations, semi-simple $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are weakly commensurable if one nontrivial eigenvalue of $\rho_1(\gamma_1)$ in *some* rational representation $\rho_1\colon G_1 \to \mathrm{GL}_{N_1'}$ equals an eigenvalue of $\rho_2(\gamma_2)$ in *some* rational representation $\rho_2\colon G_2 \to \mathrm{GL}_{N_2'}$. Consequently, weak commensurability provides a way of matching the eigenvalues of semi-simple elements in Γ_1 and Γ_2 that is independent of the choice of the original representations of G_1 and G_2. (Using the fact that a finitely generated linear group over a field of characteristic zero contains a *neat* subgroup of finite index (cf. [34, Theorem 6.11]), one shows that if Γ_1 and Γ_2 are finitely generated and $\Delta_i \subset G_i(F)$ is commensurable with Γ_i then Γ_1 and Γ_2 are weakly commensurable if and only if Δ_1 and Δ_2 are weakly commensurable (see Lemma 2.3 of [30]). Consequently, in the analysis of weak commensurability of finitely generated subgroups Γ_1 and Γ_2, one can assume that the subgroups are neat, and we would like to observe that in this case the weak commensurability of $\gamma_1 \in \Gamma_1$ and $\gamma \in \Gamma_2$ implies the weak commensurability of γ_1^m and γ_2^n for any nonzero m and n.)

Thus, the relation of weak commensurability of two Zariski-dense subgroups of semi-simple algebraic groups very loosely corresponds to the relation between two finite groups under which each column of the character table of one group contains an element that appears in the character table for the other group, and vice versa. Clearly, the latter relation is inconsequential for finite groups, viz. it may hold for infinitely many pairs of nonisomorphic groups, and therefore does not impose any significant restrictions on the finite groups at hand.

So, we find it quite remarkable that the weak commensurability of Zariski-dense subgroups enables one to recover some characteristics of Γ_1 and Γ_2 (and/or G_1 and G_2) — this is made possible by the existence in Γ_1 and Γ_2 of special elements called *generic elements*, see §4 and [33, §9] . We begin our account of the results for weakly commensurable Zariski-dense subgroups with the following.

Theorem 2.3 ([30, Theorem 1]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. Assume that there exist finitely generated Zariski-dense subgroups Γ_i of $G_i(F)$ which are weakly commensurable. Then either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$.*

By a famous theorem of Tits [41], for any semi-simple group G over a field F of characteristic zero, the group $G(F)$ contains a free Zariski-dense subgroup. So, one or both subgroups in Theorem 2.3 may very well be free, and hence carry no *structural* information about the ambient algebraic group. Nevertheless, the information about the eigenvalues of elements expressed in terms of weak commensurability allows one to see the type of the ambient algebraic group — we refer to this type of phenomenon as *eigenvalue rigidity.*

There is one more important characteristic that can be seen through the lense of weak commensurability. Given a Zariski-dense subgroup Γ of $G(F)$, where G is an absolutely almost simple algebraic group defined over a field F of characteristic zero, we let K_Γ denote the subfield of F generated by the traces $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ for all $\gamma \in \Gamma$ (the so-called *trace field*). According to a theorem of E. B. Vinberg [42], $K = K_\Gamma$ is the minimal field of definition of $\mathrm{Ad}\,\Gamma$, i.e. the minimal subfield of F such that one can pick a basis in the Lie algebra of G in which all elements of $\mathrm{Ad}\,\Gamma$ are (simultaneously) represented by matrices with entries in K.

Theorem 2.4 ([30, Theorem 2]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$, and K_{Γ_i} be the subfield of F generated by the traces $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$, in the adjoint representation, of $\gamma \in \Gamma_i$. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

What would be the strongest, hence most desirable, consequence of weak commensurability? We recall that two subgroups Δ_1 and Δ_2 of an abstract group Δ are called *commensurable* if

$$[\Delta_i : \Delta_1 \cap \Delta_2] < \infty \quad \text{for} \quad i = 1, 2.$$

In our set-up of Zariski-dense subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ in different groups, this notion needs to be modified as follows. Let $\pi_i \colon G_i \to \overline{G}_i$ be the isogeny onto the corresponding adjoint group. We say that Γ_1 and Γ_2 are *commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$* if there exists an F-isomorphism $\sigma\colon \overline{G}_1 \to \overline{G}_2$ such that the subgroups $\sigma(\pi_1(\Gamma_1))$ and $\pi_2(\Gamma_2)$ are commensurable in the usual sense (we note that the commensurability of locally symmetric spaces is consistent with this notion of commensurability for the corresponding fundamental groups). It is easy to see that Zariski-dense subgroups commensurable up to an isomorphism between the corresponding adjoint groups are always weakly commensurable. So, the central question is to determine when the converse is true. As the following example shows, the desired conclusion may not be valid even if one of the groups is arithmetic.

Example 2.5 (cf. [30, Remark 5.5])**.** Let Γ be a torsion-free Zariski-dense subgroup of $G(F)$. For an integer $m > 1$, we let $\Gamma^{(m)}$ denote the subgroup generated by the mth powers of elements of Γ. Clearly, $\Gamma^{(m)}$ is weakly commensurable to Γ for any m. On the other hand, in many situations, $\Gamma^{(m)}$ is of infinite index in Γ for all sufficiently large m. This is, for example, the case if Γ is a nonabelian free group, in particular, a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It is also the case for finite index subgroups of $\mathrm{SL}_2(\mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and for cocompact lattices in semi-simple Lie groups of $\mathbb{R}$-rank 1. In all these examples, $\Gamma^{(m)}$ (for $m \gg 0$) is weakly commensurable, but not commensurable to Γ.

This example indicates that an "ideal" result asserting that the weak commensurability of Γ_1 and Γ_2 implies their commensurability, generally speaking, is possible only if *both* subgroups Γ_1 and Γ_2 are "large" (e.g., arithmetic subgroups or at least lattices). Such results were indeed obtained in [30] (with the help of the results from [12] and [31]) for S-arithmetic groups, and we will review these results in the next section. For general Zariski-dense subgroups, one should focus on characterizing the "minimal" algebraic group containing the subgroup (in fact, this is precisely the approach that has led to the definitive results in the arithmetic situation). More precisely, let $\Gamma \subset G(F)$ be a Zariski-dense subgroup, and let $K = K_\Gamma$ be the corresponding trace field. Assuming that $G \subset \mathrm{GL}_N$ is adjoint, we know by Vinberg's theorem that one can pick a basis of the N-dimensional space such that in this basis $\Gamma \subset \mathrm{GL}_N(K)$. Then the Zariski-closure $\mathscr{G}$ of Γ is an algebraic K-group that becomes isomorphic to G over F; in other words, $\mathscr{G}$ is an F/K-form of G such that $\Gamma \subset \mathscr{G}(K)$. Moreover, if $\mathscr{G}'$ is another F/K-form with this property, there exists an F-isomorphism $\mathscr{G} \to \mathscr{G}'$ that induces the "identity" map on Γ, and then the Zariski-density of Γ implies that this isomorphism is defined over K. Thus, the F/K-form $\mathscr{G}$ is uniquely defined. We would now like to formulate the following finiteness conjecture (which can be compared with the result that there are only finitely many finite groups with a given character table).

Conjecture 2.6. *Let G_1 and G_2 be absolutely simple algebraic F-groups of adjoint type, let Γ_1 be a finitely generated Zariski-dense subgroup of $G_1(F)$ with trace field $K = K_{\Gamma_1}$. Then there exists a* finite *collection $\mathscr{G}_2^{(1)}, \ldots, \mathscr{G}_2^{(r)}$ of F/K-forms of G_2 such that if Γ_2 is a finitely generated Zariski-dense subgroup of $G_2(F)$ that is weakly commensurable to Γ_1, then it is conjugate to a subgroup of one of the*

$\mathscr{G}_2^{(i)}(K)$'s $(\subset G_2(F))$.

In §5, we will present a previously unpublished result that implies the truth of this conjecture when K is a number field. We recall that if G is a simple algebraic $\mathbb{R}$-group different from PGL_2, then for any lattice Γ of $G(\mathbb{R})$, the trace field K_Γ is a number field, so to prove this conjecture for all lattices in simple groups, it remains to consider the group $G = \mathrm{PGL}_2$. This has not been done yet, but we will discuss some related results in this direction in §4 (specifically, see Theorems 4.11 and 4.14). Of course, one is interested not only in qualitative results in the spirit of Conjecture 2.6, but also in more quantitative ones asserting that in certain situations $r = 1$, i.e. a K-form is uniquely determined by the weak commensurability class of a finitely generated Zariski-dense subgroup with trace field K. We refer the reader to Theorem 3.4 regarding results of this kind for arithmetic groups; more general cases have not been considered so far — see, however, Theorem 4.9 and Corollary 4.10.

3 Results on weak commensurability of S-arithmetic groups

3.1. The definition of arithmeticity. Our results on weakly commensurable S-arithmetic subgroups in absolutely almost simple groups rely on a specific form of their description, so we begin with a review of the relevant definitions. Let G be an algebraic group defined over a number field K, and let S be a finite subset of the set V^K of all places of K containing the set V_∞^K of archimedean places. Fix a K-embedding $G \subset \mathrm{GL}_N$, and consider the group of S-integral points

$$G(\mathscr{O}_K(S)) := G \cap \mathrm{GL}_N(\mathscr{O}_K(S)).$$

Then, for any field extension F/K, the subgroups of $G(F)$ that are commensurable (in the usual sense) with $G(\mathscr{O}_K(S))$ are called S-*arithmetic*, and in the case where $S = V_\infty^K$ simply *arithmetic* (note that $\mathscr{O}_K(V_\infty^K) = \mathscr{O}_K$, the ring of algebraic integers in K). It is well-known that the resulting class of S-arithmetic subgroups does not depend on the choice of K-embedding $G \subset \mathrm{GL}_N$ (cf. [28]). The question, however, is what we should mean by an arithmetic subgroup of $G(F)$ when G is an algebraic group defined over a field F of characteristic zero that is not equipped with a structure of K-group over some number field $K \subset F$. For example, what is an arithmetic subgroup of $G(\mathbb{R})$ where $G = \mathrm{SO}_3(f)$ and $f = x^2 + ey^2 - \pi z^2$? For absolutely almost simple groups the "right" concept that we will formalize below is given in terms of forms of G over the subfields $K \subset F$ that are number fields. In our example, we can consider the following rational quadratic forms that are equivalent to f over $\mathbb{R}$:

$$f_1 = x^2 + y^2 - 3z^2 \quad \text{and} \quad f_2 = x^2 + 2y^2 - 7z^2,$$

and set $G_i = \mathrm{SO}_3(f_i)$. Then for each $i = 1, 2$, we have an $\mathbb{R}$-isomorphism $G_i \simeq G$, so the natural arithmetic subgroup $G_i(\mathbb{Z}) \subset G_i(\mathbb{R})$ can be thought of as an "arithmetic" subgroup of $G(\mathbb{R})$. Furthermore, one can consider quadratic forms over other number subfields $K \subset \mathbb{R}$ that again become equivalent to f over $\mathbb{R}$; for example,

$$K = \mathbb{Q}(\sqrt{2}) \quad \text{and} \quad f_3 = x^2 + y^2 - \sqrt{2}z^2.$$

Then for $G_3 = \mathrm{SO}_3(f_3)$, there is an $\mathbb{R}$-isomorphism $G_3 \simeq G$ which allows us to view the natural arithmetic subgroup $G_3(\mathscr{O}_K) \subset G_3(\mathbb{R})$, where $\mathscr{O}_K = \mathbb{Z}[\sqrt{2}]$, as an "arithmetic" subgroup of $G(\mathbb{R})$. One can easily generalize such constructions from arithmetic to S-arithmetic groups by replacing the rings of integers with the rings of S-integers. So, generally speaking, by an S-arithmetic subgroup of $G(\mathbb{R})$ we mean a subgroup which is commensurable to one of the subgroups obtained through this construction for some choice of a number subfield $K \subset \mathbb{R}$, a finite set S of places of K containing all the archimedean ones, and a quadratic form $\tilde{f}$ over K that becomes equivalent to f over $\mathbb{R}$. The technical definition is as follows.

Let G be a connected absolutely almost simple algebraic group defined over a field F of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi\colon G \to \overline{G}$ be the natural isogeny. Suppose we are given the following data:

- a *number field* K with a *fixed* embedding $K \hookrightarrow F$;
- an F/K-form $\mathscr{G}$ of $\overline{G}$, which is an algebraic K-group such that there exists an F-isomorphism ${}_F\mathscr{G} \simeq \overline{G}$, where ${}_F\mathscr{G}$ is the group obtained from $\mathscr{G}$ by extension of scalars from K to F;
- a finite set S of places of K containing V_∞^K but not containing any nonarchimedean places v such that $\mathscr{G}$ is K_v-anisotropic[1].

We then have an embedding $\iota\colon \mathscr{G}(K) \hookrightarrow \overline{G}(F)$ which is well-defined up to an F-automorphism of $\overline{G}$ (note that we do *not* fix an isomorphism ${}_F\mathscr{G} \simeq \overline{G}$). A subgroup Γ of $G(F)$ such that $\pi(\Gamma)$ is commensurable with $\sigma(\iota(\mathscr{G}(\mathscr{O}_K(S))))$, for some F-automorphism σ of $\overline{G}$, will be called a $(\mathscr{G}, K, S)$-*arithmetic subgroup*[2], or an S-arithmetic subgroup described in terms of the triple $(\mathscr{G}, K, S)$. As usual, $(\mathscr{G}, K, V_\infty^K)$-arithmetic subgroups will simply be called $(\mathscr{G}, K)$-arithmetic. The key observation is that the description of S-arithmetic subgroups in terms of triples is very convenient for determining when two such subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are commensurable up to an isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

Proposition 3.2 ([30, Proposition 2.5]). *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$. Then Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if $K_1 = K_2 =: K$, $S_1 = S_2$, and $\mathscr{G}_1$ and $\mathscr{G}_2$ are K-isomorphic.*

It follows from the above proposition that the arithmetic subgroups Γ_1, Γ_2, and Γ_3 constructed above, of $G(\mathbb{R})$, where $G = \mathrm{SO}_3(f)$, are pairwise noncommensurable: indeed, Γ_3, being defined over $\mathbb{Q}(\sqrt{2})$, cannot possibly be commensurable

[1] We note that if $\mathscr{G}$ is K_v-anisotropic then $\mathscr{G}(\mathscr{O}_K(S))$ and $\mathscr{G}(\mathscr{O}_K(S \cup \{v\}))$ are commensurable, and therefore the classes of S- and $(S \cup \{v\})$-arithmetic subgroups coincide. Thus, this assumption on S is necessary if we want to recover it from a given S-arithmetic subgroup.

[2] This notion of arithmetic subgroups coincides with that in Margulis' book [22] for absolutely simple adjoint groups.

to Γ_1 or Γ_2 as these two groups are defined over $\mathbb{Q}$; at the same time, the non-commensurability of Γ_1 and Γ_2 is a consequence of the fact that $\mathrm{SO}_3(f_1)$ and $\mathrm{SO}_3(f_2)$ are not $\mathbb{Q}$-isomorphic since the quadratic form f_1 is anisotropic over $\mathbb{Q}_3$, and f_2 is not.

In view of Proposition 3.2, the central question in the analysis of weak commensurability of S-arithmetic subgroups is the following: *Suppose we are given two Zariski-dense S-arithmetic subgroups that are described in terms of triples. Which components of these triples coincide given the fact that the subgroups are weakly commensurable?* As the following result demonstrates, two of these components *must* coincide.

Theorem 3.3 ([30, Theorem 3]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. If Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroups Γ_i of $G_i(F)$ $(i = 1, 2)$ are weakly commensurable, then $K_1 = K_2$ and $S_1 = S_2$.*

In general, the forms $\mathscr{G}_1$ and $\mathscr{G}_2$ do not have to be K-isomorphic (see [30], Examples 6.5 and 6.6 as well as the general construction in §9). In the next theorem we list the cases where it can nevertheless be asserted that $\mathscr{G}_1$ and $\mathscr{G}_2$ are necessarily K-isomorphic, and then give a general finiteness result for the number of K-isomorphism classes.

Theorem 3.4 ([30, Theorem 4]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, of the* same *type different from A_n, D_{2n+1}, with $n > 1$, or E_6. If for $i = 1, 2$, $G_i(F)$ contain Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup Γ_i which are weakly commensurable to each other, then $\mathscr{G}_1 \simeq \mathscr{G}_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

In this theorem, type D_{2n} $(n \geqslant 2)$ required special consideration. The case $n > 2$ was settled in [31] using the techniques of [30] in conjunction with results on embeddings of fields with involutive automorphisms into simple algebras with involution. The remaining case of type D_4 was treated by Skip Garibaldi [12], whose argument actually applies to all n and explains the result from the perspective of Galois cohomology, providing thereby a cohomological insight (based on the notion of Tits algebras) into the difference between the types D_{2n} and D_{2n+1}. We note that the types excluded in the theorem are precisely the types for which the automorphism $\alpha \mapsto -\alpha$ of the corresponding root system is *not* in the Weyl group. More importantly, all these types are honest exceptions to the theorem — a general Galois-cohomological construction of weakly commensurable, but not commensurable, Zariski-dense S-arithmetic subgroups for all of these types is given in [30, §9].

Theorem 3.5 ([30, Theorem 5]). *Let G_1 and G_2 be two connected absolutely almost simple groups defined over a field F of characteristic zero. Let Γ_1 be a Zariski-dense $(\mathscr{G}_1, K, S)$-arithmetic subgroup of $G_1(F)$. Then the set of K-isomorphism* *classes of K-forms $\mathscr{G}_2$ of $\overline{G}_2$ such that $G_2(F)$ contains a Zariski-dense $(\mathscr{G}_2, K, S)$-arithmetic subgroup weakly commensurable to Γ_1 is finite.*

In other words, the set of all Zariski-dense (K, S)-arithmetic subgroups of $G_2(F)$ which are weakly commensurable to a given Zariski-dense (K, S)-arithmetic subgroup is a union of finitely many commensurability classes.

A noteworthy fact about weak commensurability is that it has the following implication for the existence of unipotent elements in arithmetic subgroups (even though it is formulated entirely in terms of semi-simple ones). We recall that a semi-simple K-group is called *K-isotropic* if $\mathrm{rk}_K\, G > 0$; in characteristic zero, this is equivalent to the existence of nontrivial unipotent elements in $G(K)$. Moreover, if K is a number field then G is K-isotropic if and only if every S-arithmetic subgroup contains unipotent elements, for any S.

Theorem 3.6 ([30, Theorem 6]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$. If Γ_1 and Γ_2 are weakly commensurable then $\mathrm{rk}_K \mathscr{G}_1 = \mathrm{rk}_K \mathscr{G}_2$; in particular, if $\mathscr{G}_1$ is K-isotropic, then so is $\mathscr{G}_2$.*

We note that in [30, §7] we prove a somewhat more precise result, viz. that if G_1 and G_2 are of the same type, then the Tits indices of $\mathscr{G}_1/K$ and $\mathscr{G}_2/K$ are isomorphic, but we will not get into these technical details here.

The following result asserts that a lattice[3] which is weakly commensurable with an S-arithmetic group is itself S-arithmetic.

Theorem 3.7 ([30, Theorem 7]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense lattice in $G_i(F)$. Assume that Γ_1 is a (K, S)-arithmetic subgroup of $G_1(F)$. If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is a (K, S)-arithmetic subgroup of $G_2(F)$.*

According to Theorem 2.3, if G_1 and G_2 contain weakly commensurable finitely generated Zariski-dense subgroups then either the groups are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$. Weakly commensurable S-arithmetic subgroups in the first case were analyzed in Theorem 3.4 (see also the discussion thereafter). We conclude this section with a recent result of Skip Garibaldi and the second-named author [13] which gives a criterion for two Zariski-dense S-arithmetic subgroups in the groups of type B_n and C_n to be weakly commensurable. To formulate the result we need the following definition. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be absolutely almost simple algebraic groups of types B_n and C_n with $n \geqslant 2$, respectively, over a number field K. We say that $\mathscr{G}_1$ and $\mathscr{G}_2$ are *twins* (over K) if for each $v \in V^K$, either both groups are split or both are anisotropic over the completion K_v. (We note that since groups of these types

[3] A discrete subgroup Γ of a locally compact topological group $\mathcal{G}$ is said to be a lattice in $\mathcal{G}$ if $\mathcal{G}/\Gamma$ carries a *finite* $\mathcal{G}$-invariant Borel measure.

cannot be anisotropic over K_v when v is nonarchimedean, our condition effectively says that $\mathscr{G}_1$ and $\mathscr{G}_2$ must be K_v-split for *all* nonarchimedean v.)

Theorem 3.8. ([13, Theorem 1.2]) *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero of Killing-Cartan types B_n and C_n $(n \geqslant 3)$ respectively, and let Γ_i be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$ for $i = 1, 2$. Then Γ_1 and Γ_2 are weakly commensurable if and only if the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are twins.*

(We recall that according to Theorem 3.3, if Zariski-dense $(\mathscr{G}_1, K_1, S_1)$- and $(\mathscr{G}_2, K_2, S_2)$-arithmetic subgroups are weakly commensurable then necessarily $K_1 = K_2$ and $S_1 = S_2$, so Theorem 3.8 in fact treats the most general situation.)

4 Absolutely almost simple algebraic groups having the same maximal tori

4.1. The analysis of weak commensurability leads to, and also depends on, problems of an algebraic nature that in broad terms can be described as *characterizing absolutely almost simple algebraic groups over a given (nice) field K having the same isomorphism/isogeny classes of maximal K-tori.* While these problems are not new (for example, in the context of finite-dimensional central simple algebras they can be traced back to such classical algebraic results as Amitsur's Theorem [2] on generic splitting fields — cf. §4.4 below), there has been a noticeable resurgence of interest in them in recent years. One should mention [11] and [17] where some aspects of the problem were considered over local and global fields; the local-global principles for embedding tori into absolutely almost simple algebraic groups as maximal tori (in particular, for embedding commutative étale algebras with involutive automorphisms into simple algebras with involution) have been analyzed in [5], [12], [19], [31]; some number-theoretic applications have been given in [10]. In this section, we will focus primarily on those aspects of the problem that are related to the study of weak commensurability and particularly to Conjecture 2.6. The most recent results here analyze division algebras having the same maximal subfields and/or the same splitting fields [7], [8], [14], [18], [35]. These results provide, in particular, substantial supporting evidence for the Finiteness Conjecture 4.12 about absolutely almost simple algebraic K-groups having the same isomorphism classes of maximal tori, and hence also for Conjecture 2.6. We will return to weak commensurability in the next section and present a result that indicates a unified approach to both Conjectures 2.6 and 4.12 (cf. §5).

4.2. Generic elements and the Isogeny Theorem. We begin by describing in more precise terms the connection between weak commensurability and study of absolutely almost simple algebraic groups having the same isomorphism classes of maximal tori. This connection is based on the Isogeny Theorem (see below) to formulate which we need to recall the notion of *generic tori* and *generic elements.*

Let G be a connected absolutely almost simple algebraic group defined over an infinite field K. Fix a maximal K-torus T of G, and, as usual, let $\Phi = \Phi(G, T)$

denote the corresponding root system, and let $W(G,T)$ be its Weyl group. Furthermore, we let K_T denote the (minimal) splitting field of T in a fixed separable closure $\overline{K}$ of K. Then the natural action of the Galois group $\mathrm{Gal}(K_T/K)$ on the character group $X(T)$ of T induces an injective homomorphism

$$\theta_T \colon \mathrm{Gal}(K_T/K) \to \mathrm{Aut}(\Phi(G,T)).$$

We say that T is *generic* (over K) if

$$\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G,T). \tag{3}$$

(Note that such a torus is automatically K-*irreducible*, i.e. it does not contain proper K-subtori.) For example, any maximal K-torus of $G = \mathrm{SL}_n/K$ is of the form $T = \mathrm{R}^{(1)}_{E/K}(\mathrm{GL}_1)$ for some n-dimensional commutative étale K-algebra E. Then such a torus is generic over K if and only if E is a separable field extension of K and the Galois group of the normal closure L of E over K is isomorphic to the symmetric group S_n. Furthermore, a regular semi-simple element $g \in G(K)$ is called *generic* (over K) if the K-torus $T = Z_G(g)^\circ$ (the identity component of the centralizer $Z_G(g)$ of g in G) is generic (over K) in the sense defined above. We are now in a position to formulate a result that enables one to pass from the weak commensurability of two generic elements to an isogeny, and in most cases even to an isomorphism, of the ambient tori.

Theorem 4.3 (Isogeny Theorem, [30, Theorem 4.2]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over an infinite field K, and let L_i be the minimal Galois extension of K over which G_i becomes an inner form of a split group. Suppose that for $i = 1, 2$, we are given a semi-simple element $\gamma_i \in G_i(K)$ contained in a maximal K-torus T_i of G_i. Assume that* (i) *G_1 and G_2 are either of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n,* (ii) *γ_1 has infinite order,* (iii) *T_1 is K-irreducible, and* (iv) *γ_1 and γ_2 are weakly commensurable. Then*

(1) *there exists a K-isogeny $\pi \colon T_2 \to T_1$ which carries $\gamma_2^{m_2}$ to $\gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$;*
(2) *if $L_1 = L_2 =: L$ and $\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) \supset W(G_1,T_1)$, then $\pi^* \colon X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q} \to X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ has the property that $\pi^*(\mathbb{Q} \cdot \Phi(G_1,T_1)) = \mathbb{Q} \cdot \Phi(G_2,T_2)$. Moreover, if G_1 and G_2 are of the same Killing-Cartan type different from $\mathsf{B}_2 = \mathsf{C}_2$, F_4 or G_2, then a suitable rational multiple of π^* maps $\Phi(G_1,T_1)$ onto $\Phi(G_2,T_2)$.*

It follows that in the situations where π^* can be, and has been, scaled so that $\pi^*(\Phi(G_1,T_1)) = \Phi(G_2,T_2)$, the isogeny π induces K-isomorphisms $\widetilde{\pi} \colon \widetilde{T}_2 \to \widetilde{T}_1$ and $\overline{\pi} \colon \overline{T}_2 \to \overline{T}_1$ between the corresponding tori in the simply connected and adjoint groups $\widetilde{G}_i$ and $\overline{G}_i$. Thus, in most situations, the fact that Zariski-dense torsion-free subgroups $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ are weakly commensurable implies (under some minor technical assumptions) that G_1 and G_2 have the same K-isogeny classes (and under some minor additional assumptions — even the same K-isomorphism classes) of generic maximal K-tori that nontrivially intersect Γ_1 and Γ_2, respectively. Since over finitely generated fields generic tori, and also generic elements in a given (finitely generated) Zariski-dense subgroup, exist in abundance (cf. [29], and also [33, §9]), this relates G_1 and G_2 in a significant way and leads to important results (cf., for example, Theorem 5.1). So, while the problem of understanding algebraic groups with the same isomorphism/isogeny classes is not completely equivalent to the investigation of weak commensurability of Zariski-dense subgroups (for one thing, not every maximal torus necessarily intersects a given Zariski-dense subgroup), in practice it does capture most intricacies of the latter, and in fact the connection between the problems goes both ways. The next theorem (cf. [30, Theorem 7.5] and [13, Proposition 1.3]), which is a consequence of the results on weak commensurability (cf. §3), illustrates this point.

Theorem 4.4. (1) *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a number field K, and let L_i be the smallest Galois extension of K over which G_i becomes an inner form of a split group. If G_1 and G_2 have the same K-isogeny classes of maximal K-tori then either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n, and moreover, $L_1 = L_2$.*

(2) *Fix an absolutely almost simple K-group G. Then the set of isomorphism classes of all absolutely almost simple K-groups G' having the same K-isogeny classes of maximal K-tori is finite.*

(3) *Fix an absolutely almost simple simply connected K-group G whose Killing-Cartan type is different from A_n, D_{2n+1} $(n > 1)$ or E_6. Then any K-form G' of G (in other words, any absolutely almost simple simply connected K-group G' of the* same *type as G) that has the same K-isogeny classes of maximal K-tori as G, is isomorphic to G.*

The construction described in [30, §9] shows that the types excluded in (3) are honest exceptions, i.e., for each of those types one can construct non-isomorphic absolutely almost simple simply connected K-groups G_1 and G_2 of this type over a number field K that have the same isomorphism classes of maximal K-tori.

The situation where one of the groups is of type B_n and the other is of type C_n with $n \geqslant 3$ was analyzed in [13].

Theorem 4.5 ([13, Theorem 1.4])**.** *Let G_1 and G_2 be absolutely almost simple algebraic groups over a number field K of types B_n and C_n respectively for some $n \geqslant 3$.*

(1) *The groups G_1 and G_2 have the same* isogeny *classes of maximal K-tori if and only if they are twins*[4]*.*

(2) *The groups G_1 and G_2 have the same* isomorphism *classes of maximal K-tori if and only if they are twins, G_1 is adjoint and G_2 is simply connected.*

4.6. Division algebras with the same maximal subfields. As we already mentioned, questions related to the problem of characterizing absolutely almost

[4]See the definition of twins prior to the statement of Theorem 3.8.

simple algebraic K-groups by the isomorphism/isogeny classes of their maximal K-tori were in fact raised and investigated a long time ago, particularly in the context of finite-dimensional central simple algebras. We recall that given a central simple algebra A of degree n (i.e., of dimension n^2) over a field K, a field extension F/K is called a *splitting field* if $A \otimes_K F \simeq M_n(F)$ as F-algebras; furthermore, if A is a division algebra then the splitting fields of degree n over K are precisely the maximal subfields of A. It is well-known that the splitting fields/maximal subfields of a central simple K-algebra A play a huge role in the analysis of its structure (cf., for example, [15]), which suggests the question: *to what extent do these fields actually determine* A? The answer to this question in the situation where one considers *all* splitting fields is given by the famous theorem of Amitsur [2]: *Let A_1 and A_2 be finite-dimensional central simple algebras over a field K. Assume that a field extension F/K splits A_1 if and only if it splits A_2. Then the classes $[A_1]$ and $[A_2]$ in the Brauer group* $\mathrm{Br}(K)$ *generate the same subgroup:* $\langle [A_1] \rangle = \langle [A_2] \rangle$ (the converse is obvious). The proof of Amitsur's Theorem (cf. [2], [15, Ch. 5]) uses *generic splitting fields* which are *infinite* extensions of K. At the same time, it is important to point out that the situation changes dramatically if instead of all splitting fields one considers only finite-dimensional ones or just maximal subfields.

Example 4.7. Fix $r \geqslant 2$, and pick r distinct primes $p_1, \ldots, p_r$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be any r-tuple with $\varepsilon_i = \pm 1$ such that $\sum_{i=1}^r \varepsilon_i \equiv 0 (\mathrm{mod}\ 3)$. By class field theory (the Albert-Brauer-Hasse-Noether Theorem — to be referred to as (ABHN) in the sequel, cf. [1, Ch. VII, 9.6], [26, 18.4], and also [37] for a historical perspective), corresponding to ε, we have a central cubic division algebra $D(\varepsilon)$ over $\mathbb{Q}$ with the following local invariants (considered as elements of $\mathbb{Q}/\mathbb{Z}$):

$$\mathrm{inv}_p\, D(\varepsilon) \; = \; \begin{cases} \dfrac{\varepsilon_i}{3}, & p = p_i \ \text{ for } i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \ \ (\text{including } p = \infty). \end{cases}$$

Then for any two r-tuples ε' and ε'' as above, the algebras $D(\varepsilon')$ and $D(\varepsilon'')$ have the same finite-dimensional splitting fields, hence the same maximal subfields (cf. [26, 18.4, Corollary b]), and are non-isomorphic if $\varepsilon' \neq \varepsilon''$. Obviously, the number of admissible r-tuples ε grows with r, so this method enables one to construct an *arbitrarily large* (but finite) number of pairwise nonisomorphic cubic division algebras over $\mathbb{Q}$ having the same maximal subfields.

A similar construction can be implemented for division algebras of any degree $d > 2$. On the other hand, it follows from (ABHN) that any two quaternion division algebras over a number field K with the same quadratic subfields are necessarily isomorphic. This suggests the following question:

(∗) *What can one say about two finite-dimensional central division algebras D_1 and D_2 over a field K given the fact that they have the same (isomorphism classes of) maximal subfields?*

(We say that central division K-algebras D_1 and D_2 have the same isomorphism classes of maximal subfields if they have the same degree n and a degree n field extension F/K admits a K-embedding $F \hookrightarrow D_1$ if and only if it admits a K-embedding $F \hookrightarrow D_2$.)

It should be noted that $(*)$ is closely related (although not equivalent) to the question of understanding the relationship between D_1 and D_2 when the groups $G_1 = \mathrm{SL}_{1,D_1}$ and $G_2 = \mathrm{SL}_{1,D_2}$ have the same isomorphism classes of maximal K-tori, and we will comment on this a bit later (cf. Theorem 4.14 and the discussion thereafter). Our next immediate goal, however, is to present some recent results on $(*)$, for which we need the following definition.

Definition 4.8. Let D be a central division K-algebra of degree n. The *genus* $\mathbf{gen}(D)$ is the set of all classes $[D'] \in \mathrm{Br}(K)$ represented by central division K-algebras D' having the same maximal subfields as D.

The following basic questions about the genus represent two aspects of the general question $(*)$.

Question 1. When does $\mathbf{gen}(D)$ reduce to a single class?

(This is another way of asking whether D is determined uniquely up to isomorphism by its maximal subfields.)

Question 2. When is $\mathbf{gen}(D)$ finite?

Regarding Question 1, we note that $|\mathbf{gen}(D)| = 1$ is possible only if D has exponent 2 in the Brauer group. Indeed, the opposite algebra D^{op} has the same maximal subfields as D. So, unless $D \simeq D^{\mathrm{op}}$ (which is equivalent to D being of exponent 2), we have $|\mathbf{gen}(D)| > 1$. On the other hand, as we already mentioned, it follows from (ABHN) that for any quaternion algebra D over a global field K (and hence any central simple K-algebra of exponent 2 over a global field is known to be a quaternion algebra), $\mathbf{gen}(D)$ does reduce to a single element. So, Question 1 really asks about other fields with this property. More specifically, we had asked earlier if the field of rational functions $K = \mathbb{Q}(x)$ is such a field. This question (in the context of quaternion algebras) was answered in the affirmative by D. Saltman. Then, in [14], Garibaldi and Saltman extended the result to fields of the form $K = k(x)$, where k is any number field (and also to some other situations). Recently, the following *Stability Theorem* was proved in [8] for algebras of exponent 2 (the case of quaternion algebras was considered earlier in [35]).

Theorem 4.9 ([8, Theorem 3.5]). *Let k be a field of characteristic $\neq 2$. If $|\mathbf{gen}(D)| = 1$ for any central division k-algebra D of exponent 2 then the same property holds for any central division algebra of exponent 2 over the field of rational functions $k(x)$.*

Corollary 4.10. *If k is either a number field or a finite field of* char $\neq 2$*, and $K = k(x_1, \ldots, x_r)$ is a purely transcendental extension then for any central division K-algebra D of exponent 2 we have $|\mathbf{gen}(D)| = 1$.*

Furthermore, if k is a field of char $\neq 2$ such that ${}_2\mathrm{Br}(k) = 0$, then the field of rational functions $K = k(x_1, \ldots, x_r)$ again satisfies the property described in

Theorem 4.9. The existence of various examples in which the genus of a division algebra of exponent 2 always reduces to one element naturally leads to the question of *whether the genus of a quaternion algebra can* ever *be nontrivial.* The answer is "yes", and the following construction of examples (described in [14, §2]) was offered by several people including Wadsworth, Shacher, Rost, Saltman, Garibaldi ... We will describe only the basic idea referring to [14] for the details.

We start with two nonisomorphic quaternion division algebras D_1 and D_2 over a field k of char $\neq 2$ that have a common quadratic subfield (e.g., one can take $k = \mathbb{Q}$ and $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$). If D_1 and D_2 already have the same quadratic subfields, we are done. Otherwise, there exists a quadratic extension $k(\sqrt{d})$ that embeds into D_1 but not into D_2. Then, using either properties of quadratic forms or the "index reduction formulas", one shows that there exists an extension $k^{(1)}$ of k (which is the field of rational functions on a certain quadric) such that

- $k^{(1)} \otimes_k D_1$ and $k^{(1)} \otimes_k D_2$ are non-isomorphic division algebras over $k^{(1)}$, **but**
- $k^{(1)}(\sqrt{d})$ embeds into $k^{(1)} \otimes_k D_2$.

One deals with other subfields (in the algebras obtained from D_1 and D_2 by applying the extension of scalars built at the previous step of the construction), one at a time, in a similar fashion. This process generates an ascending chain of fields

$$k^{(1)} \subset k^{(2)} \subset k^{(3)} \subset \cdots ,$$

and we let K be the union (direct limit) of this chain. Then $K \otimes_k D_1$ and $K \otimes_k D_2$ are non-isomorphic quaternion division K-algebras having the same quadratic subfields; in particular $|\mathbf{gen}(D_1 \otimes_k K)| > 1$. Note that the resulting field K has infinite transcendence degree over k, hence is infinitely generated. Furthermore, some adaptation of the above construction (cf. [23]) enables one to start with an infinite sequence $D_1, D_2, D_3, \ldots$ of division algebras over a field k of characteristic $\neq 2$ that are pairwise non-isomorphic but share a common quadratic subfield (e.g., one can take $k = \mathbb{Q}$ and consider the family of algebras of the form $\left(\frac{-1,p}{\mathbb{Q}}\right)$ where p is a prime $\equiv 3 (\mathrm{mod}\ 4)$), and then build an infinitely generated field extension K/k such that the algebras $D_i \otimes_k K$ become pairwise non-isomorphic division algebras with any two of them having the same quadratic subfields. This makes the genus $\mathbf{gen}(D_1 \otimes_k K)$ infinite, and therefore brings us to Question 2 of when one can guarantee the finiteness of the genus. Here we have the following finiteness result.

Theorem 4.11 ([7, Theorem 3]). *Let K be a finitely generated field. If D is a central division K-algebra of exponent prime to* char K, *then* $\mathbf{gen}(D)$ *is finite.*

One of the questions about the genus of a division that remains open after Theorems 4.9 and 4.11 is whether one can find a quaternion division algebra *over a finitely generated field* of characteristic $\neq 2$ with *nontrivial genus.*

4.12. The genus of an algebraic group. We will now discuss a possible generalization of the concept of the genus from finite-dimensional central division algebras to arbitrary absolutely almost simple algebraic groups obtained by replacing maximal subfields with maximal tori.

So, let G be an absolutely almost simple (simply connected or adjoint) algebraic group over a field K. We define the genus $\mathbf{gen}(G)$ to be the set of K-isomorphism classes of K-forms G' of G that have the same isomorphism classes of maximal K-tori as G. Two remarks are in order. First, if D is a finite-dimensional central division algebra over a field K and $G = SL_{1,D}$ is the corresponding group defined by elements of norm 1 in D, then only maximal *separable* subfields of D correspond to the maximal K-tori of G. So, to avoid at least the obvious discrepancies between the definitions of $\mathbf{gen}(D)$ and $\mathbf{gen}(G)$, one should probably define the former in terms of *maximal separable* subfields. We don't know however whether the definitions of $\mathbf{gen}(D)$ in terms of all and only separable maximal subfields would actually be distinct (these problems do not arise in Theorems 4.9 and 4.11 as these treat only division algebras whose degree is prime to the characteristic of the center). Second, one can give several alternative definitions of $\mathbf{gen}(G)$ by working only with maximal *generic* K-tori, and on the other hand by replacing K-isomorphisms of tori with K-isogenies. It would be interesting to determine the precise relationship between the various definitions; at this point, we will just mention without further elaboration that the definitions given in terms of generic tori and K-isomorphism vs. K-isogeny *in practice* lead to basically the same qualitative results (for the reasons contained in Theorem 4.3 and the subsequent discussion).

Building on Theorem 4.11, we would like to propose the following conjecture.

Conjecture 4.13. *Let G be an absolutely almost simple (simply connected or adjoint) algebraic group over a finitely generated field K. Assume that the characteristic of K is either zero or is "good" for G. Then the genus* $\mathbf{gen}(D)$ *is finite.*

The "bad" characteristics for each type are expected to be the following:

- A_ℓ — all prime divisors p of $(\ell+1)$, ${}^2\mathsf{A}_\ell$ — same primes and also $p = 2$;
- B_ℓ, C_ℓ, D_ℓ — $p = 2$ (and possibly $p = 3$ for ${}^{3,6}\mathsf{D}_4$);
- $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4$ and G_2 — all prime divisors of the order of the Weyl group.

The Isogeny Theorem 4.3 establishes some connections between Conjectures 2.6 and 4.13, but more importantly, we anticipate that the methods developed to deal with Conjecture 4.13 will be useful also in analyzing Conjecture 2.6 (in fact both conjectures will be consolidated in §5 into a single conjecture — see Conjecture 5.4). Now, Theorem 4.4 confirms the conjecture in the situation where K is a number field. For general fields, the conjecture is known at this point only for inner forms of type A_ℓ.

Theorem 4.14. ([8, Theorem 5.3]) *Let G be a simply connected inner form of type A_ℓ over a finitely generated field K, and assume that the characteristic of K is either zero or does not divide $(\ell+1)$. Then* $\mathbf{gen}(G)$ *is finite.*

The group G in this theorem is of the form $\mathrm{SL}_{1,A}$ for some central simple K-

algebra A of dimension n^2. While every maximal K-torus of G corresponds to some n-dimensional commutative étale subalgebra of A (and the same is true for any inner K-form G' of G), the existence of a K-isomorphism between the tori a priori may not imply the existence of an isomorphism between the étale algebras (it would be interesting to construct such examples!). For this reason, Theorem 4.14 is *not* an automatic consequence of Theorem 4.11. The proof of Theorem 4.14 uses generic tori, the isomorphisms between which after appropriate scaling do extend to an isomorphism between the étale algebras.

We mention in passing that there are other interesting approaches to the definition of the genus. For example, Krashen and McKinnie [18] defined the genus $\mathbf{gen}'(D)$ of a central division K-algebra D of prime degree based on all finite-dimensional splitting fields. Furthermore, Merkurjev proposed to define the *motivic genus* $\mathbf{gen}_m(G)$ of an absolutely almost simple algebraic K-group G along the lines suggested by Amitsur's Theorem, viz. as the set of K-isomorphism classes of K-forms G' such that for *any* field extension F/K the groups G and G' have the same F-isomorphism classes of maximal F-tori. Since this concept is less related to weak commensurability, we will not discuss it here referring the reader to [8, Remark 5.6] for the details (including an explanation of the term "motivic").

5 A finiteness result

The goal of this section is to try to establish a more direct connection between the Finiteness Conjectures 2.6 and 4.13: while such a connection undoubtedly exists, it has manifested itself so far primarily through the fact that the techniques developed for one of them are typically also useful for the other, and not through any formal implications. We begin with a new finiteness result over number fields which implies the truth of *both* conjectures in this situation. We then formulate and discuss a conjecture which says that a similar statement should be true over general fields (with some restrictions on the characteristic).

Theorem 5.1. *Let G be an absolutely almost simple algebraic group over a number field K, and let Γ be a finitely generated Zariski-dense subgroup of $G(K)$ with trace field K. Denote by $\mathbf{gen}(G,\Gamma)$ the set of isomorphism classes of K-forms G' of G having the following property: any* generic *maximal K-torus T of G that contains an element of Γ of infinite order is isogenous to some maximal K-torus T' of G'. Then $\mathbf{gen}(G,\Gamma)$ is finite.*

Proof. We begin with a statement which is valid over any finitely generated field K of characteristic zero as its proof relies only on the facts established in this generality.

Lemma 5.2. *Let $G' \in \mathbf{gen}(G,\Gamma)$, and let L (resp., L') denote the minimal Galois extension of K over which G (resp., G') is of inner type, i.e., is an inner form of a split group. Then $L = L'$, and hence G' is an inner form of G over K.*

Proof. According to [29] (cf. also [33, Theorem 9.6]), there exists a regular semisimple element $\gamma \in \Gamma$ of infinite order such that the torus $T := Z_G(\gamma)^\circ$ is generic over $\mathcal{L} := LL'$. By our assumption, there exists a maximal K-torus T' of G' for which there is a K-isogeny $\nu\colon T \to T'$. We then have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{GL}(X(T) \otimes_{\mathbb{Z}} \mathbb{Q}) \arrow[dd, "\tilde{\nu}"] \\ \mathrm{Gal}(\overline{K}/K) \arrow[ur, "\theta_T"] \arrow[dr, "\theta_{T'}"'] & \\ & \mathrm{GL}(X(T') \otimes_{\mathbb{Z}} \mathbb{Q}), \end{tikzcd}$$

where $\overline{K}$ is an algebraic closure of K and $\tilde{\nu}$ is the isomorphism induced by ν. We note that for any field extension F of K contained in $\overline{K}$, the map $\tilde{\nu}$ gives an isomorphism between the images of $\mathrm{Gal}(\overline{K}/F)$ under θ_T and $\theta_{T'}$, hence

$$|\theta_T(\mathrm{Gal}(\overline{K}/F))| = |\theta_{T'}(\mathrm{Gal}(\overline{K}/F))|. \tag{4}$$

Furthermore, since both G and G' are of inner type over $\mathcal{L}$, we have

$$\theta_T(\mathrm{Gal}(\overline{K}/\mathcal{L})) = W(G,T) \quad \text{and} \quad \theta_{T'}(\mathrm{Gal}(\overline{K}/\mathcal{L})) = W(G',T') \tag{5}$$

(cf. [30, Lemma 4.1]).

Now, assume that $L' \not\subset L$, i.e. $L \subsetneqq \mathcal{L}$. Again, since G is of inner type over L, we have

$$\theta_T(\mathrm{Gal}(\overline{K}/L)) = W(G,T). \tag{6}$$

On the other hand, it follows from (5) that $\theta_{T'}(\mathrm{Gal}(\overline{K}/L))$ contains $W(G',T')$ but in fact is strictly larger as by our assumption G' is *not* of inner type over L (cf. [30, Lemma 4.1]). Thus,

$$|\theta_{T'}(\mathrm{Gal}(\overline{K}/L))| > |W(G',T')| = |W(G,T)| = |\theta_T(\mathrm{Gal}(\overline{K}/L))|,$$

which contradicts (4) for $F = L$. Similarly, the assumption $L \not\subset L'$ would imply that

$$|\theta_T(\mathrm{Gal}(\overline{K}/L'))| > |\theta_{T'}(\mathrm{Gal}(\overline{K}/L')|,$$

contradicting (4) for $F = L'$. $\square$

By [28, Theorem 6.7], one can find a finite subset $S_1 \subset V^K$ such that G is quasi-split over K_v for any $v \in V^K \setminus S_1$. Now, let $\pi\colon \widetilde{G} \to G$ be the universal cover. It follows from [44] that one can find a finite subset $S_2 \subset V^K$ containing V_∞^K such that $\Gamma \subset G(\mathscr{O}(S_2))$ and there exists a subgroup Δ of $\pi^{-1}(\Gamma)$ of finite index contained in $\widetilde{G}(\mathscr{O}(S_2))$ whose closure in the group of S_2-adeles $\widetilde{G}(\mathbb{A}_{S_2})$ is open. Set $S = S_1 \cup S_2$.

Lemma 5.3. *Every $G' \in \mathbf{gen}(G,\Gamma)$ is quasi-split over K_v for $v \in V^K \setminus S$.*

Proof. We first make the following general observation. Let $\mathscr{G}_0$ be a quasi-split semi-simple group over a field $\mathscr{K}$, and let $\mathscr{G}$ be an *inner* $\mathscr{K}$-form of $\mathscr{G}_0$; then the fact that

$$\mathrm{rk}_{\mathscr{K}}\, \mathscr{G} \geqslant \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_0 \tag{7}$$

implies that $\mathscr{G}$ is itself quasi-split (and hence in (7) we actually have equality). Indeed, since $\mathscr{G}$ is an inner twist of $\mathscr{G}_0$, the $*$-actions on the Tits indices of $\mathscr{G}_0$ and $\mathscr{G}$ are identical (cf. [30, Lemma 4.1(a)]). Since the $\mathscr{K}$-rank of a semi-simple group equals the number of distinguished orbits under the $*$-action in its Tits index, and all the orbits in the Tits index of $\mathscr{G}_0$ are distinguished as the latter is $\mathscr{K}$-quasi-split, condition (7) implies that the same is true for $\mathscr{G}$, making it quasi-split.

Returning to the proof of the lemma, let us fix $v \in V^K \setminus S$, and set $\mathscr{K} = K_v$. By Lemma 5.2, the group G' is an inner form of G over K, and hence over $\mathscr{K}$. It follows from the construction of S that the closure of Δ in $\widetilde{G}(\mathscr{K})$ is open, and then so is the closure of Γ in $G(\mathscr{K})$. Using Theorem 3.4 in [32], we find a regular semi-simple element $\gamma \in \Gamma$ of infinite order such that the torus $T = Z_G(\gamma)^\circ$ is generic over K and contains a maximal $\mathscr{K}$-split torus of G, i.e. $\mathrm{rk}_{\mathscr{K}}\, T = \mathrm{rk}_{\mathscr{K}}\, G$. By our assumption, T is K-isogenous to a maximal K-torus T' of G'. Then we have

$$\mathrm{rk}_{\mathscr{K}}\, G' \geqslant \mathrm{rk}_{\mathscr{K}}\, T' = \mathrm{rk}_{\mathscr{K}}\, T = \mathrm{rk}_{\mathscr{K}}\, G.$$

Since by construction $\mathscr{G}_0 := G$ is quasi-split over $\mathscr{K}$, applying the remark following (7) to $\mathscr{G} := G'$, we obtain that G' is quasi-split over $\mathscr{K}$, as required. $\square$

Now, let G_0 be the quasi-split inner K-form of G. Fix an arbitrary $G' \in \mathbf{gen}(G, \Gamma)$. It follows from Lemma 5.2 that G' is an inner K-form of G and so it is an inner K-form of G_0. Hence G' is obtained by twisting G_0 by a class $\zeta \in H^1(K, G_0)$. This class lies in

$$\Sigma_S := \mathrm{Ker}\Big(H^1(K, G_0) \longrightarrow \bigoplus_{v \in V^K \setminus S} H^1(K_v, G_0)\Big),$$

since for $v \notin S$, G', being quasi-split over K_v, is K_v-isomorphic to G_0. (For this one needs to observe that the map $H^1(F, G_0) \to H^1(F, \mathrm{Aut}\, G_0)$ has trivial kernel for any field extension F/K which follows from the fact that $\mathrm{Aut}\, G_0$ is a semi-direct product of G_0 and a K-subgroup of symmetries of the Dynkin diagram.) However, Σ_S is known to be finite for any finite subset $S \subset V^K$ (cf. [38, Ch. III, §4, Theorem 7]), and the finiteness of $\mathbf{gen}(G, \Gamma)$ follows. $\square$

It is easy to see that Theorem 5.1 implies the truth of both Conjectures 2.6 and 4.13 over number fields — the connection with Conjecture 4.13 is obvious while in order to connect with Conjecture 2.6 one needs to use the Isogeny Theorem 4.3. Moreover, this kind of implication would remain valid over a general field, and we would like to end this section with the following conjecture that suggests a uniform approach to Conjectures 2.6 and 4.13.

Conjecture 5.4. *Let G be an absolutely almost simple algebraic group over a field K of "good" characteristic, and let Γ of $G(K)$ be a finitely generated Zariski-dense subgroup with field of definition*[5] *K. Let* $\mathbf{gen}(G, \Gamma)$ *be the set of isomorphism classes of K-forms G' of G having the following property: any maximal generic K-torus T of G that contains an element of Γ of infinite order is K-isogenous to some maximal K-torus T' of G'. Then* $\mathbf{gen}(G, \Gamma)$ *is finite.*

6 Back to geometry

6.1. Locally symmetric spaces. Let G be a connected adjoint semi-simple real algebraic group, let $\mathcal{G} = G(\mathbb{R})$ considered as a real Lie group, and let $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$, where $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$, be the associated symmetric space endowed with the Riemannian metric coming from the Killing form on the Lie algebra of $\mathcal{G}$. Furthermore, given a discrete torsion-free subgroup Γ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma := \mathfrak{X}/\Gamma$ denote the corresponding locally symmetric space. We say that $\mathfrak{X}_\Gamma$ is *arithmetically defined* if the subgroup $\Gamma \subset G(\mathbb{R})$ is arithmetic in the sense of §3.1. Finally, we recall that Γ is called a *lattice* if $\mathfrak{X}_\Gamma$ (or equivalently $\mathcal{G}/\Gamma$) has finite volume. As in §1, we let $L(\mathfrak{X}_\Gamma)$ denote the (weak) length spectrum of $\mathfrak{X}_\Gamma$.

Now, given two simple algebraic $\mathbb{R}$-groups G_1 and G_2, and discrete torsion-free subgroups $\Gamma_i \subset \mathcal{G}_i = G_i(\mathbb{R})$ for $i = 1, 2$, we will denote the corresponding locally symmetric spaces by $\mathfrak{X}_{\Gamma_i}$. The following statement establishes a connection between the length-commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ (i.e. the condition $\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1}) = \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2})$) and the weak commensurability of Γ_1 and Γ_2.

Theorem 6.2 ([33, Corollary 2.8])**.** *Assume that Γ_i is a* lattice *in $\mathcal{G}_i$. If the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then the subgroups Γ_1 and Γ_2 are weakly commensurable.*

Some remarks are in order. As for Riemann surfaces (cf. §1.2), closed geodesics in $\mathfrak{X}_\Gamma$ correspond to (nontrivial) semi-simple elements of Γ, but in the general case the equation for the length is significantly more complicated: instead of just the logarithm of an eigenvalue, we get basically the square root of a sum of squares of the logarithms of certain eigenvalues (see [30, Proposition 8.5] for the precise formula), although for lattices in simple groups not isogenous to $\mathrm{SL}_2(\mathbb{R})$ these eigenvalues are actually algebraic numbers. Unfortunately, at this point there are no results in transcendental number theory that would enable one to analyze expressions of this kind — most available results are for *linear* forms in terms of logarithms of algebraic numbers (cf. [4]). This forced us to base our analysis of the lengths of closed geodesics on a conjecture in transcendental number theory, known as Schanuel's conjecture, which is widely believed to be true but has been proven so far in very few situations (for the reader's convenience, we recall its statement below). The use of this conjecture is essential in the case of locally

[5] In characteristic zero the field of definition coincides with the trace field by Vinberg's theorem [42]; in positive characteristic, particularly in characteristics 2 and 3, the notion of the "right" field of definition is more tricky — see [27], but we will not get into these details here.

symmetric spaces of rank > 1, making our geometric results in this case *conditional on Schanuel's conjecture.* At the same time, the results for rank one spaces apart from the following exceptional case

($\mathcal{E}$): $G_1 = \mathrm{PGL}_2$ and Γ_1 cannot be conjugated into $\mathrm{PGL}_2(K)$ for any number field $K \subset \mathbb{R}$ while $G_2 \neq \mathrm{PGL}_2$,

rely only on the Gel'fond-Schneider Theorem (as a replacement of Schanuel's conjecture), hence are *unconditional.* Besides, a statement similar to Theorem 6.2 (and in fact more precise) can be proven under weaker conditions on Γ_1 and Γ_2 — see [33, Theorem 2.7]. A detailed discussion of these issues is contained in [33, §2] and will not be repeated here. So, we conclude simply by recalling the statement of Schanuel's conjecture.

6.3. Schanuel's conjecture. *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree (over $\mathbb{Q}$) of the field generated by*

$$z_1, \ldots, z_n; \; e^{z_1}, \ldots, e^{z_n}$$

is $\geqslant n$.

In fact, we will only need the following consequence of this conjecture: for nonzero algebraic numbers $a_1, \ldots, a_n$, (any values of) their logarithms

$$\log a_1, \ldots, \log a_n$$

are algebraically independent once they are linearly independent (over $\mathbb{Q}$).

Theorem 6.2 enables us to "translate" the algebraic results from §§2–3 about weakly commensurable Zariski-dense subgroups into the geometric setting. In particular, applying Theorems 2.3 and 2.4 we obtain the following.

Theorem 6.4. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_i}$ be a locally symmetric space of finite volume, of $\mathcal{G}_i, := G_i(\mathbb{R})$ for $i = 1, 2$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then* (i) *either G_1 and G_2 are of same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$,* (ii) *$K_{\Gamma_1} = K_{\Gamma_2}$.*

It should be pointed out that assuming Schanuel's conjecture in all cases, one can prove this theorem (in fact, a much stronger statement — see [32, Theorem 1] and [33, Theorem 8.1]) assuming only that Γ_1 and Γ_2 are finitely generated and Zariski-dense.

Next, using Theorems 3.4 and 3.5 we obtain

Theorem 6.5. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. Then the set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ of $\mathcal{G}_2$, which are length-commensurable to a given arithmetically defined locally symmetric space $\mathfrak{X}_{\Gamma_1}$ of $\mathcal{G}_1$, is a union of finitely many commensurability classes. In fact, it consists of a single commensurability class if G_1 and G_2 have the same type different from A_n, D_{2n+1}, with $n > 1$, or E_6.*

Furthermore, Theorems 3.6 and 3.7 imply the following rather surprising result which has so far defied all attempts to find a purely geometric proof.

Theorem 6.6. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be length-commensurable locally symmetric spaces of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined. Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

In fact, if one of the spaces is compact and the other is not, the weak length spectra $L(\mathfrak{X}_{\Gamma_1})$ and $L(\mathfrak{X}_{\Gamma_2})$ are quite different — see [32, Theorem 5] and [33, Theorem 8.6] for a precise statement (we note that the proof of this result uses Schanuel's conjecture in all cases).

Finally, we will describe some applications to isospectral compact locally symmetric spaces. So, in the remainder of this section, the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ as above will be assumed to be *compact*. Then, as we discussed in §1, the fact that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral implies that $L(\mathfrak{X}_{\Gamma_1}) = L(\mathfrak{X}_{\Gamma_2})$, so we can use our results on length-commensurable spaces. Thus, in particular we obtain the following.

Theorem 6.7. *If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and Γ_1 is arithmetic, then so is Γ_2.*

(Thus, the spectrum of the Laplace-Beltrami operator can see if the fundamental group is arithmetic or not — to our knowledge, no results of this kind, particularly for general locally symmetric spaces, were previously known in spectral theory.)

The following theorem settles the question "Can one hear the shape of a drum?" for arithmetically defined compact locally symmetric spaces.

Theorem 6.8. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be compact locally symmetric spaces associated with absolutely simple real algebraic groups G_1 and G_2, and assume that at least one of the spaces is arithmetically defined. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral then $G_1 = G_2 := G$. Moreover, unless G is of type A_n, D_{2n+1} $(n > 1)$, or E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.*

It should be noted that our methods based on length-commensurability or weak commensurability leave room for the following ambiguity in Theorem 6.8: either $G_1 = G_2$ or G_1 and G_2 are $\mathbb{R}$-split forms of types B_n and C_n for some $n \geqslant 3$ — and this ambiguity is unavoidable, cf. [32, Theorem 4] and the end of §7 in [33]. The fact that in the latter case the locally symmetric spaces cannot be isospectral was shown by Sai-Kee Yeung [39] by comparing the traces of the heat operator (without using Schanuel's conjecture), which leads to the statement of the theorem given above.

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