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To V. P. Platonov on the occasion of his 75th birthday

Division algebras with the same maximal subfields

V. I. Chernousov, A. S. Rapinchuk, and I. A. Rapinchuk

Abstract. This is a survey of recent results related to the problem of characterizing finite-dimensional division algebras by the set of isomorphism classes of their maximal subfields. Also discussed are various generalizations of this problem and some of its applications. In the last section the problem is extended to the context of absolutely almost simple algebraic groups.

Bibliography: 51 titles.

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Contents

1. Introduction

Our goal in this paper is to give an overview of some recent work on the problem of characterizing a division algebra in terms of its maximal subfields (and, more generally, a simple algebraic group in terms of its maximal tori). Our main focus will be on the following question, as well as some of its variations that will be introduced later:

(†) *What can one say about two finite-dimensional central division algebras D_1 and D_2 over a field K given that D_1 and D_2 have the same* (*isomorphism classes of*) *maximal subfields*?

To be precise, the condition on D_1 and D_2 means that they have the *same degree* n over K (that is, $\dim_K D_1 = \dim_K D_2 = n^2$), and a degree n field extension P/K admits a K-embedding $P \hookrightarrow D_1$ if and only if it admits a K-embedding $P \hookrightarrow D_2$.

Let us recall that for a central simple algebra A of degree n over K, a field extension F/K is called a *splitting field* of A if $A \otimes_K F \simeq M_n(F)$ as F-algebras.

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Furthermore, if A is a division algebra, then the splitting fields of degree n over K are precisely the maximal subfields of A (see, for instance, [15], Theorem 4.4). Since splitting fields and/or maximal subfields of a division K-algebra D (or, more generally, any finite-dimensional central simple algebra) are at the heart of the analysis of its structure, one is naturally led to ask to what extent these fields actually determine D. In the case when one considers *all* splitting fields, this question was answered in 1955 by Amitsur [1]:

If A_1 and A_2 are finite-dimensional central simple algebras over a field K that have the same splitting fields (that is, a field extension F/K splits A_1 if and only if it splits A_2), then the classes $[A_1]$ and $[A_2]$ in the Brauer group $\mathrm{Br}(K)$ *generate the same subgroup*: $\langle [A_1] \rangle = \langle [A_2] \rangle$.

The proof of this theorem (cf. [1], [21], Chap. 5) uses so-called *generic splitting fields*, which have *infinite* degree over K. However, the situation changes dramatically if one allows only finite-dimensional splitting fields, as seen in the following example with cubic division algebras over $\mathbb{Q}$.

We first recall the Albert–Brauer–Hasse–Noether Theorem, according to which, for a global field K there is an exact sequence

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in V^K} \mathrm{Br}(K_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0, \tag{ABHN}$$

where V^K is the set of all places of K, K_v denotes the completion of K at v, and inv_v is the so-called *invariant map* giving the isomorphism $\mathrm{Br}(K_v) \simeq \mathbb{Q}/\mathbb{Z}$ if v is a non-Archimedean place, and $\mathrm{Br}(K_v) \simeq (1/2)\,\mathbb{Z}/\mathbb{Z}$ if v is a real place (see, for instance, § 9.6 in Chap. VII of [2] and § 18.4 in [34] for number fields, § 6.5 in [21] for function fields, and [45] for a historical perspective). Now, fix an integer $r \geqslant 2$, and pick r distinct rational primes $p_1, \ldots, p_r$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be any r-tuple with $\varepsilon_i \in \{\pm 1\}$ such that $\sum_{i=1}^r \varepsilon_i \equiv 0 \pmod 3$. By (ABHN), there exists a cubic division algebra $D(\varepsilon)$ over $\mathbb{Q}$ with the following local invariants (considered as elements of $\mathbb{Q}/\mathbb{Z}$):

$$\mathrm{inv}_p\, D(\varepsilon) = \begin{cases} \dfrac{\varepsilon_i}{3}, & p = p_i \text{ for } i = 1, \ldots, r; \\ 0, & p \notin \{p_1, \ldots, p_r\} \text{ (including } p = \infty). \end{cases}$$

Then for any two r-tuples $\varepsilon' \neq \varepsilon''$ as above, the algebras $D(\varepsilon')$ and $D(\varepsilon'')$ have the same finite-dimensional splitting fields, hence the same maximal subfields (cf. [34], § 18.4, Corollary b), but are not isomorphic. Obviously, the number of such r-tuples ε grows with r, so this method enables one to construct an *arbitrarily large* (but finite) number of pairwise non-isomorphic cubic division algebras over $\mathbb{Q}$ having the same maximal subfields (at the same time, the cyclic subgroup $\langle [D(\varepsilon)] \rangle$ has order 3).

A similar construction can be carried out for division algebras of any degree $d > 2$. On the other hand, it follows from (ABHN) that a central *quaternion* division algebra D over a number field is determined up to isomorphism by its set of maximal subfields (see § 3). Thus, restricting attention to finite-dimensional splitting fields (in particular, maximal subfields) makes the question (†) rather delicate and interesting.

For the analysis of division algebras having the same maximal subfields, it is convenient to introduce the following notion. Suppose that D is a finite-dimensional central division algebra over a field K. We define the *genus* of D as

$$\mathbf{gen}(D) = \{[D'] \in \mathrm{Br}(K) \mid D' \text{ a division algebra having the same maximal subfields as } D\}.$$

Among the various questions that can be asked in relation to this definition, we will focus in this paper on the following two:

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element* (*that is*, *when is* D *determined uniquely up to isomorphism by its maximal subfields*)?

Question 2. *When is* $\mathbf{gen}(D)$ *finite*?

We will present the available results on Questions 1 and 2 in § 3, and a general technique for approaching such problems, based on an analysis of ramification of division algebras, will be outlined in § 4. Next, in § 5 we will briefly discuss several other useful notions of the genus, including the *local genus* and the *one-sided genus*. Finally, in § 6, we will give an overview of some ongoing work whose aim is to extend the analysis of division algebras with the same maximal subfields to the context of algebraic groups with the same (isomorphism or isogeny classes of) maximal tori.

Notation. Given a field K equipped with a discrete valuation v, we let K_v denote the completion of K with respect to v, $\mathscr{O}_v \subset K_v$ the valuation ring, and $\overline{K}_v$ the corresponding residue field. Furthermore, if K is a number field, then V^K will denote the set of all places of K, and V_∞^K the subset of Archimedean places. Finally, for a field K of characteristic $\neq 2$ and any pair of non-zero elements $a, b \in K^\times$, we will let $D = \left(\dfrac{a,b}{K}\right)$ be the associated quaternion algebra, that is, the 4-dimensional K-algebra with basis 1, i, j, k and multiplication determined by

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji.$$

2. Motivation

In this section we will describe two sources of motivation that naturally lead one to consider questions in the spirit of (†). The first exhibits a connection with the theory of quadratic forms, while the second (which for us was actually the deciding factor) stems from problems in differential geometry.

Let K be a field of characteristic $\neq 2$. To a quaternion algebra $D = \left(\dfrac{a,b}{K}\right)$, we associate the quadratic form

$$q(x, y, z) = ax^2 + by^2 - abz^2.$$

Note that, up to a sign, this is simply the form that gives the reduced norm of a pure quaternion, from which it follows that for any $d \in K^\times \setminus (K^\times)^2$, the quadratic extension $K(\sqrt{d})/K$ embeds into D if and only if d is represented by q (see [21], Remark 1.1.4).

Now suppose that we have two quaternion division algebras D_1 and D_2 with corresponding quadratic forms q_1 and q_2. Then these algebras have the same maximal subfields if and only if q_1 and q_2 represent the same elements over K. On the other hand, it is well known that D_1 and D_2 are K-isomorphic if and only if q_1 and q_2 are equivalent over K (cf. [34], § 1.7, Proposition). Thus, (†) leads us to the following natural question about quadratic forms:

Suppose that q_1 and q_2 are ternary forms with $\det = -1$ *that represent the same elements over K. Are q_1 and q_2 necessarily equivalent over K?*

Of course, the answer is *no* for general quadratic forms, but, as the results described in § 3 show, it may be *yes* for these special forms in certain situations.

Let us now turn to the geometric questions dealing with length-commensurable locally symmetric spaces that initially led to our interest in (†). The general philosophy in the study of Riemannian manifolds is that the isometry or commensurability class of a manifold M should to a significant extent be determined by its *length spectrum* $L(M)$ (which is the collection of the lengths of all closed geodesics). To put this into perspective, in the simplest case if M_1 and M_2 are 2-dimensional Euclidean spheres, then the closed geodesics are the great circles, and clearly these have the same lengths if and only if M_1 and M_2 are isometric. Furthermore, using the trace formula, one can relate this general idea to the problem of when two isospectral Riemannian manifolds (that is, those for which the spectra of the Laplace–Beltrami operators coincide) are isometric; informally, this is most famously expressed by Mark Kac's [24] question, "Can one hear the shape of a drum?"

To make things more concrete, and, at the same time, highlight the connections with (†), let us now consider what happens for Riemann surfaces of genus > 1 (we refer the reader to [36] for a detailed discussion of these questions for general locally symmetric spaces). Let

$$\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$$

be the upper half-plane with the standard hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$ and equipped with the usual isometric action of $\mathrm{SL}_2(\mathbb{R})$ by fractional linear transformations. Let $\pi\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical projection. Recall that any compact Riemann surface M of genus > 1 can be written as a quotient $\mathbb{H}/\Gamma$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup containing $\{\pm 1\}$ with torsion-free image $\pi(\Gamma)$ (cf., for instance, [17], Theorem 27.12). It is well known that closed geodesics in M correspond to non-trivial semisimple elements $\gamma \in \Gamma$. Furthermore, since $\pi(\Gamma)$ is discrete and torsion-free, any semisimple element $\gamma \in \Gamma$ with $\gamma \neq \pm 1$ is automatically hyperbolic, and hence $\pm\gamma$ is conjugate to a matrix of the form

$$\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix},$$

with t_γ a real number > 1. The length of the corresponding closed geodesic c_γ in M is then computed by the formula

$$\ell(c_\gamma) = \frac{2}{n_\gamma} \cdot \log t_\gamma, \tag{1}$$

where $n_\gamma \in \mathbb{Z}$ is a certain integer (in fact, a *winding number*; see [36], § 8 for further details). Note that (1) implies that

$$\mathbb{Q} \cdot L(M) = \mathbb{Q} \cdot \{\log |t_\gamma| \mid \gamma \in \Gamma \setminus \{\pm 1\} \text{ and semisimple}\}$$

(the set $\mathbb{Q} \cdot L(M)$ is sometimes called the *rational length spectrum* of M). In order to analyze this geometric set-up using algebraic and number-theoretic techniques, one considers the algebra

$$D = \mathbb{Q}[\Gamma^{(2)}] \subset M_2(\mathbb{R}),$$

where $\Gamma^{(2)} \subset \Gamma$ is the subgroup generated by squares. It turns out that D is a quaternion algebra whose centre is the *trace field* K of Γ, that is, the subfield generated over $\mathbb{Q}$ by the traces $\mathrm{Tr}(\gamma)$ for all $\gamma \in \Gamma^{(2)}$. Moreover, a non-central semisimple element $\gamma \in \Gamma^{(2)}$ gives rise to a maximal commutative étale subalgebra $K[\gamma]$ (see [31], § 3.2). We should point out that this algebra D is an important invariant of the group Γ; in particular, if Γ is an arithmetic group, then D is precisely the algebra involved in its description.

Now let $M_1 = \mathbb{H}/\Gamma_1$ and $M_2 = \mathbb{H}/\Gamma_2$ be two (compact) Riemann surfaces, with corresponding quaternion algebras $D_i = \mathbb{Q}[\Gamma_i^{(2)}]$ and trace fields $K_i = Z(D_i)$, for $i = 1, 2$. Assume that M_1 and M_2 are *length-commensurable* (L-C), that is,

$$\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2).$$

One then shows that $K_1 = K_2 =: K$ (see [36], Theorem 2).[1] Furthermore, it follows from (1) that for any non-trivial semisimple element $\gamma_1 \in \Gamma_1^{(2)}$, there exists a non-trivial semisimple element $\gamma_2 \in \Gamma_2^{(2)}$ such that

$$t_{\gamma_1}^m = t_{\gamma_2}^n$$

for some integers $m, n \geqslant 1$. Consequently, γ_1^m and $\gamma_2^n \in M_2(\mathbb{R})$ are conjugate, and hence we have an isomorphism of the corresponding étale algebras

$$K[\gamma_1] = K[\gamma_1^m] \simeq K[\gamma_2^n] = K[\gamma_2].$$

Thus, the geometric condition (L-C) translates into the algebraic condition that D_1 *and* D_2 *have the same isomorphism classes of maximal étale subalgebras intersecting* $\Gamma_1^{(2)}$ *and* $\Gamma_2^{(2)}$, *respectively.*

On the other hand, what one actually wants to prove is that (L-C) implies that M_1 and M_2 are in fact *commensurable* (that is, have a common finite-sheeted cover, or, equivalently, up to conjugation the subgroups Γ_1 and Γ_2 are commensurable in $\mathrm{GL}_2(\mathbb{R})$). If that is the case, then we necessarily have $D_1 \simeq D_2$ (see [31], Corollary 3.3.5). So our problem concerning Riemann surfaces leads to the following question about quaternion algebras:

Let D_1 *and* D_2 *be two quaternion algebras over the same field* K, *and let* $\Gamma_i \subset \mathrm{SL}(1, D_i)$, *for* $i = 1, 2$, *be Zariski-dense subgroups with trace field* K. *Assume*

[1] The equality of trace fields is actually proved assuming that Γ_1 and Γ_2 are *weakly commensurable*, which is a consequence of the length commensurability of the corresponding Riemann surfaces.

that D_1 and D_2 have the same isomorphism classes of maximal étale subalgebras that intersect Γ_1 and Γ_2, respectively. Are D_1 and D_2 isomorphic?

This is a more refined version of our original question (†). If K is a number field, then, as we have already mentioned, two quaternion division algebras with the same maximal subfields are isomorphic (see § 3 for further details). This fact was used by Reid [41] to show that any two *arithmetically defined* iso-length spectral Riemann surfaces (that is, M_1 and M_2 such that $L(M_1) = L(M_2)$) are commensurable. The general case is likely to depend on the resolution of questions like the one formulated above.

We would like to conclude our discussion with the following two remarks. First, it should be pointed out that if Γ is non-arithmetic, then it is not uniquely determined by the corresponding quaternion algebra (in fact, there may be infinitely many non-commensurable cocompact lattices having the same associated quaternion algebra; cf. [48]), so an affirmative answer to our question about algebras will not immediately yield consequences for Riemann surfaces. Second, along with the precise (quantitative) version of the question formulated above, one can consider a qualitative version, namely, whether we always have finitely many isomorphism classes of division algebras having the same maximal subfields. Let us note that some finiteness results are available even for non-arithmetic Riemann surfaces. For example, it is known that every class of isospectral compact Riemann surfaces consists of finitely many isometry classes; we recall that according to a well-known conjecture, every class of isospectral surfaces is expected to consist of a *single* commensurability class.

3. The genus of a division algebra

In this section we will give an overview of the available results on Questions 1 and 2 in § 1 about the genus of a division algebra.

We begin with a couple of remarks pertaining to Question 1. First, let us point out that $|\mathbf{gen}(D)| = 1$ is possible only if the class $[D]$ has exponent 2 in the Brauer group. Indeed, the opposite algebra D^{op} has the same maximal subfields as D. Then unless $D \simeq D^{\mathrm{op}}$ (which is equivalent to $[D]$ having exponent 2), we have $|\mathbf{gen}(D)| > 1$. At the same time, as the example in § 1 of cubic algebras over number fields shows, the genus may very well be larger than $\{[D], [D^{\mathrm{op}}]\}$. Another example of this phenomenon can be constructed as follows. Suppose that Δ_1 and Δ_2 are central division algebras over a field K that have relatively prime odd degrees, and consider the division algebras

$$D_1 = \Delta_1 \otimes_K \Delta_2 \quad \text{and} \quad D_2 = \Delta_1 \otimes_K \Delta_2^{\mathrm{op}}.$$

Then D_2 is not isomorphic to either D_1 or D_1^{op}, but D_1 and D_2 have the same splitting fields. Indeed, since the degrees of Δ_1 and Δ_2 are relatively prime, an extension splits D_1 if and only if it splits Δ_1 and Δ_2; then it also splits Δ_2^{op}, and hence D_2 (and vice versa). Thus, D_1 and D_2 have the same maximal subfields, and therefore $[D_2] \in \mathbf{gen}(D_1)$. Note that this example is in fact a consequence of the following general fact: if D is a central division algebra of degree n over a field K, then for any integer m relatively prime to n, the class $[D^{\otimes m}] = m[D] \in \mathrm{Br}(K)$

is represented by a central division algebra D_m of degree n which has the same maximal subfields as D (see [40], Lemma 3.6).

As we have already noted, the Albert–Brauer–Hasse–Noether Theorem (ABHN) enables one to give complete answers to Questions 1 and 2 when K is a number field. The precise statement is as follows.

Proposition 3.1. *Let K be a number field and let D be a finite-dimensional central division algebra over K.*

(a) *If $[D] \in \mathrm{Br}(K)$ has exponent 2 (in which case D is a quaternion algebra), then $|\mathbf{gen}(D)| = 1$.*

(b) *The genus $\mathbf{gen}(D)$ is finite for all D.*

Recall that according to (ABHN), we have an injective homomorphism

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in V^K} \mathrm{Br}(K_v),$$

where for each $v \in V^K$

$$\mathrm{Br}(K) \to \mathrm{Br}(K_v), \qquad [D] \mapsto [D \otimes_K K_v]$$

is the natural map. We say that a finite-dimensional central division algebra D over a number field K (or its class $[D] \in \mathrm{Br}(K)$) is *unramified* at $v \in V^K$ if its image in $\mathrm{Br}(K_v)$ is trivial, and *ramified* otherwise (note that this definition is consistent with the notion of ramification over general fields; see § 4, and particularly Example 4.1 below). The (finite) set of places where D is ramified will be denoted by $R(D)$.

Sketch of proof of Proposition 3.1. (a) First, we note that a division algebra D over K of exponent 2 is necessarily a quaternion algebra due to the equality of the exponent and the degree over number fields, which follows from the Grunwald–Wang Theorem (see, for instance, [33], Chap. VIII, § 2). Next, we consider the 2-torsion part of (ABHN):

$$0 \to {}_2\mathrm{Br}(K) \to \bigoplus_{v \in V^K} {}_2\mathrm{Br}(K_v).$$

Since ${}_2\mathrm{Br}(K_v)$ is either $\mathbb{Z}/2\mathbb{Z}$ or 0 for all $v \in V^K$, we see that an algebra D of exponent 2 over K is determined uniquely up to isomorphism by its set of ramified places $R(D)$. Consequently, to prove that $|\mathbf{gen}(D)| = 1$, it suffices to show that if D_1 and D_2 are two quaternion division algebras having the same maximal subfields, then $R(D_1) = R(D_2)$. This follows easily from weak approximation, together with the well-known criterion that for $d \in K^\times \setminus (K^\times)^2$,

$$L = K(\sqrt{d}\,) \text{ embeds into } D \;\Leftrightarrow\; d \notin (K_v^\times)^2 \text{ for all } v \in R(D) \tag{2}$$

(cf. [34], § 18.4, Corollary b). Indeed, suppose that there is a $v_0 \in R(D_1) \setminus R(D_2)$. Then using the openness of $(K_v^\times)^2 \subset K_v^\times$ and weak approximation, one can find $d \in K^\times \setminus (K^\times)^2$ such that

$$d \in (K_{v_0}^\times)^2, \quad \text{but} \quad d \notin (K_v^\times)^2 \text{ for all } v \in R(D_2).$$

Then according to (2), the quadratic extension $L = K(\sqrt{d}\,)$ embeds into D_2 but not into D_1, a contradiction.

(b) Let D be a division algebra over K of degree $n > 2$. We now consider the n-torsion part of (ABHN):

$$0 \to {}_n\mathrm{Br}(K) \to \bigoplus_{v \in V^K} {}_n\mathrm{Br}(K_v).$$

To establish the finiteness of $\mathbf{gen}(D)$, one first observes that if $D' \in \mathbf{gen}(D)$, then $R(D') = R(D)$ (cf. [34], § 18.4, Corollary b; in fact this is true not just over number fields — see Lemma 4.2 below). Since ${}_n\mathrm{Br}(K_v)$ is $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, or 0 for all $v \in V^K$ (see Example 4.1, (a) below), it follows that

$$|\mathbf{gen}(D)| \leqslant \left| \bigoplus_{v \in R(D)} {}_n\mathrm{Br}(K_v) \right| \leqslant n^r, \tag{3}$$

where $r = |R(D)|$.

For a concrete illustration of the argument presented above in the proof of (a), let us consider the following example.

Example 3.2. Let

$$D_1 = \left(\frac{-1, 3}{\mathbb{Q}}\right) \quad \text{and} \quad D_2 = \left(\frac{-1, 7}{\mathbb{Q}}\right).$$

One can check that $R(D_1) = \{2, 3\}$ and $R(D_2) = \{2, 7\}$, so D_1 and D_2 are non-isomorphic quaternion division algebras over $\mathbb{Q}$. Clearly, $10 \in (\mathbb{Q}_3^{\times})^2$ while $10 \notin (\mathbb{Q}_2^{\times})^2, (\mathbb{Q}_7^{\times})^2$. Thus, according to (2), the field $L = \mathbb{Q}(\sqrt{10}\,)$ embeds into D_2 but not into D_1. In other words, D_1 and D_2 are distinguished by their quadratic subfields. We would like to point out that the recent preprint [30] gives an effective way of producing, for two distinct quaternion algebras over an arbitrary number field, a quadratic field that distinguishes them by putting an explicit bound on its discriminant (Theorem 1.3 therein).

It is now natural to ask whether (and to what extent) these results for the genus carry over to general fields. Namely, can we expect the genus to be *trivial* for a quaternion division algebra and *finite* for any finite-dimensional division algebra over an *arbitrary* field K? It turns out that the answer is *no* in both cases. Several people, including Garibaldi, Rost, Saltman, Schacher, Wadsworth, and others have given a construction of quaternion algebras with non-trivial genus over certain very large fields of characteristic $\neq 2$.[2] We refer the reader to [20], § 2 for the full details, and only sketch the main ideas here.

Let D_1 and D_2 be two non-isomorphic quaternion division algebras over a field k of characteristic $\neq 2$ that have a common quadratic subfield (for example, one can take $k = \mathbb{Q}$ and the quaternion algebras D_1 and D_2 considered in Example 3.2

[2]We note that the construction was recently adapted to fields of characteristic 2 in the preprint by A. Chapman, A. Dolphin, and A. Laghribi, *Total linkage of quaternion algebras in characteristic two*, arXiv:1403.6682.

above). If D_1 and D_2 already have the same quadratic subfields, we are done. Otherwise, there exists a quadratic extension $k(\sqrt{d})$ that embeds into D_1 but not into D_2. Applying some general results on quadratic forms to the norm forms of D_1 and D_2, one shows that there exists a field extension $k^{(1)}$ of k (which is the function field of an appropriate quadric) such that

- $D_1 \otimes_k k^{(1)}$ and $D_2 \otimes_k k^{(1)}$ are non-isomorphic division algebras over $k^{(1)}$, **but**
- $k^{(1)}(\sqrt{d}\,)$ embeds into $D_2 \otimes_k k^{(1)}$.

One deals with other subfields one at a time by applying the same procedure to the algebras obtained from D_1 and D_2 by extending scalars to the field extension constructed at the previous step. This process generates an ascending chain of fields

$$k^{(1)} \subset k^{(2)} \subset k^{(3)} \subset \cdots,$$

and we let K be the union (direct limit) of this chain. Then $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are non-isomorphic quaternion division K-algebras having the same quadratic subfields; in particular $|\mathbf{gen}(D_1 \otimes_k K)| > 1$. Note that the resulting field K has infinite transcendence degree over k, and, in particular, is *infinitely generated*. Furthermore, a modification of the above construction (cf. [32]) enables one to start with an infinite sequence $D_1, D_2, D_3, \ldots$ of quaternion division algebras over a field k of characteristic $\neq 2$ that are pairwise non-isomorphic but share a common quadratic subfield (for example, one can take $k = \mathbb{Q}$ and consider the family of algebras of the form $\left(\frac{-1, p}{\mathbb{Q}}\right)$, where p is a prime $\equiv 3 \pmod 4$), and then build an infinitely generated field extension K/k such that the algebras $D_i \otimes_k K$ become pairwise non-isomorphic quaternion division algebras with any two of them having the same quadratic subfields. In particular, this yields an example of quaternion division algebras with *infinite* genus. More recently, a similar approach was used in [47] to show that for any prime p, there exist a field K and a central division algebra D over K of degree p such that $\mathbf{gen}(D)$ is infinite.

Thus, we see that while Questions 1 and 2 can be answered completely and in the affirmative over number fields, and more generally, over global fields, these questions become non-trivial over arbitrary fields. In fact, until quite recently, very little was known about the situation over fields other than global. As we mentioned in § 2, the triviality of the genus for quaternion division algebras D over number fields has consequences for arithmetically defined Riemann surfaces. In connection with their work on length-commensurable locally symmetric spaces in [36], Prasad and the second-named author of this paper asked whether $|\mathbf{gen}(D)| = 1$ for any central quaternion division algebra D over $K = \mathbb{Q}(x)$.[3] This question was answered in the affirmative by Saltman, and later, in a joint paper with Garibaldi, it was shown that any quaternion division algebra over $k(x)$, where k is an arbitrary number field, has trivial genus (in fact, they considered more generally so-called *transparent* fields of characteristic $\neq 2$; see [20]). Motivated by this result, we showed in [40] that for a given field k of characteristic $\neq 2$ the triviality of the genus for quaternion division

[3] Such questions have certainly been around informally for quite some time, at least since the work of Amitsur [1] mentioned in § 1, but to the best of our knowledge, this was the first 'official' formulation of a question along these lines, though the terminology of the genus was introduced later.

algebras over k is a property that is stable under purely transcendental extensions. Subsequently, we established a similar *Stability Theorem* for algebras of exponent 2. The precise statements are given in the following result.

Theorem 3.3 (see [40], Theorem A and [5], Theorem 3.5). *Let k be a field of characteristic $\neq 2$.*

(a) *If $|\mathbf{gen}(D)| = 1$ for any quaternion division algebra D over k, then*

$$|\mathbf{gen}(D')| = 1$$

for any quaternion division algebra D' over the field $k(x)$ of rational functions.

(b) *If $|\mathbf{gen}(D)| = 1$ for any central division k-algebra D of exponent 2, then the same property holds for any central division algebra of exponent 2 over $k(x)$.*

As an immediate consequence, we have the following.

Corollary 3.4. *If k is either a number field or a finite field of characteristic $\neq 2$, and $K = k(x_1, \ldots, x_r)$ is a purely transcendental extension, then $|\mathbf{gen}(D)| = 1$ for any central division K-algebra D of exponent 2.*

Remark 3.5. In [26] the Stability Theorem has been generalized to function fields of Severi–Brauer varieties of algebras of odd degree.

Let us now turn to Question 2. As the above discussion shows, one cannot hope to have a finiteness result for the genus over completely arbitrary fields. Nevertheless, we have obtained the following finiteness statement over finitely generated fields.

Theorem 3.6 ([4], Theorem 3). *Let K be a finitely generated field. If D is a central division K-algebra of exponent prime to* $\operatorname{char} K$, *then* $\mathbf{gen}(D)$ *is finite.*

Remark 3.7. (a) It is still an open question whether there exist quaternion division algebras over finitely generated fields with *non-trivial* genus.

(b) In [27] Krashen and McKinnie studied division algebras having the same finite-dimensional *splitting fields* (see § 5 below for the associated notion of the genus).

4. Ramification of division algebras

In this section we will describe some of the ideas involved in the proofs of Theorems 3.3 and 3.6. In broad terms, the arguments rely on many of the same general considerations that were already encountered in our discussion of the genus over number fields. More precisely, even though there is no direct counterpart of (ABHN) for arbitrary (finitely generated) fields, the analysis of *ramification* of division algebras nevertheless again plays a key role.

We begin by recalling the standard set-up used in the analysis of ramification of division algebras. Let K be a field equipped with a discrete valuation v, denote by $\mathscr{G}^{(v)} = \mathrm{Gal}(\overline{K}_v^{\,\mathrm{sep}}/\overline{K}_v)$ the absolute Galois group of the residue field $\overline{K}_v$ of K_v, and fix an integer $n > 1$. If either n is prime to $\operatorname{char} \overline{K}_v$ or $\overline{K}_v$ is perfect, then there exists a *residue map*

$$r_v \colon {}_n\mathrm{Br}(K_v) \to \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}), \tag{4}$$

where the group on the right is the group of *continuous* characters mod n of $\mathscr{G}^{(v)}$ (see [43], § 10). Furthermore, it is known that r_v is surjective. A division algebra D over K_v of exponent n (or its corresponding class $[D] \in {}_n\mathrm{Br}(K_v)$) is said to be *unramified* if $r_v([D]) = 0$, and the *unramified Brauer group at* v is defined as

$$ {}_n\mathrm{Br}(K_v)_{\{v\}} = \ker r_v. $$

Thus, by construction, we have an exact sequence

$$ 0 \to {}_n\mathrm{Br}(K_v)_{\{v\}} \to {}_n\mathrm{Br}(K_v) \xrightarrow{r_v} \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}) \to 0. \tag{5} $$

It can be shown that there is an isomorphism ${}_n\mathrm{Br}(K_v)_{\{v\}} \simeq {}_n\mathrm{Br}(\overline{K}_v)$, and moreover, the latter group is naturally identified with ${}_n\mathrm{Br}(\mathscr{O}_v)$, where $\mathscr{O}_v$ is the valuation ring of K_v (see [51], Theorem 3.2 and the subsequent discussion). In other words, the algebras that are unramified at v are precisely the ones that arise from *Azumaya algebras* over $\mathscr{O}_v$.

Next, composing the map in (4) with the natural homomorphism ${}_n\mathrm{Br}(K) \to {}_n\mathrm{Br}(K_v)$, we obtain a residue map

$$ \rho_v \colon {}_n\mathrm{Br}(K) \to \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}), \tag{6} $$

and again one says that a division algebra D over K is *unramified* if $\rho_v([D]) = 0$. For fields other than local, to get information about ${}_n\mathrm{Br}(K)$, one typically uses not just one valuation of K, but rather a suitable set of valuations. For this purpose, it is convenient to make the following definition. Fix an integer $n > 1$ and suppose that V is a set of discrete valuations of K such that the residue maps ρ_v exist for all $v \in V$. We define the *unramified* (n*-torsion*) *Brauer group with respect to* V as

$$ {}_n\mathrm{Br}(K)_V = \bigcap_{v \in V} \ker \rho_v. $$

Example 4.1. (a) Let p be a prime, $K = \mathbb{Q}_p$, and $v = v_p$ the corresponding p-adic valuation. Then since $\mathbb{F}_p$ is perfect and $\mathrm{Br}(\mathbb{F}_p) = \{0\}$, it follows from (5) that for any $n > 1$ we have an isomorphism

$$ {}_n\mathrm{Br}(\mathbb{Q}_p) \overset{r_v}{\simeq} \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{\mathbb{Z}}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \simeq (1/n)\mathbb{Z}/\mathbb{Z}. $$

Taking the direct limit over all n, we obtain an isomorphism $\mathrm{Br}(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}$, which is precisely the invariant map from local class field theory. This description of the Brauer group immediately extends to any finite extension of $\mathbb{Q}_p$. An important point here is that only the trivial algebra is unramified over such a field (with respect to its natural valuation).

(b) Let K be a number field. It follows from part (a) that a finite-dimensional central division algebra D over K is unramified at a non-Archimedean place $v \in V^K$ if and only if its image in $\mathrm{Br}(K_v)$ is trivial (which is consistent with the definition that we used in § 3). Combining this with (ABHN), we see that if $S \subset V^K$ is a finite set containing V^K_∞, then for $V = V^K \setminus S$ and any integer $n > 1$ the unramified Brauer group ${}_n\mathrm{Br}(K)_V$ is finite.

(c) Let C be a smooth connected projective curve over a field k and $K = k(C)$ the function field of C. For each closed point $P \in C$, we have a discrete valuation v_P on K. The set of discrete valuations

$$V_0 = \{v_P \mid P \in C \text{ a closed point}\}$$

is usually called the set of *geometric places* of K (note that these are precisely the discrete valuations of K that are trivial on k). If $n > 1$ is an integer that is relatively prime to $\operatorname{char} k$, then for each $v_P \in V_0$ we have a residue map

$$\rho_P = \rho_{v_P} \colon {}_n\mathrm{Br}(K) \to H^1(G_P, \mathbb{Z}/n\mathbb{Z}),$$

where G_P is the absolute Galois group of the residue field at P. In this case the corresponding unramified Brauer group ${}_n\mathrm{Br}(K)_{V_0}$ will be denoted, following tradition, by ${}_n\mathrm{Br}(K)_{\mathrm{ur}}$. Furthermore, if k is a perfect field and C is geometrically connected, then the above residue map ρ_P extends to a map $\mathrm{Br}(K) \to H^1(G_P, \mathbb{Q}/\mathbb{Z})$ defined on the entire Brauer group. We can then consider the map

$$\mathrm{Br}(K) \xrightarrow{\bigoplus \rho_P} \bigoplus_{P \in C_0} H^1(G_P, \mathbb{Q}/\mathbb{Z}),$$

where C_0 denotes the set of closed points of C. The kernel

$$\ker\Big(\bigoplus \rho_P\Big) =: \mathrm{Br}(K)_{\mathrm{ur}}$$

is known to coincide with the Brauer group $\mathrm{Br}(C)$ defined either in terms of Azumaya algebras or in terms of étale cohomology (see [21], § 6.4 and [29]). Then the group ${}_n\mathrm{Br}(K)_{\mathrm{ur}}$ is precisely the n-torsion of $\mathrm{Br}(C)$.

Returning to the general set-up, we would now like to mention a result that describes the ramification behavior of division algebras lying in the same genus. Let K be a field equipped with a discrete valuation v and $n > 1$ an integer that is relatively prime to the characteristic of the residue field $\overline{K}_v$. We have the following result.

Lemma 4.2. *Let D and D' be central division K-algebras such that*

$$[D] \in {}_n\mathrm{Br}(K) \quad \textit{and} \quad [D'] \in \mathbf{gen}(D) \cap {}_n\mathrm{Br}(K).$$

Let χ_v and $\chi'_v \in \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z})$ denote the images of $[D]$ and $[D']$, respectively, under the residue map ρ_v. Then

$$\ker \chi_v = \ker \chi'_v.$$

In particular, D is unramified at v if and only if D' is.

For the argument, let us recall that if $\mathscr{K}$ is a field that is complete with respect to a discrete valuation v and $\mathscr{D}$ is a finite-dimensional central division algebra over $\mathscr{K}$, then v extends uniquely to a discrete valuation $\widetilde{v}$ on $\mathscr{D}$ (see [44], Chap. XII, § 2, [51]). Furthermore, the corresponding valuation ring $\mathscr{O}_{\mathscr{D}}$ has a unique maximal 2-sided ideal $\mathfrak{P}_{\mathscr{D}}$ (the valuation ideal), and the quotient $\overline{\mathscr{D}} = \mathscr{O}_{\mathscr{D}}/\mathfrak{P}_{\mathscr{D}}$ is a finite-dimensional division (but not necessarily central) algebra, called the *residue algebra*, over the residue field $\overline{\mathscr{K}}$.

Proof (*Sketch*). We write

$$D \otimes_K K_v = M_\ell(\mathscr{D}) \quad \text{and} \quad D' \otimes_K K_v = M_{\ell'}(\mathscr{D}'),$$

where $\mathscr{D}$ and $\mathscr{D}'$ are central division algebras over K_v. First, one shows that $\ell = \ell'$ and $\mathscr{D}$ and $\mathscr{D}'$ have the same maximal subfields ([40], Corollary 2.4). Next, one verifies that the centres $\mathscr{E}$ and $\mathscr{E}'$ of the residue algebras $\overline{\mathscr{D}}$ and $\overline{\mathscr{D}}'$, respectively, coincide ([5], Lemma 2.3). Finally, using the fact that $\ker \chi_v$ and $\ker \chi'_v$ are precisely the subgroups of $\mathscr{G}^{(v)}$ corresponding to $\mathscr{E}$ and $\mathscr{E}'$, respectively (cf. [51], Theorem 3.5), we conclude that

$$\ker \chi_v = \ker \chi'_v,$$

as needed. □

We are now in a position to outline the proof of part (a) of Theorem 3.3 (see the proof of Theorem A in [40] for full details). Let k be a field of characteristic $\neq 2$ such that $|\mathbf{gen}(D)| = 1$ for every quaternion division algebra D over k. Viewing $k(x)$ as the function field of $\mathbb{P}^1_k$, we consider the following segment of *Faddeev's Exact Sequence* (see [21], Corollary 6.4.6):

$$0 \to {}_2\mathrm{Br}(k) \to {}_2\mathrm{Br}(k(x)) \xrightarrow{\bigoplus \rho_P} \bigoplus_{P \in (\mathbb{P}^1_k)_0} H^1(G_P, \mathbb{Z}/2\mathbb{Z}). \tag{7}$$

Suppose now that D and D' are quaternion division algebras over $k(x)$ that have the same maximal subfields. Then it follows from Lemma 4.2, together with the fact that $[D]$ and $[D']$ have exponent 2 in $\mathrm{Br}(k(x))$, that

$$\rho_P([D]) = \rho_P([D'])$$

for all $P \in (\mathbb{P}^1_k)_0$. Consequently, (7) implies that

$$[D] = [D'] \cdot [\Delta \otimes_k k(x)]$$

for some central quaternion algebra Δ over k (see [40], Corollary 4.2). A specialization argument, which relies on the assumption that quaternion division algebras over k have trivial genus, then shows that $[\Delta] = 0$, and hence $D \simeq D'$. Consequently $|\mathbf{gen}(D)| = 1$, as required.

We would now like to indicate the main elements of the proof of Theorem 3.6, which is also based on an analysis of ramification. Let K be a finitely generated field and fix an integer $n > 1$ relatively prime to $\operatorname{char} K$. We will need to consider sets V of discrete valuations of K satisfying the following two properties:

(I) *for any* $a \in K^\times$ *the set* $V(a) := \{v \in V \mid v(a) \neq 0\}$ *is finite*;

(II) *for any* $v \in V$ *the characteristic of the residue field* $\overline{K}_v$ *is prime to* n.

Note that (II) ensures the existence of a residue map

$${}_n\mathrm{Br}(K) \xrightarrow{\rho_v} \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z})$$

for each $v \in V$. When considering the case of number fields in § 3, we observed that the set of ramified places is finite for any division algebra D, which, with the help of (ABHN), led to the upper bound (3) on the size of $\mathbf{gen}(D)$. Over general fields, condition (I) again guarantees the finiteness of the set

$$R(D) = R(D, V) = \{v \in V \mid \rho_v([D]) \neq 0\}$$

of ramified places for any division algebra D of degree n over K (see [5], Proposition 2.1 for a slightly more general statement). To obtain an analogue of (3), we argue as follows, using Lemma 4.2. Suppose that D is a central division algebra over K of degree n and let $[D'] \in \mathbf{gen}(D)$. For $v \in V$, we set

$$\chi_v = \rho_v([D]) \quad \text{and} \quad \chi'_v = \rho_v([D']);$$

recall that by Lemma 4.2 we have $\ker \chi_v = \ker \chi'_v$. Note that if the character χ_v has order $m \mid n$, then any character χ'_v of $\mathscr{G}^{(v)}$ with the same kernel can be viewed as a faithful character of the cyclic group $\mathscr{G}^{(v)}/\ker \chi_v$ of order m. Consequently, there are $\varphi(m)$ possibilities for χ', and therefore

$$|\rho_v(\mathbf{gen}(D))| \leqslant \varphi(m) \leqslant \varphi(n) \tag{8}$$

for any $v \in V$ (since m divides n), and

$$\rho_v(\mathbf{gen}(D)) = \{1\},$$

if $\rho_v([D]) = 1$. Letting $\rho = (\rho_v)_{v \in V}$ be the direct sum of the residue maps for all $v \in V$, it follows that

$$|\rho(\mathbf{gen}(D))| \leqslant \varphi(n)^r, \tag{9}$$

where $r = |R(D)|$. Therefore, if $\ker \rho = {}_n\mathrm{Br}(K)_V$ is finite, then we obtain the estimate

$$|\mathbf{gen}(D)| \leqslant \varphi(n)^r \cdot |{}_n\mathrm{Br}(K)_V|. \tag{10}$$

Thus, the proof of Theorem 3.6 boils down to establishing the finiteness of the unramified Brauer group with respect to an appropriate set of discrete valuations, which is the subject matter of the next result.

Theorem 4.3 ([4], Theorem 8). *Let K be a finitely generated field and $n > 1$ an integer prime to* char K. *Then there exists a set V of discrete valuations of K that satisfies conditions* (I) *and* (II) *above and for which ${}_n\mathrm{Br}(K)_V$ is finite.*

Our proof of this theorem, which will be outlined below, relies on the analysis of the exact sequence for the Brauer group of a curve. Subsequently, it was pointed out to us by J.-L. Colliot-Thélène that the finiteness statement could also be derived from certain results in étale cohomology. More precisely, in this argument we present K as the field of rational functions on a smooth arithmetic scheme X with n invertible on X. Then one uses Deligne's finiteness theorem for the étale cohomology of constructible sheaves (Theorem 1.1 of the Chapter "Théorèmes de finitude" in [13]) to show that in this case, the n-torsion of the étale Brauer group is finite. On the other hand, by Gabber's purity theorem (see [18] for an exposition of Gabber's proof, and an account of the history of the question on p. 153 in [10] and in the discussion after Theorem 4.2 in [9]), the latter group coincides with the unramified Brauer group of K with respect to the set V of discrete valuations of

K associated with the prime divisors on X (see [4] for more details). We will use the term *divisorial* to describe such a set of valuations. Note that this argument allows quite a bit of flexibility in the choice of V: for example, X can be replaced with an open subscheme, which allows us to delete from V any *finite* set. This flexibility somewhat simplifies the proof of the finiteness of the genus. Indeed, for a given division algebra D, we can choose V so that D is unramified at all places of V. Then we do not need the full strength of Lemma 4.2 but only the fact that any $[D'] \in \mathbf{gen}(D)$ is also unramified at all $v \in V$ (see Theorem 6.7 below for an analogue of this for arbitrary algebraic groups). This immediately leads to the conclusion that

$$|\mathbf{gen}(D)| \leqslant |{}_n\mathrm{Br}(K)_V| < \infty.$$

The main disadvantage here is that this argument does not give any *explicit* estimates on the size of the genus.

On the other hand, our original proof of Theorem 4.3 *does* in principle allow one to obtain explicit estimates on the size of $\mathbf{gen}(D)$ in certain cases. We will only sketch the main ideas here; further details will be available in [6]. For simplicity, suppose that K is a finitely generated field of characteristic 0. Then K can be realized as the function field $k(C)$ of a smooth projective geometrically irreducible curve C over a field k which is a purely transcendental extension of a number field P. Let V_0 be the set of geometric places of K. It is well known (see, for instance, [29]) that the geometric Brauer group $\mathrm{Br}(K)_{\mathrm{ur}} = \mathrm{Br}(K)_{V_0}$ fits into an exact sequence

$$\mathrm{Br}(k) \xrightarrow{\iota_k} \mathrm{Br}(k(C))_{\mathrm{ur}} \xrightarrow{\omega} H^1(k, J)/\Phi(C, k), \tag{11}$$

where ι_k is the natural map, J is the Jacobian of C, and $\Phi(C, k)$ is a certain finite cyclic subgroup of $H^1(k, J)$ (in fact, if $C(k) \neq \varnothing$, then $\Phi(C, k) = 0$ and (11) becomes a split exact sequence). We take our required set V of discrete valuations of K to consist of V_0 together with a set V_1 of extensions to K of an appropriate set of valuations of the field k (if k is a number field, then V_1 consists of extensions to K of almost all non-Archimedean places of k). Then ${}_n\mathrm{Br}(K)_V \subset {}_n\mathrm{Br}(K)_{\mathrm{ur}}$ and the proof of Theorem 4.3 reduces to verifying the finiteness of $\iota_k^{-1}({}_n\mathrm{Br}(K)_V)$ and $\omega({}_n\mathrm{Br}(K)_V)$.

The proof of the finiteness of $\iota_k^{-1}({}_n\mathrm{Br}(K)_V)$ relies on certain properties of our presentation of K as $k(C)$, together with Faddeev's exact sequence (to relate the Brauer group of k to that of the number field P) and (ABHN). Our approach for proving the finiteness of $\omega({}_n\mathrm{Br}(K)_V)$ is inspired by the proof of the Weak Mordell–Weil Theorem for elliptic curves and involves the analysis of unramified cohomology classes. We should point out that this argument, although with some extra work, also allows one to delete from the constructed set V any finite subset and still retain the finiteness of ${}_n\mathrm{Br}(K)_V$.

To illustrate things in more concrete terms, we would like to conclude this section by sketching the proof of Theorem 4.3 in the case when K is the function field of an elliptic curve over a number field and $n = 2$ (see [5], § 4). The set-up that we will consider is as follows. Let k be a number field and E be an elliptic curve over k given by a Weierstrass equation

$$y^2 = f(x), \quad \text{where} \quad f(x) = x^3 + \alpha x^2 + \beta x + \gamma. \tag{12}$$

Denote by $\delta \neq 0$ the discriminant of f. We will assume that E has k-rational 2-torsion, that is, f has three (distinct) roots in k:

$$f(x) = (x-a)(x-b)(x-c).$$

Let

$$K := k(E) = k(x,y)$$

be the function field of E. Since $E(k) \neq \varnothing$ and E coincides with its Jacobian, the sequence (11) yields the exact sequence

$$0 \to {}_2\mathrm{Br}(k) \to {}_2\mathrm{Br}(K)_{\mathrm{ur}} \xrightarrow{\omega} {}_2H^1(k,E) \to 0. \tag{13}$$

In fact, this sequence is split, with a section for ω constructed as follows. The Kummer sequence

$$0 \to E[2] \to E \xrightarrow{\times 2} E \to 0$$

yields the exact sequence of cohomology

$$0 \to E(k)/2E(k) \to H^1(k,E[2]) \xrightarrow{\sigma} {}_2H^1(k,E) \to 0. \tag{14}$$

Since $E[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as Galois modules, we have

$$H^1(k,E[2]) \simeq k^\times/(k^\times)^2 \times k^\times/(k^\times)^2,$$

and we define a map

$$\nu\colon H^1(k,E[2]) \to {}_2\mathrm{Br}(k(E))_{\mathrm{ur}}, \qquad (r,s) \mapsto \left[\left(\frac{r, x-b}{K}\right) \otimes_K \left(\frac{s, x-c}{K}\right)\right].$$

One then checks that $\omega \circ \nu = \sigma$ and $\nu(\ker \sigma) = 0$, which yields the required section

$$\varepsilon\colon {}_2H^1(k,E) \to {}_2\mathrm{Br}(k(E))_{\mathrm{ur}}.$$

This leads to the following description of the geometric Brauer group, which is in fact valid over any field k of characteristic $\neq 2, 3$.

Theorem 4.4 ([3], Theorem 3.6). *Assume that the elliptic curve E given by (12) has k-rational 2-torsion, that is,*

$$f(x) = (x-a)(x-b)(x-c), \quad \text{with } a,b,c \in k.$$

Then

$${}_2\mathrm{Br}(K)_{\mathrm{ur}} = {}_2\mathrm{Br}(k) \oplus I,$$

where ${}_2\mathrm{Br}(k)$ is identified with a subgroup of ${}_2\mathrm{Br}(K)$ via the canonical map $\mathrm{Br}(k) \to \mathrm{Br}(K)$, and $I \subset {}_2\mathrm{Br}(K)_{\mathrm{ur}}$ is a subgroup such that every element of I is represented by a bi-quaternion algebra of the form

$$\left(\frac{r, x-b}{K}\right) \otimes_K \left(\frac{s, x-c}{K}\right)$$

for some $r, s \in k^\times$.

As we mentioned above, the required set V will consist of the set V_0 of geometric places together with a set V_1 of extensions of almost all non-Archimedean places of k to K which is obtained as follows. For $s \in k^\times$ we denote by $V^k(s)$ the finite set $\{v \in V^k \setminus V^k_\infty \mid v(s) \neq 0\}$. Fix a finite set of valuations $S \subset V^k$ containing $V^k_\infty \cup V^k(2) \cup V^k(\delta)$, as well as all those non-Archimedean $v \in V^k$ for which at least one of α, β, γ has a negative value. For a non-Archimedean $v \in V^k$, let $\widetilde{v}$ denote its extension to $F := k(y)$ given by

$$\widetilde{v}(p(y)) = \min_{a_i \neq 0} v(a_i), \quad \text{for} \quad p(y) = a_n y^n + \cdots + a_0 \in k[y], \quad p \neq 0. \tag{15}$$

Now K is a cubic extension of F, and one shows that for $v \in V^k \setminus S$ the valuation $\widetilde{v}$ has a *unique* extension to K, which we will denote by $w = w(v)$ (see [5], Lemma 4.5). We set

$$V_1 = \{w(v) \mid v \in V^k \setminus S\}$$

and let $V = V_0 \cup V_1$.

Proposition 4.5. *The unramified Brauer group ${}_2\mathrm{Br}(K)_V$ is finite.*

Proof (*Sketch*). We consider separately the ramification behaviour at places in V_1 of the constant and bi-quaternionic parts of elements of ${}_2\mathrm{Br}(K)_{\mathrm{ur}}$ in the decomposition given by Theorem 4.4. Let us write $[D] \in {}_2\mathrm{Br}(K)_V$ in the form

$$[D] = [\Delta \otimes_k K] \otimes_K \left[\left(\frac{r, x-b}{K} \right) \otimes_K \left(\frac{s, x-c}{K} \right) \right],$$

where $[\Delta] \in {}_2\mathrm{Br}(k)$ is a quaternion algebra and $r, s \in k^\times$. By using properties of corestriction as well as an explicit description of residue maps in this situation (see [5], Proposition 4.7 and Lemma 4.8), one shows that for any $v \in V^k \setminus S$, the quaternion algebra Δ is unramified at v and also that

$$v(r), v(s) \equiv 0 \pmod{2}. \tag{16}$$

The finiteness of ${}_2\mathrm{Br}(K)_V$ then follows. Indeed, as we saw in Example 4.1, (b), the unramified Brauer group ${}_2\mathrm{Br}(k)_{V^k \setminus S}$ is finite, and hence there are only finitely many possibilities for $[\Delta]$. On the other hand, it is a well-known consequence of the finiteness of the class number and the fact that the group of units is finitely generated that the image under the canonical map $k^\times \to k^\times/(k^\times)^2$ of the set

$$P(k, S) = \{x \in k^\times \mid v(x) \equiv 0 \pmod{2} \text{ for all } v \in V^k \setminus S\}$$

is finite, which yields the finiteness of the set of bi-quaternionic parts. $\square$

We will now sketch a cohomological proof of (16), which is similar to an argument used in the standard proof of the Weak Mordell–Weil Theorem (see, for instance, [46], Chap. VIII, § 2). First, we recall the following definition. Let $v \in V^k \setminus V^k_\infty$. We say that $x \in H^1(k, E[2])$ is *unramified* at v if

$$x \in \ker\big(H^1(k, E[2]) \xrightarrow{\mathrm{res}_v} H^1(k_v^{\mathrm{ur}}, E[2])\big),$$

where k_v^{ur} is the maximal unramified extension of the completion k_v, and res_v is the usual restriction map. Furthermore, given a set $U \subset V^k \setminus V_\infty^k$, we define the corresponding unramified cohomology group by

$$H^1(k, E[2])_U = \bigcap_{v \in U} \ker\bigl(H^1(k, E[2]) \xrightarrow{\mathrm{res}_v} H^1(k_v^{\mathrm{ur}}, E[2])\bigr).$$

One now shows that if $x \in I$ lies in ${}_2\mathrm{Br}(K)_V$, then

$$\sigma^{-1}(\omega(x)) \subset H^1(k, E[2])_{V^k \setminus S},$$

where $\sigma\colon H^1(k, E[2]) \to {}_2H(k, E)$ is the map appearing in (14). On the other hand, it is well known that in the canonical isomorphism

$$H^1(k_v^{\mathrm{ur}}, \mathbb{Z}/2\mathbb{Z}) \simeq (k_v^{\mathrm{ur}})^\times/(k_v^{\mathrm{ur}})^{\times 2} \tag{17}$$

a coset $a(k_v^{\mathrm{ur}})^{\times 2} \in (k_v^{\mathrm{ur}})^\times/(k_v^{\mathrm{ur}})^{\times 2}$ corresponds to the character

$$\chi_a\colon \mathrm{Gal}((k_v^{\mathrm{ur}})^{\mathrm{sep}}/k_v^{\mathrm{ur}}) \to \mathbb{Z}/2\mathbb{Z}$$

with kernel $\ker \chi_a = \mathrm{Gal}((k_v^{\mathrm{ur}})^{\mathrm{sep}}/k_v^{\mathrm{ur}}(\sqrt{a}\,))$ (see, for instance, [21], Proposition 4.3.6 and Corollary 4.3.9). It follows that if $a(k_v^{\mathrm{ur}})^{\times 2}$ corresponds to a cohomology class that is unramified at v, then $\sqrt{a} \in k_v^{\mathrm{ur}}$, and consequently $v(a) \equiv 0 \pmod 2$ (see [28], Proposition 1.3). In particular, this, together with the description of the geometric Brauer group given above, shows that if

$$\left[\left(\frac{r, x-b}{K}\right) \otimes_K \left(\frac{s, x-c}{K}\right)\right] \in I$$

is unramified at a place $w(v) \in V_1$, then (16) holds.

We would also like to observe that the proof sketched above not only gives the finiteness of ${}_2\mathrm{Br}(K)_V$, but in fact also yields an explicit upper bound on the size of the unramified Brauer group. More precisely, we have the following result.

Theorem 4.6. *For any finite set S as above, the unramified Brauer group ${}_2\mathrm{Br}(K)_V$ is finite and of order dividing*

$$2^{|S|-t} \cdot |{}_2\,\mathrm{Cl}_S(k)|^2 \cdot |U_S(k)/U_S(k)^2|^2,$$

where $t = c+1$ and c is the number of complex places of k, and $\mathrm{Cl}_S(k)$ and $U_S(k)$ are the class group and the group of units of the ring $\mathscr{O}_k(S)$ of S-integers, respectively.

Example 4.7. Consider the elliptic curve E over $\mathbb{Q}$ given by $y^2 = x^3 - x$. We have $\delta = 4$, so $S = \{\infty, 2\}$. Furthermore,

$$|S| - t = 1, \qquad \mathrm{Cl}_S(\mathbb{Q}) = 1, \quad \text{and} \quad U_S(\mathbb{Q}) = \{\pm 1\} \times \mathbb{Z}.$$

Then by Theorem 4.6 and (10) we have $|\mathbf{gen}(D)| \leqslant 2 \cdot 4^2 = 32$ for any quaternion division algebra D over $K = \mathbb{Q}(E)$.

5. Some other notions of the genus

In this section we would like to mention several variants of the notion of the genus of a division algebra that come up quite naturally and have proven to be useful.

First, we have the *local genus*, which already appeared implicitly in the proof of Lemma 4.2. Let K be a field and V a set of discrete valuations of K. For a central division algebra D of degree n over K, we define the *local genus* $\mathbf{gen}_V(D)$ of D with respect to V as the collection of classes $[D'] \in \mathrm{Br}(K)$ with D' a central division algebra of degree n over K such that for every $v \in V$ the following holds: if we write

$$D \otimes_K K_v = M_\ell(\mathscr{D}) \quad \text{and} \quad D' \otimes_K K_v = M_{\ell'}(\mathscr{D}'),$$

where $\mathscr{D}$ and $\mathscr{D}'$ are central division algebras over the completion K_v, then $\ell = \ell'$ and $\mathscr{D}$ and $\mathscr{D}'$ have the same maximal separable subfields. Lemma 2.3 of [40] (see also [20], Lemma 3.1) now shows that for any set V of discrete valuations of K, we have

$$\mathbf{gen}(D) \subset \mathbf{gen}_V(D).$$

Next, suppose that V is a set of discrete valuations satisfying the conditions (I) and (II) in § 4 for some integer $n > 1$. Then the argument that we sketched in § 4 shows that if *n is prime to* $\mathrm{char}\, K$ *and the unramified Brauer group* ${}_n\mathrm{Br}(K)_V$ *is finite, then* $\mathbf{gen}_V(D)$ *is finite for any central division algebra D of degree n over K.* Indeed, the proof of Lemma 4.2 implies that the characters $\chi_v = \rho_v([D])$ and $\chi'_v = \rho_v([D'])$ for $v \in V$ have the same kernel if $[D'] \in \mathbf{gen}_V(D)$. This remark eventually leads to the bound (10), with $\mathbf{gen}_V(D)$ in place of $\mathbf{gen}(D)$.

Further, as we already mentioned in Remark 3.7, (b), Krashen and McKinnie [27] have studied division algebras having the same finite-dimensional *splitting* fields. For this purpose, given a finite-dimensional central division algebra D over a field K, one defines $\mathbf{gen}'(D)$ as the collection of classes $[D'] \in \mathrm{Br}(K)$ with the property that a finite field extension L/K splits D if and only if it splits D' (the notation in [27] is slightly different). Note that we clearly have

$$\mathbf{gen}'(D) \subset \mathbf{gen}(D).$$

The main result of [27] is that if p is a prime different from $\mathrm{char}\, K$ and

$$|\mathbf{gen}'(\Delta)| < \infty$$

for all $[\Delta] \in {}_p\mathrm{Br}(K)$, then

$$|\mathbf{gen}'(D)| < \infty$$

for all $[D] \in {}_p\mathrm{Br}(K(t))$ (a similar result for $\mathbf{gen}(D)$ was obtained in [5], but technically neither result is a consequence of the other).

The third notion that we would like to discuss briefly is the so-called *one-sided* (or *asymmetric*) *genus* introduced in [26]. Following [26], we will write

$$D \leqslant D'$$

for two central division algebras D and D' of the same degree over a field K if any maximal subfield P/K of D admits a K-embedding $P \hookrightarrow D'$. For a division algebra D of degree n over K, we define the one-sided genus $\mathbf{gen}^1(D)$ to be the collection of classes $[D'] \in \mathrm{Br}(K)$, where D' is a central division algebra of degree n such that $D \leqslant D'$. We refer the reader to [26] for a detailed treatment of this notion, and only highlight here the difference in the ramification properties arising in the analysis of the two-sided and one-sided versions of the genus.

By Lemma 4.2, if $[D'] \in \mathbf{gen}(D)$, then for a discrete valuation v of K the algebra D ramifies at v if and only if D' does (assuming that the residue map ρ_v is defined). For the one-sided genus, the situation is more complicated. If K is a number field, then, as noted prior to sketching the proof of Proposition 3.1, the relation $D \leqslant D'$ implies that $R(D') \subset R(D)$ for the corresponding sets of ramified places. On the other hand, let $K = \mathbb{R}((x))$ with the standard discrete valuation v, and consider the quaternion division K-algebras

$$D_1 = \left(\frac{-1,-1}{K}\right) \quad \text{and} \quad D_2 = \left(\frac{-1,x}{K}\right).$$

Then D_1 is unramified at v with residue algebra isomorphic to the usual algebra of Hamiltonian quaternions $\mathbb{H}$ over $\mathbb{R}$, while D_2 is ramified at v. Note that any quadratic subfield L of D_1 must be unramified, and since the residue field $\overline{K}_v = \mathbb{R}$ has $\mathbb{C}$ as its only non-trivial finite extension, we conclude that L is isomorphic to $K(i) = \mathbb{C}((x))$ (where $i^2 = -1$). Since $\mathbb{C}((x))$ is contained in D_2, we see that any maximal subfield of D_1 can be embedded into D_2. This means that $D_1 \leqslant D_2$, hence $[D_2] \in \mathbf{gen}^1(D_1)$. Nevertheless, D_1 is unramified at v while D_2 is ramified.

The results of [26] show that this construction provides essentially the main instance where such ramification behavior appears. To give precise statements, we need to introduce some terminology and fix notations. In [26] a finite-dimensional central division algebra D over a field K is said to be *varied* if there is no non-trivial cyclic extension P/K contained isomorphically in every maximal subfield of D (note that it suffices to check this property for P/K of prime degree). For example, it is known that if K is a field that is finitely generated over a global field, then any central division algebra D over K is varied (see [26], Theorem 1). For a central division algebra D over a field K that is complete with respect to a discrete valuation, we will denote by $\overline{D}$ the residue algebra and by $\overline{E}_D$ the centre of $\overline{D}$; we will also let E_D be the unique unramified subfield of D with residue field $\overline{E}_D$ (in [26] the latter is referred to as the *ramification field* of D).

Proposition 5.1 ([26], Proposition 4). *Let K be a field that is complete with respect to a discrete valuation v, and let D and D' be central division algebras over K of the same degree such that $D \leqslant D'$. If $\overline{D}/\overline{E}_D$ is varied, then $E_D = E_{D'}$.*

To relate this result to the above discussion of ramification, suppose that there exists a residue map

$$\rho_v \colon {}_n\mathrm{Br}(K) \to \mathrm{Hom}(\mathscr{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}),$$

where $n = \deg D$. As we mentioned earlier, it is well known that if $\chi_v = \rho_v([D])$, then $\overline{E}_D$ is precisely the subfield of the separable closure of $\overline{K}_v$ corresponding to $\ker \chi_v$. Thus, Proposition 5.1 asserts that under the assumption that $\overline{D}/\overline{E}_D$ is varied, the conclusion of Lemma 4.2 holds already if $D \leqslant D'$, and in particular,

D and D' are simultaneously either ramified or unramified at v. Of course, the above example of quaternion algebras over $K = \mathbb{R}((x))$ shows that this assumption cannot be omitted—note that $E_{D_1} = \mathbb{R}((x))$, while $E_{D_2} = \mathbb{C}((x))$. On the other hand, it turns out that all situations where a division algebra is not varied are in some sense of this nature.

Proposition 5.2 ([26], § 3). (a) *Suppose that D is a finite-dimensional central division algebra over a field K that is not varied. Then K is a Pythagorean field and*

$$D = \left(\frac{-1,-1}{K}\right) \otimes_K D'$$

for some finite-dimensional central division K-algebra D'.

(b) *Suppose that K is a field that is complete with respect to a discrete valuation and let D, D' be central division algebras over K of the same degree such that $D \leqslant D'$. Then $E_D E_{D'} \subseteq E_D(\sqrt{-1}\,)$. If $E_D \subsetneq E_{D'} = E_D(\sqrt{-1}\,)$, then the degree $[E_D : K]$ is odd and $K = F((x))$ for some Pythagorean field F.*

(Recall that a field F is said to be *Pythagorean* if every sum of two squares in F is a square.)

6. The genus of an algebraic group

We would like to conclude this article with a brief overview of ongoing work whose goal is to extend the techniques and results developed in the context of division algebras (which we have outlined in §§ 3–5) to absolutely almost simple algebraic groups of all types. In this case the notion of division algebras having the same maximal subfields is replaced with the notion of algebraic groups having the same maximal tori. More precisely, let G_1 and G_2 be absolutely almost simple algebraic groups defined over a field K. We say that G_1 and G_2 have the same *K-isomorphism* (respectively, *K-isogeny*) *classes of maximal K-tori* if every maximal K-torus T_1 of G_1 is K-isomorphic (respectively, K-isogenous) to some maximal K-torus T_2 of G_2, and vice versa. Furthermore, let G be an algebraic K-group and K^{sep} a separable closure of K. We recall that an algebraic K-group G' is called a K-form (or, more precisely, a K^{sep}/K-form) of G if G and G' become isomorphic over K^{sep} (see, for instance, [45], Chap. III, § 1 or [35], Chap. II, § 2). For example, for any central division algebra D of degree n over K, there exists a K^{sep}-isomorphism $D \otimes_K K^{\mathrm{sep}} \simeq M_n(K^{\mathrm{sep}})$, which means that the algebraic K-group $G = \mathrm{SL}_{1,D}$ associated with the group of elements in D having reduced norm 1 is a K-form of SL_n.

Definition 6.1. Let G be an absolutely almost simple algebraic group over a field K. The (K-)*genus* $\mathbf{gen}_K(G)$ (or simply $\mathbf{gen}(G)$ if there is no risk of confusion) of G is the set of K-isomorphism classes of K-forms G' of G that have the same K-isomorphism classes of maximal K-tori as G.

We should point out that for a finite-dimensional central division K-algebra D, only maximal *separable* subfields of D give rise to maximal K-tori of the corresponding group $G = \mathrm{SL}_{1,D}$. Thus, in hindsight, to make the definition of $\mathbf{gen}(D)$ consistent with that of $\mathbf{gen}(G)$, one should probably reformulate the former in terms of maximal *separable* subfields. This change would not affect the results that were discussed in §§ 3 and 4, since these dealt with the case where the degree of the algebra is prime to char K, but its potential impact on the general case has not yet been investigated. We also observe that while we will be interested primarily in simple algebraic groups with the same *isomorphism* classes of maximal tori, the analysis of weakly commensurable Zariski-dense subgroups, which is related to geometric applications (see [36]), sometimes requires one to consider simple groups with the same *isogeny* classes of maximal tori.

As in the case of division algebras, the focus of our current work is on the following two questions.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element*?

(This means that among K-groups of the same type, G is defined up to a K-isomorphism by the isomorphism classes of its maximal K-tori.)

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite*?

At this point, only the case of absolutely almost simple algebraic groups over number fields has been considered in full.

Theorem 6.2 (cf. [36], Theorem 7.5). (a) *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a number field K, and let L_i be the smallest Galois extension of K over which G_i becomes an inner form of a split group. If G_1 and G_2 have the same K-isogeny classes of maximal K-tori, then either G_1 and G_2 are of the same Killing–Cartan type, or one of them is of type B_n and the other is of type C_n $(n \geqslant 3)$, and moreover, $L_1 = L_2$.*

(b) *Fix an absolutely almost simple K-group G. Then the set of isomorphism classes of all absolutely almost simple K-groups G' with the same K-isogeny classes of maximal K-tori as G is finite.*

(c) *Fix an absolutely almost simple simply connected K-group G whose Killing–Cartan type is different from A_n, D_{2n+1} $(n > 1)$, and E_6. Then any K-form G' of G (in other words, any absolutely almost simple simply connected K-group G' of the same type as G) that has the same K-isogeny classes of maximal K-tori as G is K-isomorphic to G.*

Regarding the types excluded in (c), the construction in [36], § 9, shows that they are honest exceptions, that is, for each of those types one can construct non-isomorphic absolutely almost simple simply connected K-groups G_1 and G_2 of this type over a number field K that have the same isomorphism classes of maximal K-tori. The case where G_1 and G_2 are of types B_n and C_n, respectively, has been analyzed fully in [19].

We now observe that the investigation of $\mathbf{gen}(G)$ presents additional challenges even for groups of the form $G = \mathrm{SL}_{m,D}$, where D is a central division algebra of degree n (we recall that G is a simply connected inner form of type A_ℓ with $\ell = mn - 1$, and that all inner forms of this type are obtained in this fashion; see [35], Proposition 2.17). The reason is that while every maximal K-torus of such a group G is a norm 1 torus

$$\mathrm{R}^{(1)}_{E/K}(G_m) = \mathrm{R}_{E/K}(G_m) \cap G$$

for some étale subalgebra E of $M_m(D)$, where G_m is the 1-dimensional split torus and $\mathrm{R}_{E/K}$ is the functor of restriction of scalars, the fact that two such tori $\mathrm{R}^{(1)}_{E_1/K}(G_m)$ and $\mathrm{R}^{(1)}_{E_2/K}(G_m)$ are K-isomorphic as algebraic groups does not in general imply that the algebras E_1 and E_2 are isomorphic, even when these algebras are field extensions of K. This makes it rather difficult to relate $\mathbf{gen}(G)$ and $\mathbf{gen}(D)$ and apply the results of §§ 3–4 directly. Nevertheless, with some extra work one can prove the following theorem, which parallels Theorems 3.3 and 3.6 for division algebras.

Theorem 6.3 (cf. [5], Theorem 5.3). (a) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \dots, x_r)$, where k is either a number field or a finite field of characteristic $\neq 2$. Then for any $m \geqslant 1$, the genus of $G = \mathrm{SL}_{m,D}$ reduces to a single element.*

(b) *Let G be an absolutely almost simple simply connected algebraic group of inner type A_ℓ over a finitely generated field K whose characteristic is either zero or does not divide $\ell + 1$. Then $\mathbf{gen}(G)$ is finite.*

The additional input that is needed to prove Theorem 6.3 involves so-called *generic tori*. Since these are becoming increasingly useful in a variety of contexts, we will quickly recall here some relevant definitions and results.

Let G be a semisimple algebraic group over a field K. Fix a maximal K-torus T of G and let $\Phi = \Phi(G, T)$ denote the corresponding root system. Furthermore, let K_T be the minimal splitting field of T and $\Theta_T = \mathrm{Gal}(K_T/K)$ its Galois group. Then the natural action of Θ_T on the character group $X(T)$ gives rise to an injective group homomorphism

$$\theta_T \colon \Theta_T \to \mathrm{Aut}(\Phi),$$

and we say that T is *generic* over K if the image of θ_T contains the Weyl group $W(\Phi) = W(G, T)$. For example, for $G = \mathrm{SL}_{m,D}$ as above, a maximal K-torus $T = \mathrm{R}^{(1)}_{E/K}(G_m)$ is generic if and only if E is a (separable) field extension of K of degree mn and the Galois group of its normal closure is the symmetric group S_{mn}.

The following result shows that when K is finitely generated, one can always find a generic K-torus with prescribed local properties.

Proposition 6.4 (cf. [37], Corollary 3.2). *Let G be an absolutely almost simple algebraic group over a finitely generated field K. For a given discrete valuation v of K and a maximal K_v-torus T_v of G, there exists a maximal K-torus T of G which is generic over K and is conjugate to T_v by an element of $G(K_v)$.*

The second result that we would like to mention is a rigidity property for isomorphisms between generic tori. More precisely, let G_1 and G_2 be two simply connected inner K-forms of type A_ℓ, and let T_i be a generic maximal K-torus of G_i for $i = 1, 2$. Then any K-isomorphism $\varphi \colon T_1 \to T_2$ extends to a K^{sep}-isomorphism $\widetilde{\varphi} \colon G_1 \to G_2$. (This is a consequence of the so-called *Isogeny Theorem*; see Theorem 4.2 and Remark 4.4 in [36] and Theorem 9.8 in [38].) It follows that for *generic* $T_i = R^{(1)}_{E_i/K}(G_m)$ as above, the existence of a K-isomorphism of tori $T_1 \to T_2$ *does imply* the existence of a K-isomorphism of algebras $E_1 \to E_2$. We are now in a position to discuss the proof of the last theorem.

Sketch of the proof of Theorem 6.3. Assume that K is a finitely generated field and let $G' \in \mathbf{gen}_K(G)$. One first shows that G' is an inner form over K, and in fact $G' = \mathrm{SL}_{m,D'}$ for some central division K-algebra D' of degree n. Using the results on generic tori outlined above, one proves that for any discrete valuation v of K the algebras $D \otimes_K K_v$ and $D' \otimes_K K_v$ have the same isomorphism classes of maximal étale subalgebras. This means that for any set V of discrete valuations of K the class $[D']$ lies in the local genus $\mathbf{gen}_V(D)$. On the other hand, according to Theorem 4.3, for a finitely generated K of characteristic prime to n, there exists a set V of discrete valuations of K satisfying conditions (I) and (II) and such that ${}_n\mathrm{Br}(K)_V$ is finite. As we pointed out in § 5, this implies finiteness of the local genus $\mathbf{gen}_V(D)$, which completes the proof of part (b) of Theorem 3.6. To prove part (a), we write $K = \ell(x_1)$, where $\ell = k(x_2, \ldots, x_r)$, and we let V be the set of discrete valuations of K that are trivial on ℓ. Then as in the proof of Theorem 3.3, the fact that $[D'] \in \mathbf{gen}_V(D)$ in conjunction with Faddeev's exact sequence implies that

$$[D] = [D'][\Delta \otimes_\ell K] \quad \text{in } \mathrm{Br}(K)$$

for some central division algebra Δ over ℓ. Finally, to prove that Δ is trivial, we pick a place $v_0 \in V$ of degree 1 so that both D and D' are unramified at v_0, and we write

$$D \otimes_K K_{v_0} = M_s(\mathscr{D}) \quad \text{and} \quad D' \otimes_K K_{v_0} = M_{s'}(\mathscr{D}'),$$

with $\mathscr{D}$ and $\mathscr{D}'$ central division K_{v_0}-algebras. Then $s = s'$, and $\mathscr{D}$ and $\mathscr{D}'$ have the same maximal (separable) subfields. It follows that the residue algebras $\overline{\mathscr{D}}$ and $\overline{\mathscr{D}'}$ are central division ℓ-algebras having the same maximal subfields. Since $\overline{\mathscr{D}}$ has exponent 2, it has trivial genus by Theorem 3.3. It follows that $[\overline{\mathscr{D}}] = [\overline{\mathscr{D}'}]$, and hence $[\Delta]$ is trivial. Thus, $D \simeq D'$, and consequently $G \simeq G'$.

Remark 6.5. The nature of the argument that we have just sketched suggests that it makes sense to consider an alternative definition of $\mathbf{gen}_K(G)$ over a finitely generated field K in terms of generic maximal K-tori.

On the basis of the finiteness result for the genus of an inner form of type A_ℓ over a finitely generated field (Theorem 6.3, (b)), it is natural to propose the following.

Conjecture 6.6. *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field K of characteristic 0 or of good*[4] *characteristic relative to G. Then $\mathbf{gen}_K(G)$ is finite.*

The proof of Theorem 6.3 indicates that to approach this conjecture, one needs to extend the techniques based on ramification and the analysis of unramified division algebras to absolutely almost simple groups of all types. An adequate replacement of the notion of an unramified central division algebra is the notion of a group with *good reduction.* Suppose that G is an absolutely almost simple algebraic group over a field K. One says that G has good reduction at a discrete valuation v of K

[4]For each type, the following characteristics are defined to be *bad*: for type A_ℓ, all primes dividing $(\ell+1)$ and also $p = 2$ for outer forms; for types B_ℓ, C_ℓ, and D_ℓ, $p = 2$ and also $p = 3$ for ${}^{3,6}\mathsf{D}_4$; for type E_6, $p = 2, 3, 5$; for types E_7 and E_8, $p = 2, 3, 5, 7$; for types F_4 and G_2, $p = 2, 3$. All other characteristics for a given type are *good*.

if there exists a reductive group scheme[5] $\mathscr{G}$ over the valuation ring $\mathscr{O}_v \subset K_v$ whose generic fibre $\mathscr{G} \otimes_{\mathscr{O}_v} K_v$ is isomorphic to $G \otimes_K K_v$. We then let $\underline{G}^{(v)}$ denote the reduction $\mathscr{G} \otimes_{\mathscr{O}_v} \overline{K}_v$. The following result extends Lemma 4.2 to simple algebraic groups of all types.

Theorem 6.7 [7]. *Let G be an absolutely almost simple simply connected group over a field K, and let v be a discrete valuation of K. Assume that the residue field $\overline{K}_v$ is finitely generated and that G has good reduction at v. Then any $G' \in \mathbf{gen}_K(G)$ also has good reduction at v. Furthermore, the reduction $\underline{G'}^{(v)}$ lies in the genus $\mathbf{gen}_{\overline{K}_v}(\underline{G}^{(v)})$.*

(We should point out that the proof of this result again makes use of generic tori.)

Assume now that the field K is equipped with a set V of discrete valuations that satisfies condition (I) of § 4 and also the following condition:

(III) for any $v \in V$ the residue field $\overline{K}_v$ is finitely generated.

Corollary 6.8. *If K and V satisfy conditions* (I) *and* (III), *then for any absolutely almost simple simply connected algebraic K-group G, there exists a finite subset $V_0 \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus V_0$.*

It follows from Corollary 6.8 that in order to prove Conjecture 6.6, it would suffice to show that every finitely generated field K can be equipped with a set V of discrete valuations satisfying conditions (I) and (III) and having the following property:

(Φ) For any absolutely almost simple algebraic K-group G such that $\operatorname{char} K$ is good for G and any finite subset $V_0 \subset V$, the set of K-isomorphism classes of (inner) K-forms G' of G having good reduction at all $v \in V \setminus V_0$ is finite.

(Obviously, in this formulation one can assume G to be quasi-split over K.) One expects that *divisorial* sets of valuations that appeared in the discussion of Theorem 4.3 (that is, discrete valuations of K arising from the prime divisors on an appropriate regular arithmetic scheme X with function field K) will also work for general algebraic groups.

Conjecture 6.9. *Any divisorial set V of discrete valuations of a finitely generated field K satisfies the property* (Φ).

Over a number field K, the assertion of Conjecture 6.9 is an easy consequence of the finiteness result for Galois cohomology (see [45], Chap. III, § 4.6), since a semisimple group over a finite extension of $\mathbb{Q}_p$ that has good reduction is necessarily quasi-split (cf. [35], Theorem 6.7). (Interestingly, there are non-split groups over $\mathbb{Q}$ that have good reduction at *all* primes (see [22], [11]), but there are no abelian varieties over $\mathbb{Q}$ with smooth reduction everywhere [16].) Furthermore, the finiteness of ${}_n\mathrm{Br}(K)_V$ implies Conjecture 6.9 for inner forms of type A_ℓ. We also have the following conditional results for spinor groups.

[5]Let R be a commutative ring and $S = \operatorname{Spec} R$. Recall that a *reductive R-group scheme* is a smooth affine group scheme $G \to S$ such that the geometric fibres $G_{\overline{s}}$ are connected reductive algebraic groups (see [14], Exp. XIX, Definition 2.7 or [12], Definition 3.1.1).

Let $\mu_2 = \{\pm 1\}$. Then for any discrete valuation v of K such that $\mathrm{char}\,\overline{K}_v \neq 2$ and for any $i \geqslant 1$, one can define a residue map in Galois cohomology

$$\rho_v^i \colon H^i(K, \mu_2) \to H^{i-1}(\overline{K}_v, \mu_2)$$

extending the map (6) introduced in § 4 (see, for instance, [8], § 3.3 or [21], § 6.8 for the details). Then for any set V of discrete valuations of K one defines the unramified part $H^i(K, \mu_2)_V$ to be $\bigcap_{v \in V} \ker \rho_v^i$ (of course, $H^2(K, \mu_2)_V = {}_2\mathrm{Br}(K)_V$).

Theorem 6.10 ([7]). *Let K and V be as in Conjecture 6.9. Assume that for any finite set $V_0 \subset V$, the unramified cohomology groups $H^i(K, \mu_2)_{V \setminus V_0}$ are finite for all $i \geqslant 1$. Then for any $n \geqslant 5$ and any finite subset $V_0 \subset V$, the set of K-isomorphism classes of the spinor groups $\mathrm{Spin}_n(q)$, where q is a non-degenerate n-dimensional quadratic form, that have good reduction at all $v \in V \setminus V_0$ is finite.*

It is important to point out that the range of potential consequences of Conjecture 6.9 goes beyond the finiteness of the genus (Conjecture 6.6) and includes, for example, the Finiteness Conjecture for weakly commensurable Zariski-dense subgroups (see [39], Conjecture 6.1) as well the finiteness of the Tate–Shafarevich set in certain situations. More precisely, let G be an absolutely almost simple simply connected algebraic group over a field K of good characteristic, and let V be a divisorial set of discrete valuations of K. Denote by

$$Ш(\overline{G}) := \ker\left(H^1(K, \overline{G}) \to \prod_{v \in V} H^1(K_v, \overline{G}) \right)$$

the Tate-Shafarevich set for the corresponding adjoint group $\overline{G}$. We can pick a finite subset $V_0 \subset V$ such that G has good reduction at all $v \in V \setminus V_0$. Suppose that $\xi \in Ш(\overline{G})$ and let $G' = {}_\xi G$ be the corresponding twisted group. By our assumption, $G' \simeq G$ over K_v for all $v \in V$, and consequently G' has good reduction at all $v \in V \setminus V_0$. Assuming Conjecture 6.9, we can now conclude that the groups ${}_\xi G$ for $\xi \in Ш(\overline{G})$ form finitely many K-isomorphism classes; in other words, the image of $Ш(\overline{G})$ under the canonical map

$$H^1(K, \overline{G}) \xrightarrow{\lambda} H^1(K, \mathrm{Aut}\, G)$$

is finite. But since $\overline{G} \simeq \mathrm{Int}\, G$ has finite index in $\mathrm{Aut}\, G$, the map λ has finite fibres, which would give the finiteness of $Ш(\overline{G})$.

We note that Theorem 6.7 can be used not only to investigate the finiteness of the genus, but also to prove that in some situations the genus is trivial. For example, we have the following.

Theorem 6.11 ([7]). *Let G be a simple group of type G_2 over a field of rational functions $K = k(x)$, where k is a global field of characteristic $\neq 2$. Then $\mathbf{gen}_K(G)$ consists of a single element.*

We also note the following stability result.

Theorem 6.12 ([7]). *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field k of characteristic zero. Then for the field $k(x)$ of rational functions, every $G' \in \mathbf{gen}_{k(x)}(G \otimes_k k(x))$ is of the form $H \otimes_k k(x)$ with $H \in \mathbf{gen}_k(G)$.*

Combining this theorem with Theorem 6.2, we conclude that if G is an absolutely almost simple simply connected algebraic group of type different from A_ℓ ($\ell > 1$), $\mathsf{D}_{2\ell+1}$ ($\ell > 1$), and E_6 over a number field k, then $\mathbf{gen}_{k(x)}(G \otimes_k k(x))$ consists of a single element.

We began the article by mentioning the result of Amitsur [1] on finite-dimensional central division algebras having the same splitting fields, and explaining the additional features one encounters if one considers only finite-dimensional splitting fields or just maximal subfields; this eventually led to our definition of the genus of a division algebra and, later, of a simple algebraic group. We would now like to conclude with a different notion of the genus, which is also based on the consideration of maximal étale subalgebras or maximal tori, but at the same time incorporates the availability of infinite-dimensional splitting fields, which was the key in Amitsur's theorem. This (more functorial) notion was proposed by A. S. Merkurjev. One defines the *motivic genus* $\mathbf{gen}_m(G)$ of an absolutely almost simple algebraic K-group G as the set of K-isomorphism classes of K-forms G' of G that have the same isomorphism classes of maximal tori not only over K but also over any field extension F/K. Then Amitsur's theorem implies that for $G = \mathrm{SL}_{1,D}$ the motivic genus is always finite, and it reduces to one element for D of exponent 2. Furthermore, according to a result of Izhboldin [23], for given non-degenerate quadratic forms q and q' of *odd* dimension n over a field K of characteristic $\neq 2$ the condition

($*$) *q and q' have the same Witt index over any extension F/K*

implies that q and q' are scalar multiples of each other (this conclusion being false for even-dimensional forms). It follows that $|\mathbf{gen}_m(G)| = 1$ for $G = \mathrm{Spin}_n(q)$ with n odd. We note that condition ($*$) is equivalent to the fact that the motives of q and q' in the category of Chow motives are isomorphic (Vishik [49], and also Vishik [50], Theorem 4.18, and Karpenko [25]), which motivated the choice of terminology for this version of the genus. Other groups have not yet been investigated.

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Vladimir I. Chernousov
University of Alberta, Edmonton, Canada
E-mail: vladimir@ualberta.ca

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Andrei S. Rapinchuk
University of Virginia, Charlottesville, USA
E-mail: asr3x@virginia.edu

Igor A. Rapinchuk
Harvard University, Cambridge, USA
E-mail: rapinch@math.harvard.edu